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A Detector Geometry for Measuring Both the Gamma Flux and One Component of the Gamma Current

I. Introduction

As discussed in Sensor and Simulation Notes (SSN) IX and X, it is desirable for nuclear EMP purposes to measure not only the gamma-ray flux, γ, but also the gamma-ray current, $\vec{\gamma}$, since this latter parameter can be correlated with the Compton current density in the air. The previous notes on this subject have described possible detectors for measuring one or more vector components of the gamma current, one device using the Compton diode technique and the other using the SEMIRAD technique. These devices would be classed as insensitive detectors applicable in the higher ranges of interest. In this note we shall describe a possible detector geometry for measuring a vector component of the gamma current, and, if desired, the gamma flux. If sensitive detectors (such as fluors or semiconductors) are used in this geometry, then we may have a low level $\vec{\gamma}$ detector.

The first element of the detector geometry we are considering consists of a sphere of radius r_o constructed, with uniform density, of a material with γ-ray mean free path r_γ (normally a function of the γ-ray energy). Since we will consider only single scattering of the γ rays, i.e., exponential attenuation of the γ rays through this spherical attenuator, r_γ can be considered to be based on the total scattering cross section of γ rays in the material. We may be able to improve our results (based on the above assumption) to include the effects of multiple scattering of the γ-rays (or buildup) by calculating r_γ from an "effective" cross section (e.g., an energy loss cross section).

The second element of this detector geometry consists of an array of isotropic gamma detectors distributed over the surface of the spherical attenuator. The characteristic dimensions of the individual detectors are assumed to be small compared to r_o. By this assumption we are confining the individual detectors to a thin layer on the spherical surface. We also assume that there are sufficient numbers of these detectors so that we may think of detector density distribution functions or of sensitivity per unit area of the spherical surface. Alternatively we could have some sort of continuous detector on the spherical surface whose sensitivity per unit area we could vary. For small discrete detectors perhaps semiconductor detectors would be appropriate.[1]

Referring to figure 1A we see a cross section of such a detector. In order to measure a vector component of $\vec{\gamma}$, we weight the signal by the cosine of the angle, θ, between a gamma ray, $\gamma(\theta)$, and a given axis, say the z axis, by choosing some appropriate distribution function for the gamma detectors so that a factor of $\cos\theta$ will appear in the net signal output. This is obtained by summing and/or differencing the electrical outputs of the individual detectors on the spherical surface. From symmetry considerations we can establish some general characteristics of the required distribution of the individual detectors. First, the distribution should be symmetric about the axis determining the particular component

1. The possible use of semiconductor detectors for this application was suggested by Mr. James Doyle of EG&G, Santa Barbara.

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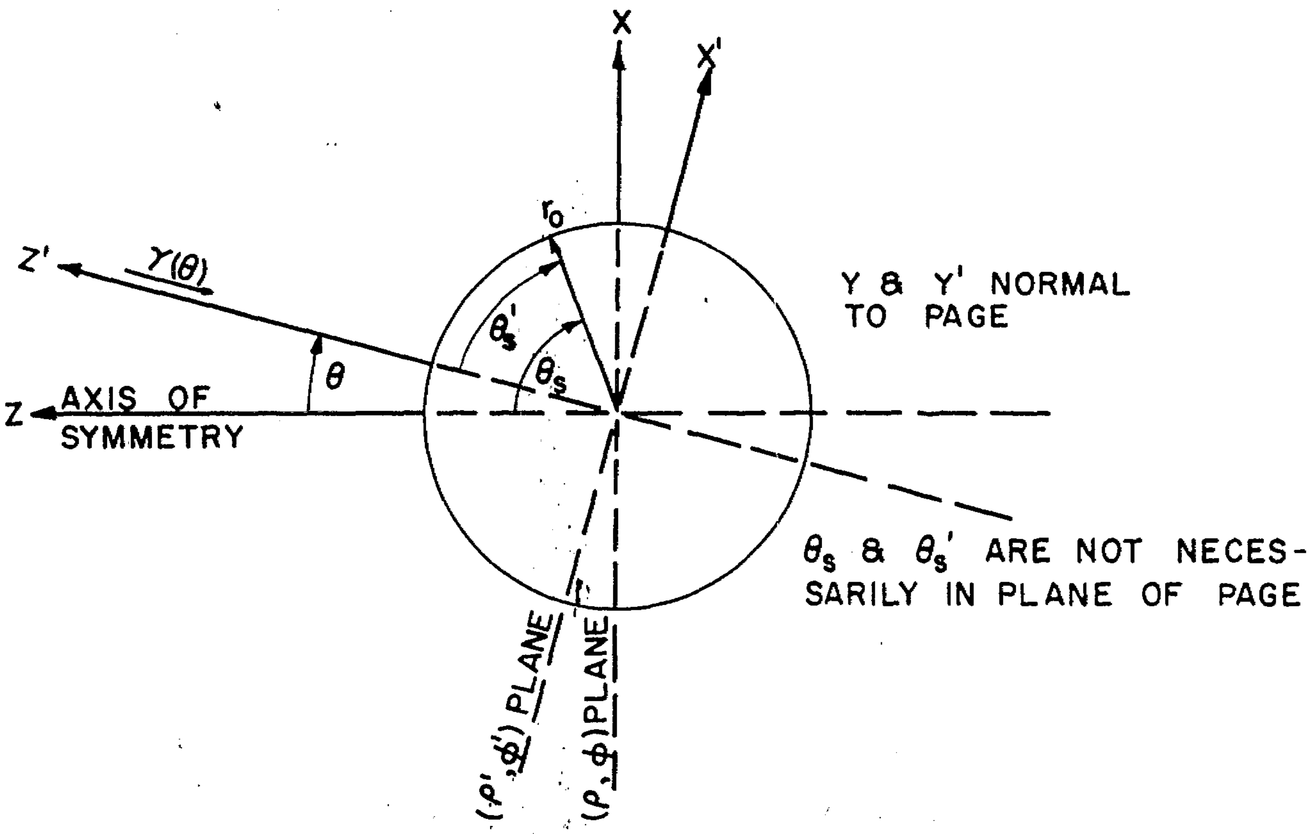
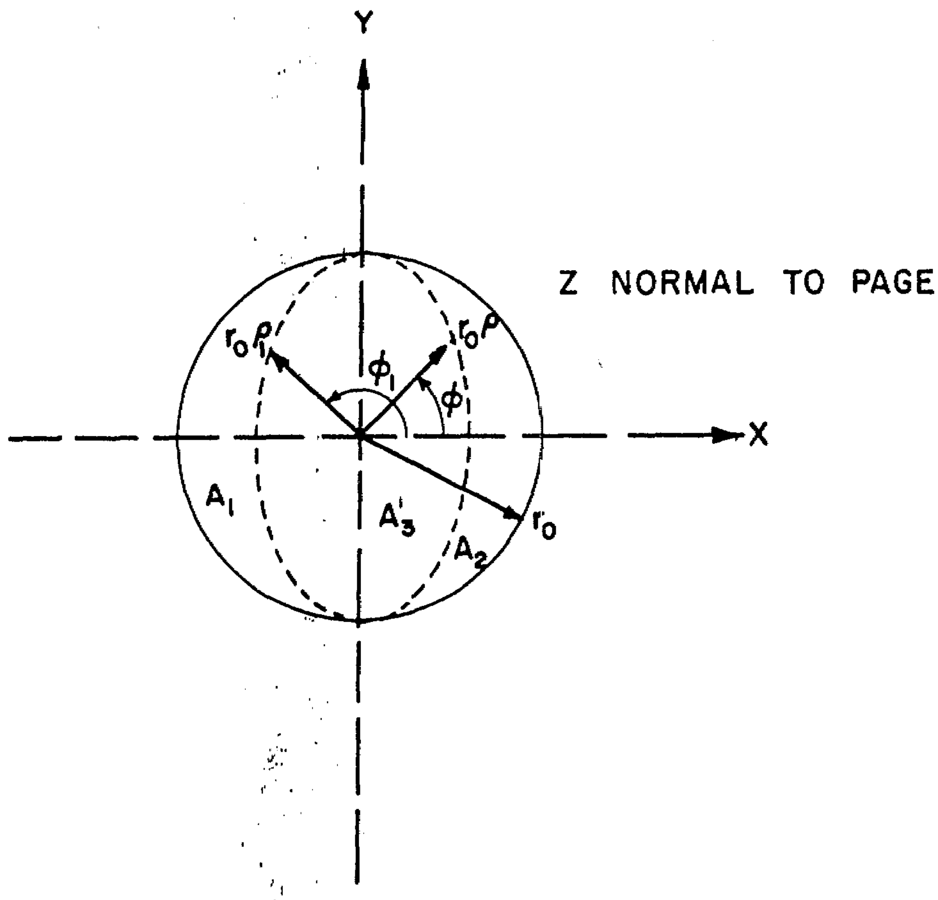


A. DETECTOR CROSS SECTION



B. (ρ,ϕ) PLANE

Figure 1. COS θ DETECTOR GEOMETRY

of $\vec{\gamma}$ which we wish to measure (the z axis in this case). Second, since cos θ, the desired angular dependence of the electrical output, is antisymmetric about $\theta = \pi/2$, the detector distribution must be effectively antisymmetric about the (x,y) plane (also labeled the (ρ,ϕ) plane). This can be accomplished by using detectors with oppositive (positive and negative) sensitivities, in opposite positions with respect to the (x,y) plane, and electrically adding together (in parallel) all the detector outputs. An alternate method is to make the detector distribution symmetric about the (x,y) plane but electrically add the detector outputs for positive or negative z separately and then subtract the two resulting signals. This last method has the advantage that the two resulting signals can also be electrically added in an attempt to measure the gamma flux (i.e., hopefully the response may be independent of θ for the sum of these two signals), and therefore, in this note we will follow the latter method. However, any results for the differential signal using this method can be directly converted to the case of an antisymmetric detector distribution and a single electrical signal by simply reversing the sign of the sensitivity of the detectors for either positive or negative z (but not both).

The ideal case for this detector, as for the Compton diode in SSN IX, is for $r_o \gg r_\gamma$, i.e., we assume that no gamma rays can penetrate the spherical attenuator. With this assumption we can then use a geometrical construction to find a detector distribution which will satisfy our requirement of having the differential output proportional to cos θ and which will, in addition, make the common mode output independent of θ. Having used this simplified case to obtain a detector distribution function we can then investigate the effects of a finite r_o/r_γ assuming exponential attenuation of the gamma rays through the spherical attenuator. These results can also be compared to the analogous results for the Compton diode of SSN IX.

II. Thick Spherical Attenuator

Consider the case in which we assume that $\gamma(\theta)$ does not penetrate through the spherical attenuator, i.e., $r_o \gg r_\gamma$. Referring to figure 1 again we can find an appropriate detector distribution for our purposes. Suppose we look at the sphere from a vantage point on the positive z axis. As indicated we assume that unidirectional gamma rays, $\gamma(\theta)$, are uniformly incident on the sphere at an angle θ with the z axis. The x axis is then (arbitrarily) defined so that the direction of the gamma rays is parallel to the (x,z) plane.

If, for conceptual purposes, we replace the gamma rays by a plane light source from the same direction we would see, as indicated in figure 1B, that only part of the sphere corresponding to regions A_2 and A_3 was illuminated. Of course, what we are "seeing" is not a spherical surface but a plane surface, or better, the projection on a plane surface which we can take to be the (x,y) or (ρ,ϕ) plane as indicated. Even though we do not see all of the illuminated portion of the sphere, we can still mentally project this area onto the (ρ,ϕ) plane. If we project the boundary of the illuminated spherical surface (which defines the (x',y') or (ρ',ϕ') plane) onto the (ρ,ϕ) plane we obtain an ellipse with semimajor axis r_o, semiminor axis $r_o \cos\theta$, and area $\pi r_o^2 \cos\theta$. This ellipse forms the boundary of region A_3. If we can make the electrical output of the detector array proportional to the area of this ellipse we will have the desired result, i.e., an output proportional to cos θ.

Suppose then we choose our detector distribution function so that the number of detectors or sensitivity per unit area as projected on the (ρ,ϕ) plane is a constant. Since we have required that the detector distribution be symmetric with respect to the (ρ,ϕ) plane then the detector distribution as projected on the (ρ,ϕ) plane is the same for the projections from each hemisphere. Considering only the uniform projected distributions on the (ρ,ϕ) plane we can now determine the θ dependence of the sensitivity of the detector array.

For region A_2 in the (ρ,ϕ) plane we have two identical sets of detectors, one for positive and one for negative z, which give identical outputs because both sets (consisting of isotropic detectors) are illuminated by the γ rays. When we difference the outputs of the detectors on the two hemispherical surfaces, the signals corresponding to region A_2 will cancel leaving only the signal corresponding to region A_3. Since the density of detectors as projected onto region A_3 is a constant and the output of each of the detectors is the same, then the differential signal will be proportional to the area of region A_3 and, therefore, proportional to cos θ as desired.

Since we now have a detector distribution which gives us the desired differential output let us look at the sum of the outputs from the detectors on the two hemispheres. In this case, the signals from the two sets of detectors projected onto region A_2 do not cancel. Instead, the double signal compensates for the lack of a signal from region A_1 because the projected detector distribution is the same in both regions. Thus, the sum of the signals from the two hemispheres is independent of θ. Therefore, with a relatively simple detector distribution function we can measure both one vector component of $\vec{\gamma}$, in this case γ_z, and the gamma flux, γ. The units in which these are measured depend on the manner in which the sensitivity of the individual detectors depends on the energy of the gamma rays.

For use in subsequent sections of this note we will now put the results of the previous analysis in a more formal mathematical form. Consider a normalized density function, $\eta(\rho)$, which gives the detector distribution as projected on the (ρ,ϕ) plane with normalization condition that the average value of $\eta(\rho)$ be unity. Thus

$$\pi r_o^2 = \int_0^{2\pi} \int_0^1 \eta(\rho) \quad r_o^2 \rho d\rho d\phi \tag{1}$$

or

$$1 = 2 \int_0^1 \eta(\rho)\rho d\rho \tag{2}$$

Note that $\eta(\rho)$ is independent of ϕ because of the symmetry of the detector distribution about the z axis. If from our previous analysis we constrain $\eta(\rho)$ to be a constant, independent of ρ, then equation (2) is solved if

$$\eta(\rho) = 1 \tag{3}$$

For convenience now define two distribution functions:

$$\eta_1(\rho) = 1 \tag{4}$$

which applies for the detectors projected from positive z and

$$\eta_2(\rho) = -\ \eta_1(\rho) = -1 \tag{5}$$

which applies for the detectors projected from negative z. Thus, when considering the differential signal we can add the contributions of the two hemispheres using $\eta_1(\rho)$ and $\eta_2(\rho)$ directly because of their opposite signs. However, when considering the common mode signal we need only take the magnitudes of the η's and again add the contributions to obtain the correct results.

Next we can analytically describe the boundary which separates the three regions of interest on the (ρ,ϕ) plane since it is an ellipse with semimajor axis r_o and semiminor axis $r_o\cos\theta$. Thus, if (x_1, y_1) and (ρ_1, ϕ_1) describe the elliptical boundary

$$\left(\frac{x_1}{r_o\cos\theta}\right)^2 + \left(\frac{y_1}{r_o}\right)^2 = 1 \tag{6}$$

and

$$x_1 = r_o\rho_1\cos\phi_1 \tag{7}$$

$$y_1 = r_o\rho_1\ \sin\phi_1$$

then

$$\rho_1^{\ 2}\ \frac{\cos^2\phi_1}{\cos^2\theta} + \rho_1^{\ 2}\sin^2\phi_1 = 1 \tag{8}$$

or

$$\rho_1^{\ 2}\left[\frac{\cos^2\phi_1(1-\cos^2\theta)}{\cos^2\theta} + \cos^2\phi_1 + \sin^2\phi_1\right] = 1 \tag{9}$$

or

$$\rho_1 = \left[1 + \cos^2\phi_1\ \tan^2\theta\right]^{-1/2} \tag{10}$$

since ρ_1 is assumed positive.

If we have a sensitivity per unit area, S_A, (coulombs per square meter per unit of the incident gamma rays) as projected on the (ρ,ϕ) plane (i.e., the unit area is in the (ρ,ϕ) plane) we can calculate gamma sensitivities. Thus, for the differential signal, using standard convention,

$$S_{dif}(\theta) = \frac{S_A}{2}\left\{\int_{A_3}\eta_1(\rho)dA + \int_{A_2}\eta_1(\rho)dA + \int_{A_2}\eta_2(\rho)dA\right\} \tag{11}$$

Since from equation (5) η_2 and η_1 are of opposite sign the contributions from A_2 cancel. Thus

$$S_{dif}(\theta) = S_A\frac{r_o^2}{2}\int_0^{2\pi}\int_0^{(1+\cos^2\phi\tan^2\theta)^{-1/2}} \eta_1(\rho)\rho d\rho d\phi \tag{12}$$

For convenience we can define

$$S_{dif}(\theta) = S_A\pi\frac{r_o^2}{2}\ f_{dif}(\theta) \tag{13}$$

so that

$$f_{dif}(\theta) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{(1+\cos^2\phi\tan^2\theta)^{-1/2}} \eta_1(\rho)\rho d\rho d\phi \tag{14}$$

From equations (14) and (2) we can see that for $\theta = 0$

$$f_{dif}(0) = 1 \tag{15}$$

It is not until this point that we are constrained to choose a functional dependence for $\eta_1(\rho)$. Using the choice of equation (4) we have

$$f_{dif}(\theta) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{(1+\cos^2\phi\tan^2\theta)^{-1/2}} \rho d\rho d\phi \tag{16}$$

However, as discussed before, this double integral represents an ellipse of major axis 1.0 and minor axis $\cos\theta$. Thus

$$f_{dif}(\theta) = \frac{1}{\pi}(\pi\cos\theta) = \cos\theta \tag{17}$$

For the common mode sensitivity we have

$$S_{com}(\theta) = S_A\left\{\int_{A_3} \eta_1(\rho)\ dA + \int_{A_2}\eta_1(\rho)dA + \int_{A_2}(-\eta_2(\rho))dA\right\} \tag{18}$$

but again from equation (5)

$$S_{com}(\theta) = S_A\left\{\int_{A_3} \eta_1(\rho)dA + \int_{A_2}\eta_1(\rho)dA + \int_{A_2}\eta_1(\rho)dA\right\} \tag{19}$$

By rotating ϕ through π radians we can match A_2 point for point with A_1. Thus

$$S_{com}(\theta) = S_A\left\{\int_{A_3} \eta_1(\rho)dA + \int_{A_2}\eta_1(\rho)dA + \int_{A_1}\eta_1(\rho)dA\right\} \tag{20}$$

which combines into one integral over the whole circular area as

$$S_{com}(\theta) = S_A r_o^2\int_0^{2\pi}\int_0^{1} \eta_1(\rho)\rho d\rho d\phi \tag{21}$$

Defining

$$S_{com}(\theta) = S_A \pi r_o^2 f_{com}(\theta) \tag{22}$$

we have

$$f_{com}(\theta) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \eta_1(\rho)\rho d\rho d\phi = 2 \int_0^1 \eta_1(\rho)\rho d\rho \tag{23}$$

Thus from equation (2)

$$f_{com}(\theta) = 1 \tag{24}$$

Note that this last result did not depend on the functional form of $\eta_1(\rho)$ but only on the general symmetry conditions which we have required. We only need the particular functional dependence of equation (4) for the desired differential signal.

We have now solved for the necessary detector distribution function, but only assuming that $r_o >> r_\gamma$. Let us then use this detector distribution function and determine the effect of finite r_o/r_γ on $f_{dif}(\theta)$ and $f_{com}(\theta)$ which describe the performance of this detector geometry in nondimensional form. As expected, the analysis will be more complex as will be the results.

III. Transformation of Coordinates

To include the contribution of the gamma rays which penetrate the spherical attenuator to the response of the detector array we will assume an exponential attenuation of the gamma rays. To do this conveniently we will need to express the thickness of the spherical attenuator in the direction of the gamma rays. As illustrated in figure 1A, we define a new coordinate system, the (x', y', z') system, where the z' axis is parallel to the assumed direction of travel of the gamma rays. Thus, if we can transform the detector distribution function from the (x,y) or (ρ, ϕ) plane to the (x',y') or (ρ', ϕ') plane we can use the thickness of the spherical attenuator in the z' direction to compute the contribution (to the output of the detector array) of the gamma rays which pass through the attenuator.

To transform the projected detector distribution function from the (ρ, ϕ) plane to the (ρ', ϕ') plane let us determine the detector distribution function on the spherical surface. This basically involves transforming a unit area on the (ρ, ϕ) plane to the spherical surface. On the (ρ, ϕ) plane define an incremental area, da, (normalized) as

$$da = \rho d\rho d\phi \tag{25}$$

This area is the projection (along a line parallel to the z axis) of an incremental area, da_s, from the spherical surface of

$$da_s = \frac{\rho d\rho d\phi}{|\cos\theta_s|} \tag{26}$$

where θ_s is the polar angle from the z axis to a point on the spherical surface.

we only need the polar angle, θ_s, and the azimuthal angle, ϕ, to describe any point on the spherical surface.

Given the distribution functions, $\eta_1(\rho)$ and $\eta_2(\rho)$, on the (ρ,ϕ) plane (from equations (4) and (5)), the manner of transforming the incremental areas, and the fact that the surface area is increased a factor of two in going from the circle in the (ρ,ϕ) plane to either of the hemispherical surfaces, we have a detector distribution function, η_s, for the spherical surface as

$$\eta_s(\theta_s) = \begin{cases} 2\ \eta_1(\rho)\ \dfrac{da}{da_s} = 2\cos\theta_s & \left(\text{for } 0 \leq \theta_s \leq \frac{\pi}{2}\right) \\[2ex] 2\ \eta_2(\rho)\ \dfrac{da}{da_s} = 2\ (-1)(-\cos\theta_s) = 2\cos\theta_s & \left(\text{for } \frac{\pi}{2} \leq \theta_s \leq \pi\right) \end{cases} \tag{27}$$

However, we should recognize that if we are using the differencing scheme for the detectors on the two hemispheres the distribution function is always positive, but note that due to the present mathematical manner of defining $\eta_1(\rho)$ and $\eta_2(\rho)$, η_s is given by one simple analytic function over the range $0 \leq \theta_s \leq \pi$. Conveniently, the average value of η_s is 1.0 over the hemisphere corresponding to positive z and -1.0 over the hemisphere corresponding to negative z, i.e.,

$$1 = \int_0^{\pi/2} \eta_s(\theta_s)\sin\theta_s d\theta_s = -\int_{\pi/2}^{\pi} \eta_s(\theta_s)\sin\theta_s d\theta_s \tag{28}$$

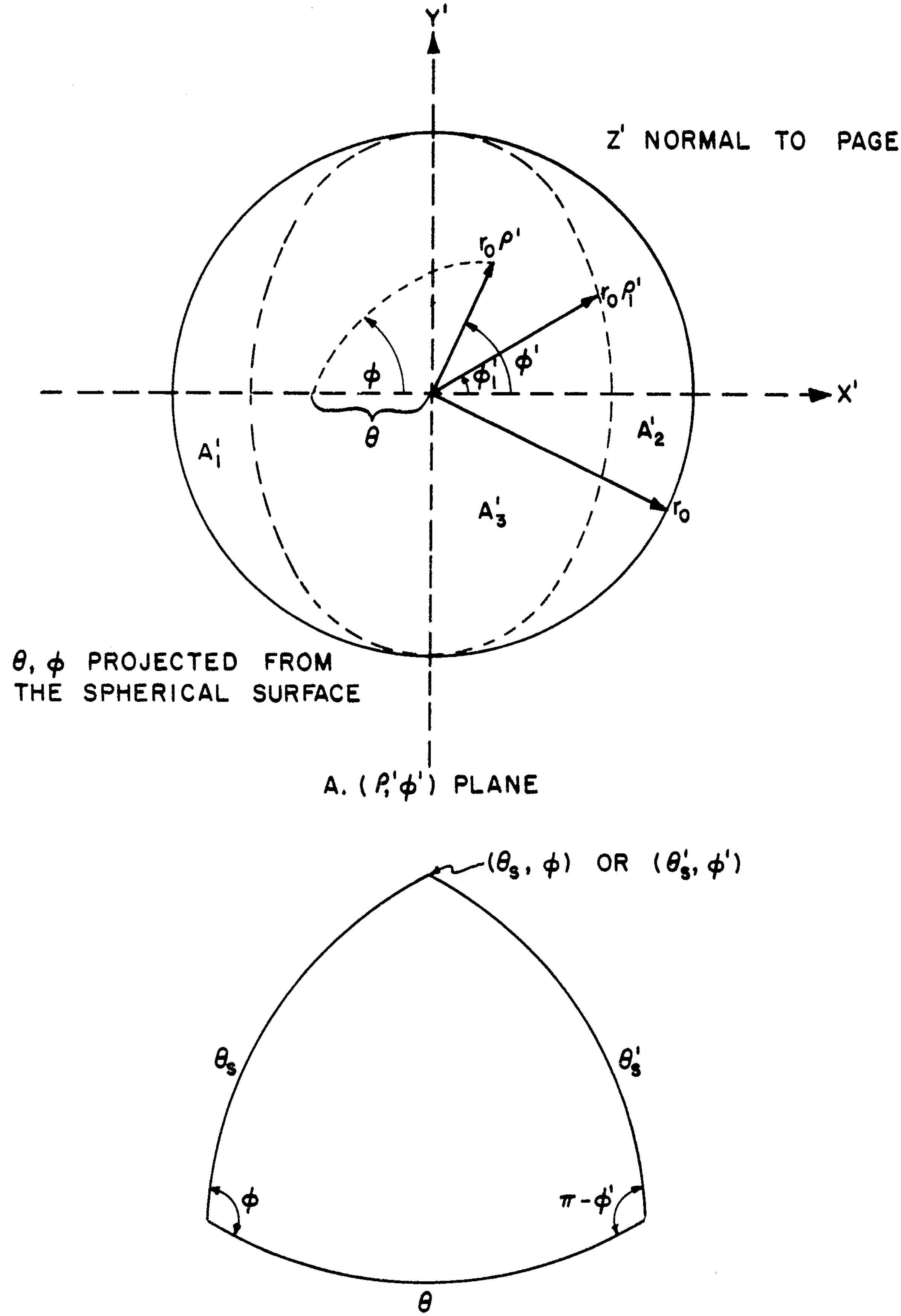
Now that we have transformed the detector distribution function to (θ_s,ϕ) coordinates on the spherical surface, let us transform the distribution function to (θ_s',ϕ') coordinates where θ_s' is the polar angle from the z' axis (as in figure 1A) and ϕ' is the azimuthal angle in the (ρ',ϕ') plane (as in figure 2A). As illustrated in figure 2B, a point (θ_s,ϕ) or (θ_s',ϕ') on the spherical surface determines a geometric construction from which θ_s (contained in η_s) can be related to θ_s' and ϕ'. Each of the angles θ, θ_s, and θ_s' are great circular arcs on the spherical surface and each of the angles ϕ and $\pi-\phi'$ are angles between pairs of the great circular arcs. Figure 2A also gives a projection of this geometric construction on the (ρ',ϕ') plane. From figure 2B, using the law of cosines for an oblique spherical triangle[2], we have

$$\cos\theta_s = \cos\theta_s' \cos\theta + \sin\theta_s' \sin\theta \cos(\pi-\phi') \tag{29}$$

or

$$\cos\theta_s = \cos\theta_s' \cos\theta - \sin\theta_s' \sin\theta \cos\phi' \tag{30}$$

2. Burington, Mathematical Tables and Formulas, Third Edition, (1958), p. 23.



B. COORDINATE TRANSFORMATION ON SPHERICAL SURFACE

Figure 2. COORDINATE TRANSFORMATION

Thus, equation (27) can be changed to give $\eta_s'(\theta_s',\phi')$ in the new spherical coordinates as

$$\eta_s'(\theta_s',\phi') = 2\left\{\cos\theta_s' \cos\theta - \sin\theta_s' \sin\theta \cos\phi'\right\} \tag{31}$$

Note that the detector distribution function in the new spherical coordinates is no longer independent of the azimuthal angle, ϕ'.

Now let us project the detector distribution function as in equation (31) onto the (ρ',ϕ') plane. The incremental area, da', in the (ρ',ϕ') plane is

$$da' = \rho' d\rho' d\phi' \tag{32}$$

which is a projection (along a line parallel to the z' axis) of an incremental area, da_s', from the spherical surface of

$$da_s' = \frac{\rho' d\rho' d\phi'}{|\cos\theta_s'|} \tag{33}$$

Including the factor of two in the areas between the hemispherical surface and the circular surface (as before) we then have

$$\eta_1'(\rho',\phi') = \frac{\eta_s'(\theta_s',\phi')}{2} \frac{da_s'}{da'} = \frac{\eta_s'(\theta_s',\phi')}{2\cos\theta_s'} \tag{34}$$

(for $0 \le \theta_s' \le \pi/2$)

and

$$\eta_2'(\rho',\phi') = \frac{\eta_s'(\theta_s',\phi')}{2} \frac{da_s'}{da} = -\frac{\eta_s'(\theta_s',\phi')}{2\cos\theta_s'} \tag{35}$$

(for $\pi/2 \le \theta_s' \le \pi$)

Thus

$$\eta_1'(\rho',\phi') = \cos\theta - \sin\theta \frac{\sin\theta_s'}{\cos\theta_s'} \cos\phi' \tag{36}$$

(for $0 \le \theta_s' \le \pi/2$)

and

$$\eta_2'(\rho',\phi') = -\cos\theta + \sin\theta \frac{\sin\theta_s'}{\cos\theta_s'} \cos\phi' \tag{37}$$

(for $\pi/2 \le \theta_s' \le \pi$)

Now we can remove θ_s' from the previous equations by

$$\sin\theta_s' = \rho' \tag{38}$$

and

$$\cos\theta_s' = \begin{cases} +\sqrt{1-\rho'^2} & (\text{for } 0 \le \theta_s' \le \pi/2) \\ -\sqrt{1-\rho'^2} & (\text{for } \pi/2 \le \theta_s' \le \pi) \end{cases} \tag{39}$$

Thus

$$\eta_1'(\rho',\phi') = \cos\theta - \sin\theta \frac{\rho'}{\sqrt{1-\rho'^2}} \cos\phi' \quad (\text{for } 0 \le \theta_s' \le \pi/2) \tag{40}$$

and

$$\eta_2'(\rho',\phi') = -\cos\theta - \sin\theta \frac{\rho'}{\sqrt{1-\rho'^2}} \cos\phi' \quad (\text{for } \pi/2 \le \theta_s' \le \pi) \tag{41}$$

Now that we have transformed the detector distribution to the (ρ',ϕ') plane we can also find, for later use, the projection on the (ρ',ϕ') plane of the intersection of the spherical surface with the (ρ,ϕ) plane. By symmetry one can obtain this curve by changing the variables in equation (10). Thus

$$\rho_1' = [1 + \cos^2\phi_1' \tan^2\theta]^{-1/2} \tag{42}$$

as illustrated in figure 2A.

This result can also be obtained by setting either η_1' or η_2' equal to zero. In other words, when we calculate the common mode signal and need to take the absolute value of η_1' and η_2' (as discussed in Section II) equation (42) will be needed to break up the integral over the (ρ',ϕ') plane so that signs can be placed to make the contributions always positive. For the differential signal we will not need equation (42) for a boundary because η_1' and η_2' can be both positive and negative.

IV. General Spherical Attenuator

With the detector distribution function in the (ρ',ϕ') coordinates we can calculate $f_{dif}(\theta)$ and $f_{com}(\theta)$ by appropriate integrals. For positive z' all detectors directly "see" the gamma rays. However, for negative z' the gamma rays are attenuated by an amount determined by the thickness of the attenuator in the direction of the gamma rays. Corresponding to a (ρ',ϕ') the spherical thickness, $\Delta z'$, is

$$\Delta z' = 2\, r_o \sqrt{1-\rho'^2} \tag{43}$$

The shadowed detectors will then give a contribution, to the output of the detector array, which is weighted by a factor $e^{-\frac{\Delta z'}{r_\gamma}}$.

A. Differential Signal.

Using the definition of equation (13) for $f_{dif}(\theta)$ but with the coordinates of equation (14) shifted to the (ρ',ϕ') plane, and including the contribution for finite r_o/r_γ, we have

$$f_{dif}(\theta) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \eta_1'(\rho',\phi')\rho' d\rho' d\phi' + \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \eta_2'(\rho',\phi') \, e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} \rho' d\rho' d\phi' \tag{44}$$

If we had not defined our detector distribution function so that the differential signal is obtained by adding all the contributions, instead of subtracting them, we would have had more terms to equation (44) because of the necessity of splitting up the region of integration in the (ρ',ϕ') plane. (This would have also made the integration limits more complex.) Using equations (40) and (41) and rearranging terms then

$$f_{dif}(\theta) = \cos\theta \, \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left(1-e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}}\right)\rho' d\rho' d\phi' - \sin\theta \, \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left(1+e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}}\right) \frac{\rho'^2}{\sqrt{1-\rho'^2}} \cos\phi' \, d\rho' d\phi' \tag{45}$$

If in the second double integral we integrate first over ϕ' the only term dependent on ϕ' (i.e., $\cos\phi'$) will make the integral zero because of the limits of 0 and 2π. Thus,

$$f_{dif}(\theta) = \cos\theta \, \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left(1-e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}}\right) \rho' d\rho' d\phi' \tag{46}$$

This is an important result in that under our assumption of first order scattering the differential signal still maintains the desired $\cos\theta$ dependence.

Let us then rewrite equation (46) as

$$f_{dif}(\theta) = f_{dif}(0) \cos\theta \tag{47}$$

where

$$f_{dif}(0) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left(1-e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}}\right) \rho' d\rho' d\phi' \tag{48}$$

This is solved in SSN IX (equations (16) and (19) through (27)) except in that note it describes f_{com} for a Compton diode which we might call $f_{com_{C.D.}}$. Thus,

$$f_{dif}(0) = 1 - \frac{1}{2}\left(\frac{r_\gamma}{r_o}\right)^2 + \left(\frac{r_\gamma}{r_o} + \frac{1}{2}\left(\frac{r_\gamma}{r_o}\right)^2\right) e^{-2\frac{r_o}{r_\gamma}} \tag{49}$$

Asymptotic expressions for this result were also derived in SSN IX. Thus, for

$\frac{r_o}{r_\gamma} \gg 1$ we have

$$f_{dif}(0) \simeq 1 - \frac{1}{2}\left(\frac{r_\gamma}{r_o}\right)^2 \tag{50}$$

and for $\frac{r_o}{r_\gamma} \ll 1$

$$f_{dif}(0) \simeq \frac{4}{3}\frac{r_o}{r_\gamma} \tag{51}$$

For comparison with the results for the differential signal of the Compton diode we obtain from equation (55) of SSN IX

$$f_{dif_{C.D.}}(0) = 1 - \frac{7}{2}\left(\frac{r_\gamma}{r_o}\right)^2 + 4\left(\frac{r_\gamma}{r_o} + \left(\frac{r_\gamma}{r_o}\right)^2\right)e^{-\frac{r_o}{r_\gamma}} - \left(\frac{r_\gamma}{r_o} + \frac{1}{2}\left(\frac{r_\gamma}{r_o}\right)^2\right)e^{-2\frac{r_o}{r_\gamma}} \tag{52}$$

The results of equations (49) and (52) are plotted in figure 3.

B. Common Mode Signal.

Using the definition of equation (22) and the absolute values of η_1' and η_2' from equations (40) and (41) we can now compute $f_{com}(\theta)$. However, we will have to break up the surface integrals in the (ρ',ϕ') plane into the three regions illustrated in figure 2A because η_1' and η_2' change sign, in some cases, between these regions. Remember that the boundary line between these regions is a projection of the great circle (on the spherical surface) along which the detector distribution function is zero. In the following analysis we will assume that $0 \le \theta \le \pi/2$ so that we know the sign of the distribution functions. The results apply for $0 \le \theta \le \pi$ if we use the symmetry relation

$$f_{com}(\pi-\theta) = f_{com}(\theta) \tag{53}$$

which arises from the symmetry of the detector distribution function on the sphere, i.e., from equation (27)

$$|\eta_s(\pi-\theta_s)| = |\eta_s(\theta_s)| \tag{54}$$

Therefore, we can set up the expression for the common mode angular dependence as

$$\begin{aligned} f_{com}(\theta) = {} & \frac{1}{\pi}\iint_{A_1'} \eta_1'(\rho',\phi')\rho'd\rho'd\phi' + \frac{1}{\pi}\iint_{A_2'}(-\eta_1'(\rho',\phi'))\rho'd\rho'd\phi' \\ & + \frac{1}{\pi}\iint_{A_3'} \eta_1'(\rho',\phi')\rho'd\rho'd\phi' + \frac{1}{\pi}\iint_{A_1'}\eta_2'(\rho',\phi')e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}}\rho'd\rho'd\phi' \\ & + \frac{1}{\pi}\iint_{A_2'}(-\eta_2'(\rho',\phi'))e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}}\rho'd\rho'd\phi' + \frac{1}{\pi}\iint_{A_3'}(-\eta_2'(\rho',\phi'))e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}}\rho'd\rho'd\phi' \end{aligned} \tag{55}$$

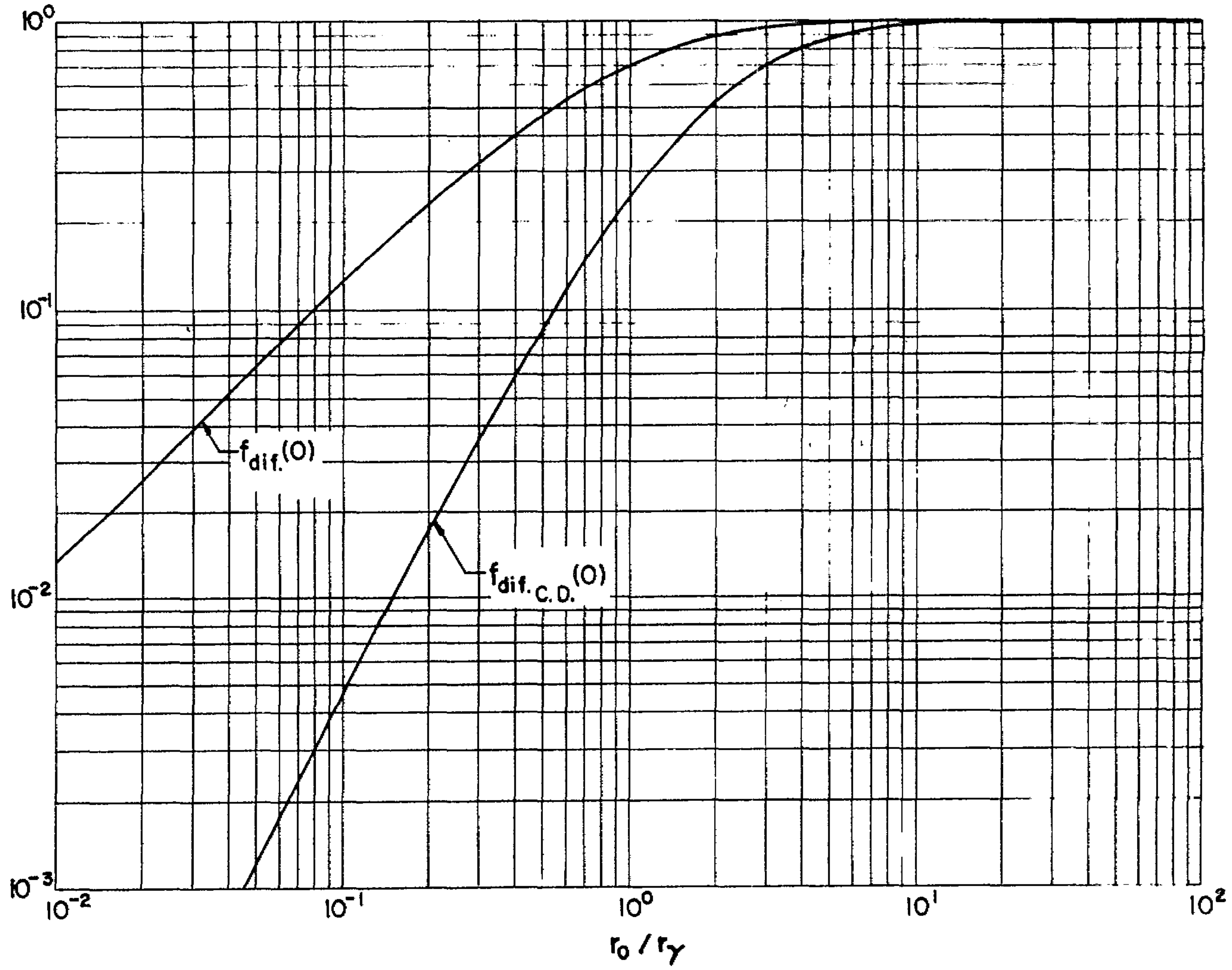


Figure 3. DIFFERENTIAL DETECTOR-ARRAY RESPONSE

For convenience define the first three of the above integrals as I_1 and the second three as I_2. Then

$$I_1 = \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{(1+\cos^2\phi'\tan^2\theta)^{-1/2}}^{1} [\cos\theta - \sin\theta \frac{\rho'}{\sqrt{1-\rho'^2}} \cos\phi'] \, \rho' d\rho' d\phi'$$

$$+ \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{(1+\cos^2\phi'\tan^2\theta)^{-1/2}}^{1} [-\cos\theta + \sin\theta \frac{\rho'}{\sqrt{1-\rho'^2}} \cos\phi'] \rho' d\rho' d\phi'$$

$$+ \frac{1}{\pi} \int_{0}^{2\pi} \int_{0}^{(1+\cos^2\phi'\tan^2\theta)^{-1/2}} [\cos\theta - \sin\theta \frac{\rho'}{\sqrt{1-\rho'^2}} \cos\phi'] \, \rho' d\rho' d\phi' \qquad (56)$$

Collecting similar terms in the integrands together and combining the integrals over the second terms in the integrands we have

$$I_1 = \frac{\cos\theta}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{(1+\cos^2\phi'\tan^2\theta)^{-1/2}}^{1} \rho' d\rho' d\phi' - \frac{\cos\theta}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{(1+\cos^2\phi'\tan^2\theta)^{-1/2}}^{1} \rho' d\rho' d\phi'$$

$$+ \frac{\cos\theta}{\pi} \int_{0}^{2\pi} \int_{0}^{(1+\cos^2\phi'\tan^2\theta)^{-1/2}} \rho' d\rho' d\phi' - \frac{\sin\theta}{\pi} \int_{0}^{2\pi} \int_{0}^{1} \frac{\rho'^2}{\sqrt{1-\rho'^2}} \cos\phi' \, d\rho' d\phi'$$

$$+ \frac{2\sin\theta}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{(1+\cos^2\phi'\tan^2\theta)^{-1/2}}^{1} \frac{\rho'^2}{\sqrt{1-\rho'^2}} \cos\phi' \, d\rho' d\phi' \qquad (57)$$

In this last representation the first two integrals cancel one another because by shifting ϕ' in one of the integrals by π, the form of the two integrals becomes the same. As before (equation (17)) the third integral (exclusive of coefficients) equals $\pi \cos\theta$. On integrating over ϕ' the fourth integral vanishes because of the symmetry of $\cos\phi'$. Thus

$$I_1 = \cos^2\theta + \sin\theta \, \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{(1+\cos^2\phi' \tan^2\theta)^{-1/2}}^{1} \frac{\rho'^2}{\sqrt{1-\rho'^2}} \cos\phi' \, d\rho' d\phi' \tag{58}$$

To solve this integral first divide the integral over ϕ' to give two equal integrals so that

$$I_1 = \cos^2\theta + \sin\theta \, \frac{4}{\pi} \int_{0}^{\frac{\pi}{2}} \int_{(1+\cos^2\phi' \tan^2\theta)^{-1/2}}^{1} \frac{\rho'^2}{\sqrt{1-\rho'^2}} \cos\phi' \, d\rho' d\phi' \tag{59}$$

Equation (42) for the boundary between regions in the (ρ', ϕ') plane can be rewritten as

$$\cos^2\phi_1' = \cot^2\theta \left[\frac{1}{\rho'^2} - 1\right] \tag{60}$$

or

$$\sin^2\phi_1' = 1 - \cot^2\theta \left[\frac{1}{\rho'^2} - 1\right]$$

$$= 1 - \cot^2\theta \left[\frac{1-\rho'^2}{\rho'^2}\right] \tag{61}$$

Then interchanging the order of integration

$$I_1 = \cos^2\theta + \sin\theta \, \frac{4}{\pi} \int_{\cos\theta}^{1} \int_{0}^{\arcsin\left[1-\cot^2\theta\left(\frac{1-\rho'^2}{\rho'^2}\right)\right]^{1/2}} \frac{\rho'^2}{\sqrt{1-\rho'^2}} \cos\phi' \, d\phi' d\rho'$$

$$= \cos^2\theta + \sin\theta \, \frac{4}{\pi} \int_{\cos\theta}^{1} \frac{\rho'^2}{\sqrt{1-\rho'^2}} \sin\phi' \Bigg|_{0}^{\arcsin\left[1-\cot^2\theta\left(\frac{1-\rho'^2}{\rho'^2}\right)\right]^{1/2}} d\rho'$$

$$= \cos^2\theta + \sin\theta \frac{4}{\pi} \int_{\cos\theta}^{1} \frac{\rho'^2}{\sqrt{1-\rho'^2}} \left[1-\cot^2\theta\left(\frac{1-\rho'^2}{\rho'^2}\right)\right]^{1/2} d\rho'$$

$$= \cos^2\theta + \sin\theta \frac{4}{\pi} \int_{\cos\theta}^{1} \frac{\rho'^2}{\sqrt{1-\rho'^2}} \left[\frac{\rho'^2(1+\cot^2\theta)-\cot^2\theta}{\rho'^2}\right]^{1/2} d\rho'$$

$$= \cos^2\theta + \frac{4}{\pi} \int_{\cos\theta}^{1} \frac{\rho'}{\sqrt{1-\rho'^2}} (\rho'^2-\cos^2\theta)^{1/2} d\rho' \qquad (62)$$

Letting

$$\psi = \sqrt{1-\rho'^2} \qquad (63)$$

and thus

$$\psi d\psi = -\rho' d\rho' \qquad (64)$$

we have

$$I_1 = \cos^2\theta + \frac{4}{\pi} \int_{0}^{\sin\theta} (\sin^2\theta-\psi^2)^{1/2}\, d\psi \qquad (65)$$

Next letting

$$\psi = \xi \sin\theta \qquad (66)$$

we have

$$I_1 = \cos^2\theta + \sin^2\theta \frac{4}{\pi} \int_{0}^{1} \sqrt{1-\xi^2}\, d\xi \qquad (67)$$

However, this last integral (neglecting coefficients) represents the area of a quadrant of a unit circle, i.e., $\pi/4$. Therefore

$$I_1 = \cos^2\theta + \sin^2\theta = 1 \qquad (68)$$

Since I_1 represents that part of $f_{com}(\theta)$ due to the unshielded detectors this result is not unexpected (as in equation (24)) and serves as a check on our distribution function transformations.

The second three integrals of equation (55) represent that part of $f_{com}(\theta)$ due to the shielded detectors. Combining these integrals we have

$$I_2 = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 [\cos\theta + \sin\theta \frac{\rho'}{\sqrt{1-\rho'^2}} \cos\phi'] \, e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} \rho' d\rho' d\phi'$$

$$- \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{(1+\cos^2\phi'\tan^2\theta)^{-1/2}}^{1} [\cos\theta + \sin\theta \frac{\rho'}{\sqrt{1-\rho'^2}} \cos\phi'] \, e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} \rho' d\rho' d\phi' \qquad (69)$$

In the first of these integrals the second term goes to zero when integrated over ϕ'. The limits on ϕ' in the second integral can be shifted by π for convenience, changing the sign of the second term in the integrand. Thus we are left with three basic integrals:

$$I_2 = \cos\theta \; \frac{1}{\pi}\int_0^{2\pi}\int_0^1 e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} \rho' d\rho' d\phi'$$

$$- \cos\theta \; \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{(1+\cos^2\phi'\tan^2\theta)^{-1/2}}^{1} e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} \rho' d\rho' d\phi'$$

$$+ \sin \; \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{(1+\cos^2\phi'\tan^2\theta)^{-1/2}}^{1} \frac{\rho'^2}{\sqrt{1-\rho'^2}} \cos\phi' \, e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} d\rho' d\phi' \qquad (70)$$

Define these terms, in sequence, as I_3, I_4, and I_5.

In evaluating $f_{dif}(0)$ in equations (48) and (49) we have already evaluated I_3. Thus

$$I_3 = \cos\theta \; \frac{1}{\pi}\int_0^{2\pi}\int_0^1 e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} \rho' d\rho' d\phi'$$

$$= \cos\theta \left[\frac{1}{2}\left(\frac{r_\gamma}{r_o}\right)^2 - \left(\frac{r_\gamma}{r_o} + \frac{1}{2}\left(\frac{r_\gamma}{r_o}\right)^2\right) e^{-2\frac{r_o}{r_\gamma}}\right] \qquad (71)$$

Next we have

$$I_4 = -\cos\theta \, \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{(1+\cos^2\phi'\tan^2\theta)^{-1/2}}^{1} e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} \rho' d\rho' d\phi' \tag{72}$$

Dividing the integral over ϕ' at $\phi' = 0$ to give two equal integrals, and interchanging the order of integration (using equation (60)) we have

$$I_4 = -\cos\theta \, \frac{4}{\pi} \int_{\cos\theta}^{1} \int_{0}^{\arccos\left(\cot\theta \frac{\sqrt{1-\rho'^2}}{\rho'}\right)} e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} \rho' d\phi' d\rho'$$

$$= -\cos\theta \, \frac{4}{\pi} \int_{\cos\theta}^{1} e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} \arccos\left(\cot\theta \frac{\sqrt{1-\rho'^2}}{\rho'}\right) \rho' d\rho' \tag{73}$$

Using the substitutions of equations (63) and (64)

$$I_4 = -\cos\theta \, \frac{4}{\pi} \int_{0}^{\sin\theta} e^{-2\frac{r_o}{r_\gamma}\psi} \arccos\left(\cot\theta \frac{\psi}{\sqrt{1-\psi^2}}\right) \psi d\psi \tag{74}$$

and using the substitution of equation (66)

$$I_4 = -\cos\theta \, \sin^2\theta \, \frac{4}{\pi} \int_{0}^{1} e^{-2\left(\frac{r_o}{r_\gamma}\sin\theta\right)\xi} \arccos\left(\frac{\cos\theta\,\xi}{\sqrt{1-\sin^2\theta\,\xi^2}}\right) \xi d\xi \tag{75}$$

This last form may or may not be the most convenient. However, it is similar in form (for comparison) to the next result.

Finally, we have

$$I_5 = \sin\theta \, \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{(1+\cos^2\phi'\tan^2\theta)^{-1/2}}^{1} \frac{\rho'^2}{\sqrt{1-\rho'^2}} \cos\phi' \, e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} d\rho' d\phi' \tag{76}$$

We can follow precisely the steps used in equations (59) through (62) in solving for I_1 because no integration is performed over ρ'. Thus

$$I_5 = \frac{4}{\pi} \int_{\cos\theta}^{1} \frac{\rho'}{\sqrt{1-\rho'^2}} (\rho'^2 - \cos^2\theta)^{1/2} e^{-2\frac{r_o}{r_\gamma}\sqrt{1-\rho'^2}} d\rho' \tag{77}$$

Using the substitutions of equations (63) and (64) we have

$$I_5 = \frac{4}{\pi} \int_0^{\sin\theta} (\sin^2\theta - \psi^2)^{1/2} e^{-2\frac{r_o}{r_\gamma}\psi} d\psi \tag{78}$$

Then using the substitution of equation (66) we have

$$I_5 = \sin^2\theta \frac{4}{\pi} \int_0^1 \sqrt{1-\xi^2}\, e^{-2(\frac{r_o}{r_\gamma}\sin\theta)\xi} d\xi \tag{79}$$

Combining the results of equations (68), (71), (75), and (79) we have the results for the common mode signal. Including the symmetry condition expressed in equation (53) we have

$$f_{com}(\theta) = 1 + |\cos\theta| \left[\frac{1}{2}\left(\frac{r_\gamma}{r_o}\right)^2 - \left(\frac{r_\gamma}{r_o} + \frac{1}{2}\left(\frac{r_\gamma}{r_o}\right)^2\right) e^{-2\frac{r_o}{r_\gamma}}\right]$$

$$-|\cos\theta| \sin^2\theta \frac{4}{\pi} \int_0^1 e^{-2(\frac{r_o}{r_\gamma}\sin\theta)\xi} \arccos\left(\frac{|\cos\theta|\, \xi}{\sqrt{1-\sin^2\theta\,\xi^2}}\right) \xi d\xi$$

$$+ \sin^2\theta \frac{4}{\pi} \int_0^1 \sqrt{1-\xi^2}\; e^{-2(\frac{r_o}{r_\gamma}\sin\theta)\xi} d\xi \tag{80}$$

This now applies over the region $0 \leq \theta \leq \pi$. Figures 4 and 5 illustrate the dependence of $f_{com}(\theta)$ on r_o/r_γ and θ, respectively. In figure 4 we include the result for the Compton diode (SSN IX, equations (26) and (27)) which is

$$f_{com_{C.D.}} = 1 - \frac{1}{2}\left(\frac{r_\gamma}{r_o}\right)^2 + \left(\frac{r_\gamma}{r_o} + \frac{1}{2}\left(\frac{r_\gamma}{r_o}\right)^2\right) e^{-2\frac{r_o}{r_\gamma}} \tag{81}$$

Note that this is precisely the result for $f_{dif}(0)$ in equation (49).

The results of equations (49) and (80) and figures 3, 4, and 5 can now be used to determine how large we should make r_o/r_γ if we wish, for example, to prevent a significant fraction of the differential or common mode signal from coming from "shielded" detectors.

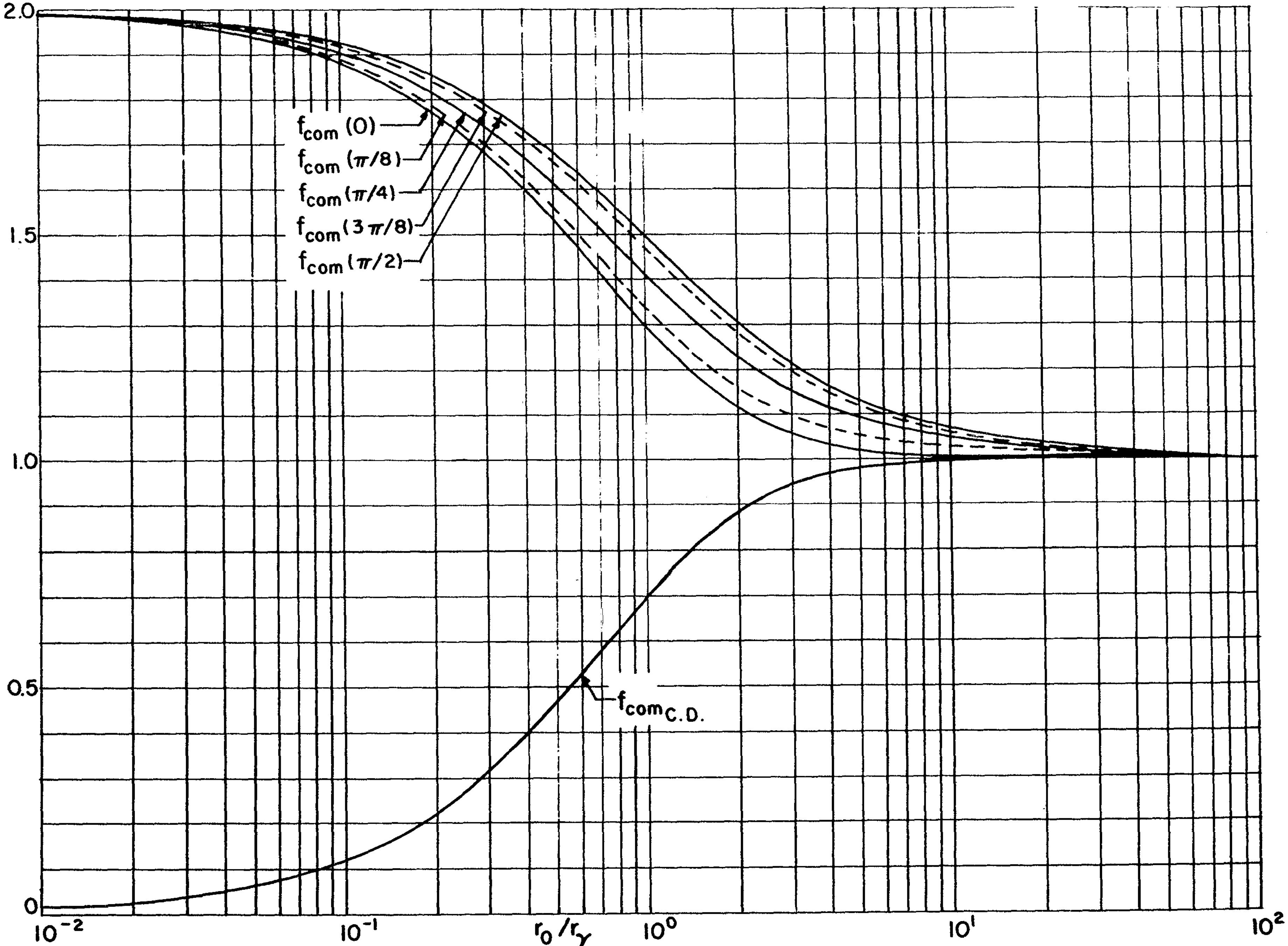
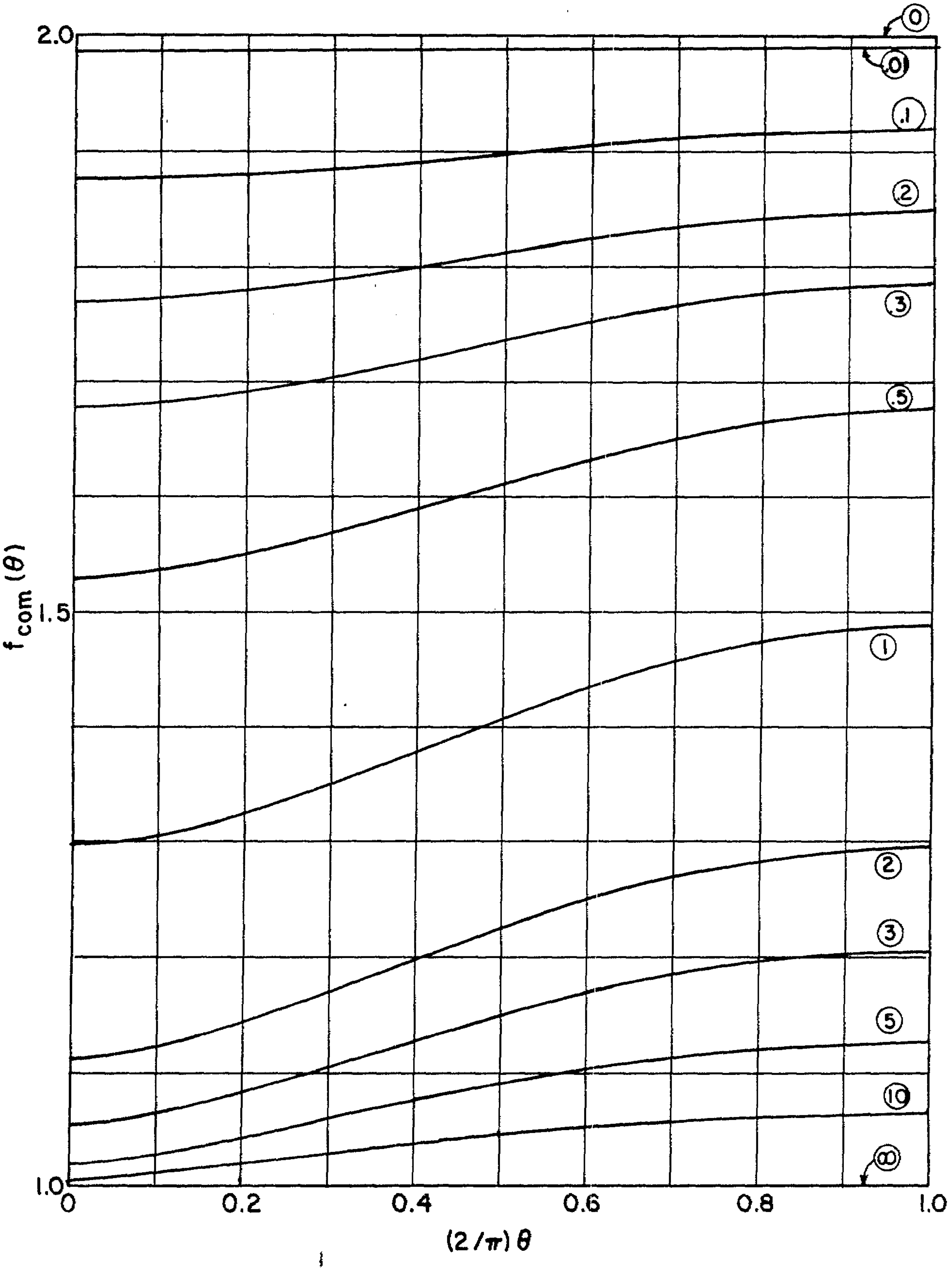


Figure 4. DEPENDENCE OF f_{com} ON r_0/r_γ



NOTE: ① DENOTES $r_o/r_\gamma = 1$, ETC.

Figure 5. DEPENDENCE OF f_{com} ON θ

V. Summary.

Based on the results of the previous sections we can conclude that by placing isotropic gamma detectors in a rather uncomplicated distribution around the surface of a spherical attenuator and by appropriately summing and differencing the outputs of the detectors, we can obtain signals proportional to both the gamma flux and one component of the gamma current. Assuming exponential attenuation of the gamma rays through the spherical attenuator, the differential signal maintains its $\cos\theta$ dependence on the direction of the incident gamma rays, independent of r_o/r_γ. However, the common mode signal is not independent of θ for all r_o/r_γ. To make this signal independent of θ we need either large or small r_o/r_γ (compared to one). In general it seems better to make $r_o/r_\gamma >> 1.0$, minimizing the signal from "shielded" detectors because r_o/r_γ is a function of the gamma-ray energy. Thus for large r_o/r_γ the differential signal is independent of this parameter (see equation (49)) eliminating an unwanted extra dependence of the differential signal on the gamma-ray energy.

The basic sensitivity of the detector array depends on the type of detectors used so there should be a fairly wide range of sensitivities possible. The dependence of the sensitivity of the detector array on the gamma-ray energy (assuming $r_o/r_\gamma >> 1.0$) is also governed by the choice of the individual detector elements.

Of course, there are other problems associated with making a practical device of this nature so we should exercise a general caution.

We would like to thank Mrs. Linda Crosby for her calculational assistance in the preparation of this note.

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