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Sensor and Simulation Notes

Note 160

December, 1972

Fields at the Center of a Full Circular TORUS  
and a Vertically Oriented TORUS  
on a Perfectly Conducting Earth

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Abstract

In this note a special toroidal coordinate system is introduced in order to derive general solutions for the electric and magnetic fields in space and on the toroidal surface. These solutions depend on the current distribution on the toroid and are in integral forms. Some Fourier expansion techniques have been used in order to simplify these integral equations. The surface current is found under the assumption that the thickness of the toroid is thin compared to the wavelength, which leads to an analytic solution for the fields at the center. The frequency response with a delta-gap source is shown and the transient behavior of the fields with a step-function excitation is also discussed. Some kinds of uniform loading impedances are used with the purpose of producing a plane-wave-like field at the center of the toroid.

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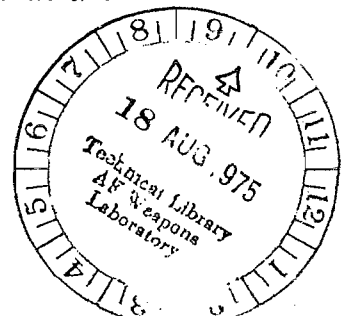
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## I. Introduction

There are two types of TORUS (Transient Omnidirectional Radiating Unidistant and Static) simulators. One of them is shown in figure 1 in a full toroid<sup>1</sup> configuration as appropriate for simulating an incident plane wave without reflections from the earth surface. Such a simulator will be held by some kind of dielectric stand (such as a trestle) high above the earth surface in order to reduce the effect of earth reflections. Another type of TORUS includes a ground reflection as part of the simulator.<sup>2</sup>

Figure 2 shows an example of this class. It consists of only a half toroid connected to a ground or water surface. The plane of the toroid can be angled with respect to the vertical plane. Both types of TORUS simulators would have some impedance per unit length  $Z'$  distributed around the antenna for wave shaping purposes in order to simulate the electromagnetic pulse (EMP) from a nuclear burst.

It has been suggested that a section of bicone<sup>3</sup> or a distributed source<sup>4</sup> be inserted in the toroid as a generator such that the field of radiation is a biconical wave at early time. The distributed source is assumed configured to reproduce the bicone electric field distribution or some other appropriate electric field distribution tangential to an appropriate surface, such as a circular cylinder.

In this note, we will consider the field from a full circular toroid with a pulse generator configured to initiate electromagnetic fields from

a position on the circumference. A half toroid on a vertical plane above a perfectly conducting earth, by the method of images, also forms a full toroid. Therefore, the derivations in this note are applicable to both types of simulators. It is assumed that the toroid has a mean major diameter of  $2a$  and a minor diameter of  $2b$ , and  $a^2 \gg b^2$  in order to simplify the analysis. The system under test lies roughly in the center of the toroid so that the distance of the system from the center of the generator is "a", and is independent of the orientation of the toroid and generator.

In the study of the frequency response of TORUS an impulse generator will be used for the purpose of mathematical convenience. However, an impulse generator is not appropriate for simulating the strong electromagnetic pulse of a nuclear burst. A high power generator may be achieved by charging a large capacitance to an initial voltage  $V_0$ . If the source decay time is sufficiently long it may be considered as a step function generator in the time range of interest. In any event, the step response will illustrate features of both early time and late time response.

The source, in this note, will be regarded as occupying a delta-gap only, and located at  $\phi = 0$ . For a vertically oriented half toroid, the total field at the test volume then is the superposition of the field radiation from the source and its image. Techniques developed in this note may be extended to more complicated sources, such as an approximation of a biconical source, in the future.



A toroidal coordinate system is convenient in performing calculations for fields on the surface of TORUS. However, rectangular and cylindrical coordinate systems have some advantage in calculating the fields in the test volume. So, some switching back and forth in the three coordinate systems will be done in order to simplify the problem. The relationships among the coordinate systems are given in the following chapter.

Very general solutions of the fields in space and on the toroid are derived in Chapter III. These solutions involve some integral equations which can be simplified by Fourier expansion techniques, discussed in Chapter IV. For uniform impedance loading the Fourier coefficients of the current are solved in Chapter V. The fields at the center of the loop are calculated using simpler techniques in Chapter VI. An important part of this chapter is the search for a loading impedance such that the ratio of E to H is that of free space in order to simulate plane waves at the center of the toroid. Extensive numerical results for the fields at the center with various impedances are shown in Chapter VII. Finally, a discussion is given on extensions of this work for future studies.

## II. Special Toroidal Coordinate System

A special right-handed coordinate system based on the center of the toroid's minor circle is defined as a toroidal coordinate system and will be represented by the symbol  $(\lambda, \xi, \phi)$ . Its relations to the rectangular  $(x, y, z)$  and cylindrical  $(\Psi, \phi, z)$  coordinate systems are shown in figure 3. A unit vector  $\vec{e}$  of a particular coordinate in terms of the others may be formulated, as well as the vector identities in terms of these three coordinate systems. The relation between the rectangular and cylindrical coordinate systems are well known,

$$\begin{aligned}x &= \Psi \cos(\phi) \\y &= \Psi \sin(\phi) \\ \vec{e}_x &= \vec{e}_\Psi \cos(\phi) - \vec{e}_\phi \sin(\phi) \\ \vec{e}_y &= \vec{e}_\Psi \sin(\phi) + \vec{e}_\phi \cos(\phi) \\ \Psi^2 &= x^2 + y^2 \\ \phi &= \arctan\left(\frac{y}{x}\right) \\ \vec{e}_\Psi &= \vec{e}_x \cos(\phi) + \vec{e}_y \sin(\phi) \\ \vec{e}_\phi &= -\vec{e}_x \sin(\phi) + \vec{e}_y \cos(\phi)\end{aligned}\tag{2.1}$$

where  $\vec{e}$  with a subscript is a unit vector in the appropriate coordinate direction.

The toroidal coordinates in terms of them are,

$$\lambda^2 = (\Psi - a)^2 + z^2 = \left( \sqrt{x^2 + y^2} - a \right)^2 + z^2$$

$$\tan(\xi) = \frac{-z}{\Psi - a} = \frac{-z}{\sqrt{x^2 + y^2} - a}$$

$$\vec{e}_\lambda = \vec{e}_\Psi \cos(\xi) - \vec{e}_z \sin(\xi) \quad (2.2)$$

$$\vec{e}_\xi = -\vec{e}_\Psi \sin(\xi) - \vec{e}_z \cos(\xi)$$

$$\vec{e}_\lambda = \vec{e}_x \cos(\xi) \cos(\phi) + \vec{e}_y \cos(\xi) \sin(\phi) - \vec{e}_z \sin(\xi)$$

$$\vec{e}_\xi = -\vec{e}_x \sin(\xi) \cos(\phi) - \vec{e}_y \sin(\xi) \sin(\phi) - \vec{e}_z \cos(\xi)$$

The other coordinate systems can be written in terms of toroidal coordinates as given below,

$$\Psi = a + \lambda \cos(\xi)$$

$$z = -\lambda \sin(\xi)$$

$$x = \left[ a + \lambda \cos(\xi) \right] \cos(\phi) \quad (2.3)$$

$$y = \left[ a + \lambda \cos(\xi) \right] \sin(\phi)$$

$$\vec{e}_\Psi = \vec{e}_\lambda \cos(\xi) - \vec{e}_\xi \sin(\xi)$$

$$\vec{e}_z = -\vec{e}_\lambda \sin(\xi) - \vec{e}_\xi \cos(\xi)$$

$$\vec{e}_x = \vec{e}_\lambda \cos(\phi) \cos(\xi) - \vec{e}_\xi \cos(\phi) \sin(\xi) - \vec{e}_\phi \sin(\phi)$$

$$\vec{e}_y = \vec{e}_\lambda \sin(\phi) \cos(\xi) - \vec{e}_\xi \sin(\phi) \sin(\xi) + \vec{e}_\phi \cos(\phi)$$

Note that the restriction  $\Psi \geq 0$  is made.

Using these formulas, one obtains the scalar product of two unit vectors as follows,

$$\vec{e}'_x \cdot \vec{e}_x = \vec{e}'_y \cdot \vec{e}_y = \vec{e}'_z \cdot \vec{e}_z = 1$$

$$(\vec{e}'_\Psi)_\Psi = \vec{e}'_\Psi \cdot \vec{e}_\Psi = \cos(\phi - \phi')$$

$$(\vec{e}'_\Psi)_\phi = \vec{e}'_\Psi \cdot \vec{e}_\phi = -\sin(\phi - \phi')$$

$$(\vec{e}'_\Psi)_z = 0$$

$$(\vec{e}'_\phi)_\Psi = \sin(\phi - \phi')$$

$$(\vec{e}'_\phi)_\phi = \cos(\phi - \phi')$$

(2.4)

$$(\vec{e}'_\phi)_z = 0$$

$$(\vec{e}'_\lambda)_\lambda = \cos(\xi) \cos(\xi') \cos(\phi - \phi') + \sin(\xi) \sin(\xi')$$

$$(\vec{e}'_\lambda)_\xi = -\cos(\xi') \sin(\xi) \cos(\phi - \phi') + \sin(\xi') \cos(\xi)$$

$$(\vec{e}'_\lambda)_\phi = \cos(\xi') \sin(\phi' - \phi)$$

$$(\vec{e}'_\xi)_\lambda = -\cos(\xi) \sin(\xi') \cos(\phi - \phi') + \sin(\xi) \cos(\xi')$$

$$(\vec{e}'_\xi)_\xi = \sin(\xi) \sin(\xi') \cos(\phi - \phi') + \cos(\xi) \cos(\xi')$$

$$(\vec{e}'_\xi)_\phi = \sin(\xi') \sin(\phi - \phi')$$

$$(\vec{e}'_\phi)_\lambda = \cos(\xi) \sin(\phi - \phi')$$

$$(\vec{e}'_\phi)_\xi = \sin(\xi) \sin(\phi' - \phi)$$

$$(\vec{e}'_\phi)_\phi = \cos(\phi - \phi')$$

Another interesting set of components of the unit vectors is for the relationship between the toroidal  $(\lambda', \xi', \phi')$  and cylindrical  $(\Psi, \phi, z)$

coordinate systems.

$$\begin{aligned}
 (\vec{e}'_{\lambda})_{\Psi} &= \vec{e}'_{\lambda} \cdot \vec{e}_{\Psi} = \cos(\xi') \cos(\phi' - \phi) \\
 (\vec{e}'_{\lambda})_{\phi} &= \cos(\xi') \sin(\phi' - \phi) \\
 (\vec{e}'_{\lambda})_z &= -\sin(\xi') \\
 (\vec{e}'_{\xi})_{\Psi} &= -\sin(\xi') \cos(\phi - \phi') \\
 (\vec{e}'_{\xi})_{\phi} &= \sin(\xi') \sin(\phi - \phi') \\
 (\vec{e}'_{\xi})_z &= -\cos(\xi') \\
 (\vec{e}'_{\phi})_{\Psi} &= \sin(\phi - \phi') \\
 (\vec{e}'_{\phi})_{\phi} &= \cos(\phi - \phi') \\
 (\vec{e}'_{\phi})_z &= 0
 \end{aligned} \tag{2.5}$$

The distance between  $r$  and  $r'$  in terms of rectangular coordinates is,

$$\vec{r} - \vec{r}' = (x - x')\vec{e}_x + (y - y')\vec{e}_y + (z - z')\vec{e}_z \tag{2.6}$$

There is a particular surface of interest on the toroid itself described by  $(b, \xi, \phi)$  in toroidal coordinates or  $(a + b \cos(\xi), \phi, -b \sin(\xi))$  in cylindrical coordinates. With  $r'$  taken on this surface  $S'$  and using eq. (2.3),

$$\begin{aligned}
 \vec{r} - \vec{r}' &= \left[ (a + \lambda \cos(\xi)) \cos(\phi) - (a + b \cos(\xi')) \cos(\phi') \right] \vec{e}_x \\
 &+ \left[ (a + \lambda \cos(\xi)) \sin(\phi) - (a + b \cos(\xi')) \sin(\phi') \right] \vec{e}_y \\
 &+ \left[ -\lambda \sin(\xi) + b \sin(\xi') \right] \vec{e}_z
 \end{aligned} \tag{2.7}$$

By simple algebraic manipulations,

$$D^2 = \left| \vec{r} - \vec{r}' \right|^2 = \lambda^2 + b^2 - 2b\lambda \left[ \cos(\xi) \cos(\xi') \cos(\phi - \phi') + \sin(\xi) \sin(\xi') \right] + 4a \sin^2 \left( \frac{\phi - \phi'}{2} \right) \left[ a + \lambda \cos(\xi) + b \cos(\xi') \right] \quad (2.8)$$

If, instead, the cylindrical coordinate system is used with  $(b, \xi', \phi')$  on  $S'$ , then,

$$\begin{aligned} \vec{r} - \vec{r}' &= \left[ \Psi \cos(\phi) - (a + b \cos(\xi')) \cos(\phi') \right] \vec{e}_x \\ &\quad + \left[ \Psi \sin(\phi) - (a + b \cos(\xi')) \sin(\phi') \right] \vec{e}_y + \left[ z + b \sin(\xi') \right] \vec{e}_z \end{aligned} \quad (2.9)$$

and

$$D^2 = \left[ \Psi - a - b \cos(\xi') \right]^2 + \left[ z + b \sin(\xi') \right]^2 + 4\Psi \sin^2 \left( \frac{\phi - \phi'}{2} \right) \left[ a + b \cos(\xi') \right] \quad (2.10)$$

The vector  $\vec{r} - \vec{r}'$  can also be written by using various unit vectors.

In the cylindrical coordinates, this is,

$$\begin{aligned} \vec{r} - \vec{r}' &= \Psi \vec{e}_\Psi + z \vec{e}_z - \left[ a + b \cos(\xi') \right] \vec{e}'_\Psi + b \sin(\xi') \vec{e}'_z \\ &= \left[ \Psi - (a + b \cos(\xi')) \cos(\phi - \phi') \right] \vec{e}_\Psi \\ &\quad + \left[ a + b \cos(\xi') \right] \sin(\phi - \phi') \vec{e}_\phi + \left[ z + b \sin(\xi') \right] \vec{e}_z \end{aligned} \quad (2.11)$$

In the toroidal coordinate system  $(\lambda, \xi, \phi)$ ,

$$\begin{aligned}
\vec{r} - \vec{r}' &= \left[ (a + \lambda \cos(\xi)) - (a + b \cos(\xi')) \cos(\phi - \phi') \right] \vec{e}_\Psi \\
&\quad + \left[ a + b \cos(\xi') \right] \sin(\phi - \phi') \vec{e}_\phi - \left[ \lambda \sin(\xi) - b \sin(\xi') \right] \vec{e}_z \\
&= \left\{ \lambda + 2a \cos(\xi) \sin^2\left(\frac{\phi - \phi'}{2}\right) - b \left[ \cos(\xi) \cos(\xi') \cos(\phi - \phi') \right. \right. \\
&\quad \left. \left. + \sin(\xi) \sin(\xi') \right] \right\} \vec{e}_\lambda + \left\{ -2a \sin(\xi) \sin^2\left(\frac{\phi - \phi'}{2}\right) \right. \\
&\quad \left. + b \left[ \sin(\xi) \cos(\xi') \cos(\phi - \phi') - \cos(\xi) \sin(\xi') \right] \right\} \vec{e}_\xi \\
&\quad + \left\{ \left[ a + b \cos(\xi') \right] \sin(\phi - \phi') \right\} \vec{e}_\phi
\end{aligned} \tag{2.12}$$

A line element may be represented as,

$$\begin{aligned}
d\vec{l} &= h_1 du_1 \vec{e}_1 + h_2 du_2 \vec{e}_2 + h_3 du_3 \vec{e}_3 \\
&= dx \vec{e}_x + dy \vec{e}_y + dz \vec{e}_z
\end{aligned} \tag{2.13}$$

The magnitude is,

$$|d\vec{l}|^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{2.14}$$

In this expression  $u_1, u_2,$  and  $u_3$  are orthogonal such that,

$$\frac{du_i}{du_j} = \delta_{i,j} \tag{2.15}$$

and the  $h_i$  are scale factors which can be found from eqs. (2.14) and

(2.15) as follows,

$$h_i^2 = \left( \frac{dx}{du_i} \right)^2 + \left( \frac{dy}{du_i} \right)^2 + \left( \frac{dz}{du_i} \right)^2 \tag{2.16}$$

Using the relationship among the coordinate systems it can be shown that

$$\begin{aligned}
 h_x &= h_y = h_z = h_\Psi = h_\lambda = 1 \\
 h_\phi &= \Psi = a + \lambda \cos(\xi) \\
 h_\xi &= \lambda
 \end{aligned} \tag{2.17}$$

With these scale factors defined, the vector operators may be determined.

The Gradient Operator is defined as,

$$\begin{aligned}
 \nabla &= \frac{\vec{e}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\vec{e}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\vec{e}_3}{h_3} \frac{\partial}{\partial u_3} \\
 &= \vec{e}_1 \nabla_1 + \vec{e}_2 \nabla_2 + \vec{e}_3 \nabla_3
 \end{aligned} \tag{2.18}$$

therefore,

$$\begin{aligned}
 \nabla_\Psi &= \frac{\partial}{\partial \Psi} \\
 \nabla_\phi &= \frac{\partial}{\Psi \partial \phi} = \frac{\partial}{[a + \lambda \cos(\xi)] \partial \phi} \\
 \nabla_\lambda &= \frac{\partial}{\partial \lambda} \\
 \nabla_\xi &= \frac{\partial}{\lambda \partial \xi}
 \end{aligned} \tag{2.19}$$

The Divergence Operator is given in toroidal coordinates by,

$$\begin{aligned}
 \nabla \cdot \vec{A} &= \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_3 h_1 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right] \\
 &= \frac{1}{\lambda [a + \lambda \cos(\xi)]} \left\{ \frac{\partial}{\partial \lambda} [\lambda (a + \lambda \cos(\xi)) A_\lambda] + \frac{\partial}{\partial \xi} [(a + \lambda \cos(\xi)) A_\xi] \right. \\
 &\quad \left. + \frac{\partial}{\partial \phi} [\lambda A_\phi] \right\}
 \end{aligned} \tag{2.20}$$



The Curl Operator is defined as,

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{e}_1 & h_2 \vec{e}_2 & h_3 \vec{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$= \frac{\vec{e}_1}{h_2 h_3} \left\{ \frac{\partial(h_3 A_3)}{\partial u_2} - \frac{\partial(h_2 A_2)}{\partial u_3} \right\} + \frac{\vec{e}_2}{h_3 h_1} \left\{ \frac{\partial(h_1 A_1)}{\partial u_3} - \frac{\partial(h_3 A_3)}{\partial u_1} \right\}$$

$$+ \frac{\vec{e}_3}{h_1 h_2} \left\{ \frac{\partial(h_2 A_2)}{\partial u_1} - \frac{\partial(h_1 A_1)}{\partial u_2} \right\} \quad (2.21)$$

So, in cylindrical coordinates this becomes,

$$\nabla \times \vec{A} = \vec{e}_\Psi \left\{ \frac{\partial A_z}{\Psi \partial \phi} - \frac{\partial A_\phi}{\partial z} \right\} + \vec{e}_\phi \left\{ \frac{\partial A_\Psi}{\partial z} - \frac{\partial A_z}{\partial \Psi} \right\} + \frac{\vec{e}_z}{\Psi} \left\{ \frac{\partial(\Psi A_\phi)}{\partial \Psi} - \frac{\partial A_\Psi}{\partial \phi} \right\} \quad (2.22)$$

and in toroidal coordinates,

$$\nabla \times \vec{A} = \frac{\vec{e}_\lambda}{a + \lambda \cos(\xi)} \left\{ \frac{\partial [a + \lambda \cos(\xi)] A_\phi}{\lambda \partial \xi} - \frac{\partial A_\xi}{\partial \phi} \right\}$$

$$+ \frac{\vec{e}_\xi}{a + \lambda \cos(\xi)} \left\{ \frac{\partial A_\lambda}{\partial \phi} - \frac{\partial [a + \lambda \cos(\xi)] A_\phi}{\partial \lambda} \right\}$$

$$+ \frac{\vec{e}_\phi}{\lambda} \left\{ \frac{\partial(\lambda A_\xi)}{\partial \lambda} - \frac{\partial A_\lambda}{\partial \xi} \right\} \quad (2.23)$$

The scale factors can also be used to determine the surface integral.

Using primes for variables on the toroidal surface one has  $\xi'$ ,  $\phi'$  and the three unit vectors  $\vec{e}'_\lambda$ ,  $\vec{e}'_\xi$ , and  $\vec{e}'_\phi$  on the surface. Here  $\vec{e}'_\lambda$  is the surface

normal (pointing outward) and the other two unit vectors are parallel to the surface. The scaling factors on the toroid surface, according to eq. (2.17) are,

$$\begin{aligned}h'_{\xi} &= b \\h'_{\phi} &= a + b \cos(\xi')\end{aligned}\tag{2.24}$$

The increment on this surface is then,

$$dS' = b \left[ a + b \cos(\xi') \right] d\xi' d\phi'\tag{2.25}$$

The formulas derived in this chapter will be used in the calculations in succeeding chapters.

### III. Fields and Currents

It is well known that the vector potential  $\vec{A}$  and the scalar potential  $\Phi$  can be calculated from the current density  $\vec{J}$  and the charge density  $\rho$ .

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \iiint_{V'} \vec{J}(\vec{r}') \frac{e^{-\gamma|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} dV'$$

$$\Phi(\vec{r}) = \frac{i\omega}{4\pi(\sigma + i\omega\epsilon)} \iiint_{V'} \rho(\vec{r}') \frac{e^{-\gamma|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} dV' \quad (3.1)$$

Here, the frequency dependent factor  $e^{i\omega t}$  is omitted (or equivalently the two-sided Laplace transform of the quantities is used) and

$$\gamma = \sqrt{i\omega\mu(\sigma + i\omega\epsilon)} \quad (3.2)$$

In general, an antenna is made from an impedance loaded toroidal surface (including conductors) which makes the current and charge concentrate on the surface. Actually, knowing the current distribution will be sufficient in determining the potentials since, by the continuity equation, the surface charge density  $\rho_s$  can be obtained from the surface current density  $\vec{J}_s$  by,

$$\rho_s = -\frac{1}{i\omega} \nabla \cdot \vec{J}_s \quad (3.3)$$

Define Green's function as,

$$G(\vec{r}, \vec{r}') = \frac{e^{-\gamma|\vec{r} - \vec{r}'|}}{4\pi|\vec{r} - \vec{r}'|} \quad (3.4)$$

Then, the potentials become,

$$\begin{aligned} \vec{A}(\vec{r}) &= \mu \iint_{S'} \vec{J}_S(\vec{r}') G(\vec{r}, \vec{r}') dS' \\ \Phi(\vec{r}) &= - \frac{1}{\sigma + i\omega\epsilon} \iint_{S'} \left| \nabla'_S \cdot \vec{J}_S(\vec{r}') \right| G(\vec{r}, \vec{r}') dS' \end{aligned} \quad (3.5)$$

The electric field can be found from the potentials by,

$$\begin{aligned} \vec{E} &= -\nabla\Phi - i\omega\vec{A} \\ &= \frac{\nabla}{\sigma + i\omega\epsilon} \iint_{S'} G(\vec{r}, \vec{r}') \left[ \nabla'_S \cdot \vec{J}_S(\vec{r}') \right] dS' - i\omega\mu \iint_{S'} G(\vec{r}, \vec{r}') \vec{J}_S(\vec{r}') dS' \end{aligned} \quad (3.6)$$

On the toroid there are only  $\xi'$  and  $\phi'$  components of surface current,

i. e.,  $J'_\lambda = 0$ . From eq. (2.20),

$$\begin{aligned} \nabla'_S \cdot \vec{J}_S(\vec{r}') &= \frac{1}{b[a + b \cos(\xi')]} \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) J_{S_{\xi'}}(\xi', \phi') \right] \right. \\ &\quad \left. + \frac{\partial}{\partial \phi'} \left[ b J_{S_{\phi'}}(\xi', \phi') \right] \right\} \end{aligned} \quad (3.7)$$

Using the scalar product relationships in eq. (2.4) and the vector identities of eq. (2.19), one obtains,

$$\begin{aligned}
\frac{\gamma^2}{i\omega\mu} E_\lambda = \frac{\gamma}{Z_0} E_\lambda = \frac{1}{\partial\lambda} \iint_{S'} \frac{G(\vec{r}, \vec{r}')}{b[a + b \cos(\xi')] } \left\{ \frac{\partial}{\partial\xi'} \left[ (a + b \cos(\xi')) J_{S_{\xi'}}(\xi', \phi') \right] \right. \\
+ \left. \frac{\partial}{\partial\phi'} \left[ b J_{S_{\phi'}}(\xi', \phi') \right] \right\} b[a + b \cos(\xi')] d\xi' d\phi' \\
- \gamma^2 \iint_{S'} G(\vec{r}, \vec{r}') \left\{ \left[ -\cos(\xi) \sin(\xi') \cos(\phi - \phi') + \sin(\xi) \cos(\xi') \right] J_{S_{\xi'}}(\xi', \phi') \right. \\
+ \left. \cos(\xi) \sin(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} b[a + b \cos(\xi')] d\xi' d\phi' \quad (3.8)
\end{aligned}$$

where  $Z_0 = \sqrt{i\omega\mu / (\sigma + i\omega\epsilon)}$  is the characteristic impedance. Similarly,

$$\begin{aligned}
\frac{\gamma^2}{i\omega\mu} E_\xi = \frac{\gamma}{Z_0} E_\xi = \frac{1}{\lambda} \frac{\partial}{\partial\xi} \iint_{S'} \frac{G(\vec{r}, \vec{r}')}{b[a + b \cos(\xi')] } \left\{ \frac{\partial}{\partial\xi'} \left[ (a + b \cos(\xi')) J_{S_{\xi'}}(\xi', \phi') \right] \right. \\
+ \left. \frac{\partial}{\partial\phi'} \left[ b J_{S_{\phi'}}(\xi', \phi') \right] \right\} b[a + b \cos(\xi')] d\xi' d\phi' \\
- \gamma^2 \iint_{S'} G(\vec{r}, \vec{r}') \left\{ \left[ \sin(\xi) \sin(\xi') \cos(\phi - \phi') + \cos(\xi) \cos(\xi') \right] J_{S_{\xi'}}(\xi', \phi') \right. \\
+ \left. \sin(\xi) \sin(\phi' - \phi) J_{S_{\phi'}}(\xi', \phi') \right\} b[a + b \cos(\xi')] d\xi' d\phi' \quad (3.9)
\end{aligned}$$

and,

$$\begin{aligned}
\frac{\gamma^2}{i\omega\mu} E_\phi &= \frac{\gamma}{Z_0} E_\phi = \frac{1}{a + \lambda \cos(\xi)} \frac{\partial}{\partial \phi} \iint_{S'} \frac{G(\vec{r}, \vec{r}')}{b[a + b \cos(\xi')] } \\
&\left\{ \frac{\partial}{\partial \xi'} [(a + b \cos(\xi')) J_{S_{\xi'}}(\xi', \phi')] + \frac{\partial}{\partial \phi'} [b J_{S_{\phi'}}(\xi', \phi')] \right\} \\
&b[a + b \cos(\xi')] d\xi' d\phi' - \gamma^2 \iint_{S'} G(\vec{r}, \vec{r}') \left\{ \left[ \sin(\xi') \sin(\phi - \phi') J_{S_{\xi'}}(\xi', \phi') \right] \right. \\
&\left. + \left[ \cos(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right] \right\} b[a + b \cos(\xi')] d\xi' d\phi' \quad (3.10)
\end{aligned}$$

Rearranging these equations,

$$\begin{aligned}
\frac{\gamma}{Z_0} E_\lambda &= \frac{\partial}{\partial \lambda} \iint_{S'} G(\vec{r}, \vec{r}') \left\{ \frac{\partial}{\partial \xi'} [(a + b \cos(\xi')) J_{S_{\xi'}}(\xi', \phi')] + b \frac{\partial}{\partial \phi'} [J_{S_{\phi'}}(\xi', \phi')] \right\} d\xi' d\phi' \\
&- \gamma^2 b \iint_{S'} G(\vec{r}, \vec{r}') [a + b \cos(\xi')] \left\{ \left[ -\cos(\xi) \sin(\xi') \cos(\phi - \phi') \right] \right. \\
&\left. + \left[ \sin(\xi) \cos(\xi') \right] J_{S_{\xi'}}(\xi', \phi') + \left[ \cos(\xi) \sin(\phi - \phi') \right] J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \quad (3.11)
\end{aligned}$$

$$\begin{aligned}
\frac{\gamma}{Z_0} E_\xi &= \frac{1}{\lambda} \frac{\partial}{\partial \xi} \iint_{S'} G(\vec{r}, \vec{r}') \left\{ \frac{\partial}{\partial \xi'} [(a + b \cos(\xi')) J_{S_{\xi'}}(\xi', \phi')] \right. \\
&\left. + b \frac{\partial}{\partial \phi'} [J_{S_{\phi'}}(\xi', \phi')] \right\} d\xi' d\phi' - \gamma^2 b \iint_{S'} G(\vec{r}, \vec{r}') [a + b \cos(\xi')] \\
&\left\{ \left[ \sin(\xi) \sin(\xi') \cos(\phi - \phi') + \cos(\xi) \cos(\xi') \right] J_{S_{\xi'}}(\xi', \phi') \right. \\
&\left. - \left[ \sin(\xi) \sin(\phi - \phi') \right] J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \quad (3.12)
\end{aligned}$$

and,

$$\begin{aligned}
\frac{\gamma}{Z_0} E_\phi = & \frac{1}{a + \lambda \cos(\xi)} \frac{\partial}{\partial \phi} \iint_{S'} G(\vec{r}, \vec{r}') \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) J_{S_{\xi'}}(\xi', \phi') \right] \right. \\
& + b \frac{\partial}{\partial \phi'} \left[ J_{S_{\phi'}}(\xi', \phi') \right] \left. \right\} d\xi' d\phi' - \gamma^2 b \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \\
& \left\{ \left[ \sin(\xi') \sin(\phi - \phi') J_{S_{\xi'}}(\xi', \phi') \right] + \left[ \cos(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right] \right\} d\xi' d\phi'
\end{aligned} \tag{3.13}$$

Equations (3.11) to (3.13) are the exact solutions for the electric field in terms of the surface currents on the toroid.

Wu<sup>5</sup> obtained equations (3.12) and (3.13) in the same form as above. But there exists a discrepancy in the results. There is an opposite sign in the  $\sin(\xi) \sin(\phi - \phi') J_{S_{\phi'}}(\xi', \phi')$  and in the  $\sin(\xi') \sin(\phi - \phi') J_{S_{\xi'}}(\xi', \phi')$  terms. This difference has been carefully checked and it has been concluded that the two signs in Wu's results are in error. However, this error does not affect his calculations on current distribution. This is because Wu considered only eq. (3.13) and neglected the  $J_{S_{\xi'}}$  component. Eq. (3.12) in some cases does not contribute new information; this point will be clarified in the next chapter. Without the  $J_{S_{\xi'}}$  terms, eqs. (3.11) to (3.13) have precisely the opposite sign as obtained by Fante et al<sup>6</sup> for all terms.

The electric field components are expressed in cylindrical coordinates, but with the surface current density components still expressed in toroidal coordinates, are,

$$\begin{aligned}
\frac{\gamma}{Z_0} E_{\Psi} &= \frac{\partial}{\partial \Psi} \iint_{S'} G(\vec{r}, \vec{r}') \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) J_{s_{\xi'}}(\xi', \phi') \right] + \left[ b \frac{\partial}{\partial \phi'} J_{s_{\phi'}}(\xi', \phi') \right] \right\} d\xi' d\phi' \\
&\quad - \gamma^2 b \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \left\{ -\sin(\xi') \cos(\phi - \phi') J_{s_{\xi'}}(\xi', \phi') \right. \\
&\quad \left. + \sin(\phi - \phi') J_{s_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \\
\frac{\gamma}{Z_0} E_{\phi} &= \frac{1}{\Psi} \frac{\partial}{\partial \phi} \iint_{S'} G(\vec{r}, \vec{r}') \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) J_{s_{\xi'}}(\xi', \phi') \right] + \left[ b \frac{\partial}{\partial \phi'} J_{s_{\phi'}}(\xi', \phi') \right] \right\} d\xi' d\phi' \\
&\quad - \gamma^2 b \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \left\{ \sin(\xi') \sin(\phi - \phi') J_{s_{\xi'}}(\xi', \phi') \right. \\
&\quad \left. + \cos(\phi - \phi') J_{s_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \\
\frac{\gamma}{Z_0} E_z &= \frac{\partial}{\partial z} \iint_{S'} G(\vec{r}, \vec{r}') \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) J_{s_{\xi'}}(\xi', \phi') \right] + \left[ b \frac{\partial}{\partial \phi'} J_{s_{\phi'}}(\xi', \phi') \right] \right\} d\xi' d\phi' \\
&\quad - \gamma^2 b \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \left[ -\cos(\xi') J_{s_{\xi'}}(\xi', \phi') \right] d\xi' d\phi' \quad (3.14)
\end{aligned}$$

The magnetic field can be derived from the vector potential of eq. (3.5), which gives,

$$\vec{H} = \frac{1}{\mu} \nabla \times \vec{A} = \nabla \times \iint_{S'} \vec{J}_s(\vec{r}') G(\vec{r}, \vec{r}') dS' \quad (3.15)$$



Using eq. (2.23) and the scalar products of eq. (2.4),

$$\begin{aligned}
\nabla \times \vec{A} = & \frac{\vec{e}_\lambda}{a + \lambda \cos(\xi)} \left\{ \frac{\partial [a + \lambda \cos(\xi)]}{\lambda \partial \xi} \left[ \sin(\xi') \sin(\phi - \phi') A_{\xi'} + \cos(\phi - \phi') A_{\phi'} \right] \right. \\
& - \frac{\partial}{\partial \phi} \left( \left[ \sin(\xi) \sin(\xi') \cos(\phi - \phi') + \cos(\xi) \cos(\xi') \right] A_{\xi'} \right. \\
& \left. \left. + \sin(\xi) \sin(\phi' - \phi) A_{\phi'} \right) \right\} + \frac{\vec{e}_\xi}{a + \lambda \cos(\xi)} \left\{ \frac{\partial}{\partial \phi} \left( \left[ -\cos(\xi) \sin(\xi') \cos(\phi - \phi') \right. \right. \right. \\
& \left. \left. + \sin(\xi) \cos(\xi') \right] A_{\xi'} + \cos(\xi) \sin(\phi - \phi') A_{\phi'} \right) - \frac{\partial}{\partial \lambda} \left( \left[ a + \lambda \cos(\xi) \right] \right. \\
& \left. \left. \left[ \sin(\xi') \sin(\phi - \phi') A_{\xi'} + \cos(\phi - \phi') A_{\phi'} \right] \right) \right\} \\
& + \frac{\vec{e}_\phi}{\lambda} \left\{ \frac{\partial}{\partial \lambda} \left( \lambda \left[ \sin(\xi) \sin(\xi') \cos(\phi - \phi') + \cos(\xi) \cos(\xi') \right] A_{\xi'} \right. \right. \\
& \left. \left. + \lambda \sin(\xi) \sin(\phi' - \phi) A_{\phi'} \right) - \frac{\partial}{\partial \xi} \left( \left[ -\cos(\xi) \sin(\xi') \cos(\phi - \phi') \right. \right. \right. \\
& \left. \left. + \sin(\xi) \cos(\xi') \right] A_{\xi'} + \cos(\xi) \sin(\phi - \phi') A_{\phi'} \right) \right\} \quad (3.16)
\end{aligned}$$

and thus, the magnetic field components are obtained as follows,

$$\begin{aligned}
H_\lambda &= \frac{b}{a + \lambda \cos(\xi)} \left\{ \frac{1}{\lambda} \frac{\partial}{\partial \xi} \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + \lambda \cos(\xi) \right] \left[ a + b \cos(\xi') \right] \right. \\
&\quad \left. \left\{ \sin(\xi') \sin(\phi - \phi') J_{S_{\xi'}}(\xi', \phi') + \cos(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \right. \\
&\quad - \frac{\partial}{\partial \phi} \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \left\{ \left[ \sin(\xi) \sin(\xi') \cos(\phi - \phi') \right. \right. \\
&\quad \left. \left. + \cos(\xi) \cos(\xi') \right] J_{S_{\xi'}}(\xi', \phi') + \sin(\xi) \sin(\phi' - \phi) J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \\
H_\xi &= \frac{b}{a + \lambda \cos(\xi)} \left\{ \frac{\partial}{\partial \phi} \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \left\{ \left[ -\cos(\xi) \sin(\xi') \cos(\phi - \phi') \right. \right. \right. \\
&\quad \left. \left. + \sin(\xi) \cos(\xi') \right] J_{S_{\xi'}}(\xi', \phi') + \cos(\xi) \sin(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \\
&\quad - \frac{\partial}{\partial \lambda} \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + \lambda \cos(\xi) \right] \left[ a + b \cos(\xi') \right] \left\{ \sin(\xi') \sin(\phi - \phi') J_{S_{\xi'}}(\xi', \phi') \right. \\
&\quad \left. + \cos(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \\
H_\phi &= \frac{b}{\lambda} \left\{ \frac{\partial}{\partial \lambda} \iint_{S'} G(\vec{r}, \vec{r}') \lambda \left[ a + b \cos(\xi') \right] \left\{ \sin(\xi) \sin(\xi') \cos(\phi - \phi') \right. \right. \\
&\quad \left. \left. + \cos(\xi) \cos(\xi') \right] J_{S_{\xi'}}(\xi', \phi') - \sin(\xi) \sin(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \\
&\quad - \frac{\partial}{\partial \xi} \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \left\{ \left[ -\cos(\xi) \sin(\xi') \cos(\phi - \phi') \right. \right. \\
&\quad \left. \left. + \sin(\xi) \cos(\xi') \right] J_{S_{\xi'}}(\xi', \phi') + \cos(\xi) \sin(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi'
\end{aligned} \tag{3.17}$$

The magnetic field components expressed in cylindrical  $(\Psi, \phi, z)$  coordinates, but with toroidal coordinates  $(\lambda', \xi', \phi')$  for the surface current components, are,

$$\begin{aligned}
H_{\Psi} &= \frac{b}{\Psi} \frac{\partial}{\partial \phi} \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \left\{ -\cos(\xi') J_{S_{\xi'}}(\xi', \phi') \right\} d\xi' d\phi' \\
&\quad - b \frac{\partial}{\partial z} \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \left\{ \sin(\xi') \sin(\phi - \phi') J_{S_{\xi'}}(\xi', \phi') \right. \\
&\quad \left. + \cos(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \\
H_{\phi} &= b \frac{\partial}{\partial z} \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \left\{ -\sin(\xi') \cos(\phi - \phi') J_{S_{\xi'}}(\xi', \phi') \right. \\
&\quad \left. + \sin(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' - b \frac{\partial}{\partial \Psi} \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \\
&\quad \left\{ -\cos(\xi') J_{S_{\xi'}}(\xi', \phi') \right\} d\xi' d\phi' \\
H_z &= \frac{b}{\Psi} \frac{\partial}{\partial \Psi} \iint_{S'} G(\vec{r}, \vec{r}') \Psi \left[ a + b \cos(\xi') \right] \left\{ \sin(\xi') \sin(\phi - \phi') J_{S_{\xi'}}(\xi', \phi') \right. \\
&\quad \left. + \cos(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' - \frac{b}{\Psi} \frac{\partial}{\partial \phi} \iint_{S'} G(\vec{r}, \vec{r}') \left[ a + b \cos(\xi') \right] \\
&\quad \left\{ -\sin(\xi') \cos(\phi - \phi') J_{S_{\xi'}}(\xi', \phi') + \sin(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \quad (3.18)
\end{aligned}$$

Therefore, knowing the current distributions on this special configuration of TORUS, the electric and magnetic field in space can be calculated either in the toroidal or in the cylindrical coordinate systems.

Unfortunately, the current in general is unknown. It will be shown in the next chapter how to solve for the current distribution from boundary conditions.

#### IV. Solution of the Integral Equations

If the sheet impedance  $\vec{\bar{Z}}$  is given, the electric field on the toroidal surface can be formulated in terms of the source field  $E_g$  and the current distribution as follows:

$$\vec{E}_S(\xi, \phi) = \vec{E}_g(\xi, \phi) + \vec{\bar{Z}}(\xi, \phi) \cdot \vec{J}_S(\xi, \phi) \quad (4.1)$$

It is also known from eqs. (3.12) and (3.13) that the surface electric field is,

$$\begin{aligned} \frac{\gamma}{Z_0} E_{S_\phi} = & \frac{1}{a + b \cos(\xi)} \frac{\partial}{\partial \phi} \iint_{S'_S} G_S(\vec{r}_S, \vec{r}'_S) \left\{ \frac{\partial}{\partial \xi'} [a + b \cos(\xi')] J_{S_{\xi'}}(\xi', \phi') \right. \\ & \left. + b \frac{\partial}{\partial \phi'} J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' - \gamma^2 b \iint_{S'_S} G_S(\vec{r}_S, \vec{r}'_S) [a + b \cos(\xi')] \\ & \left\{ \sin(\xi') \sin(\phi - \phi') J_{S_{\xi'}}(\xi', \phi') + \cos(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} \frac{\gamma}{Z_0} E_{S_\xi} = & \frac{1}{b} \frac{\partial}{\partial \xi} \iint_{S'_S} G_S(\vec{r}_S, \vec{r}'_S) \left\{ \frac{\partial}{\partial \xi'} [a + b \cos(\xi')] J_{S_{\xi'}}(\xi', \phi') \right. \\ & \left. + b \frac{\partial}{\partial \phi'} J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' - \gamma^2 b \iint_{S'_S} G_S(\vec{r}_S, \vec{r}'_S) [a + b \cos(\xi')] \\ & \left\{ \left[ \sin(\xi) \sin(\xi') \cos(\phi - \phi') + \cos(\xi) \cos(\xi') \right] J_{S_{\xi'}}(\xi', \phi') \right. \\ & \left. - \sin(\xi) \sin(\phi - \phi') J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' \end{aligned} \quad (4.3)$$

where  $G_s(\vec{r}_s, \vec{r}'_s)$  is the Green's function in eq. (3.4) with  $\vec{r}_s = (b, \xi, \phi)$  and  $\vec{r}'_s = (b, \xi', \phi')$  on the toroid surface.

Integral equations (4.2) and (4.3) can be reduced to a simpler form by Fourier expansions of the integrand. The surface current on the toroid may be expressed in terms of a Fourier series expansion as follows,

$$\vec{J}_s(\xi, \phi) = \vec{\alpha}_0(\xi) + 2 \sum_{n=1}^{\infty} \left[ \vec{\alpha}_n(\xi) \cos(n\phi) + \vec{\alpha}'_n(\xi) \sin(n\phi) \right] \quad (4.4)$$

where,

$$\vec{\alpha}_n(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \vec{J}_s(\xi, \phi) \cos(n\phi) d\phi \quad \text{for } n = 0, 1, 2, \dots$$

$$\vec{\alpha}'_n(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \vec{J}_s(\xi, \phi) \sin(n\phi) d\phi \quad \text{for } n = 1, 2, \dots \quad (4.5)$$

The coefficients  $\vec{\alpha}_n$  and  $\vec{\alpha}'_n$  are vectors (with  $\xi$  and  $\phi$  components) and, in general, are functions of  $\xi$ . Similarly, the tangential electric field on the toroid surface,

$$\vec{E}_s(\vec{r}_s) = E_{s\xi}(\xi, \phi) \vec{e}_\xi + E_{s\phi}(\xi, \phi) \vec{e}_\phi \quad (4.6)$$

can be expanded in the form

$$\vec{E}_s(\xi, \phi) = \vec{\beta}_0(\xi) + 2 \sum_{n=1}^{\infty} \left[ \vec{\beta}_n(\xi) \cos(n\phi) + \vec{\beta}'_n(\xi) \sin(n\phi) \right] \quad (4.7)$$

with,

$$\bar{\beta}_n(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \bar{E}_s(\xi, \phi) \cos(n\phi) d\phi \quad \text{for } n = 0, 1, 2, \dots$$

$$\bar{\beta}'_n(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \bar{E}_s(\xi, \phi) \sin(n\phi) d\phi \quad \text{for } n = 1, 2, \dots \quad (4.8)$$

The idea, of course, is to relate the  $\bar{\alpha}_n$ ,  $\bar{\alpha}'_n$ ,  $\bar{\beta}_n$ , and  $\bar{\beta}'_n$ . Now, the Green's function on the toroid surface is defined by,

$$G_s(\bar{r}_s, \bar{r}'_s) = \frac{1}{4\pi D_s} e^{-\gamma D_s} \quad (4.9)$$

where  $D_s = |\bar{r}_s - \bar{r}'_s|$  can be obtained from eq. (2.8) by letting  $\lambda = b$ . For this case,

$$\begin{aligned} D_s^2 = 2b^2 \left[ 1 - \cos(\xi) \cos(\xi') \cos(\phi - \phi') - \sin(\xi) \sin(\xi') \right] \\ + 4a \sin^2 \left( \frac{\phi - \phi'}{2} \right) \left[ a + b \cos(\xi) + b \cos(\xi') \right] \end{aligned} \quad (4.10)$$

Note that  $G_s$  is even in  $\phi - \phi'$ . Thus,  $G_s$  can be Fourier expanded with respect to  $\phi - \phi'$  as follows,

$$G_s = G_0 + 2 \sum_{n=1}^{\infty} G_n \cos[n(\phi - \phi')] \quad (4.11)$$

Here the  $G_n$  are functions of  $\xi$  and  $\xi'$  but not of  $\phi$  or  $\phi'$ . The coefficients are given by,

$$G_n = \frac{1}{2\pi} \int_0^{2\pi} G_s \cos [n(\phi - \phi')] d(\phi - \phi') \quad (4.12)$$

Substituting these Fourier expansions into eq. (4.2), one obtains,

$$\begin{aligned} & \frac{\gamma}{Z_0} \left[ a + b \cos(\xi) \right] \left\{ \beta_{\phi_0} + 2 \sum_{m=1}^{\infty} \left[ \beta_{\phi_m} \cos(m\phi) + \beta'_{\phi_m} \sin(m\phi) \right] \right\} \\ &= \frac{\partial}{\partial \phi} \iint_{S'} \left[ G_0 + 2 \sum_{m=1}^{\infty} G_{m+1} \cos(m(\phi - \phi')) \right] \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) J_{S_{\xi'}}(\xi', \phi') \right] \right. \\ & \quad \left. + b \frac{\partial}{\partial \phi'} J_{S_{\phi'}}(\xi', \phi') \right\} d\xi' d\phi' - \gamma^2 b \left[ a + b \cos(\xi) \right] \iint_{S'} \left[ a + b \cos(\xi') \right] \\ & \quad \left\{ \sin(\xi') J_{S_{\xi'}}(\xi', \phi') \left[ G_0 \sin(\phi - \phi') + 2 \sum_{m=1}^{\infty} G_m \cos(m(\phi - \phi')) \sin(\phi - \phi') \right] \right. \\ & \quad \left. + J_{S_{\phi'}}(\xi', \phi') \left[ G_0 \cos(\phi - \phi') + 2 \sum_{m=1}^{\infty} G_m \cos(m(\phi - \phi')) \cos(\phi - \phi') \right] \right\} d\xi' d\phi' \quad (4.13) \end{aligned}$$

Hereafter, for simplicity,  $J_{S_{\xi}}$  and  $J_{S_{\phi}}$  instead of  $J_{S_{\xi'}}$  and  $J_{S_{\phi'}}$  will be used to represent the surface current.

Multiplying both sides of eq. (4.13) by  $\frac{1}{2\pi} \cos(n\phi)$  and using the orthogonality relations,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \cos(n\phi) \cos(m\phi) d\phi &= \frac{1}{2} \delta_{n,m} (1 + \delta_{0,n}) \\ \frac{1}{2\pi} \int_0^{2\pi} \sin(n\phi) \sin(m\phi) d\phi &= \frac{1}{2} \delta_{n,m} (1 - \delta_{0,n}) \\ \frac{1}{2\pi} \int_0^{2\pi} \cos(n\phi) \sin(m\phi) d\phi &= 0 \end{aligned} \quad (4.14)$$

as well as the geometric rules,

$$\cos[m(\phi - \phi')] = \cos(m\phi) \cos(m\phi') + \sin(m\phi) \sin(m\phi')$$

$$\frac{\partial}{\partial \phi} \cos[m(\phi - \phi')] = -m \sin(m\phi) \cos(m\phi') + m \cos(m\phi) \sin(m\phi')$$

$$\begin{aligned} \cos[m(\phi - \phi')] \sin(\phi - \phi') &= \frac{1}{2} \left[ -\sin((m-1)\phi) \cos((m-1)\phi') \right. \\ &\quad + \cos((m-1)\phi) \sin((m-1)\phi') + \sin((m+1)\phi) \cos((m+1)\phi') \\ &\quad \left. - \cos((m+1)\phi) \sin((m+1)\phi') \right] \end{aligned}$$

$$\begin{aligned} \cos[m(\phi - \phi')] \cos(\phi - \phi') &= \frac{1}{2} \left[ \cos((m-1)\phi) \cos((m-1)\phi') \right. \\ &\quad + \sin((m-1)\phi) \sin((m-1)\phi') + \cos((m+1)\phi) \cos((m+1)\phi') \\ &\quad \left. + \sin((m+1)\phi) \sin((m+1)\phi') \right] \end{aligned} \quad (4.15)$$

it can be shown after integrating over  $0 \leq \phi \leq 2\pi$  that



$$\begin{aligned}
\frac{\gamma}{Z_0} [a + b \cos(\xi)] \beta_{\phi_n} = & \iint_{S'} G_n \sin(n\phi') \left\{ \frac{\partial}{\partial \xi'} [(a + b \cos(\xi')) J_{S_\xi}(\xi', \phi')] \right. \\
& + b \frac{\partial}{\partial \phi'} J_{S_\phi}(\xi', \phi') \left. \right\} d\xi' d\phi' - \frac{1}{2} \gamma^2 b [a + b \cos(\xi)] \iint_{S'} [a + b \cos(\xi')] \\
& \left\{ \sin \xi' J_{S_\xi}(\xi', \phi') [G_{n+1} - G_{n-1}] \sin(n\phi') + J_{S_\phi}(\xi', \phi') [G_{n+1}(1 + \delta_{0,n}) \right. \\
& \left. + G_{n-1}(1 - \delta_{0,n})] \cos(n\phi') \right\} d\xi' d\phi' \quad \text{for } n = 0, 1, 2, \dots \quad (4.16)
\end{aligned}$$

Similarly, multiplying by  $\frac{1}{2\pi} \sin(n\phi)$  and integrating over  $0 \leq \phi \leq 2\pi$ ,

$$\begin{aligned}
\frac{\gamma}{Z_0} [a + b \cos(\xi)] \beta'_{\phi_n} = & \iint_{S'} -G_n \cos(n\phi') \left\{ \frac{\partial}{\partial \xi'} [(a + b \cos(\xi')) J_{S_\xi}(\xi', \phi')] \right. \\
& + b \frac{\partial}{\partial \phi'} J_{S_\phi}(\xi', \phi') \left. \right\} d\xi' d\phi' - \frac{1}{2} \gamma^2 b [a + b \cos(\xi)] \iint_{S'} [a + b \cos(\xi')] \\
& \left\{ -\sin \xi' J_{S_\xi}(\xi', \phi') [G_{n+1} - G_{n-1}] \cos(n\phi') + J_{S_\phi}(\xi', \phi') [G_{n+1} \right. \\
& \left. + G_{n-1}] \sin(n\phi') \right\} d\xi' d\phi' \quad \text{for } n = 1, 2, \dots \quad (4.17)
\end{aligned}$$

Now, using eq. (4.4) and integrating with respect to  $\phi'$ , eq. (4.16) becomes

$$\begin{aligned}
\frac{\gamma}{Z_0} [a + b \cos(\xi)] \beta_{\phi_n} = & 2\pi \int_0^{2\pi} G_n \left\{ \frac{\partial}{\partial \xi'} [(a + b \cos(\xi')) \alpha'_{\xi_n}(\xi')] \right. \\
& \left. - b n \alpha'_{\phi_n}(\xi') \right\} d\xi' - \pi \gamma^2 b [a + b \cos(\xi)] \int_0^{2\pi} [a + b \cos(\xi')] \\
& \left\{ \sin(\xi') [G_{n+1} - G_{n-1}] \alpha'_{\xi_n}(\xi) + [G_{n+1}(1 + \delta_{0,n}) \right. \\
& \left. + G_{n-1}(1 - \delta_{0,n})] \alpha'_{\phi_n}(\xi) \right\} d\xi' \quad (4.18)
\end{aligned}$$

Eq. (4.17) becomes,

$$\begin{aligned}
\frac{\gamma}{Z_0} \left[ a + b \cos(\xi) \right] \beta'_{\phi_n} = & -2\pi \int_0^{2\pi} G_{n,n} \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) \alpha_{\xi_n}(\xi') \right] \right. \\
& \left. + b n \alpha'_{\phi_n}(\xi') \right\} d\xi' - \pi \gamma^2 b \left[ a + b \cos(\xi) \right] \int_0^{2\pi} \left[ a + b \cos(\xi') \right] \\
& \left\{ -\sin \xi' \left[ G_{n+1} - G_{n-1} \right] \alpha_{\xi_n}(\xi') + \left[ G_{n+1} + G_{n-1} \right] \alpha'_{\phi_n}(\xi') \right\} d\xi' \quad (4.19)
\end{aligned}$$

Here,  $\beta_{\phi_n}$  is linked to  $\alpha'_{\xi_n}$  and  $\alpha_{\phi_n}$ , with  $\beta'_{\phi_n}$  linked to  $\alpha_{\xi_n}$  and  $\alpha'_{\phi_n}$ , in a somewhat complementary fashion between  $\beta_{\phi_n}$  and  $\beta'_{\phi_n}$  for  $n \geq 1$ . The  $\beta'_{\phi_n}$  equation can be obtained from the  $\beta_{\phi_n}$  equation by substituting for  $\beta_{\phi_n}$ ,  $\alpha'_{\xi_n}$ , and  $\alpha_{\phi_n}$  by  $\beta'_{\phi_n}$ ,  $-\alpha_{\xi_n}$ , and  $\alpha'_{\phi_n}$  respectively,

In the same way, the  $E_{S\xi}$  equation in (4.3) expanded in  $\phi$  is

$$\begin{aligned}
\frac{\gamma b}{Z_0} \left\{ \beta_{\xi_0} + 2 \sum_{m=1}^{\infty} \left[ \beta_{\xi_m} \cos(m\phi) + \beta'_{\xi_m} \sin(m\phi) \right] \right\} = & \frac{\partial}{\partial \xi} \iint_{S'} \left[ G_0 \right. \\
& \left. + 2 \sum_{m=1}^{\infty} G_m \cos(m(\phi - \phi')) \right] \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) J_{S\xi}(\xi', \phi') \right] \right. \\
& \left. + b \frac{\partial}{\partial \phi'} \left[ J_{S\phi}(\xi', \phi') \right] \right\} d\xi' d\phi' - \gamma^2 b^2 \iint_{S'} \left[ a + b \cos(\xi') \right] \\
& \left\{ \sin(\xi) \sin(\xi') J_{S\xi}(\xi', \phi') \left[ G_0 \cos(\phi - \phi') + 2 \sum_{m=1}^{\infty} G_m \cos(m(\phi - \phi')) \cos(\phi - \phi') \right] \right. \\
& \left. + \cos(\xi) \cos(\xi') J_{S\xi}(\xi', \phi') \left[ G_0 + 2 \sum_{m=1}^{\infty} G_m \cos(m(\phi - \phi')) \right] - \sin(\xi) J_{S\phi}(\xi', \phi') \right. \\
& \left. \left[ G_0 \sin(\phi - \phi') + 2 \sum_{m=1}^{\infty} G_m \cos(m(\phi - \phi')) \sin(\phi - \phi') \right] \right\} \quad (4.20)
\end{aligned}$$

Multiply both sides by  $\frac{1}{2\pi} \cos(n\phi)$  or  $\frac{1}{2\pi} \sin(n\phi)$  and integrate over

$0 \leq \phi \leq 2\pi$ ; one gets,

$$\begin{aligned} \frac{\gamma b}{Z_0} \beta_{\xi_n} &= \frac{\partial}{\partial \xi} \iint_{S'} G_n \cos(n\phi') \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) J_{S_\xi}(\xi', \phi') \right] \right. \\ &+ b \frac{\partial}{\partial \phi'} J_{S_\phi}(\xi', \phi') \left. \right\} d\xi' d\phi' - \frac{\gamma^2 b^2}{2} \iint_{S'} \left[ a + b \cos(\xi') \right] \left\{ \sin(\xi) \sin(\xi') J_{S_\xi}(\xi', \phi') \right. \\ &\left[ G_{n+1} (1 + \delta_{0,n}) + G_{n-1} (1 - \delta_{0,n}) \right] \cos(n\phi') + 2 \cos(\xi) \cos(\xi') J_{S_\xi}(\xi', \phi') \\ &\left. G_n \cos(n\phi') - \sin(\xi) J_{S_\phi}(\xi', \phi') \left[ G_{n+1} - G_{n-1} \right] \sin(n\phi') \right\} d\xi' d\phi' \end{aligned}$$

$$\text{for } n = 0, 1, 2, \dots \quad (4.21)$$

and,

$$\begin{aligned} \frac{\gamma b}{Z_0} \beta'_{\xi_n} &= \frac{\partial}{\partial \xi} \iint_{S'} G_n \sin(n\phi') \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) J_{S_\xi}(\xi', \phi') \right] \right. \\ &+ \left[ b \frac{\partial}{\partial \phi'} J_{S_\phi}(\xi', \phi') \right] \left. \right\} d\xi' d\phi' - \frac{\gamma^2 b^2}{2} \iint_{S'} \left[ a + b \cos(\xi') \right] \left\{ \sin(\xi) \sin(\xi') J_{S_\xi}(\xi', \phi') \right. \\ &\left[ G_{n+1} + G_{n-1} \right] \sin(n\phi') + 2 \cos(\xi) \cos(\xi') J_{S_\xi}(\xi', \phi') G_n \sin(n\phi') \\ &+ \sin(\xi) J_{S_\phi}(\xi', \phi') \left[ G_{n+1} - G_{n-1} \right] \cos(n\phi') \left. \right\} d\xi' d\phi' \text{ for } n=1, 2, \dots \end{aligned} \quad (4.22)$$

Now, integrate over  $\phi'$ ,

$$\begin{aligned}
\frac{\gamma b}{Z_0} \beta_{\xi_n} &= 2\pi \frac{\partial}{\partial \xi} \int_0^{2\pi} G_n \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) \alpha_{\xi_n}(\xi') \right] + \left[ b n \alpha'_{\phi_n}(\xi') \right] \right\} d\xi' \\
&\quad - \pi \gamma^2 b^2 \int_0^{2\pi} \left[ a + b \cos(\xi') \right] \left\{ \sin(\xi) \sin(\xi') \alpha_{\xi_n}(\xi') \left[ G_{n+1} (1 + \delta_{0,n}) \right. \right. \\
&\quad \left. \left. + G_{n-1} (1 - \delta_{0,n}) \right] + 2 \cos(\xi) \cos(\xi') \alpha_{\xi_n}(\xi') G_n \right. \\
&\quad \left. - \sin(\xi) \alpha'_{\phi_n} \left[ G_{n+1} - G_{n-1} \right] \right\} d\xi' \tag{4.23}
\end{aligned}$$

and,

$$\begin{aligned}
\frac{\gamma b}{Z_0} \beta'_{\xi_n} &= 2\pi \frac{\partial}{\partial \xi} \int_0^{2\pi} G_n \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) \alpha'_{\xi_n}(\xi') \right] - \left[ b n \alpha_{\phi_n}(\xi') \right] \right\} d\xi' \\
&\quad - \pi \gamma^2 b^2 \int_0^{2\pi} \left[ a + b \cos(\xi') \right] \left\{ \sin(\xi) \sin(\xi') \alpha'_{\xi_n}(\xi') \left[ G_{n+1} + G_{n-1} \right] \right. \\
&\quad \left. + 2 \cos(\xi) \cos(\xi') \alpha'_{\xi_n}(\xi') G_n + \sin(\xi) \alpha_{\phi_n}(\xi') \left[ G_{n+1} - G_{n-1} \right] \right\} d\xi' \tag{4.24}
\end{aligned}$$

Again,  $\beta_{\xi_n}$ , linked to  $\alpha_{\xi_n}$  and  $\alpha'_{\phi_n}$ , is somewhat complementary to  $\beta'_{\xi_n}$  which linked to  $\alpha'_{\xi_n}$  and  $\alpha_{\phi_n}$ . The  $\beta'_{\xi_n}$  equation can be obtained from the  $\beta_{\xi_n}$  equation if  $\beta_{\xi_n}$ ,  $\alpha_{\xi_n}$ , and  $\alpha'_{\phi_n}$  are substituted by  $\beta'_{\xi_n}$ ,  $\alpha'_{\xi_n}$ , and  $-\alpha_{\phi_n}$  respectively. This is not true for  $n = 0$  since  $\beta'_{\xi_n} = 0$ .

Equation (4.1) can also be expanded by Fourier series techniques. Equating it with eqs. (4.18), (4.19) and (4.23), (4.24), the current, in principle, can be solved. This should be a general solution since no assumptions or restrictions have been made so far. Unfortunately, these

equations are still in the integral form and, for techniques familiar to the authors,  $G_n$  can be obtained only by numerical methods. It is rather complicated to solve for an integral equation with numerical parameters.

This study will be left for future investigation. In a later part of this note we will assume that "a" is much greater than "b". This way, the integral equation can be reduced to an algebraic equation and thus an analytical approach would be possible. This assumption in reality is true for most cases. Also, an analytical result could give a check with the existing formulas for the extremely high and low frequencies. It also provides a general feeling of the TORUS behavior for future complete investigations.

## V. Current on TORUS Loaded with Uniform Impedance

According to eqs. (2.8) and (3.4) the surface Green's function depends on  $D_s$  in such a way that

$$\int_{-\pi}^{\pi} \frac{\partial}{\partial \xi} G_s(\xi) d\xi = G_s(\pi) - G_s(-\pi) = 0 \quad (5.1)$$

Therefore,

$$\int_{-\pi}^{\pi} \frac{\partial}{\partial \xi} \iint_{-\pi}^{\pi} G_s \left\{ \frac{\partial}{\partial \xi'} \left[ (a + b \cos(\xi')) J_{s_{\xi}}(\xi', \phi') \right] + \left[ b \frac{\partial}{\partial \phi'} J_{s_{\phi}}(\xi', \phi') \right] \right\} d\xi' d\phi' d\xi = 0 \quad (5.2)$$

Also from eq. (2.8),  $D(\xi, \xi') = D(-\xi, -\xi')$ , which makes

$$G(\xi, \xi') = G(-\xi, -\xi') \quad (5.3)$$

From this relation, it can be shown that,

$$\begin{aligned} \int_{-\pi}^{\pi} G(\xi, \xi') d\xi &= \int_{-\pi}^{\pi} G(\xi, -\xi') d\xi \\ \int_{-\pi}^{\pi} G(\xi, \xi') \cos(\xi) d\xi &= \int_{-\pi}^{\pi} G(\xi, -\xi') \cos(\xi) d\xi \\ \int_{-\pi}^{\pi} G(\xi, \xi') \sin(\xi) d\xi &= -\int_{-\pi}^{\pi} G(\xi, -\xi') \sin(\xi) d\xi \end{aligned} \quad (5.4)$$

In other words,  $\int_{-\pi}^{\pi} G(\xi, \xi') d\xi$  and  $\int_{-\pi}^{\pi} G(\xi, \xi') \cos(\xi) d\xi$  are even, and

$\int_{-\pi}^{\pi} G(\xi, \xi') \sin(\xi) d\xi$  is odd with respect to  $\xi'$ .

For a uniform impedance loading ( $\vec{Z}_s$  reducing to a constant scalar  $Z_s$ ), with the source at  $\phi = 0$ , symmetry<sup>7</sup> implies that

$$\begin{aligned}
 J_{S\phi}(\xi, \phi) &= J_{S\phi}(-\xi, \phi) \\
 J_{S\xi}(\xi, \phi) &= -J_{S\xi}(-\xi, \phi) \\
 E_{S\phi}(\xi, \phi) &= E_{S\phi}(-\xi, \phi) \\
 E_{S\xi}(\xi, \phi) &= -E_{S\xi}(-\xi, \phi)
 \end{aligned} \tag{5.5}$$

Thus,

$$\begin{aligned}
 \int_{-\pi}^{\pi} J_{S\xi}(\xi, \phi) d\xi &= 0 \\
 \int_{-\pi}^{\pi} E_{S\xi}(\xi, \phi) d\xi &= 0
 \end{aligned} \tag{5.6}$$

Using these relations it can be easily shown that,

$$\begin{aligned}
 -\gamma^2 b \iiint_{-\pi}^{\pi} G_S [a + b \cos(\xi')] \sin(\xi) \sin(\xi') \cos(\phi - \phi') J_{S\xi}(\xi', \phi') d\xi' d\phi' d\xi &= 0 \\
 -\gamma^2 b \iiint_{-\pi}^{\pi} G_S [a + b \cos(\xi')] \cos(\xi) \cos(\xi') J_{S\xi}(\xi', \phi') d\xi' d\phi' d\xi &= 0 \\
 -\gamma^2 b \iiint_{-\pi}^{\pi} G_S [a + b \cos(\xi')] \sin(\xi) \sin(\phi - \phi') J_{S\phi}(\xi', \phi') d\xi' d\phi' d\xi &= 0
 \end{aligned} \tag{5.7}$$

Each term in the integration of eq. (4.3) with respect to  $\xi$  and  $\xi'$  from  $-\pi$  to  $\pi$  is zero before any  $\phi'$  integration is taken. Thus, eq. (4.3) does

not give any useful information and therefore can be discarded if the load is uniform. In this case eq. (4.2) is the only fundamental equation for solving the current distribution. This equation can be simplified by assuming that

$$a^2 \gg b^2 \quad (5.8)$$

and

$$kb \ll 1 \quad (5.9)$$

The fields calculated under this assumption therefore are valid only for  $kb$  less than one. For higher frequencies, the charges will be accumulated near the source. The known<sup>8,9</sup> far field formula of an infinite cylindrical antenna can be used to represent the high frequency field near the center of TORUS. Under these assumptions of eqs. (5.8) and (5.9) the current on the TORUS can be considered as uniformly distributed with respect to  $\xi$  and we may assume,

$$\begin{aligned} J_{S\xi}(\phi, \xi) &\approx 0 \\ J_{S\phi}(\phi, \xi) &\approx \frac{I(\phi)}{2\pi b} \end{aligned} \quad (5.10)$$

If the TORUS is located in free space where

$$\begin{aligned} \gamma^2 &= \left(\frac{i\omega}{c}\right)^2 = (ik)^2 \\ Z_0 &= \sqrt{\mu_0/\epsilon_0} \approx 377 \text{ ohms} \end{aligned} \quad (5.11)$$



then eq. (4.2) is reduced to

$$\begin{aligned} \frac{4\pi a i k}{Z_0} E_{s\phi} &= \frac{\partial}{\partial \phi} \int_0^{2\pi} K(\phi - \phi') \frac{\partial}{\partial \phi'} I(\phi') d\phi' \\ &+ k^2 a^2 \int_0^{2\pi} K(\phi - \phi') \cos(\phi - \phi') I(\phi') d\phi' \end{aligned} \quad (5.12)$$

with

$$\begin{aligned} K(\phi) &= 2 \int_{-\pi}^{\pi} G_s d\xi' = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ (2a \sin(\frac{\phi}{2}))^2 + (2b \sin(\frac{\xi}{2}))^2 \right]^{-1/2} \\ &\exp \left\{ -ik \left[ (2a \sin(\frac{\phi}{2}))^2 + (2b \sin(\frac{\xi}{2}))^2 \right]^{1/2} \right\} d\xi \end{aligned} \quad (5.13)$$

Instead of using eqs. (4.4), (4.7), and (4.11) we may expand the current, Green's function, and surface electric field in the following ways in order to compare our derivations with some of the previous works.

$$\begin{aligned} I(\phi) &= \sum_{m=-\infty}^{\infty} I_m e^{-im\phi} \\ K(\phi) &= \sum_{m=-\infty}^{\infty} K_m e^{-im\phi} \\ E_{s\phi}(\phi) &= \sum_{m=-\infty}^{\infty} E_m e^{-im\phi} \end{aligned} \quad (5.14)$$

where

$$\begin{aligned} I_m &= \frac{1}{2\pi} \int_0^{2\pi} I(\phi) e^{im\phi} d\phi \\ K_m &= \frac{1}{2\pi} \int_0^{2\pi} K(\phi) e^{im\phi} d\phi \end{aligned}$$

$$E_m = \frac{1}{2\pi} \int_0^{2\pi} E_s(\phi) e^{im\phi} d\phi \quad (5.15)$$

The Fourier coefficients defined in the last chapter are related to them as follows,

$$\begin{aligned} I_0 &= 2\pi b_{\alpha} \phi_0 \\ \frac{1}{2}(I_{-m} + I_m) &= 2\pi b_{\alpha} \phi_m, \quad m \neq 0 \\ \frac{i}{2}(I_{-m} - I_m) &= 2\pi b_{\alpha}' \phi_m, \quad m \neq 0 \\ E_0 &= \beta \phi_0 \\ \frac{1}{2}(E_{-m} + E_m) &= \beta \phi_m, \quad m \neq 0 \\ \frac{i}{2}(E_{-m} - E_m) &= \beta' \phi_m, \quad m \neq 0 \end{aligned} \quad (5.16)$$

Substituting the above equations into eq. (5.13)

$$\begin{aligned} \frac{\partial}{\partial \phi} \int_0^{2\pi} d\phi' \sum_{m=-\infty}^{\infty} K_m e^{-im(\phi - \phi')} \frac{\partial}{\partial \phi'} I(\phi') + k^2 a^2 \int_0^{2\pi} d\phi' \sum_{m=-\infty}^{\infty} K_m e^{-im(\phi - \phi')} \cos(\phi - \phi') I(\phi') \\ = \frac{4\pi aik}{Z_0} \sum_{m=-\infty}^{\infty} E_m e^{-im\phi} \end{aligned} \quad (5.17)$$

Multiplying by  $\frac{1}{2\pi} e^{in\phi}$  and integrating with respect to  $\phi$  from 0 to  $2\pi$  gives,

$$\begin{aligned}
& \frac{1}{2\pi} \int_0^{2\pi} d\phi' \frac{\partial}{\partial \phi'} I(\phi') \sum_{m=-\infty}^{\infty} e^{im\phi'} \int_0^{2\pi} (-im) K_m e^{-im\phi} e^{in\phi} d\phi \\
& + \frac{k^2 a^2}{4\pi} \int_0^{2\pi} d\phi' I(\phi') \sum_{m=-\infty}^{\infty} \left[ e^{i(m+1)\phi'} \int_0^{2\pi} K_m e^{-i(m+1)\phi} e^{in\phi} d\phi \right. \\
& \left. + e^{i(m-1)\phi'} \int_0^{2\pi} K_m e^{-i(m-1)\phi} e^{in\phi} d\phi \right] \\
& = i \frac{4\pi ak}{2\pi Z_0} \int_0^{2\pi} \sum_{m=-\infty}^{\infty} E_m e^{-im\phi} e^{in\phi} d\phi
\end{aligned} \tag{5.18}$$

This reduces to

$$\begin{aligned}
& \int_0^{2\pi} \left[ \frac{\partial}{\partial \phi'} I(\phi') e^{in\phi'} (-inK_n) \right] d\phi' + \frac{k^2 a^2}{2} \int_0^{2\pi} I(\phi') \left[ e^{in\phi'} K_{n-1} + e^{in\phi'} K_{n+1} \right] d\phi' \\
& = i \frac{4\pi ak}{Z_0} E_n
\end{aligned} \tag{5.19}$$

Now, substitute eq. (3.15)

$$\begin{aligned}
& - \int_0^{2\pi} \left[ \sum_{m=-\infty}^{\infty} im I_m e^{-im\phi'} e^{in\phi'} (-inK_n) \right] d\phi' \\
& + \frac{k^2 a^2}{2} \int_0^{2\pi} \left[ \sum_{m=-\infty}^{\infty} I_m e^{-im\phi'} e^{in\phi'} (K_{n-1} + K_{n+1}) \right] d\phi' \\
& = \frac{i4\pi ak}{Z_0} E_n
\end{aligned} \tag{5.20}$$

It becomes

$$I_n \left[ (-n^2 K_n) + \frac{k^2 a^2}{2} (K_{n-1} + K_{n+1}) \right] = \frac{i2ak}{Z_0} E_n \quad (5.21)$$

Let

$$\lambda_n = a \left[ \frac{(-n^2 K_n)}{ka} + \frac{ka}{2} (K_{n-1} + K_{n+1}) \right] \quad (5.22)$$

then

$$I_n \lambda_n = \frac{i2a}{Z_0} E_n \quad (5.23)$$

Thus, knowing the Fourier coefficients of the surface electric field, the current can be found. Here  $E_n$  is also a function of  $I_n$  if the toroid is loaded.

From eqs. (5.13) and (5.15),

$$K_n = \frac{1}{4\pi^2} \int_0^{2\pi} e^{-in\phi} d\phi \int_{-\pi}^{\pi} \left[ (2a \sin(\frac{\phi}{2}))^2 + (2b \sin(\frac{\theta}{2}))^2 \right]^{-1/2} \exp \left\{ -ik \left[ (2a \sin(\frac{\phi}{2}))^2 + (2b \sin(\frac{\theta}{2}))^2 \right] \right\} d\theta \quad (5.24)$$

Now, let,

$$A = 2b \sin(\theta/2) \quad (5.25)$$

such that,

$$dA = b \cos(\frac{\theta}{2}) d\theta = b \sqrt{1 - \sin^2(\frac{\theta}{2})} d\theta = b \sqrt{1 - \frac{A^2}{4b^2}} d\theta = \frac{1}{2} \sqrt{4b^2 - A^2} d\theta \quad (5.26)$$

and define a new function

$$M_n(A) = \int_0^{2\pi} e^{in\phi} \left[ (2a \sin(\frac{\phi}{2}))^2 + A^2 \right]^{-1/2} \exp \left\{ -ik \left[ (2a \sin(\frac{\phi}{2}))^2 + A^2 \right]^{1/2} \right\} d\phi \quad (5.27)$$

then,

$$\begin{aligned} K_n &= \frac{1}{2\pi} \int_{\theta=-\pi}^{\theta=\pi} (4b^2 - A^2)^{-1/2} M_n(A) dA = \frac{1}{2} \int_{\theta=0}^{\theta=\pi} (4b^2 - A^2)^{-1/2} M_n(A) dA \\ &= \frac{1}{2} \int_0^{2b} (4b^2 - A^2)^{-1/2} M_n(A) dA \end{aligned} \quad (5.28)$$

$M_n(A)$  can be simplified for small  $A$ . If  $0 < A^2 \ll b^2$  and  $kA \ll 1$ , then

$$M_n(A) = \int_0^{2\pi} e^{in\phi} \frac{\exp \left[ -ik(2a \sin(\frac{\phi}{2})) \right]}{2a \sin(\frac{\phi}{2})} d\phi \quad (5.29)$$

The differences are,

$$\begin{aligned} \Delta_n &\equiv M_{n+1}(A) - M_n(A) = \int_0^{2\pi} \frac{\exp \left[ -2ika \sin(\frac{\phi}{2}) \right]}{2a \sin(\frac{\phi}{2})} \left[ e^{i(n+1)\phi} - e^{in\phi} \right] d\phi \\ &= \int_0^{2\pi} \frac{\exp \left[ -2ika \sin(\frac{\phi}{2}) \right]}{2a \sin(\frac{\phi}{2})} e^{i(n+\frac{1}{2})\phi} \left[ e^{i\frac{\phi}{2}} - e^{-i\frac{\phi}{2}} \right] d\phi \\ &= \int_0^{2\pi} \frac{2i \exp \left[ -2ika \sin(\frac{\phi}{2}) + i(n+\frac{1}{2})\phi \right]}{a} d\phi \\ &= \int_0^{\pi} \frac{2i}{a} \exp \left[ -2ika \sin(\theta) + i(2n+1)\theta \right] d\theta \end{aligned} \quad (5.30)$$

Now, let  $n = 0$ ,

$$\begin{aligned}
 M_0(A) &= \int_0^{2\pi} \frac{\exp\left\{-ik\left[(2a \sin(\frac{\phi}{2}))^2 + A^2\right]^{1/2}\right\}}{\left[(2a \sin(\frac{\phi}{2}))^2 + A^2\right]^{1/2}} d\phi \\
 &\approx \int_0^{2\pi} \frac{\exp\left[-ik(2a \sin(\frac{\phi}{2}))\right] - 1}{2a \sin(\frac{\phi}{2})} d\phi + \int_0^{2\pi} \frac{d\phi}{\sqrt{\left[2a \sin(\frac{\phi}{2})\right]^2 + A^2}} \quad (5.31)
 \end{aligned}$$

The first term is

$$\begin{aligned}
 &\int_0^{2\pi} \frac{\exp\left[-ik(2a \sin(\frac{\phi}{2}))\right] - 1}{2a \sin(\frac{\phi}{2})} d\phi = \int_0^{\pi} \frac{\exp[-i2ka \sin(\theta)] - 1}{a \sin(\theta)} d\theta \\
 &= \frac{-i}{a} \int_0^{\pi} d\theta \int_0^{2ka} \exp[-ix \sin(\theta)] dx = \frac{-i}{a} \int_0^{2ka} dx \int_0^{\pi} \exp[-ix \sin(\theta)] d\theta \\
 &= \frac{-i\pi}{a} \int_0^{2ka} \left[ J_0(x) + i E_0(x) \right] dx \quad (5.32)
 \end{aligned}$$

The second term is<sup>10,11</sup>, for  $A/a \rightarrow 0$ ,

$$\int_0^{2\pi} \frac{d\phi}{\sqrt{\left[2a \sin(\frac{\phi}{2})\right]^2 + A^2}} = \frac{2}{a} \ln\left(\frac{8a}{A}\right) + \text{terms } O\left(\frac{A^2}{a^2}\right) \quad (5.33)$$

Here,

$$J_{\nu}(z) = \frac{1}{\pi} \int_0^{\pi} \cos[\nu\theta - z \sin(\theta)] d\theta \quad (5.34)$$

is the Anger's function which reduces to the Bessel function  $J_\nu(z)$  if  $\nu$  is an integer;

$$\mathbb{E}_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin[\nu\theta - z \sin(\theta)] d\theta \quad (5.35)$$

is called Weber's function. This is also written as  $-\Omega(z)$  where  $\Omega(z)$  is the Lommel-Weber function referred by Wu.<sup>5</sup> The integral of the Anger-Weber function,  $S_m(z)$  is defined as

$$S_m(z) \equiv \frac{i}{2} \int_0^{2z} \left[ \mathbb{J}_m(x) + i\mathbb{E}_m(x) \right] dx \quad (5.36)$$

and has been discussed in detail in Mathematics Note 25.<sup>12</sup> Therefore,

$$M_0(A) = \frac{2}{a} \ln\left(\frac{8a}{A}\right) - \frac{2\pi}{a} S_0(ka) + O\left(\frac{A^2}{a}\right) \quad (5.37)$$

From eq. (5.30),

$$\begin{aligned} M_n(A) &= M_0 + \sum_0^{n-1} \Delta_m = M_0 + \sum_0^{n-1} \int_0^\pi \frac{2i}{a} \exp[-2ika \sin(\theta)] \exp[i(2m+1)\theta] d\theta \\ &= M_0 + \frac{2i}{a} \int_0^\pi \exp[-2ika \sin(\theta)] \left\{ \sum_0^{n-1} e^{i(2m+1)\theta} \right\} d\theta \end{aligned} \quad (5.38)$$

Here

$$\sum_0^{n-1} e^{i(2m+1)\theta} = e^{i\theta} \frac{1 - e^{i2n\theta}}{1 - e^{i2\theta}} = \frac{1 - e^{i2n\theta}}{-2i \sin(\theta)} \quad (5.39)$$

Therefore,

$$\begin{aligned} M_n(A) &= M_0 - \int_0^\pi \frac{\exp[-2ika \sin(\theta)][1 - e^{i2n\theta}]}{a \sin(\theta)} d\theta \\ &= \frac{2}{a} \ln\left(\frac{8a}{A}\right) + \int_0^\pi \frac{\exp[-i2ka \sin(\theta)] - 1}{a \sin(\theta)} d\theta - \int_0^\pi \frac{\exp[-2ika \sin(\theta)][1 - e^{i2n\theta}]}{a \sin(\theta)} d\theta \\ &= \frac{2}{a} \ln\left(\frac{8a}{A}\right) + \int_0^\pi \frac{[\exp(-i2ka \sin(\theta)) - 1]e^{i2n\theta}}{a \sin(\theta)} d\theta + \int_0^\pi \frac{[e^{i2n\theta} - 1]}{a \sin(\theta)} d\theta \\ &= \frac{2}{a} \ln\left(\frac{8a}{A}\right) - \frac{i}{a} \int_0^{2ka} dx \int_0^\pi \exp[i2n\theta - ix \sin(\theta)] d\theta \\ &\quad + \frac{2i}{a} \int_0^\pi \left[ \sum_0^{n-1} e^{i(2m+1)\theta} \right] d\theta \\ &= \frac{2}{a} \ln\left(\frac{8a}{A}\right) - \frac{i\pi}{a} \int_0^{2ka} \left[ \mathcal{J}_{2n}(x) + i\mathcal{E}_{2n}(x) \right] dx - \frac{4}{a} \sum_{m=0}^{n-1} \frac{1}{2m+1} \\ &= \frac{2}{a} \ln\left(\frac{8a}{A}\right) - \frac{2\pi}{a} S_{2n}(ka) - \frac{4}{a} \sum_{m=0}^{n-1} \frac{1}{2m+1} \end{aligned} \quad (5.40)$$

Define,

$$N_n(A) = \int_{-\infty}^{\infty} e^{in\phi} \frac{\exp\left[-ik\left[a^2\phi^2 + A^2\right]^{1/2}\right]}{\left[a^2\phi^2 + A^2\right]^{1/2}} d\phi \quad (5.41)$$



Let  $\phi = \frac{A}{a} \sinh(t)$  such that  $d\phi = \frac{A}{a} \cosh(t) dt = \sqrt{\phi^2 + \left(\frac{A}{a}\right)^2} dt$ ,

then,

$$\begin{aligned} N_n(A) &= \frac{1}{a} \int_{-\infty}^{\infty} \exp \left\{ -ikA \cosh(t) + in \frac{A}{a} \sinh(t) \right\} dt \\ &= \frac{1}{a} \int_{-\infty}^{\infty} \exp \left\{ -iA \left[ k \cosh(t) - \frac{n}{a} \sinh(t) \right] \right\} dt \end{aligned} \quad (5.42)$$

Change variables by letting  $k = B \cosh(x)$  and  $n/a = B \sinh(x)$ , then,

$$\begin{aligned} N_n(A) &= \frac{1}{a} \int_{-\infty}^{\infty} \exp \left\{ -iAB \left[ \cosh(x) \cosh(t) - \sinh(x) \sinh(t) \right] \right\} dt \\ &= \frac{1}{a} \int_{-\infty}^{\infty} \exp \left\{ -iAB \cosh(t - x) \right\} dt \end{aligned} \quad (5.43)$$

Where  $B^2 = B^2 \cosh^2(x) - B^2 \sinh^2(x) = k^2 - (n/a)^2$ . Let  $v = t - x$  such that  $dv = dt$ , then,

$$N_n(A) = \frac{1}{a} \int_{-\infty}^{\infty} \exp \left\{ -iAB \cosh(v) \right\} dv \quad (5.44)$$

Referring to Gradshteyn and Ryzhik<sup>13</sup>

$$\begin{aligned} N_n(A) &= \frac{-\pi i}{a} H_0^{(2)} \left[ \sqrt{k^2 - \left(\frac{n}{a}\right)^2} A \right] \\ &= \frac{2}{a} K_0 \left[ \sqrt{\left(\frac{n}{a}\right)^2 - k^2} A \right] \end{aligned} \quad (5.45)$$

where  $H_0^{(2)}$  is the zero order of the Hankel function of the second kind and  $K_0$  is the zero order of the modified Bessel function of the second kind.

For small  $A$ ,

$$N_n(A) \approx -\frac{2}{a} \left\{ \ln \left[ \sqrt{\left(\frac{n}{a}\right)^2 - k^2 \frac{A}{2}} \right] + \gamma \right\} \quad (5.46)$$

where  $\gamma$  is Euler's constant,<sup>14</sup> numerically about 0.5772157.

At least for  $n$  not too large, the difference between  $M_n(A)$  and  $N_n(A)$  is approximately independent of  $A$ . In other words,

$$M_n(A) - N_n(A) \Big|_{\text{for any } A} \approx M_n(A) - N_n(A) \Big|_{\text{for small } A} \quad (5.47)$$

The right-hand side can be obtained from eqs. (5.40) and (5.46). The exact  $N_n(A)$  is given in eq. (5.45). Thus, the general solution of  $M_n(A)$  for any  $A$  is approximately

$$M_n(A) = \frac{2}{a} K_0 \left[ \sqrt{\left(\frac{n}{a}\right)^2 - k^2 A} \right] + \frac{2}{a} \ln\left(\frac{8a}{A}\right) - \frac{2\pi}{a} S_{2n}(ka) - \frac{4}{a} \sum_{m=0}^{n-1} \frac{1}{2m+1} + \frac{2}{a} \left\{ \ln \left[ \sqrt{\left(\frac{n}{a}\right)^2 - k^2 \frac{A}{2}} \right] + \gamma \right\} \quad (5.48)$$

If we define,

$$C_n = \gamma - 2 \sum_{m=0}^{n-1} \frac{1}{2m+1} + \ln(4n) \quad (5.49)$$

we obtain

$$aM_n(A) = 2K_0 \left[ \sqrt{\left(\frac{n}{a}\right)^2 - k^2} A \right] + \ln \left[ 1 - \left(\frac{ka}{n}\right)^2 \right] + 2C_n - 2\pi S_{2n}(ka) \quad (5.50)$$

By the same argument, we may assume that the difference of,

$$\begin{aligned} K_0 \left[ \sqrt{\left(\frac{n}{a}\right)^2 - k^2} A \right] - K_0 \left(\frac{nA}{a}\right) &\approx \lim_{A \rightarrow 0} \left\{ K_0 \left[ \sqrt{\left(\frac{n}{a}\right)^2 - k^2} A \right] - K_0 \left(\frac{nA}{a}\right) \right\} \\ &= -\gamma - \ln \left[ \sqrt{\left(\frac{n}{a}\right)^2 - k^2} \frac{A}{2} \right] + \gamma + \ln \left(\frac{nA}{2a}\right) \\ &= -\frac{1}{2} \ln \left[ 1 - \left(\frac{ka}{n}\right)^2 \right] \end{aligned} \quad (5.51)$$

Thus,

$$aM_n(A) = 2K_0 \left(\frac{nA}{a}\right) + 2C_n - 2\pi S_{2n}(ka) \quad (5.52)$$

Now, <sup>15, 16</sup>

$$\begin{aligned} \frac{1}{\pi} \int_0^{2b} \frac{dA}{\sqrt{4b^2 - A^2}} &= \frac{1}{2\pi} \\ \frac{1}{\pi} \int_0^{2b} \frac{K_0 \left(\frac{nA}{a}\right) dA}{\sqrt{4b^2 - A^2}} &= \frac{1}{\pi} \int_0^{\pi/2} K_0 \left[ \frac{2nb}{a} \sin(\theta) \right] d\theta \\ &= \frac{1}{2\pi} I_0 \left(\frac{nb}{a}\right) K_0 \left(\frac{nb}{a}\right) \end{aligned} \quad (5.53)$$

where  $I_0$  is the zero order of the modified Bessel function of the first kind

and  $A = 2b \sin(\theta)$  is used so that  $d\theta = \frac{dA}{2b \cos(\theta)} = \frac{dA}{\sqrt{4b^2 - A^2}}$ .

The Fourier coefficients of the Green's function can be calculated from eq. (5.28), which becomes,

$$K_n = \frac{1}{a\pi} \left\{ \kappa_0 \left( \frac{nb}{a} \right) I_0 \left( \frac{nb}{a} \right) + C_n - \pi S_{2n}(ka) \right\} \quad (5.54)$$

Now, consider the case for  $n = 0$ . Eq. (5.37) shows that for  $A^2 \ll a^2$

$$M_0(A) = \frac{2}{a} \ln \left( \frac{8a}{A} \right) - \frac{2\pi}{a} S_0(ka) \quad (5.55)$$

Referring to Dwight,<sup>16</sup>

$$\begin{aligned} \frac{2}{\pi^2 a} \int_0^{2b} \frac{dA \ln \left( \frac{8a}{A} \right)}{\sqrt{4b^2 - A^2}} &= \frac{1}{\pi^2 a} \int_0^1 \frac{dx \ln \left( \frac{4a}{bx} \right)}{\sqrt{1-x^2}} \\ &= \frac{2}{\pi^2 a} \left[ \ln \left( \frac{4a}{b} \right) \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \int_0^1 \frac{dx \ln \left( \frac{1}{x} \right)}{\sqrt{1-x^2}} \right] \\ &= \frac{2}{\pi^2 a} \left[ \ln \left( \frac{4a}{b} \right) \sin^{-1} x \Big|_0^1 + \frac{\pi}{2} \ln(2) \right] \\ &= \frac{1}{\pi a} \ln \left( \frac{8a}{b} \right) \end{aligned} \quad (5.56)$$

Using eq. (5.28), (5.55), (5.53), and (5.56), we obtain

$$K_0 = \frac{1}{a\pi} \left\{ \ln \left( \frac{8a}{b} \right) - \pi S_0(ka) \right\} \quad (5.57)$$

Equations (5.54) and (5.57) agree with Colling and Zucker.<sup>17</sup>

Current distributions on TORUS can be found from eq. (5.23) if the surface electric field is known. This equation is valid only if the antenna is loaded with a uniform impedance. For a load of  $Z'$  ohms per unit length the surface electric field is the sum of the potential drop in  $Z'$  and the source field  $E_g$  which is assumed a function of  $\phi$ . That is,

$$E_{S\phi} = E_g(\phi) + Z'I(\phi) \quad (5.58)$$

Note that both  $E_g$  and the drop  $Z'I$  are taken in the positive  $\phi$  direction and add to give the net  $E_S$  on the toroid. The  $E_S$  expanded in Fourier series is,

$$E_{S\phi} = \sum_{n=-\infty}^{\infty} (g_n + Z'I_n) e^{-in\phi} \quad (5.59)$$

and so,

$$E_n = g_n + Z'I_n \quad (5.60)$$

From eq. (5.23),

$$I_n \lambda_n = i \left( \frac{2\pi a}{\pi Z_0} \right) (g_n + Z'I_n) \quad (5.61)$$

Thus we obtain the Fourier coefficients of the current distribution in terms of the total impedance on the TORUS,

$$Z = 2\pi a Z' \quad (5.62)$$

and the source of excitation  $g_n$ ,

$$I_n = \frac{-2\pi a g_n}{Z + i\pi Z_0 \lambda_n} \quad (5.63)$$

If the source is a delta-gap voltage  $V_0$  located at  $\phi = 0$ , the source field on the surface of the loop is,<sup>17</sup>

$$\begin{aligned} E_g &= -V_0 \delta(\phi)/a \\ &= -\sum_{n=-\infty}^{\infty} \frac{V_0}{2\pi a} e^{-in\phi} \end{aligned} \quad (5.64)$$

Therefore,

$$I_n = \frac{V_0}{Z + i\pi Z_0 \lambda_n} \quad (5.65)$$

This is the Fourier coefficient for the surface current shown in eq. (5.14).

Since  $K_n = K_{-n}$  given in eq. (5.24), it can be easily shown that  $\lambda_n = \lambda_{-n}$  from eq. (5.22) and  $I_n = I_{-n}$  from eq. (5.62). One may also express the current in the form of eq. (4.4) whose Fourier coefficients can be found from eq. (5.16). They are,

$$\alpha_{\phi n} = \frac{I_n}{2\pi b} \quad (5.66)$$

and

$$\alpha'_{\phi n} = 0 \quad (5.67)$$

Clearly, the surface current is symmetry with respect to  $\phi$ . We have neglected the current in  $\xi$  direction so that both  $\alpha_{\xi n}$  and  $\alpha'_{\xi n}$  are zero.

## VI. Fields at the Center

In order to minimize the testing volume, equipment will be located near the center of the TORUS for testing the nuclear EMP effects. In this chapter we will study the possibility of simulating a plane-wave at the center of the TORUS using a uniform loading impedance. The source of excitation will be a delta-gap generator of constant voltage  $V_0$ , an impulse in the time domain.

Fields in space can be calculated by the integral equations in Chapter III. They, in general, can only be computed numerically. Here, an analytical approach will be tried to find the field at a particular point of interest, namely the center of TORUS.

The current on the toroid has two components; the  $\phi$  component  $J_{S\phi}$  and the  $\xi$  component  $J_{S\xi}$ . As was discussed in Chapter V,  $J_{S\xi}$  may be neglected if  $kb \ll 1$ . However, as long as  $kb$  is not too large,  $J_{S\xi}$  still is small compared to  $J_{S\phi}$  in most portions of the toroid and  $J_{S\phi}$  is approximately independent of  $\phi$ . The contribution to the fields at the center under the condition  $b^2 \ll a^2$  would be mainly due to the  $J_{S\phi}$  component even if  $kb$  is close to 1. The  $\phi$  surface current density thus can be regarded as uniformly distributed in such a way that the total current  $I(\phi) = 2\pi b J_{S\phi}(\phi)$ . Omitting the time dependent factor  $e^{i\omega t}$ , the vector potential may be calculated as follows,

$$\begin{aligned}
\vec{A} &= \frac{\mu}{4\pi} \iint_{S'} \frac{\vec{J}_{S\phi}(\phi', \xi') e^{-\frac{i\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \text{ad}\phi' \text{bd}\xi' \\
&= \frac{\mu a}{4\pi} \iint_{S'} \frac{I(\phi') \vec{e}'_{\phi} e^{-\frac{i\omega}{c}|\vec{r}-\vec{r}'|}}{2\pi |\vec{r}-\vec{r}'|} \text{d}\phi' \text{d}\xi'
\end{aligned} \tag{6.1}$$

and the scalar potential,

$$\begin{aligned}
\Phi &= \frac{1}{4\pi\epsilon} \iint_{S'} \frac{\rho_s(\vec{r}') e^{-\frac{i\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \text{ad}\phi' \text{bd}\xi' \\
&= \frac{-1}{4\pi\epsilon} \iint_{S'} \left( \frac{1}{i\omega} \right) \frac{\partial I(\phi')}{2\pi \partial \phi'} e^{-\frac{i\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \text{d}\phi' \text{d}\xi'
\end{aligned} \tag{6.2}$$

Here the continuity relation for the charge accumulation  $\rho_s$  and current  $\vec{J}_{S\phi}$  has been used as follows,

$$\frac{\partial}{\partial t} \rho_s(\vec{r}') = i\omega \rho_s(\vec{r}') = -\nabla' \cdot \vec{J}_s(\vec{r}') = -\frac{\partial}{\partial \phi'} \left[ \frac{I(\phi')}{2\pi b} \right] \tag{6.3}$$

A vector identity gives,

$$\nabla \times \left( \frac{e^{-\frac{i\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \vec{e}'_{\phi} \right) = \nabla \left( \frac{e^{-\frac{i\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \right) \times \vec{e}'_{\phi} + \left( \frac{e^{-\frac{i\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \right) \nabla \times \vec{e}'_{\phi} \tag{6.4}$$

Here,  $\nabla \times \vec{e}'_{\phi} = 0$  for a fixed  $\vec{r}'$  which is not a function of  $\vec{r}$ .



In rectangular coordinates,

$$|\vec{r} - \vec{r}'| = \left[ (x - x')^2 + (y - y')^2 + (z - z')^2 \right]^{1/2}$$

$$\nabla \frac{1}{|\vec{r} - \vec{r}'|} = \frac{\vec{e}_x(x - x') + \vec{e}_y(y - y') + \vec{e}_z(z - z')}{|\vec{r} - \vec{r}'|^3} \quad (6.5)$$

So,

$$\nabla \frac{e^{-\frac{i\omega}{c}|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} = \frac{\left[ \vec{e}_x(x - x') + \vec{e}_y(y - y') + \vec{e}_z(z - z') \right] \left[ |\vec{r} - \vec{r}'| \left( \frac{-i\omega}{c} \right) - 1 \right]}{|\vec{r} - \vec{r}'|^3}$$

$$= e^{-\frac{i\omega}{c}|\vec{r} - \vec{r}'|} \cdot \left[ \frac{\vec{e}_x(x - x') + \vec{e}_y(y - y') + \vec{e}_z(z - z')}{|\vec{r} - \vec{r}'|^2} - \frac{1}{|\vec{r} - \vec{r}'|} \left( \frac{-i\omega}{c} \right) \right] \quad (6.6)$$

At the center where  $\vec{r} = 0$ ,

$$\nabla \frac{e^{-\frac{i\omega}{c}|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} = \frac{\left[ \vec{e}_x x' + \vec{e}_y y' + \vec{e}_z z' \right] \left[ 1 + \frac{i\omega}{c}|\vec{r}'| \right] e^{-\frac{i\omega}{c}|\vec{r}'|}}{|\vec{r}'|^2}$$

$$= \left[ \vec{e}_x \cos(\phi') + \vec{e}_y \sin(\phi') + \vec{e}_z \frac{z'}{|\vec{r}'|} \right] \frac{\left[ 1 + \frac{i\omega}{c}|\vec{r}'| \right] e^{-\frac{i\omega}{c}|\vec{r}'|}}{|\vec{r}'|^2}$$

$$= \left[ \vec{e}'_{\Psi} + \vec{e}'_z \frac{z'}{|\vec{r}'|} \right] \frac{\left[ 1 + \frac{i\omega}{c}|\vec{r}'| \right] e^{-\frac{i\omega}{c}|\vec{r}'|}}{|\vec{r}'|^2} \quad (6.7)$$

The fields at the center are now obtained:

$$\begin{aligned}
\vec{H}_c &= \frac{1}{\mu} \nabla \times \vec{A} = \frac{a}{4\pi} \iint_{S'} \frac{I(\phi')}{2\pi} \left[ \vec{e}'_{\Psi} + \vec{e}'_z \frac{z'}{|\vec{r}'|} \right] \times \vec{e}'_{\phi} \frac{\left[ 1 + \frac{i\omega}{c} |\vec{r}'| \right] e^{-\frac{i\omega}{c} |\vec{r}'|}}{|\vec{r}'|^2} d\phi' d\xi' \\
\vec{E}_c &= -\nabla \phi - \frac{\partial \vec{A}}{\partial t} \\
&= \frac{a}{4\pi} \iint_{S'} \frac{1}{2\pi} \left\{ -i\omega\mu I(\phi') \vec{e}'_{\phi} \frac{e^{-\frac{i\omega}{c} |\vec{r}'|}}{|\vec{r}'|} \right. \\
&\quad \left. + \frac{1}{ia\omega\epsilon} \frac{\partial I(\phi')}{\partial \phi'} \left[ \vec{e}'_{\Psi} + \vec{e}'_z \frac{z'}{|\vec{r}'|} \right] \frac{\left[ 1 + \frac{i\omega}{c} |\vec{r}'| \right] e^{-\frac{i\omega}{c} |\vec{r}'|}}{|\vec{r}'|^2} \right\} d\phi' d\xi' \quad (6.8)
\end{aligned}$$

It has been shown in eqs. (2.10) and (2.3) that  $|\vec{r}'|$  is even and  $z'$  is odd with respect to  $\xi'$ . The terms involving the integration of  $z'$  with respect to  $\xi'$  are therefore zero.

With a uniform load, the current will be evenly distributed with respect to the location of source,  $\phi = 0$ . The Fourier expansion in eq. (5.14) can now be written as

$$\begin{aligned}
I(\phi') &= I_0 + 2 \sum_{n=1}^{\infty} I_n \cos(n\phi') \\
\frac{\partial I(\phi')}{\partial \phi'} &= -2 \sum_{n=1}^{\infty} n I_n \sin(n\phi') \quad (6.9)
\end{aligned}$$

and therefore,

$$\int_{-\pi}^{\pi} \vec{e}'_{\phi} I(\phi') d\phi' = \int_{-\pi}^{\pi} \vec{e}'_y \cos(\phi') I(\phi') d\phi'$$

$$\int_{-\pi}^{\pi} \vec{e}'_{\psi} \frac{\partial I(\phi')}{\partial \phi'} d\phi' = \int_{-\pi}^{\pi} \vec{e}'_y \sin(\phi') \frac{\partial I(\phi')}{\partial \phi'} d\phi' \quad (6.10)$$

Substituting eqs. (6.9) and (6.10) into (6.8),

$$\vec{E}_c = -\vec{e}'_y \frac{I_1 Z_0 a}{4\pi} \int_{-\pi}^{\pi} \left\{ \frac{i\omega}{c} \frac{e^{-\frac{i\omega}{c}|\vec{r}'|}}{|\vec{r}'|} + \frac{c}{ia\omega} \frac{\left[1 + \frac{i\omega}{c}|\vec{r}'|\right] e^{-\frac{i\omega}{c}|\vec{r}'|}}{|\vec{r}'|^2} \right\} d\xi'$$

$$\vec{H}_c = \vec{e}'_z \frac{a}{4\pi} I_0 \int_{-\pi}^{\pi} \frac{\left[1 + \frac{i\omega}{c}|\vec{r}'|\right] e^{-\frac{i\omega}{c}|\vec{r}'|}}{|\vec{r}'|^2} d\xi' \quad (6.11)$$

Since  $a^2 \gg b^2$ ,  $kb \ll 1$ , and  $|r'| \approx a$  we finally obtain,

$$\vec{E}_c \approx -\vec{e}'_y \frac{I_1 Z_0}{2a} \left( \frac{i\omega a}{c} + 1 + \frac{c}{i\omega a} \right) e^{-\frac{i\omega a}{c}}$$

$$\vec{H}_c \approx \vec{e}'_z \frac{I_0}{2a} \left( 1 + \frac{i\omega a}{c} \right) e^{-\frac{i\omega a}{c}} \quad (6.12)$$

With  $k = \omega/c$ , the fields at the center of the toroid are,

$$\begin{aligned}\vec{E}_c &= -\vec{e}_y \frac{I_1 Z_0}{2a} \left( ika + 1 + \frac{1}{ika} \right) e^{-ika} \\ \vec{H}_c &= \vec{e}_z \frac{I_0}{2a} (1 + ika) e^{-ika}\end{aligned}\quad (6.13)$$

Instead, one may let  $|\vec{r}'| \approx a + b \sin(\xi)$ . Then,

$$\int_{-\pi}^{\pi} e^{-\frac{i\omega}{c}|\vec{r}'|} d\xi' \approx e^{-\frac{i\omega a}{c}} \int_{-\pi}^{\pi} e^{-\frac{i\omega}{c} b \sin \xi} d\xi = 2\pi e^{-\frac{i\omega a}{c}} J_0\left(\frac{\omega b}{c}\right)$$

That is, we should multiply eq. (6.12) by a factor of  $J_0\left(\frac{\omega b}{c}\right)$ . However, numerical computations of the field strength show larger errors with the Bessel function than without this factor. It is decided, therefore, to use eq. (6.12) or eq. (6.13) for the field calculation at low frequencies.

The ratio of electric to magnetic field can now be calculated,

$$\Lambda \equiv \frac{-E_c}{Z_0 H_c} = \frac{I_1}{I_0} \frac{(1 + ika - k^2 a^2)}{(ika - k^2 a^2)} = \Lambda' \frac{(1 + ika - k^2 a^2)}{(ika - k^2 a^2)} \quad (6.14)$$

With the total loading impedance  $Z$ , it is shown in eq. (5.65) that,

$$\Lambda' = \frac{I_1}{I_0} = \frac{Z + i\pi Z_0 \lambda_0}{Z + i\pi Z_0 \lambda_1} \quad (6.15)$$

Here  $\lambda_0$  and  $\lambda_1$  are derived in Chapter V. For  $n = 0$ ,  $nb/a \ll 1$ . The asymptotic approximations for the  $K_0$  and  $I_0$  can be used.<sup>18</sup> These are,

$$I_0(x) \approx 1$$

$$K_0(x) \approx -\ln\left(\frac{x}{2}\right) - \gamma \quad (6.16)$$

From eq. (5.54)

$$K_1 = \frac{1}{a\pi} \left\{ \left[ \ln\left(\frac{8a}{b}\right) - 2 \right] - \pi S_2(ka) \right\}$$

$$K_2 = \frac{1}{a\pi} \left\{ \left[ \ln\left(\frac{8a}{b}\right) - \frac{8}{3} \right] - \pi S_4(ka) \right\} \quad (6.17)$$

These equations in association with eqs. (5.57) and (5.22) give

$$\lambda_0 = ka \left\{ \frac{1}{\pi} \left[ \ln\left(\frac{8a}{b}\right) - 2 \right] - S_2(ka) \right\}$$

$$\lambda_1 = \frac{ka}{2} \left\{ \frac{1}{\pi} \left[ 2 \ln\left(\frac{8a}{b}\right) - \frac{8}{3} \right] - S_0(ka) - S_4(ka) \right\}$$

$$- \frac{1}{ka} \left\{ \frac{1}{\pi} \left[ \ln\left(\frac{8a}{b}\right) - 2 \right] - S_2(ka) \right\} \quad (6.18)$$

The main object of this study is to find a loading impedance such that the ratio of electric to magnetic field at the center is that of free space, i. e.,  $\Lambda = 1$ . According to eqs. (6.14), (6.16), and (6.18) this impedance  $Z_i$  is related to frequencies in the following manner,

$$\frac{Z_i}{Z_0} = \left[ \ln\left(\frac{8a}{b}\right) - 2 \right] - \pi S_2(ka) - \left[ (ka)^2 + i(ka)^3 \right]$$

$$\left\{ \frac{2}{3} - \frac{\pi}{2} \left[ S_0(ka) - 2S_2(ka) + S_4(ka) \right] \right\} \quad (6.19)$$

This impedance contains not only passive elements but also active elements. It is rather impractical and therefore  $Z_i$  will be referred to as an ideal impedance. However, it is possible that some kind of combination of the passive elements might come close to this value. For example, at extremely low frequencies the condition of  $\Lambda = 1$  can be accomplished by a loading resistance of

$$R_0 = Z_0 \left[ \ln \left( \frac{8a}{b} \right) - 2 \right] \quad (6.20)$$

Actually, when  $ka$  is large,

$$\lim_{ka \rightarrow \infty} \Lambda = \lim_{ka \rightarrow \infty} \Lambda' = \lim_{ka \rightarrow \infty} \frac{Z + i\pi Z_0 \lambda_0}{Z + i\pi Z_0 \lambda_1} \quad (6.21)$$

as can be seen from eqs. (6.14) and (6.15). Using the asymptotic expansion of  $S_m(ka)^{12}$  and eq. (6.18), one can show

$$\lim_{ka \rightarrow \infty} \lambda_0 = \lim_{ka \rightarrow \infty} \lambda_1 = -\frac{ka}{\pi} \ln(ka)$$

Therefore,  $\Lambda$  is always one at large  $ka$  and is independent of load. Thus, a load of  $R_0$  could produce plane waves at the center of the toroidal loop antenna at low and high frequencies. At the intermediate frequencies the ratio of E to H fields fluctuates around one. As "a" increases the center is farther away from the current on the loop, which is the source of the field; the field is closer to a plane wave.

With load  $R_0$ ,

$$\begin{aligned} \lim_{ka \rightarrow 0} \lambda_0 &= \frac{ka}{\pi} \left[ \ln \left( \frac{8a}{b} \right) - 2 \right] \\ \lim_{ka \rightarrow 0} \lambda_1 &= -\frac{1}{\pi ka} \left[ \ln \left( \frac{8a}{b} \right) - 2 \right] \\ \lim_{ka \rightarrow 0} I_0 &= \frac{V_0}{R_0} \\ \lim_{ka \rightarrow 0} I_1 &= \frac{-V_0 ka}{iR_0} \end{aligned} \tag{6.22}$$

The fields at low frequencies now become

$$\begin{aligned} \vec{E}_0 &= -\vec{e}_y \frac{V_0 Z_0}{2aR_0} \\ \vec{H}_0 &= \vec{e}_z \frac{V_0}{2aR_0} \end{aligned} \tag{6.23}$$

This magnetic field agrees with the static field<sup>19</sup> at the center of a loop since  $V_0/R_0$  is the D.C. current on the loop. At very high frequencies, it can also be shown that eqs. (6.13) agree with the fields from an infinitely long cylindrical antenna excited by a delta-gap voltage.

Let's assume that  $ka \gg 1 \gg kb$ . The asymptotic formula for the integral of the Lommel-Weber function at large  $ka$ <sup>12</sup> gives,

$$\begin{aligned}
S_0(ka) &= \frac{i}{2} + \frac{1}{\pi} \ln(ka) - \frac{1}{\pi} \psi\left(\frac{1}{2}\right) \\
S_4(ka) &= \frac{i}{2} + \frac{1}{\pi} \ln(ka) - \frac{1}{\pi} \psi\left(\frac{5}{2}\right)
\end{aligned} \tag{6.24}$$

From the psi function relation<sup>18</sup>,

$$\begin{aligned}
\psi\left(\frac{1}{2}\right) &= -\gamma - 2\ln 2 \\
\psi(z+1) &= \psi(z) + \frac{1}{z}
\end{aligned} \tag{6.25}$$

one gets  $\psi(5/2) = \psi(1/2) + 8/3$ . Therefore, from eq. (6.18),

$$\begin{aligned}
\lim_{ka \rightarrow \infty} \lambda_1 &= \frac{ka}{2} \left\{ \frac{1}{\pi} \left[ 2 \ln\left(\frac{8a}{b}\right) - \frac{8}{3} \right] - S_0(ka) - S_4(ka) \right\} \\
&= -\frac{ka}{2} \left\{ \frac{2}{\pi} \left[ \ln\left(\frac{kb}{2}\right) + \gamma \right] + i \right\}
\end{aligned} \tag{6.26}$$

Therefore, under the condition that  $ka \gg 1 \gg kb$ ,

$$\begin{aligned}
I_1 &= \frac{V_0}{Z + i\pi Z_0 \lambda_1} \\
&\approx \frac{V_0}{Z + \frac{\pi Z_0 ka}{2} \left\{ 1 - \frac{2i}{\pi} \left[ \ln\left(\frac{kb}{2}\right) + \gamma \right] \right\}}
\end{aligned} \tag{6.27}$$

and from eq. (6.13)



$$\begin{aligned}\vec{E}_c &\approx -\vec{e}_y \frac{iI_1 Z_0 ka}{2a} e^{-ika} \\ &\approx -\vec{e}_y \left(\frac{V_0 i}{\pi}\right) \left(\frac{e^{-ika}}{a}\right) \frac{1}{\left\{1 - \frac{2}{\pi} i \left[\ln\left(\frac{kb}{2}\right) + \gamma\right] + \frac{2Z}{Z_0 \pi ka}\right\}}\end{aligned}\quad (6.28)$$

For  $kb \ll 1$ , the Hankel function of the second kind can be represented as follows:<sup>14</sup>

$$\begin{aligned}H_0^{(2)}(kb) &= J_0(kb) - iY_0(kb) \approx 1 - \frac{2}{\pi} i \left[\ln\left(\frac{kb}{2}\right) + \gamma\right] \\ H_1^{(2)}(kb) &= J_1(kb) - iY_1(kb) \approx \frac{2i}{\pi kb}\end{aligned}\quad (6.29)$$

So, eq. (6.28) can now be written in terms of Hankel functions as

$$\vec{E}_c \approx -\vec{e}_y \left(\frac{V_0 i}{\pi}\right) \left(\frac{e^{-ika}}{a}\right) \left[ \frac{1}{H_0^{(2)}(kb) - i\left(\frac{Z}{Z_0}\right)\left(\frac{b}{a}\right)H_1^{(2)}(kb)} \right]\quad (6.30)$$

which agrees with the radiation field from an infinitely long cylindrical antenna.<sup>8,9</sup> This is certainly understandable. At very high frequencies, the currents are largest near the generator. The other part of the antenna, whether it is curved or straight, has very little effect on the far field radiation. It is found numerically that eq. (6.30) agrees very well with eq. (6.13) somewhere around  $kb = 0.3$  for  $b/a$  from 0.01 to 0.001. Therefore, we may use eq. (6.13) for fields at  $kb < 0.3$  and then switch

to eq. (6.30) for higher frequencies. This way, we obtain fields at the center of TORUS at all frequencies. Although the assumptions were made that not only  $a^2 \gg b^2$ , but also that  $kb \ll 1$  in the derivations to eq. (6.13), the results indicate that this formula actually is still good for  $kb$  close to one.

If  $kb$  is not very small, then as long as  $b^2 \ll a^2$ , the local geometry around the source delta-gap closely approximates that near a delta-gap in an infinite cylindrical antenna of radius  $b$ . The gap electric field (almost uniformly distributed in  $\xi$ ) will result in an almost uniformly distributed  $J_{s\phi}$  with a small  $J_{s\xi}$  by comparison. The infinite cylindrical antenna then provides the approximate fields at the center of TORUS for  $|ka| \gg 1$  provided  $b^2 \ll a^2$ .

## VII. Numerical Results

We have obtained an ideal loading impedance  $Z_i$  for generating a plane-wave at the center of the toroid at all frequencies. The ratio of  $Z_i$  to the characteristic impedance of free space  $Z_0$  is given in eq. (6.19) which is,

$$\frac{Z_i}{Z_0} = \left[ \ln\left(\frac{8a}{b}\right) - 2 \right] - \pi S_2(ka) - \left[ (ka)^2 + i(ka)^3 \right] \left\{ \frac{2}{3} - \frac{\pi}{2} \left[ S_0(ka) - 2S_2(ka) + S_4(ka) \right] \right\} \quad (7.1)$$

and is plotted in figure 4 with respect to  $ka$ . This impedance contains not only passive elements but also active elements as shown in the phase plot. It is shown in eq. (6.20) that  $Z_i = R_0$  at low frequency, where

$$R_0 = Z_0(\ln(8a/b) - 2) \quad (7.2)$$

is independent of frequency.

From eqs. (6.14) and (6.15) we have already obtained the ratio of  $E_c$  to  $Z_0 H_c$ ,

$$\Lambda = \frac{-E_c}{Z_0 H_c} = \left( \frac{Z + i\pi Z_0 \lambda_0}{Z + i\pi Z_0 \lambda_1} \right) \left( \frac{1 + ika - k^2 a^2}{ika - k^2 a^2} \right) \quad (7.3)$$

which is plotted in figures 5 through 7 with a purely resistive load  $Z = \chi R_0$ . When  $\chi = 0$ ,  $\Lambda$  has a peak near  $ka = 1$ . The resonance is

sharper for a thinner toroid. This is what one would expect from a circular loop antenna. The peak decreases as resistances are loaded to the antenna. In order to simulate a plane-wave-like field, the amplitude of  $\Lambda$  must be close to one. From these figures, one finds that the optimum choice for  $\chi$  is one for which  $\Lambda$  is in general about one at all frequencies, although  $\Lambda$  may be closer to one at some particular frequency with other values of  $\chi$ . For  $\chi = 1$ , the field at the center is near a plane wave at low and high frequencies as shown in the plots. This characteristic is also proved analytically in Chapter VI. In the intermediate frequencies the maximum fluctuation of  $|\Lambda - 1|$  is about 50% for  $b/a = 0.1$ . This fluctuation drops to 25% for  $b/a = 0.01$  and becomes only 17% at  $b/a = 0.001$ .

One can not make the exact ideal impedance  $Z_i$  with passive elements. But some other kind of combination of the passive elements might be able to produce an impedance close to  $Z_i$ . As shown in figure 4,  $Z_i$  is purely resistive at low frequencies and becomes inductive at intermediate frequencies. So, one may choose a load  $Z$  in the following form:  
 $Z = R_0 + ika\alpha Z_0$ . The amplitude and phase of  $\Lambda$  are shown in figures 8 to 10 for various  $b$  over  $a$  ratios. It appears that the maximum fluctuation of  $|\Lambda - 1|$  has a minimum for  $\alpha$  between 1 and 2. The optimum  $\alpha$  is given in an expanded scale for  $|\Lambda - 1|$  in figure 11b. For  $b/a = 0.1$ , the maximum fluctuation of  $|\Lambda - 1|$  is about 36% with  $\alpha = 1.556$ . For  $b/a = 0.01$ ,  $\alpha = 1.811$  and  $b/a = 0.001$ ,  $\alpha = 1.951$ , the maximum fluctuation is about 22% and 16% respectively. This shows some improvement

in the intermediate frequency as compared to the pure resistive case (figure 11a). Figure 12 gives the comparison for peak  $|\Lambda - 1|$  for a load  $Z = R_0$  and  $Z = R_0 + ika\alpha Z_0$  with optimum  $\alpha$ . One should bear in mind that EMP usually has a very wide frequency range. Since an inductance becomes an open circuit at very high frequencies, this kind of load may not simulate EMP properly at high frequencies.

This optimum  $\alpha$  is a function of  $b/a$ , and it only applies to the E/H ratio. For good high frequency efficiency a series inductance is undesirable. However, other more reasonable impedance functions can also be investigated.

There is an interesting phenomenon in the pure resistive cases. The magnitude of  $\Lambda$  is one at  $ka \approx 0.7, 3, 5.7, 6.3$ , etc. At these frequencies it is independent of load as well as  $b/a$  ratio. The impedance  $Z$  appears in the formula for  $\Lambda$ . If it is real then at certain frequencies the other terms are pure imaginary but conjugate with respect to numerator and denominator. This gives magnitude one but does shift phase, the amount of phase shift being proportional to the imaginary part.

The electric and magnetic fields will be shown in terms of normalized factors  $E_0$  and  $H_0$  given in eq. (6.23). As was discussed in the last chapter, one should choose eq. (6.13) for the fields at low frequencies and then switch to eq. (6.30) at high frequencies. That is, for low frequencies,

$$\begin{aligned}\vec{E}_N &= \frac{\vec{E}_c}{E_0} = \vec{e}_y \left( \frac{R_0 e^{-ikb}}{Z + iZ_0 \lambda_1 \pi} \right) \left( ika + 1 + \frac{1}{ika} \right) \\ \vec{H}_N &= \frac{\vec{H}_c}{H_0} = \vec{e}_z \left( \frac{R_0 e^{-ikb}}{Z + iZ_0 \lambda_0 \pi} \right) (1 + ika)\end{aligned}\quad (7.4)$$

and for high frequencies,

$$E_N \vec{e}_y = \frac{-2R_0 e^{-ikb}}{i\pi Z_0} \left[ \frac{1}{H_0^{(2)}(kb) - i\left(\frac{Z}{Z_0}\right)\left(\frac{b}{a}\right)H_1^{(2)}(kb)} \right] = H_N \vec{e}_z \quad (7.5)$$

The source is turned on at  $t = -\frac{a-b}{c}$ .

Equation (7.4) and (7.5) are plotted in figures (13) to (16) for comparison. For  $\frac{b}{a} = 0.001$ , they agree very well, although the current oscillations along the loop do not appear in the asymptotic formula. The portion of agreement decreases as  $\frac{b}{a}$  increases. Since  $|kb| \ll 1$  and  $a^2 \gg b^2$  were assumed in the derivations of eq. (7.4) and eq. (7.5) is good for  $\frac{b}{a} \rightarrow 0$ , one would expect some disagreements in these two formulas for large  $\frac{b}{a}$  ratios. A typical example is shown in figure (13) for  $\frac{b}{a} = 0.1$ . From these figures, the best switching points seem to be at  $ka = 9$  for  $\frac{b}{a} = 0.1$  and  $kb = 0.3$  for  $\frac{b}{a} \leq 0.02$ . This way, we obtain a complete frequency response for fields at the center as shown in figures (17) and (18). Although, there is a clear discontinuity in the switching point for  $\frac{b}{a} = 0.1$ , it does not appear too bad really. So, the above formulas give us the

fields at all frequencies for  $\frac{b}{a} < 0.1$  and  $\frac{b}{a} = 0.1$  can be considered the limiting case. More accurate studies are proposed in the next chapter for extending the range of validity to a thicker toroidal structure.

Since the field strength increases at large  $ka$ , one needs to consider a very wide frequency range in order to obtain an impulse response. However, as it is discussed in the Introduction that a unit step function is a more adequate simple function for giving an EMP-like response. The frequency response from a unit step source is that from impulse multiplying  $(ika)^{-1}$ . Thus, the field will now drop to zero at high frequencies and we need only some finite frequency ranges in computing the inverse Fourier transform. The transient behavior is shown in figures (19) and (20). The decay is extremely fast in the early time and approaches one at later time. There is no great variation in the results for various  $b/a$  except  $b/a = 0.1$  which is not quite accurate in the frequency domain.

With various loading resistances, the frequency and time behaviors of E and H are shown in figures (21) to (36) for some typical values of  $b/a$ , namely 0.1, 0.02, 0.01 and 0.001. Without a load, the electric field has a resonance at  $ka = 1$  and becomes sharper for a thinner loop. At this frequency, the circumference equals the wavelength. The phase will not be changed even with some resistive loads. Therefore the field at the center is enhanced. At low frequencies, the electric field is mainly due to the voltage drop from the source spread out over the toroid

to give a charge separation. The normalized E field at the center is independent of the load and will approach one at late time. On the other hand, the magnetic field depends on the current on the loop at low frequencies. Therefore the normalized H field shown in these figures is a function of load and becomes infinity when  $\chi = 0$ . The transient behavior of the E field shows oscillations when the loading resistance is zero or smaller. The oscillations are damped as the load increases and finally approach one at later time. For the transient H field, it shows a rise after an initial decay. This increase of field strength is more significant for small loading resistances. The field finally approaches a constant steady state value equal to  $1/\chi$ .



## VIII. Summary

This note is one of a series which discuss and study the electromagnetic performance of the TORUS type of simulator. In this note, a special coordinate system was introduced to depict toroidal geometry and thereby facilitating the derivation of solutions for the electric and magnetic fields. Some special techniques were employed, such as Fourier expansions, to simplify the integral equations. Some assumptions and restrictions were made, like  $a^2 \gg b^2$ , to further simplify the expressions. These assumptions may be considered valid for most cases. As a result of this study a fairly extensive investigation of the Anger-Weber function was done and documented.<sup>12</sup> General solutions have been derived for fields in space and on the toroidal structure. Extensive numerical results for fields at the center of TORUS in both the frequency and time domains are given.

This work can be extended in the following order according to the authors' preferences.

1. Considering the first together with the next higher order terms in the Fourier Expansion with respect to  $\phi$ , the field near the center can be found.

2. Using eq. (5.62) the fields at the center from various types of source can be calculated. The transient behavior can be obtained either by inverse Laplace transforms of the frequency response or by the convolution of the source function with the impulse response.

3. Instead of using eq. (5.10), one may assume that

$$J_{\xi}(\phi, \xi) = \frac{a}{a + b \cos(\xi)} J_{\xi_a}(\phi) \vec{e}_{\phi} \quad (8.1)$$

to account for the decrease of current density and electric field with increasing cylindrical radius. This way, one may be able to obtain solutions for larger  $b$  over  $a$  ratio.

4. For any value of  $a$  and  $b$  (but  $a/b > 1$ ) one may try to solve the integral formula in Chapter IV analytically or numerically to obtain the surface current density not only in the  $\phi$  direction but also the  $\xi$  component.

5. By numerical methods the fields anywhere in space can be computed from eqs. (3.14) and (3.18).

6. Non-uniform loading impedances can be considered for improving the field distribution in some sense in various frequency bands.

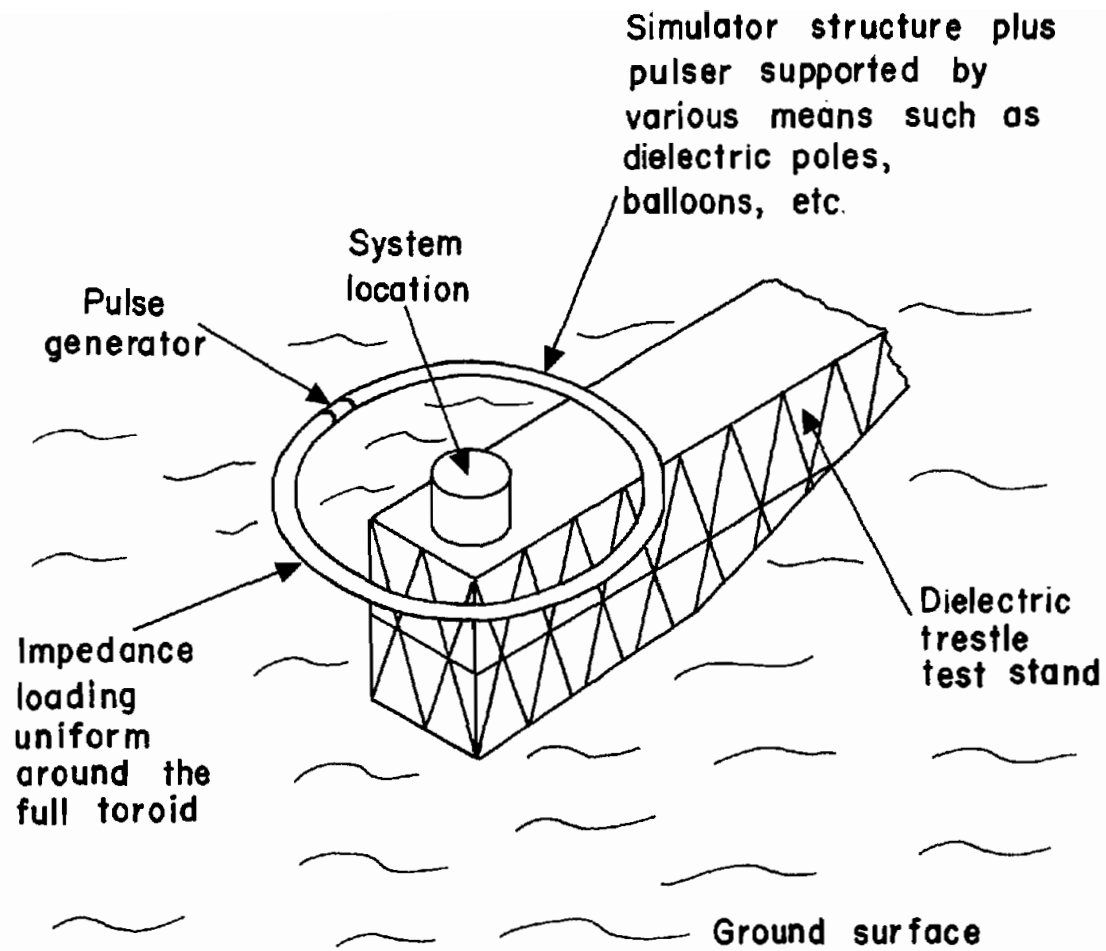
7. A tilted TORUS above a perfectly conducting ground is an interesting topic for future study.

8. One may investigate the behavior of TORUS for the case that the ground has finite conductivity.

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FIGURE 1. TORUS IN A FULL TOROID CONFIGURATION

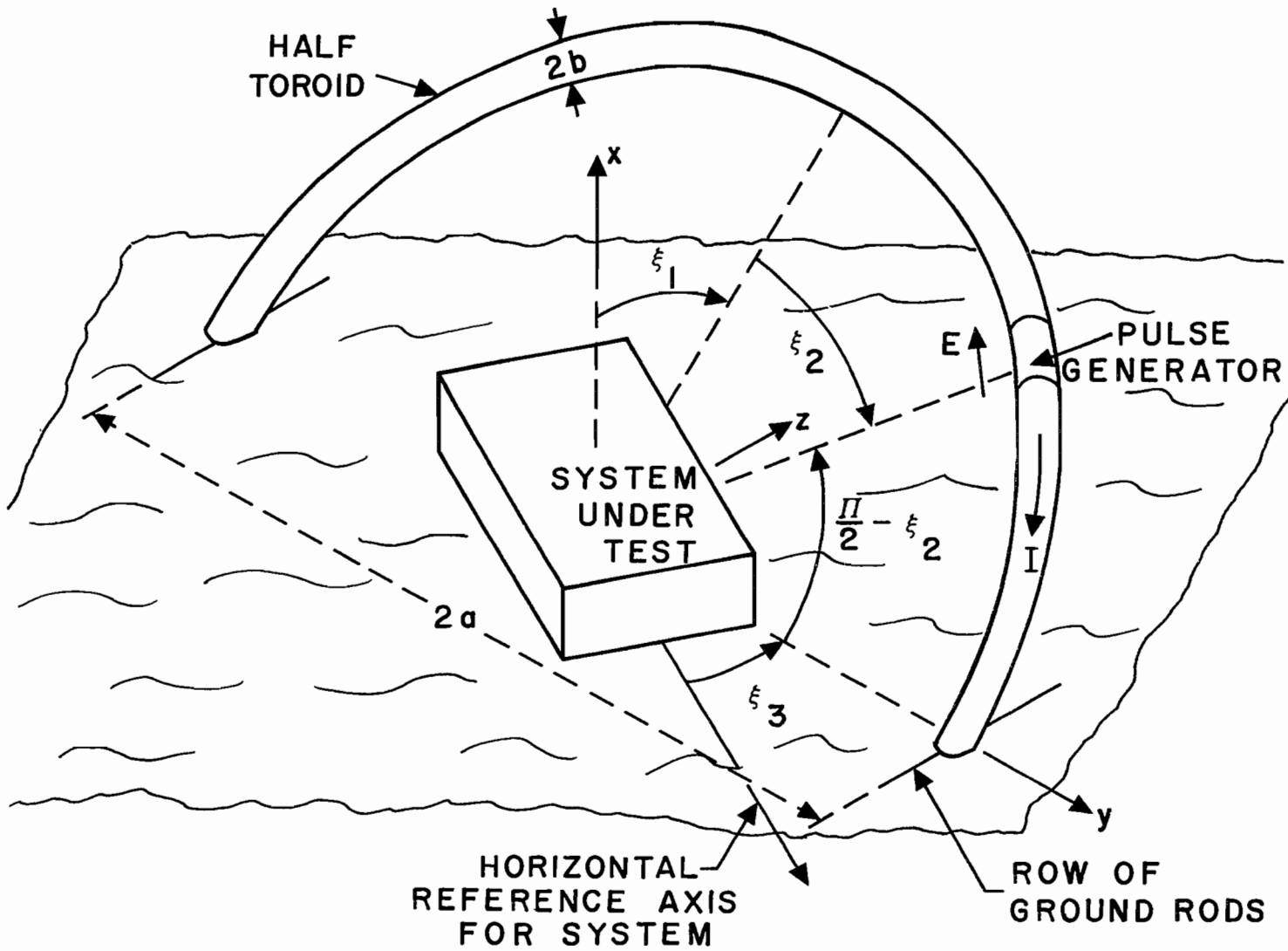
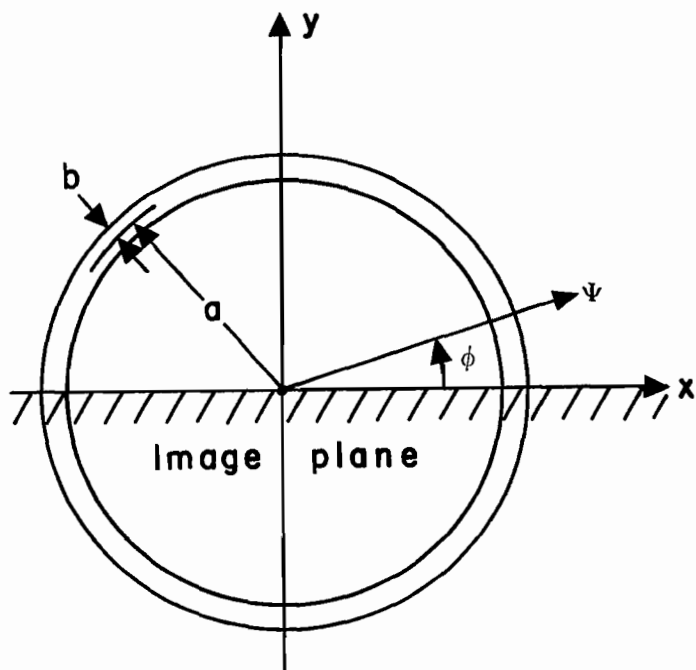
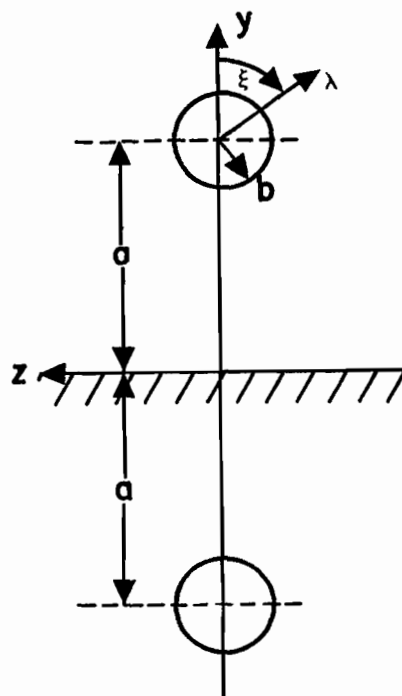


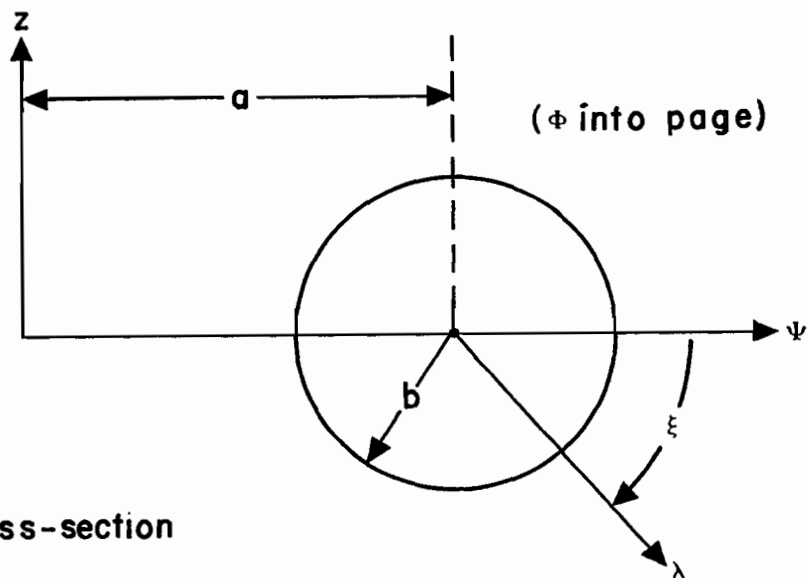
FIGURE 2. GEOMETRY OF TORUS SIMULATOR FOR USE WITH SYSTEM ON THE EARTH SURFACE



(A) Front view

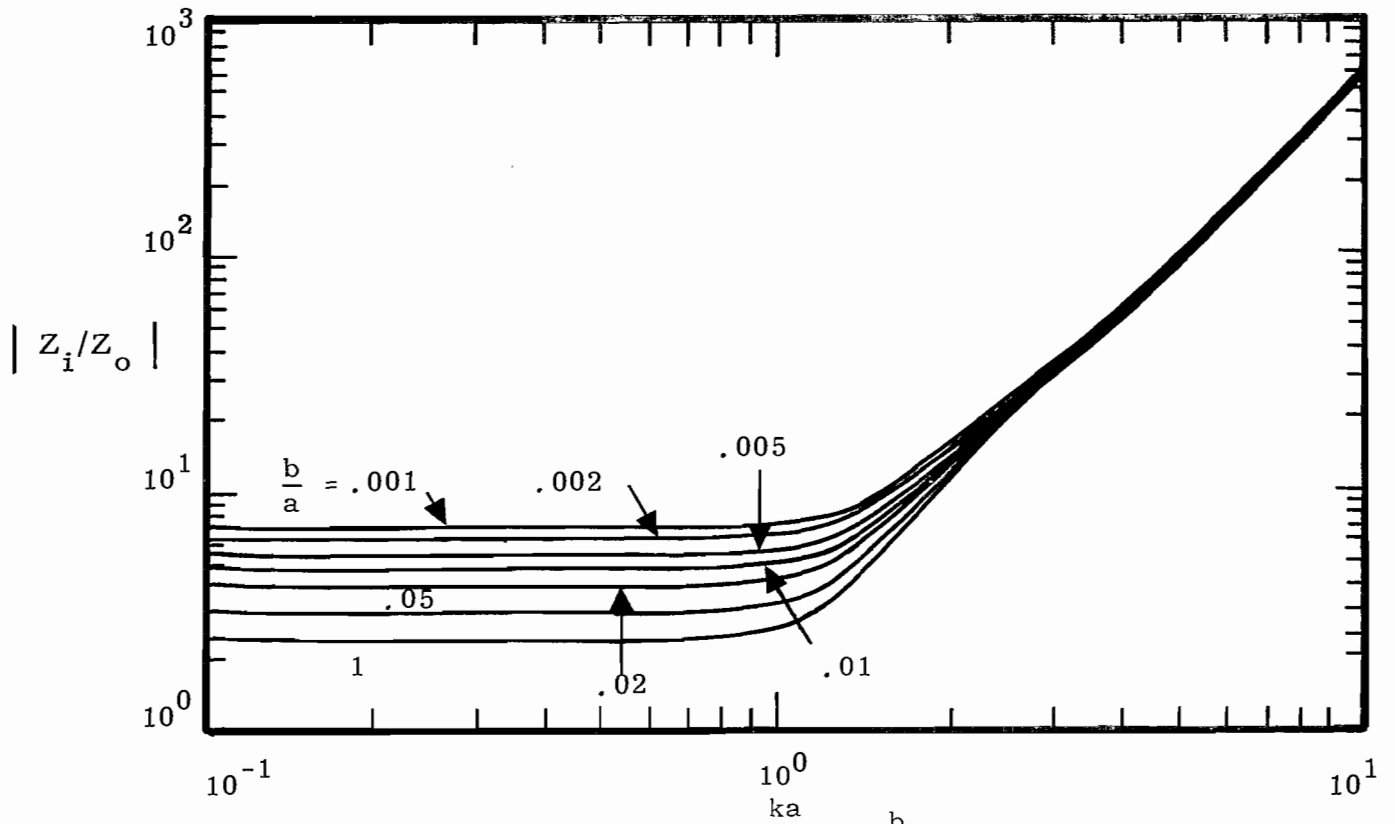


(B) Side view

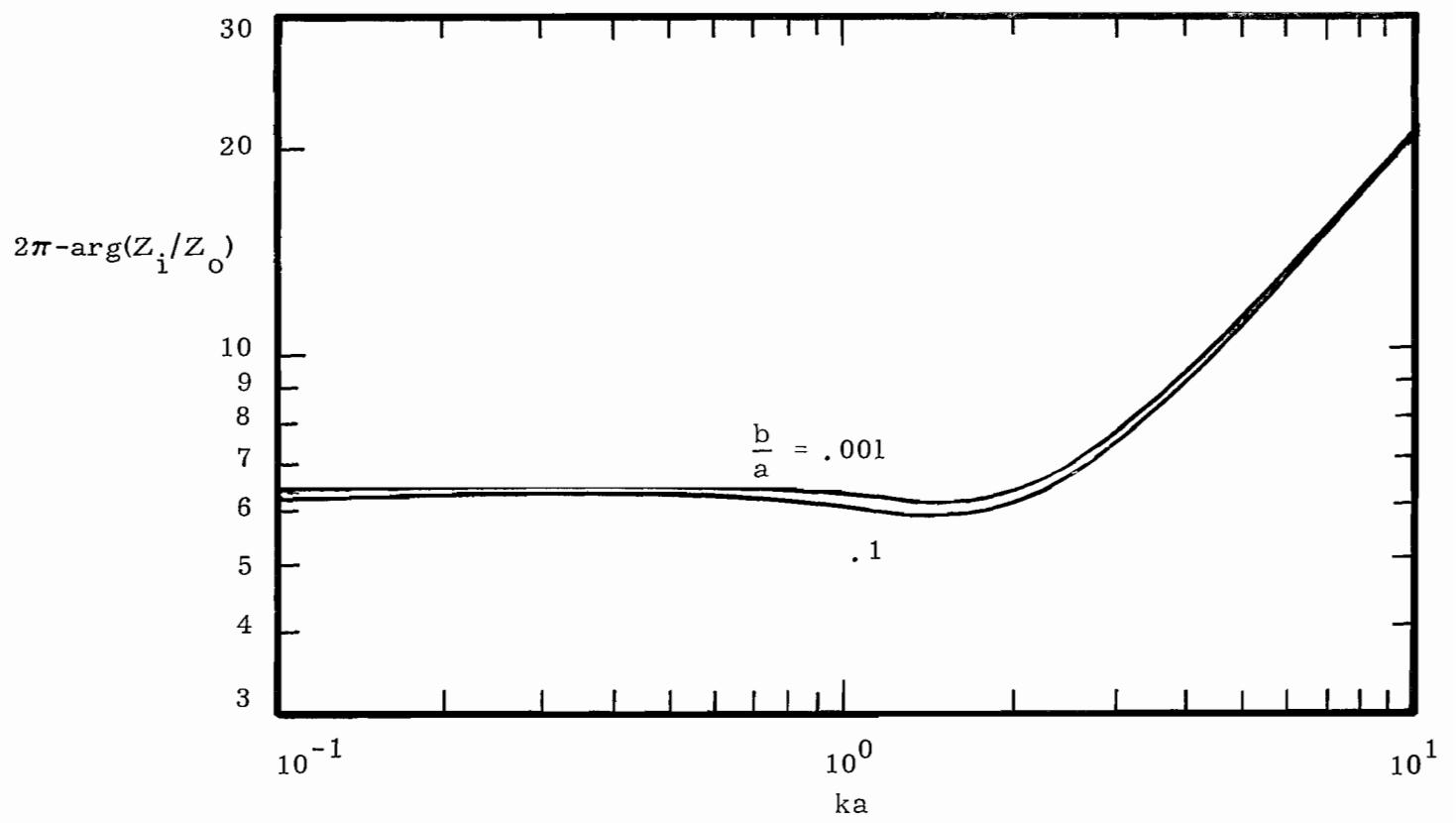


(C) Cross-section

FIGURE 3. RELATION OF A VERTICALLY ORIENTED TORUS ON A PERFECTLY CONDUCTING GROUND PLANE TO A FULL CIRCULAR TORUS WITH THREE COORDINATE SYSTEMS



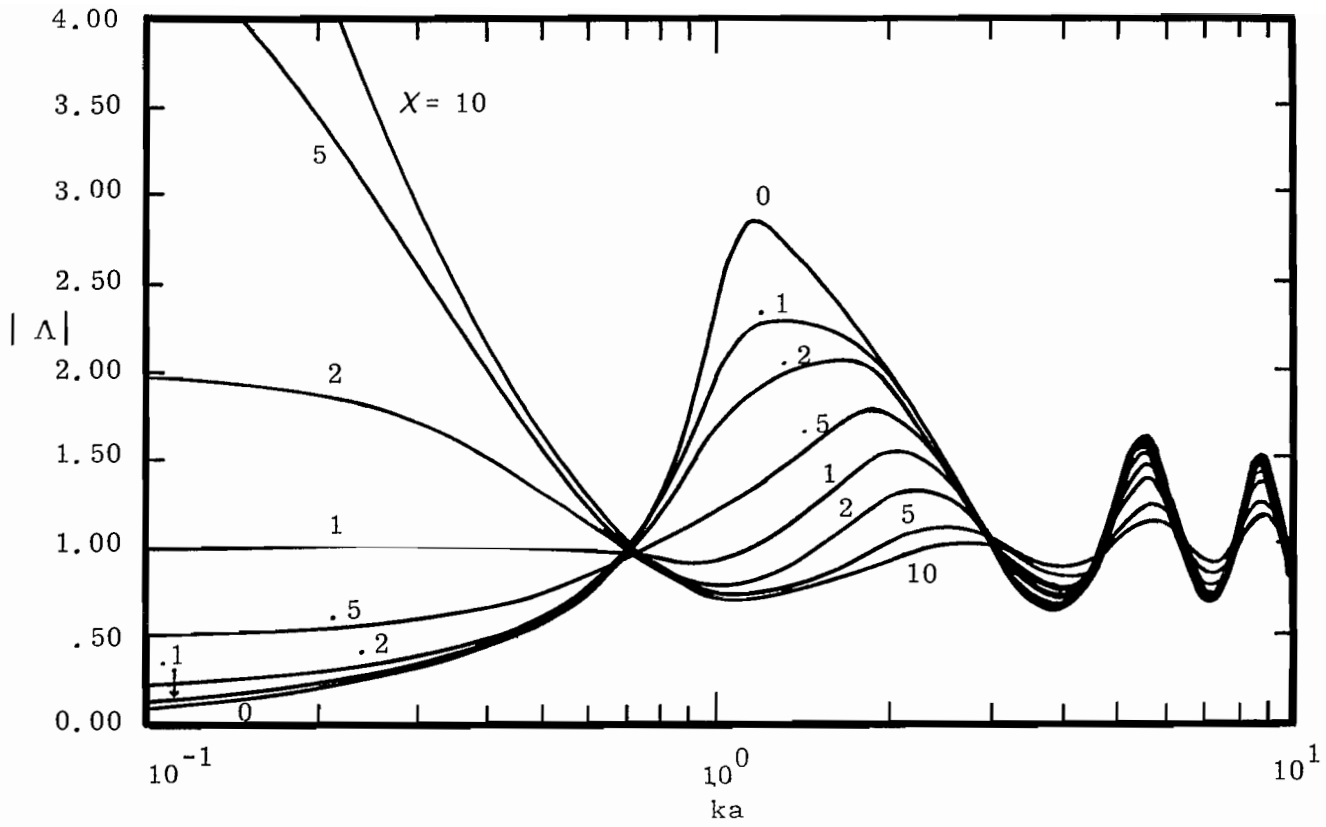
A. MAGNITUDE OF  $Z_i/Z_o$  FOR VARIOUS  $\frac{b}{a}$



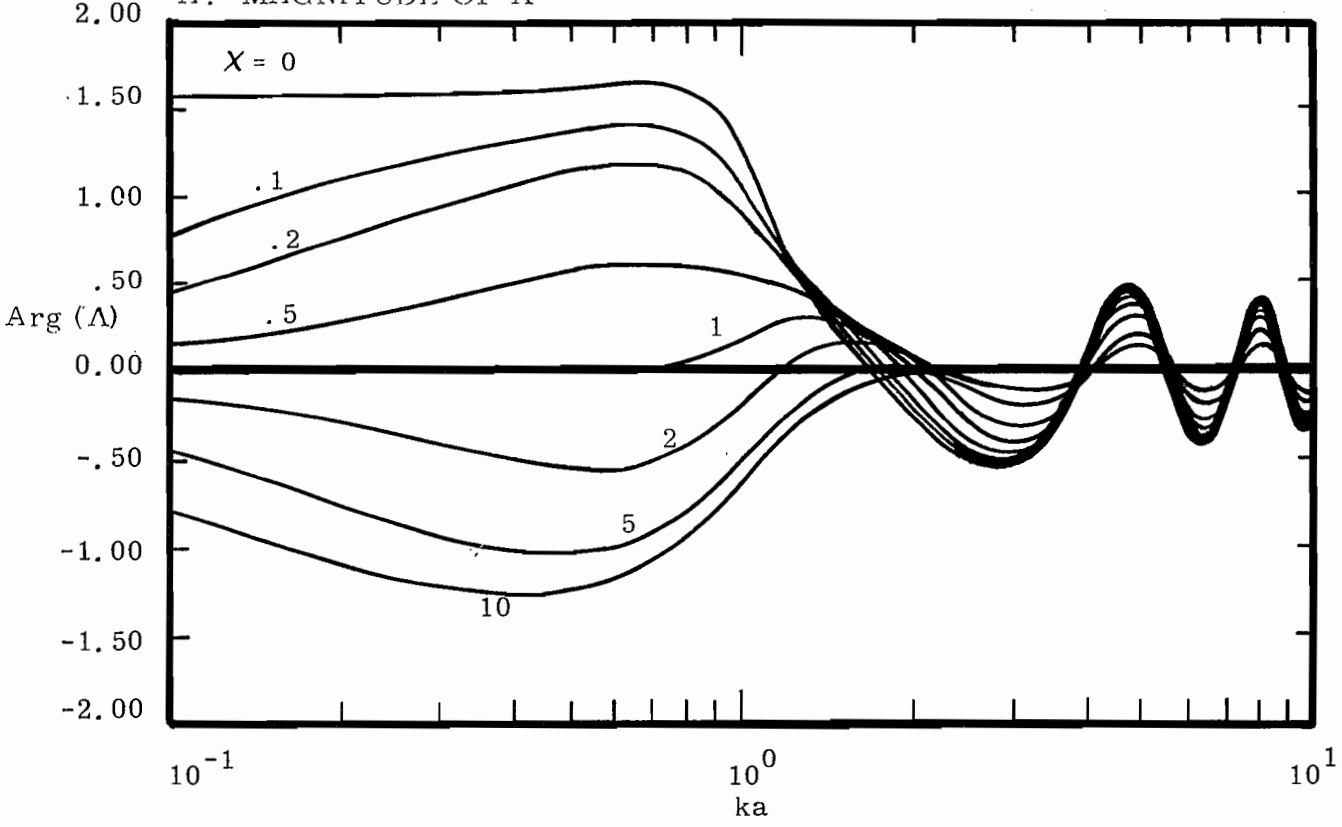
B. PHASE OF  $Z_i/Z_o$  FOR VARIOUS  $\frac{b}{a}$

FIGURE 4. IDEAL IMPEDANCE



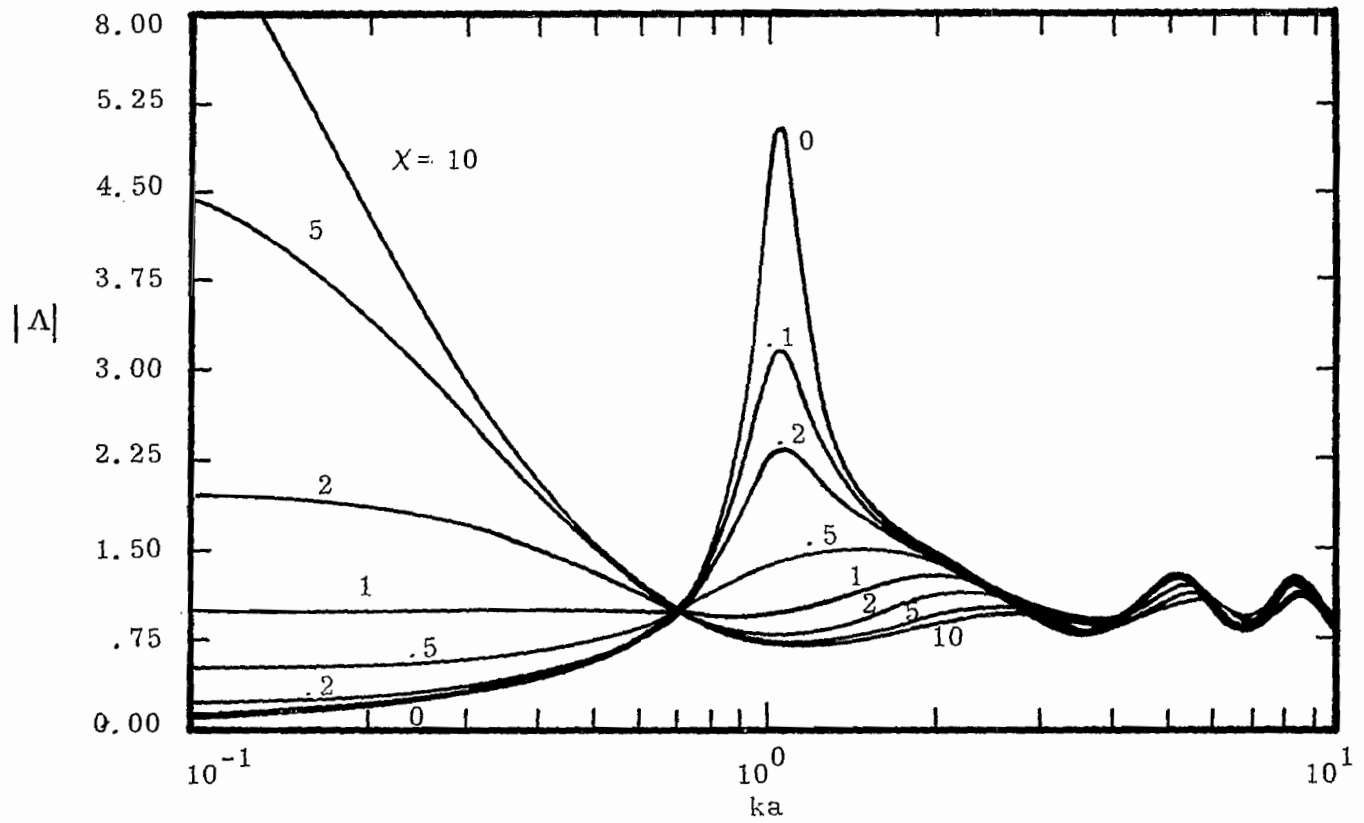


A. MAGNITUDE OF  $\Delta$

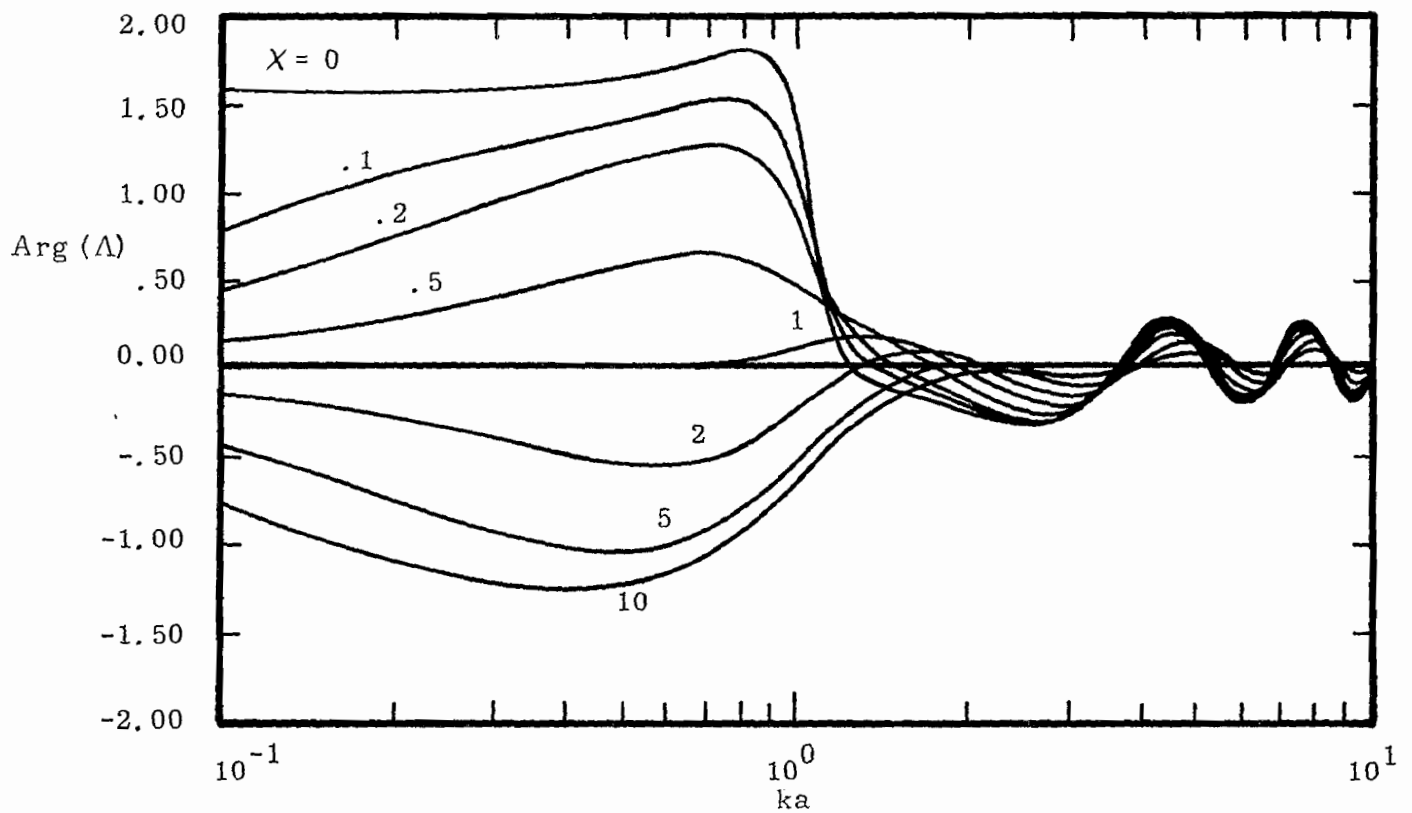


B. PHASE OF  $\Delta$

FIGURE 5.  $\Delta$  FOR  $\frac{b}{a} = 0.1$  WITH  $Z/Z_0 = \dot{X} \left[ \ln \left( 8 \frac{a}{b} \right) - 2 \right]$  FOR VARIOUS  $X$

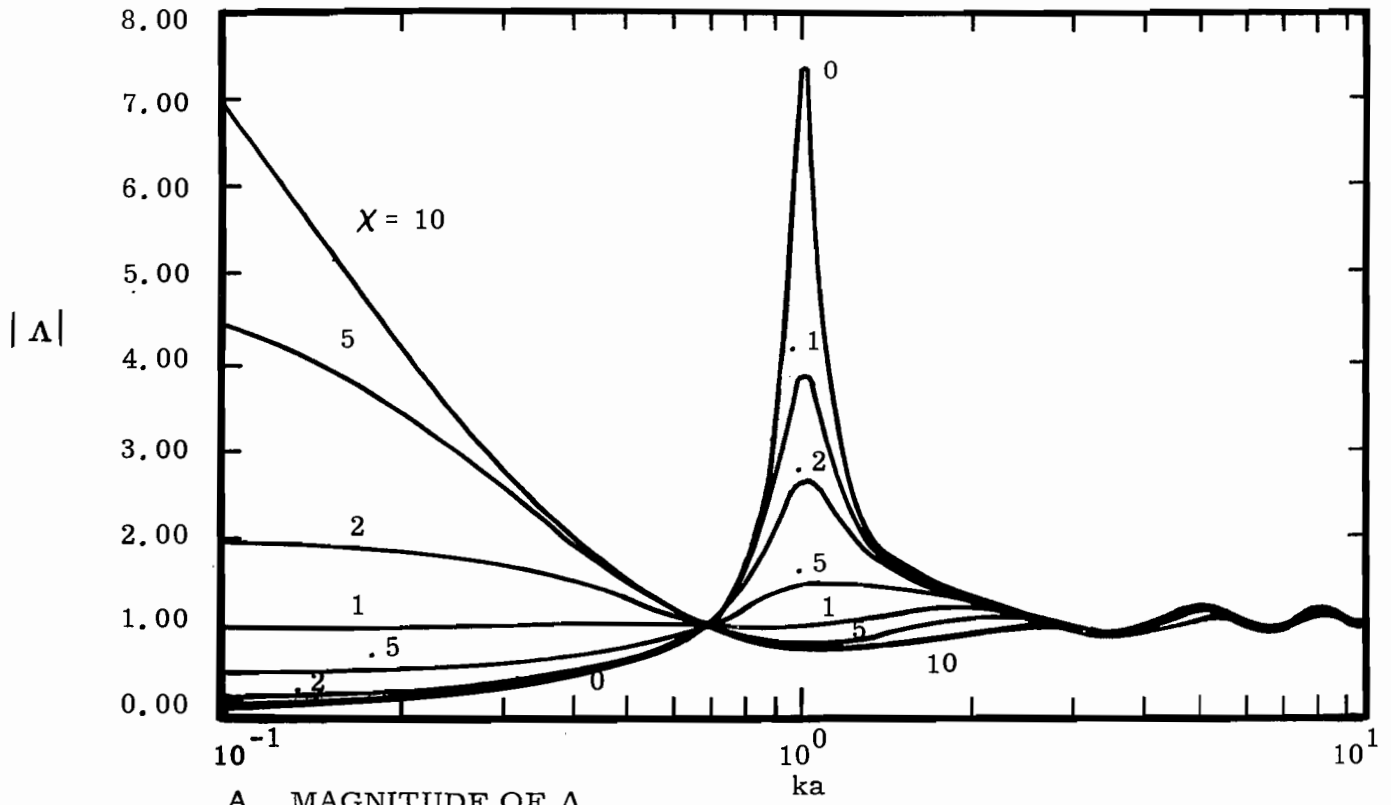


A. MAGNITUDE OF  $\Delta$

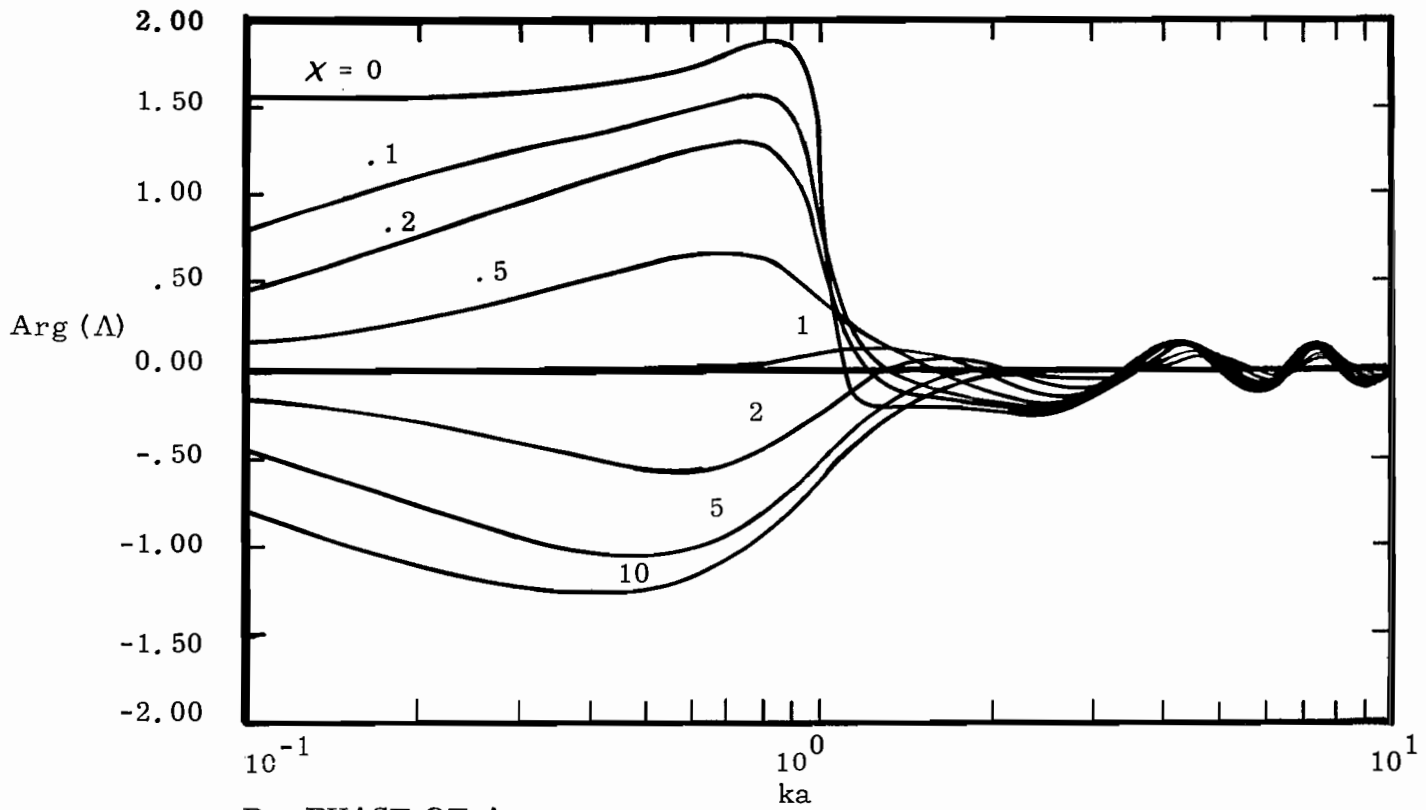


B. PHASE OF  $\Delta$

FIGURE 6.  $\Delta$  FOR  $\frac{b}{a} = 0.01$  WITH  $Z/Z_0 = \chi \left[ \ln \left( 8 \frac{a}{b} \right) - 2 \right]$  FOR VARIOUS  $\chi$

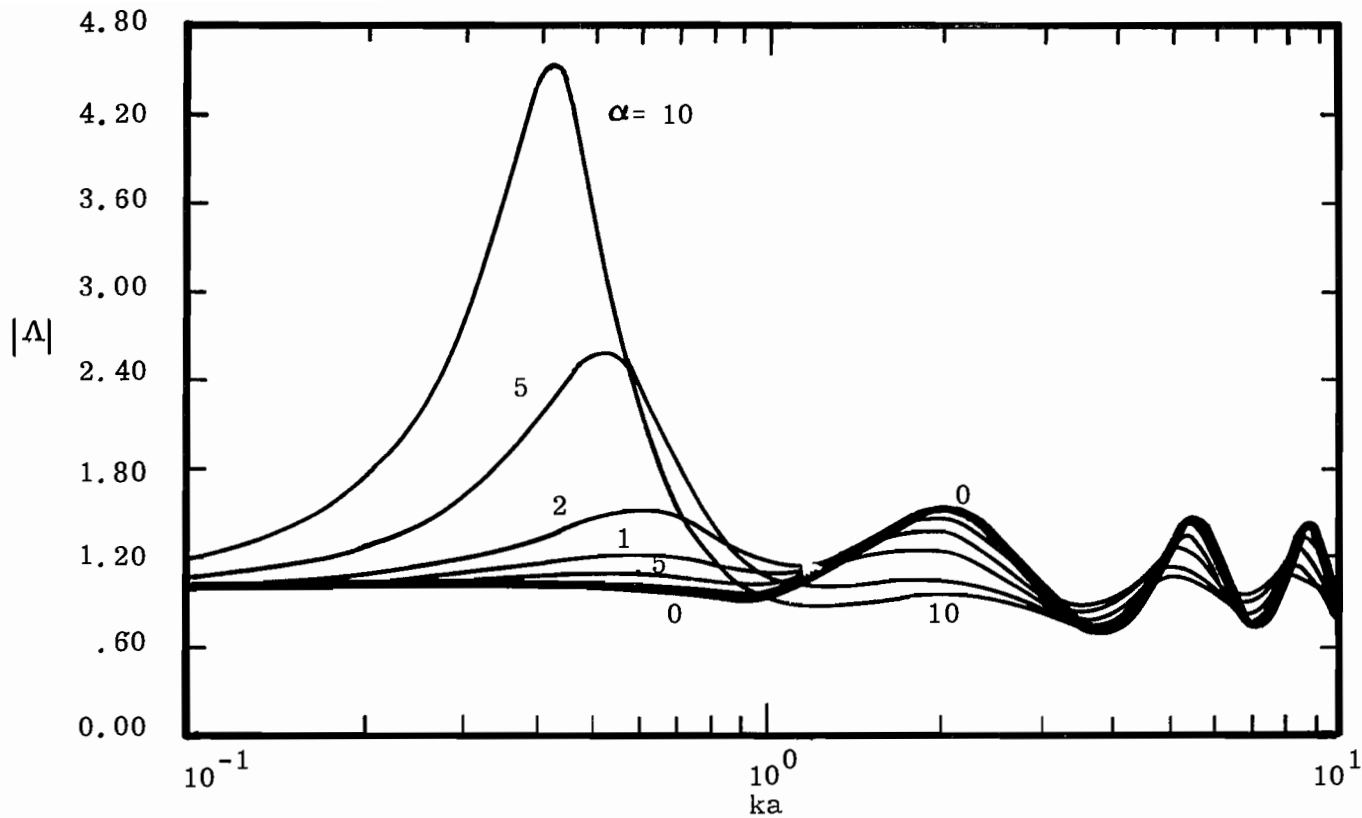


A. MAGNITUDE OF  $\Delta$

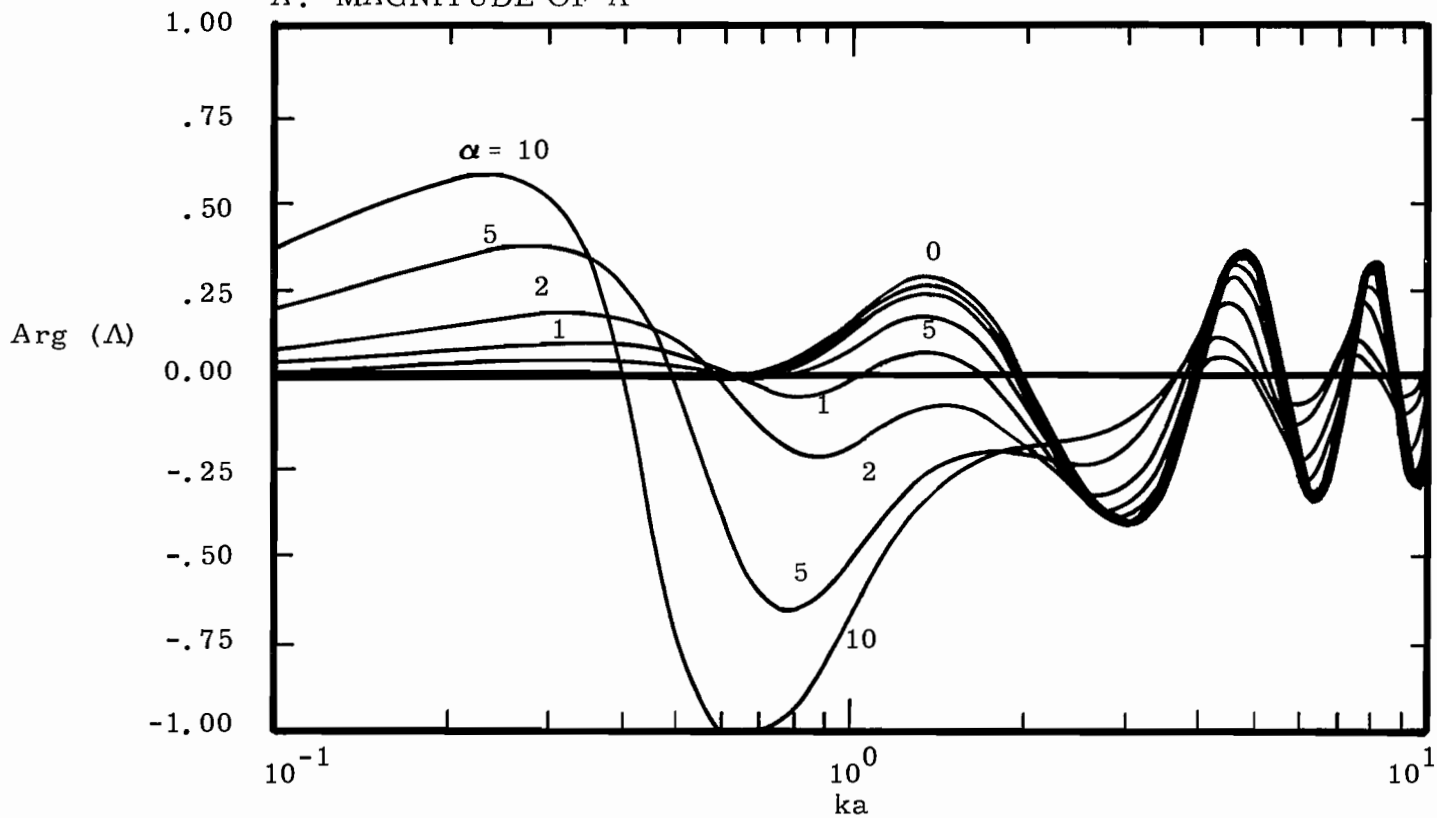


B. PHASE OF  $\Delta$

FIGURE 7.  $\Delta$  FOR  $\frac{b}{a} = 0.001$  WITH  $Z/Z_0 = X \left[ \ln\left(8\frac{a}{b}\right) - 2 \right]$  FOR VARIOUS  $X$ .

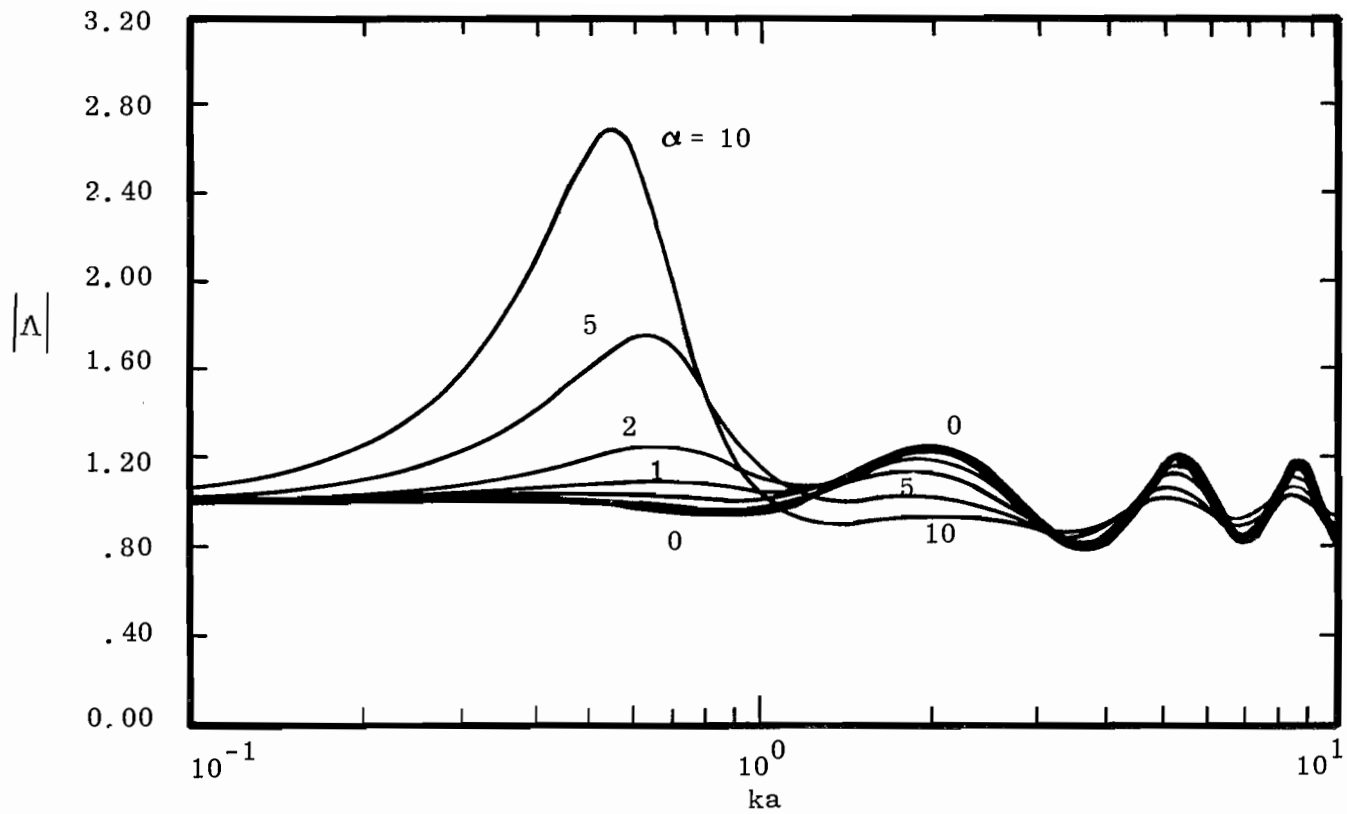


A. MAGNITUDE OF  $\Lambda$

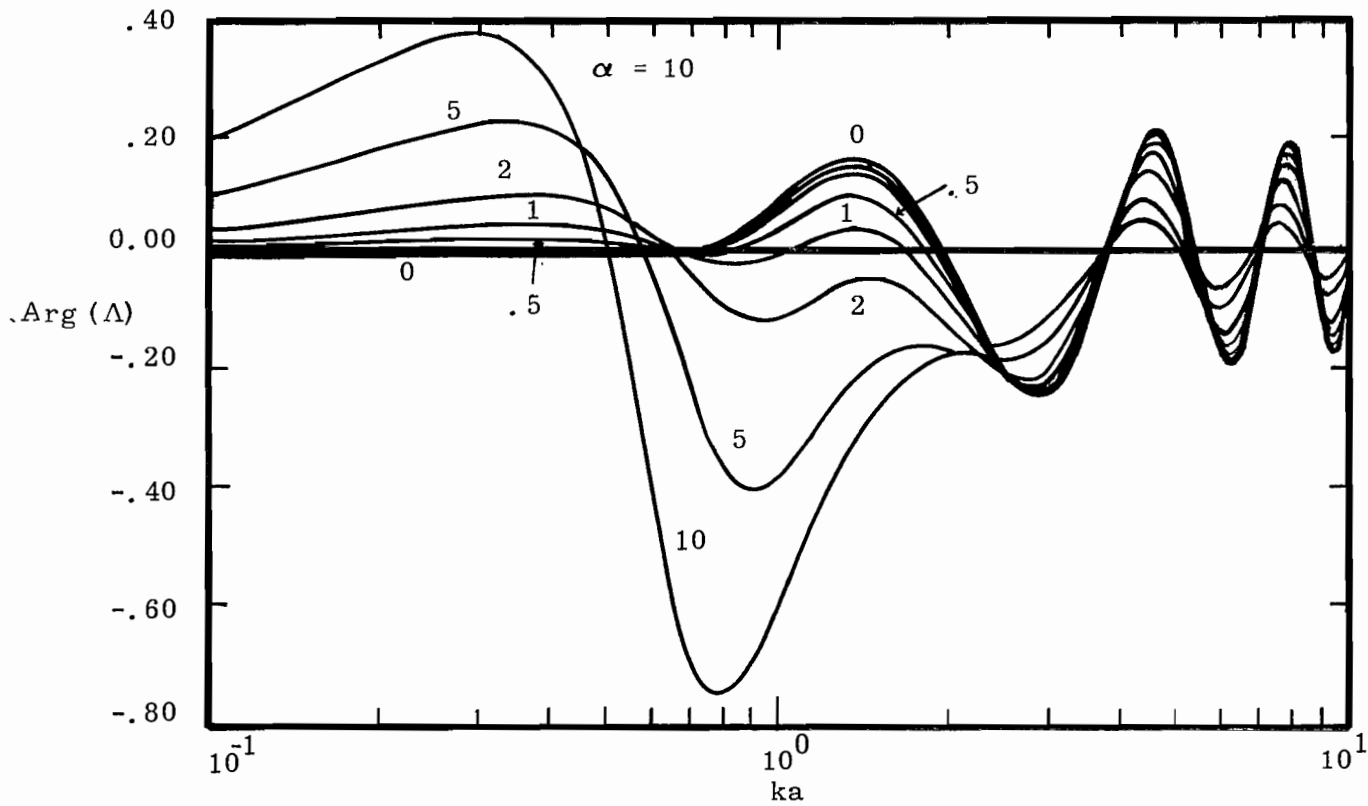


B. PHASE OF  $\Lambda$

FIGURE 8.  $\Lambda$  FOR  $\frac{b}{a} = 0.1$  WITH  $Z/Z_0 = \ln(8\frac{a}{b}) - 2 + ika\alpha$  FOR VARIOUS  $\alpha$ .

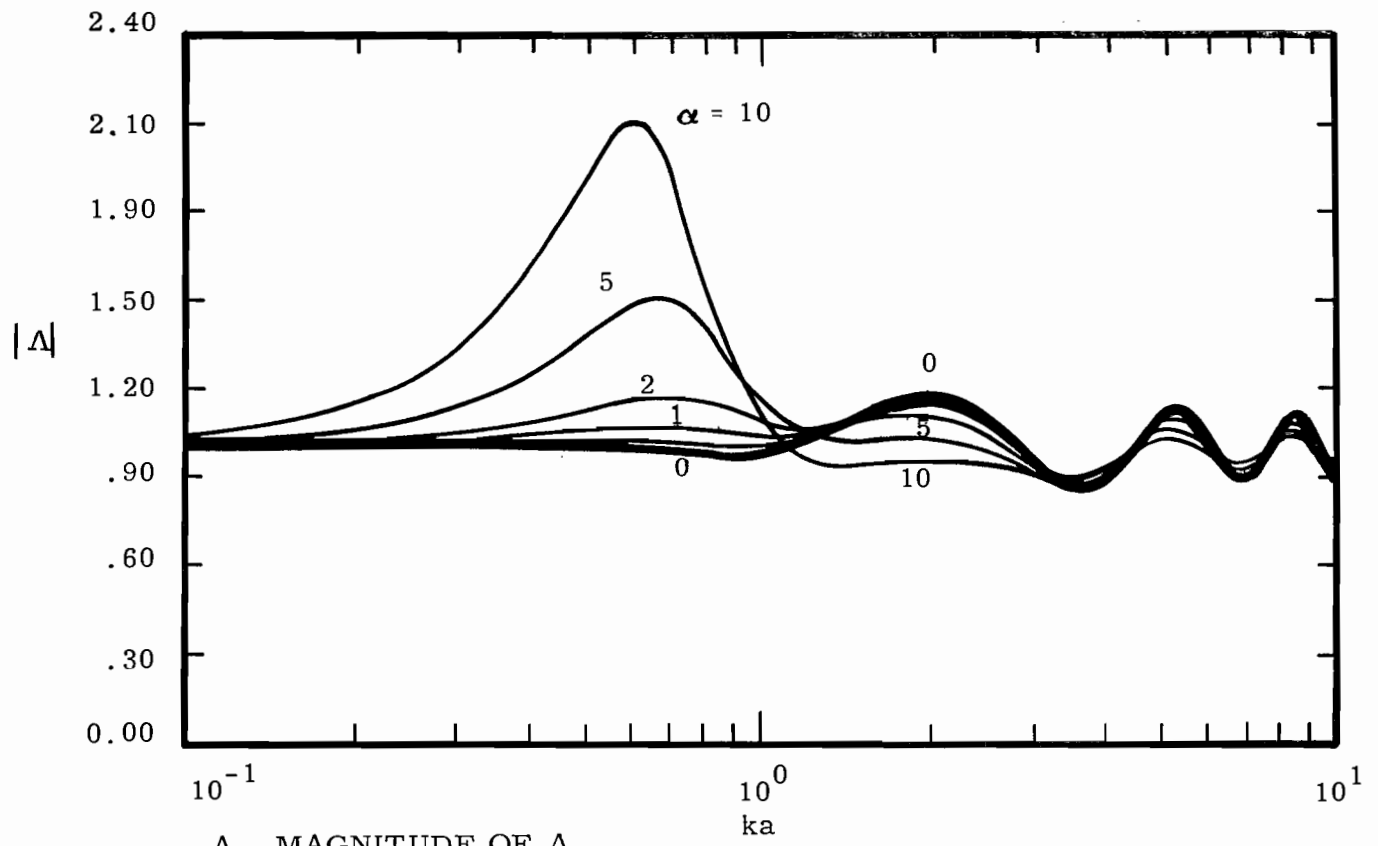


A. MAGNITUDE OF  $\Delta$

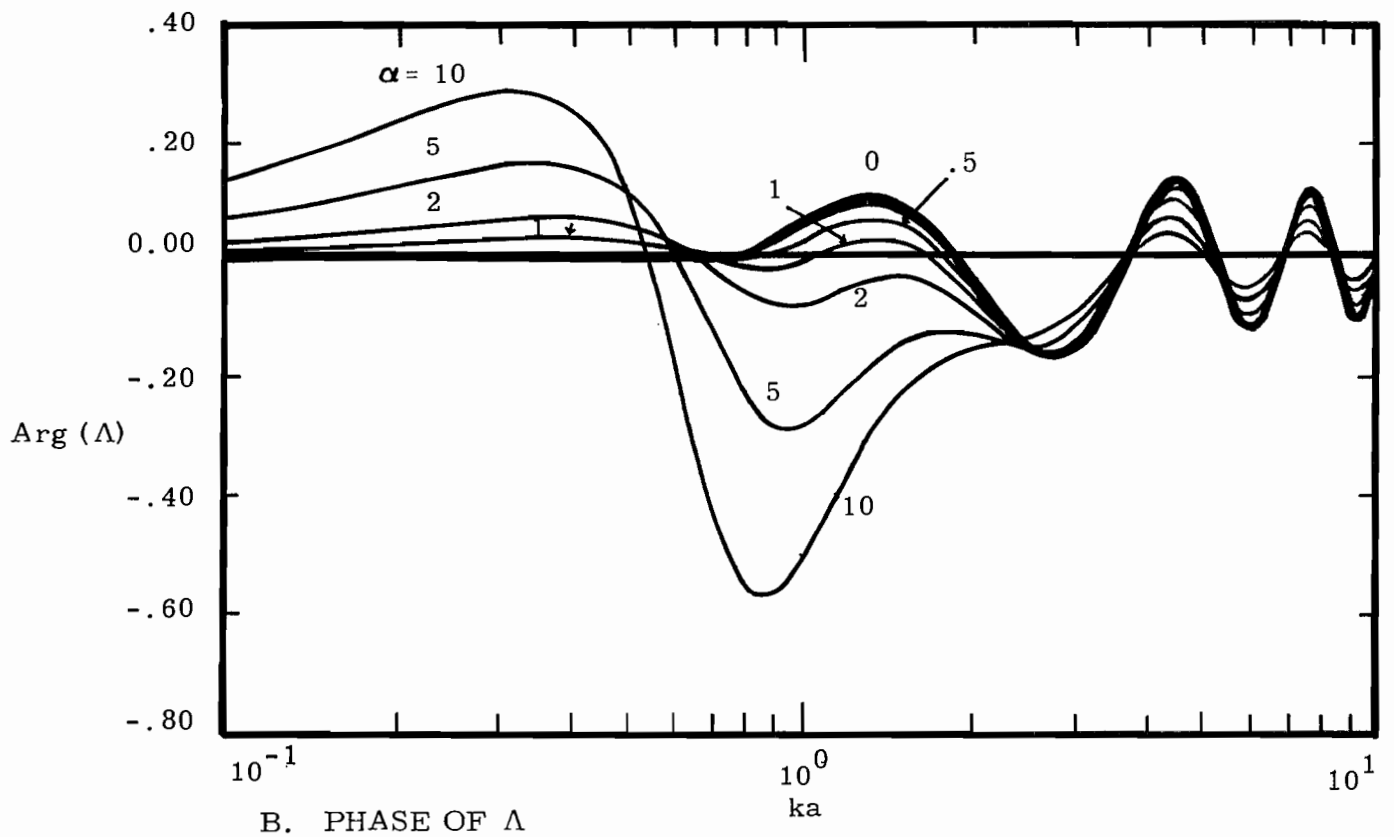


B. PHASE OF  $\Delta$

FIGURE 9.  $\Delta$  FOR  $\frac{b}{a} = 0.01$  WITH  $Z/Z_0 = \ln(8 \frac{a}{b}) - 2 + ika\alpha$  FOR VARIOUS  $\alpha$ .

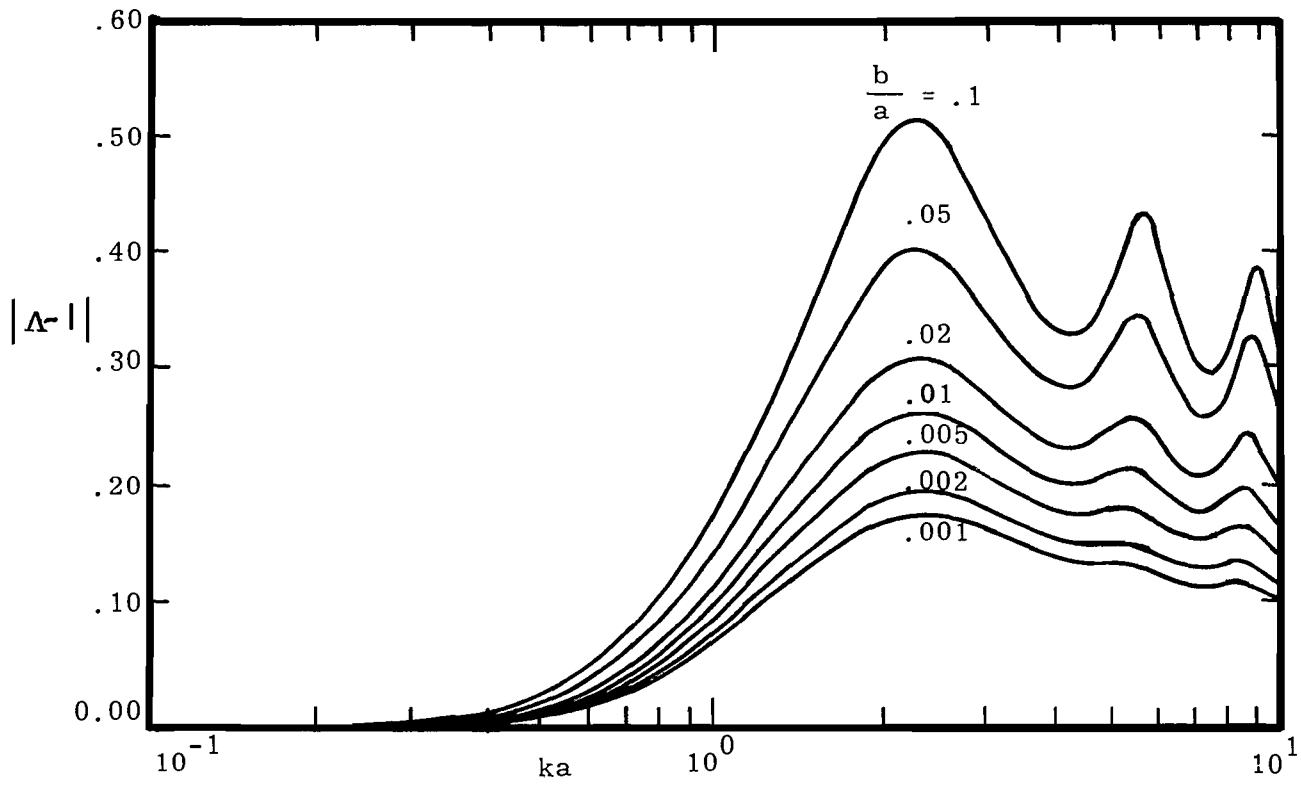


A. MAGNITUDE OF  $\Delta$

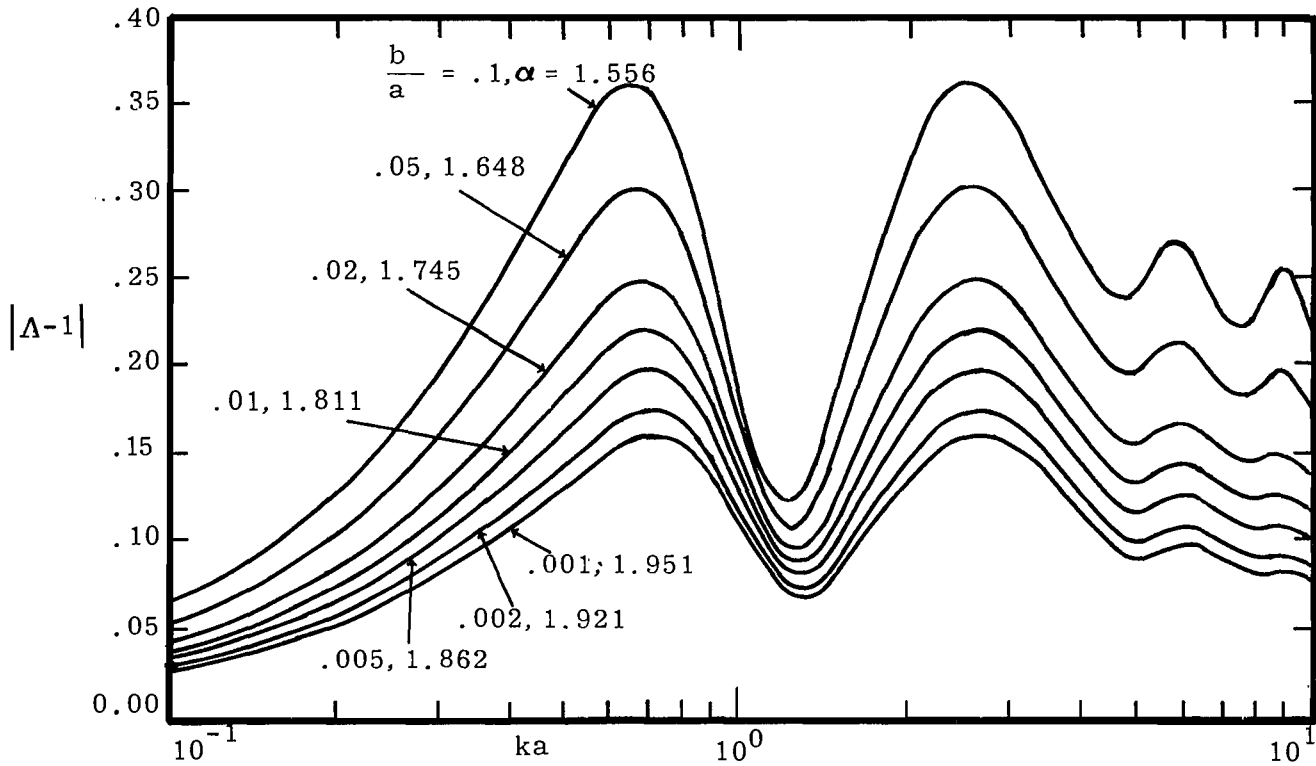


B. PHASE OF  $\Delta$

FIGURE 10.  $\Delta$  FOR  $b/a = 0.001$  WITH  $Z/Z_0 = \ln(8 \frac{a}{b}) - 2 + ika\alpha$  FOR VARIOUS  $\alpha$ .

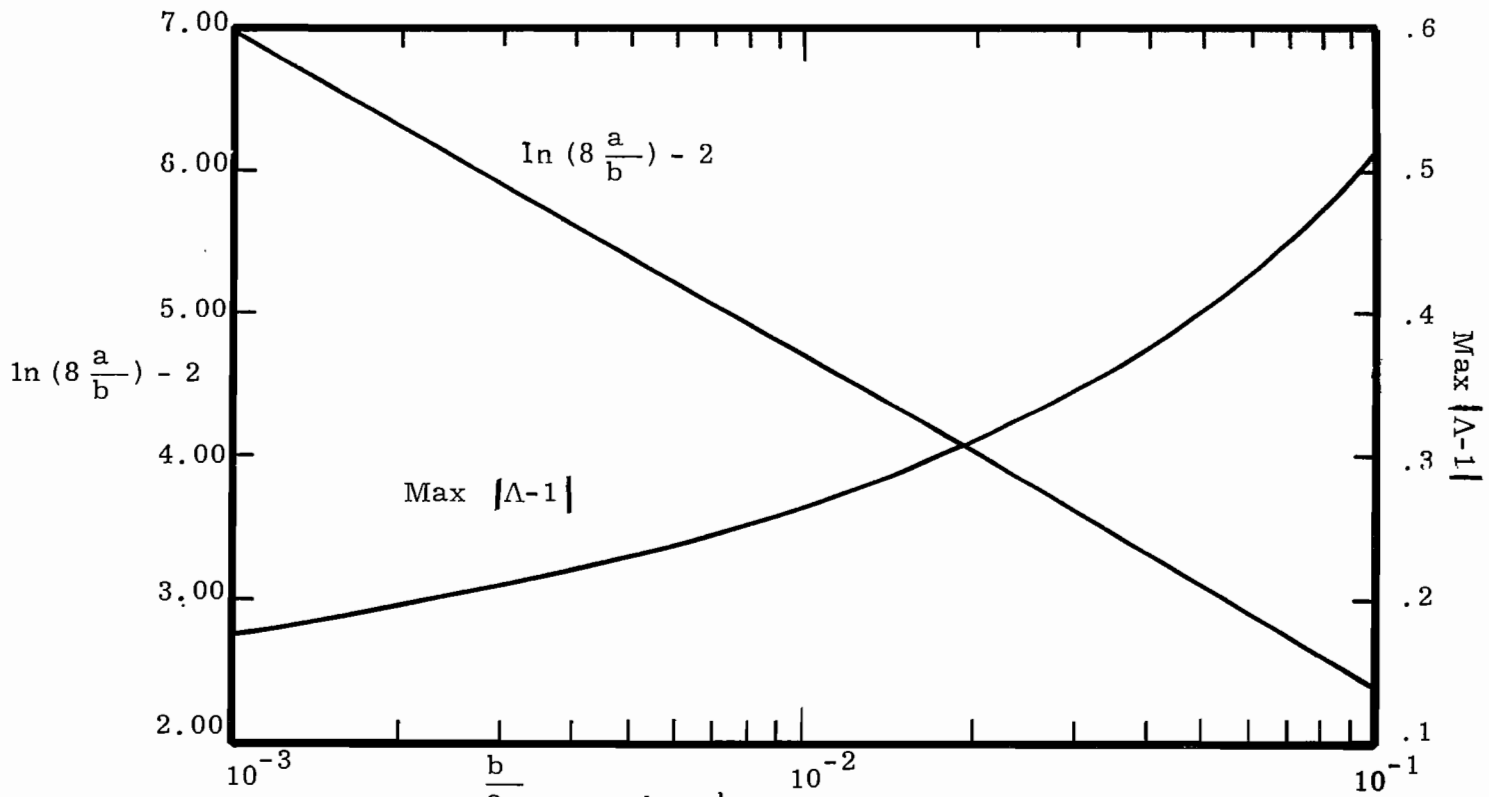


A. MAGNITUDE OF  $(\Lambda-1)$  FOR  $Z/Z_o = \ln(8 \frac{a}{b}) - 2$  FOR VARIOUS  $\frac{b}{a}$

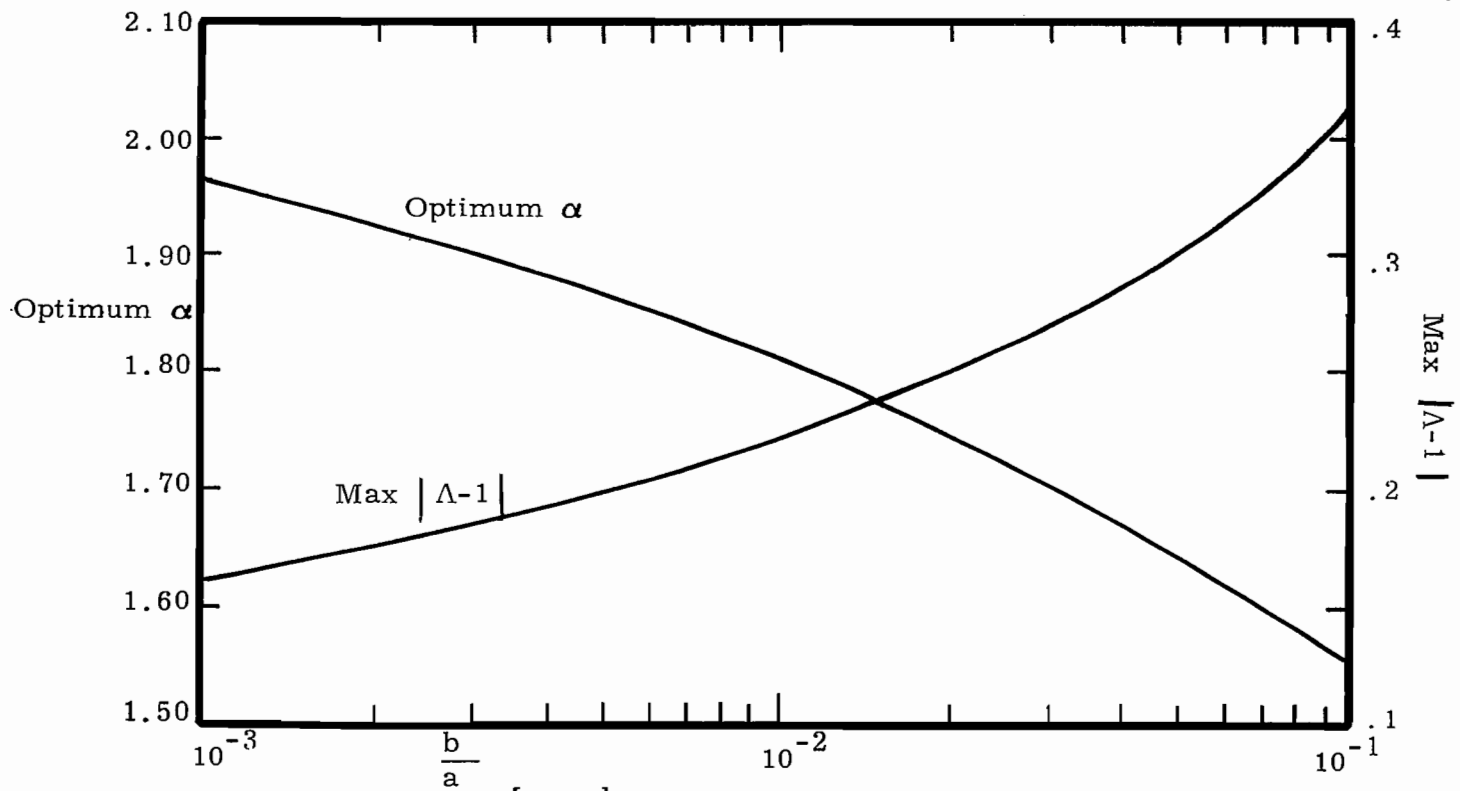


B. MAGNITUDE OF  $(\Lambda-1)$  FOR  $Z/Z_o = \ln(8 \frac{a}{b}) - 2 + ika\alpha$  FOR VARIOUS  $\frac{b}{a}$  WITH THEIR OPTIMUM  $\alpha$

FIGURE 11. MAGNITUDE OF  $(\Lambda-1)$



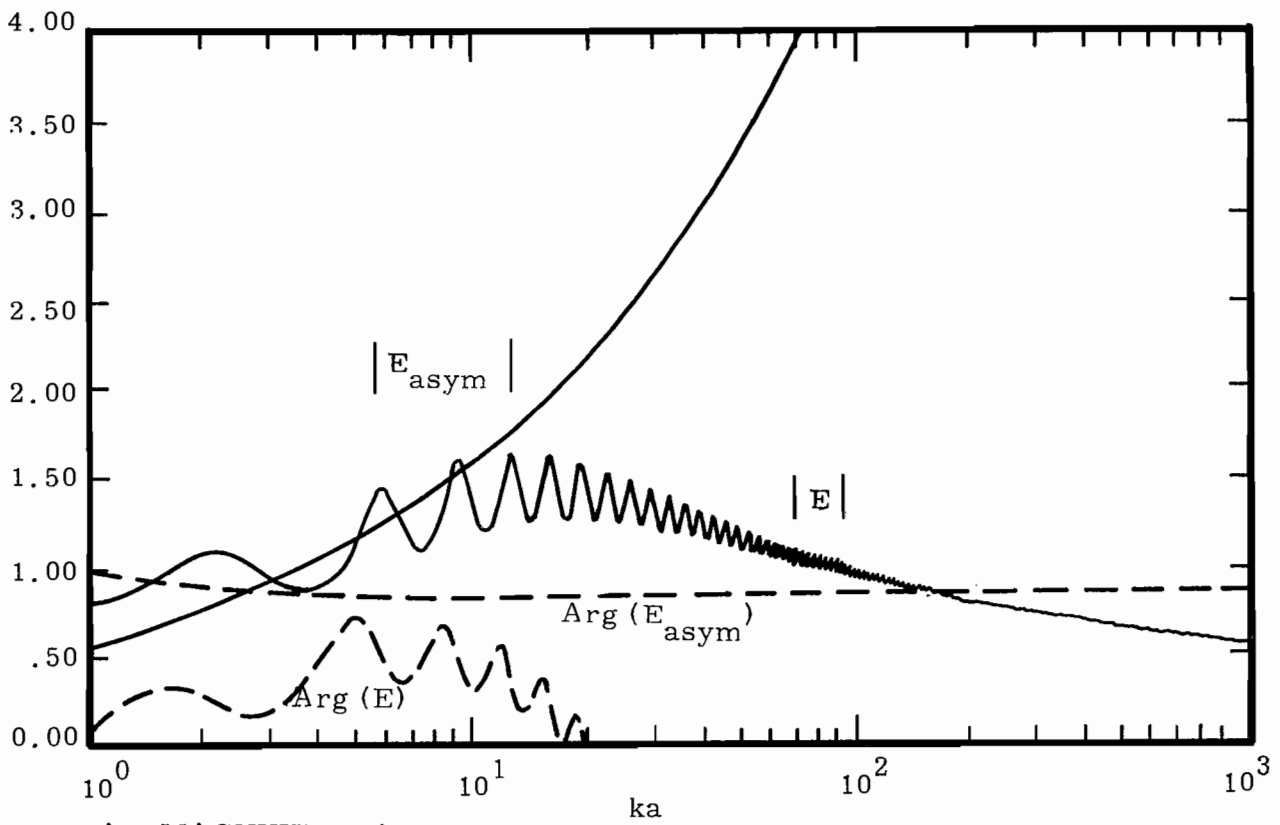
A. MAXIMUM  $|\Lambda - 1|$  WITH  $Z/Z_0 = \ln(8 \frac{a}{b}) - 2$  AS A FUNCTION OF  $\frac{b}{a}$



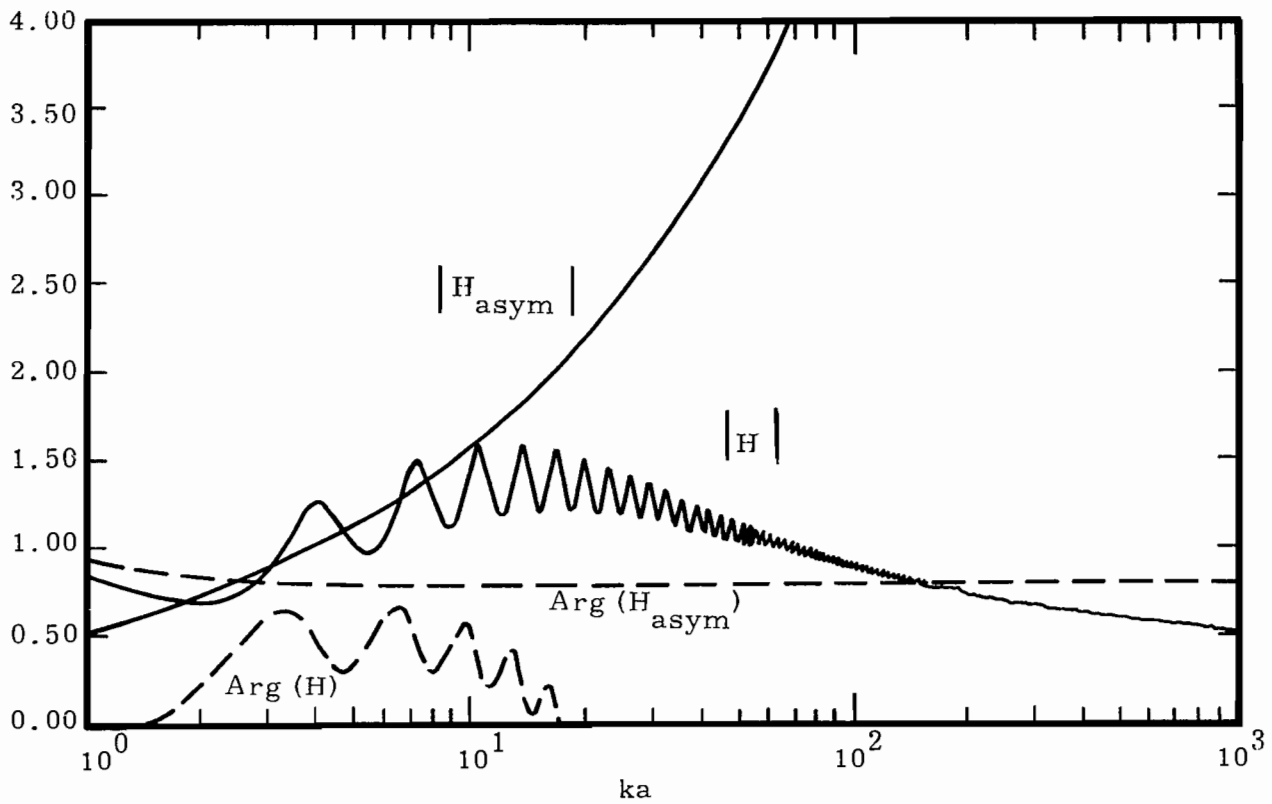
B. MAXIMUM  $|\Lambda - 1|$  WITH  $Z/Z_0 = \ln(8 \frac{a}{b}) - 2 + ika\alpha$  FOR OPTIMUM  $\alpha$  AS A FUNCTION OF  $\frac{b}{a}$

FIGURE 12. MAXIMUM  $|\Lambda - 1|$  FOR "BEST" IMPEDANCES.





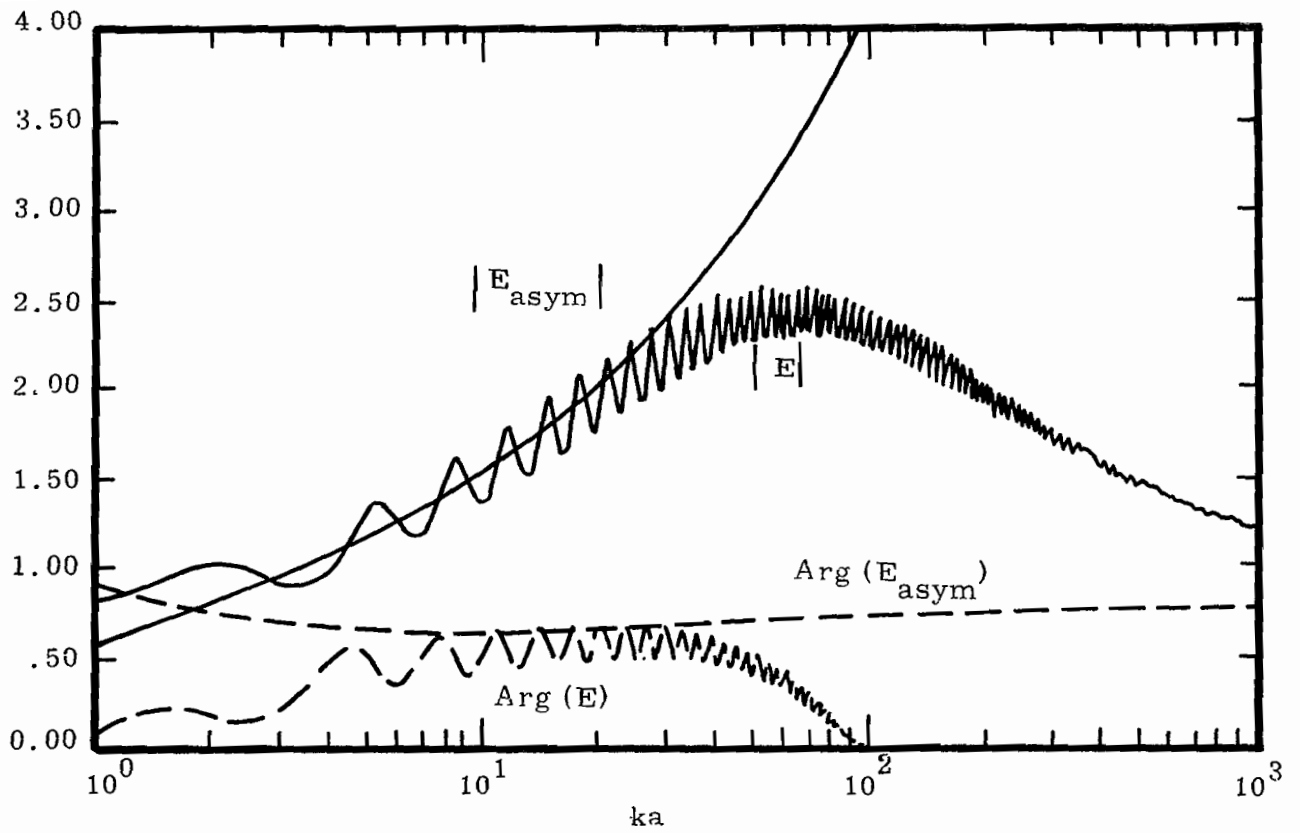
A. MAGNITUDE AND PHASE OF E



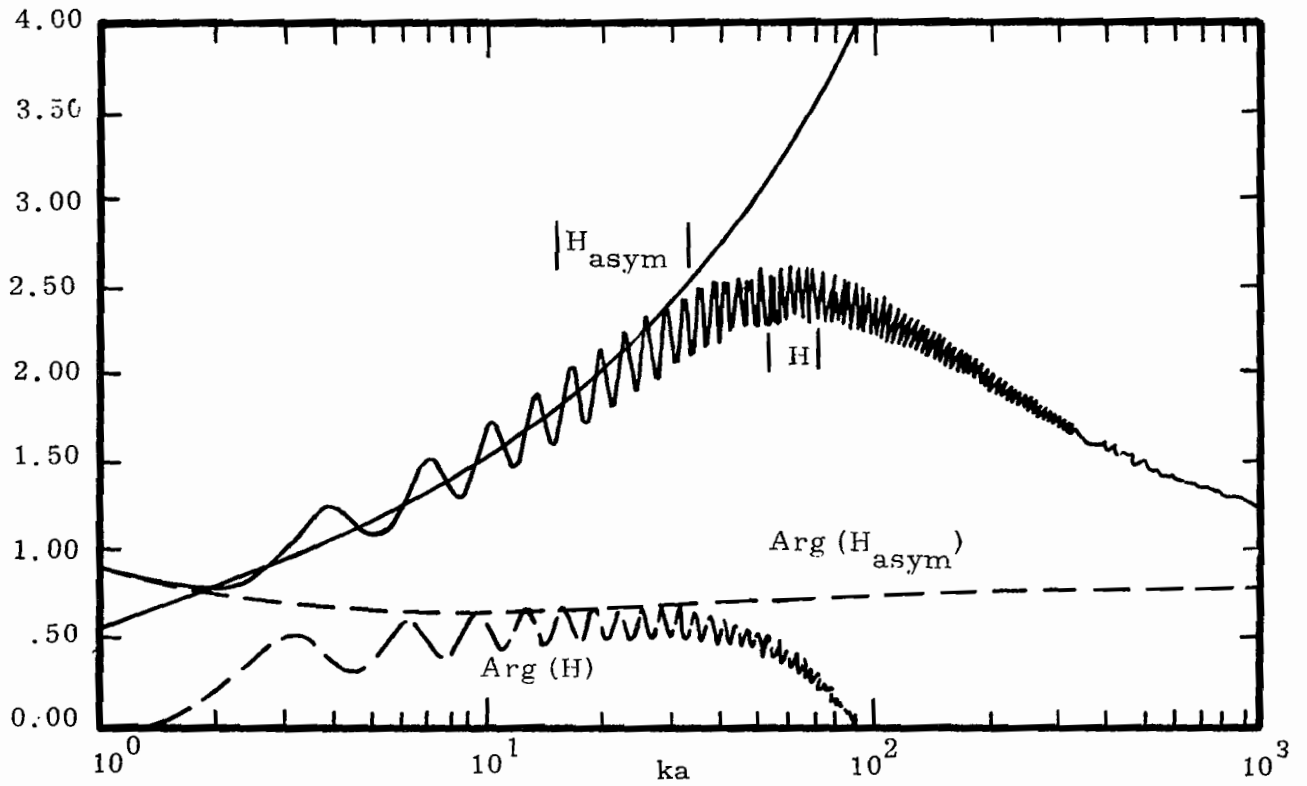
B. MAGNITUDE AND PHASE OF H

FIGURE 13. FIELDS AND ASYMPTOTIC FORMS FOR

$$Z/Z_0 = \ln(8 \frac{a}{b}) - 2 \text{ AND } \frac{b}{a} = 0.1.$$



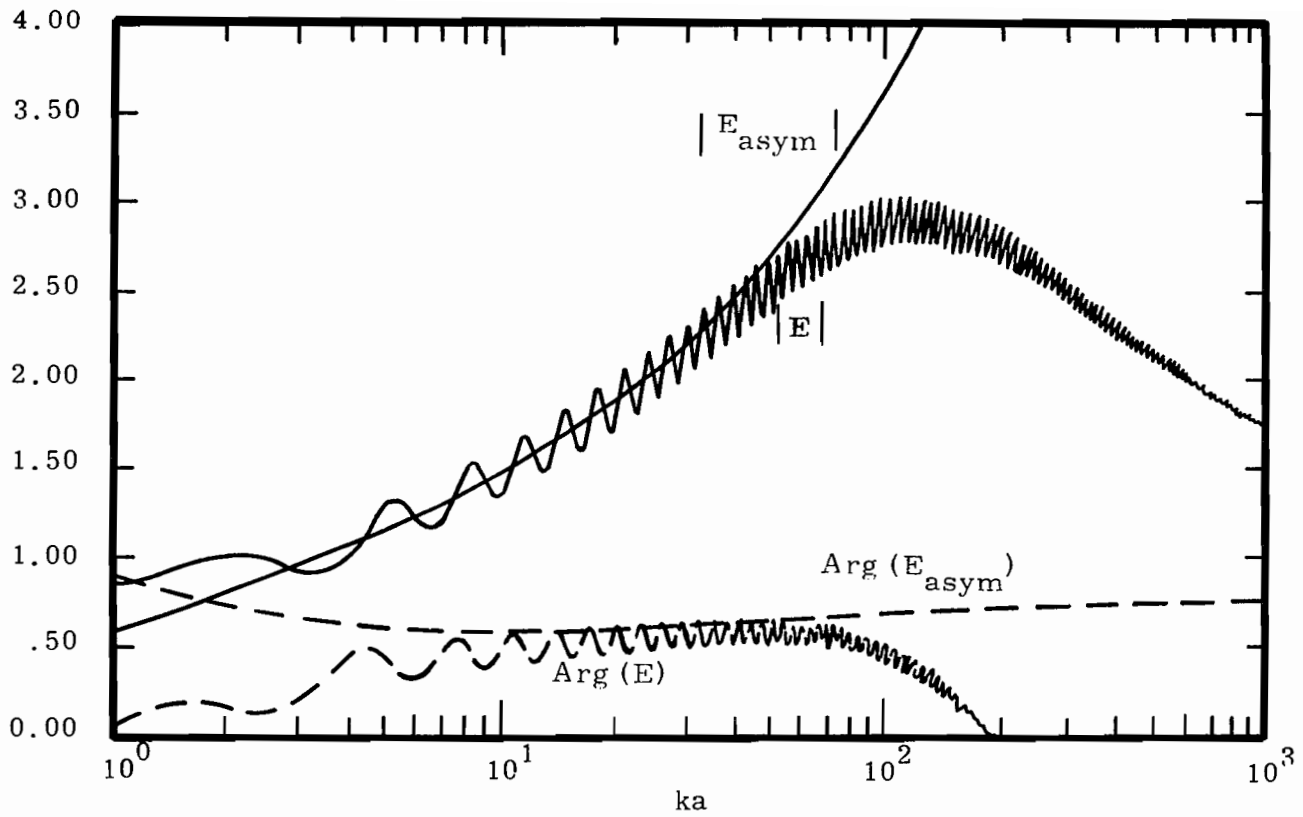
A. MAGNITUDE AND PHASE OF E



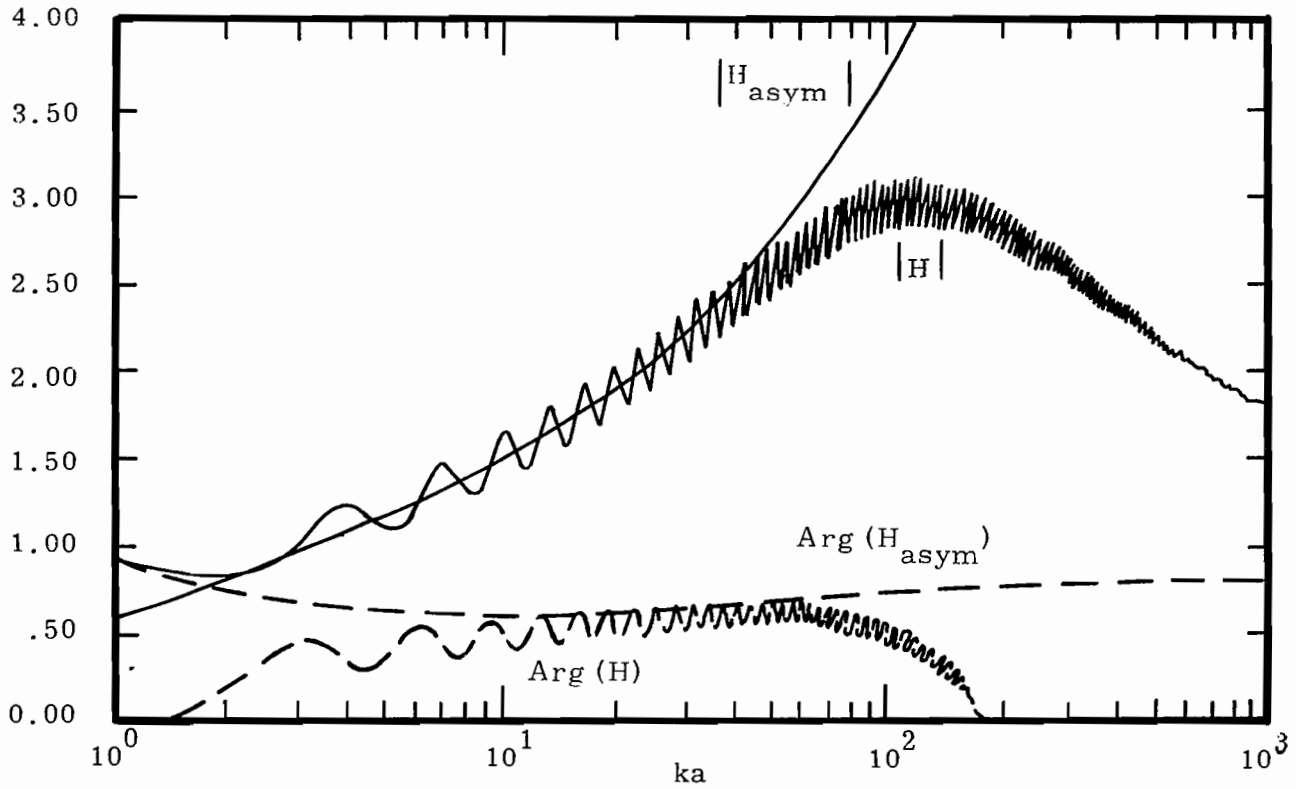
B. MAGNITUDE AND PHASE OF H

FIGURE 14. FIELDS AND ASYMPTOTIC FORMS FOR

$$Z/Z_0 = \ln(8\frac{a}{b}) - 2 \text{ AND } \frac{b}{a} = 0.02.$$



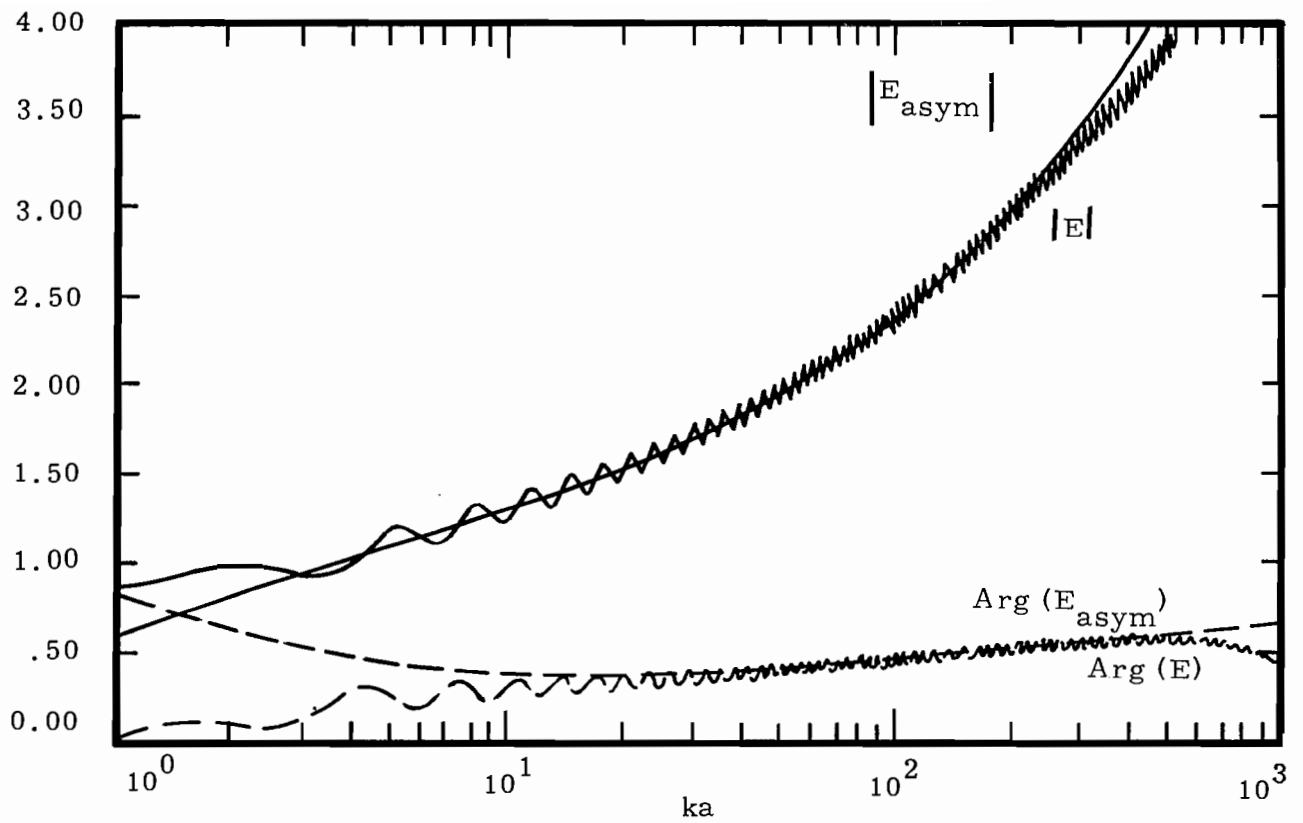
A. MAGNITUDE AND PHASE OF  $E$



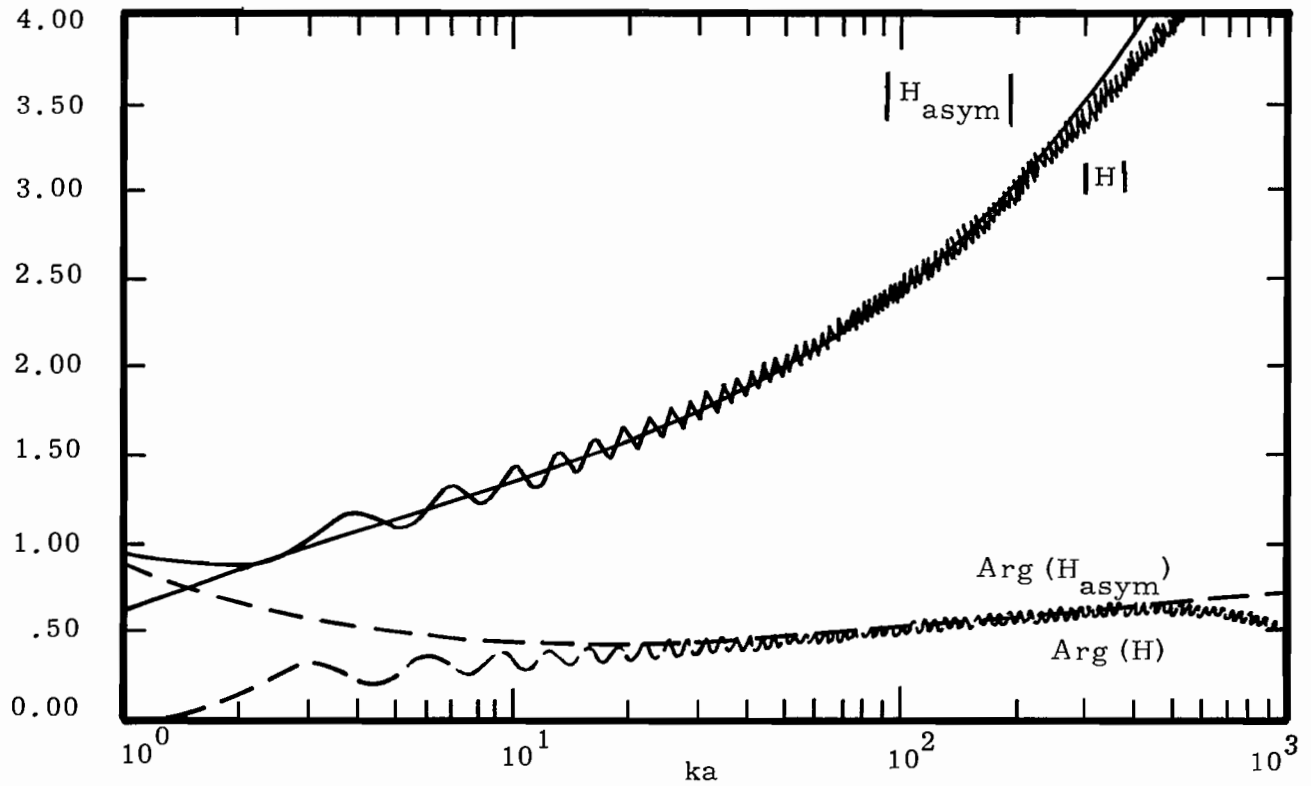
B. MAGNITUDE AND PHASE OF  $H$

FIGURE 15. FIELDS AND ASYMPTOTIC FORMS FOR

$$Z/Z_0 = \ln\left(8\frac{a}{b}\right) - 2 \text{ AND } \frac{b}{a} = 0.01.$$



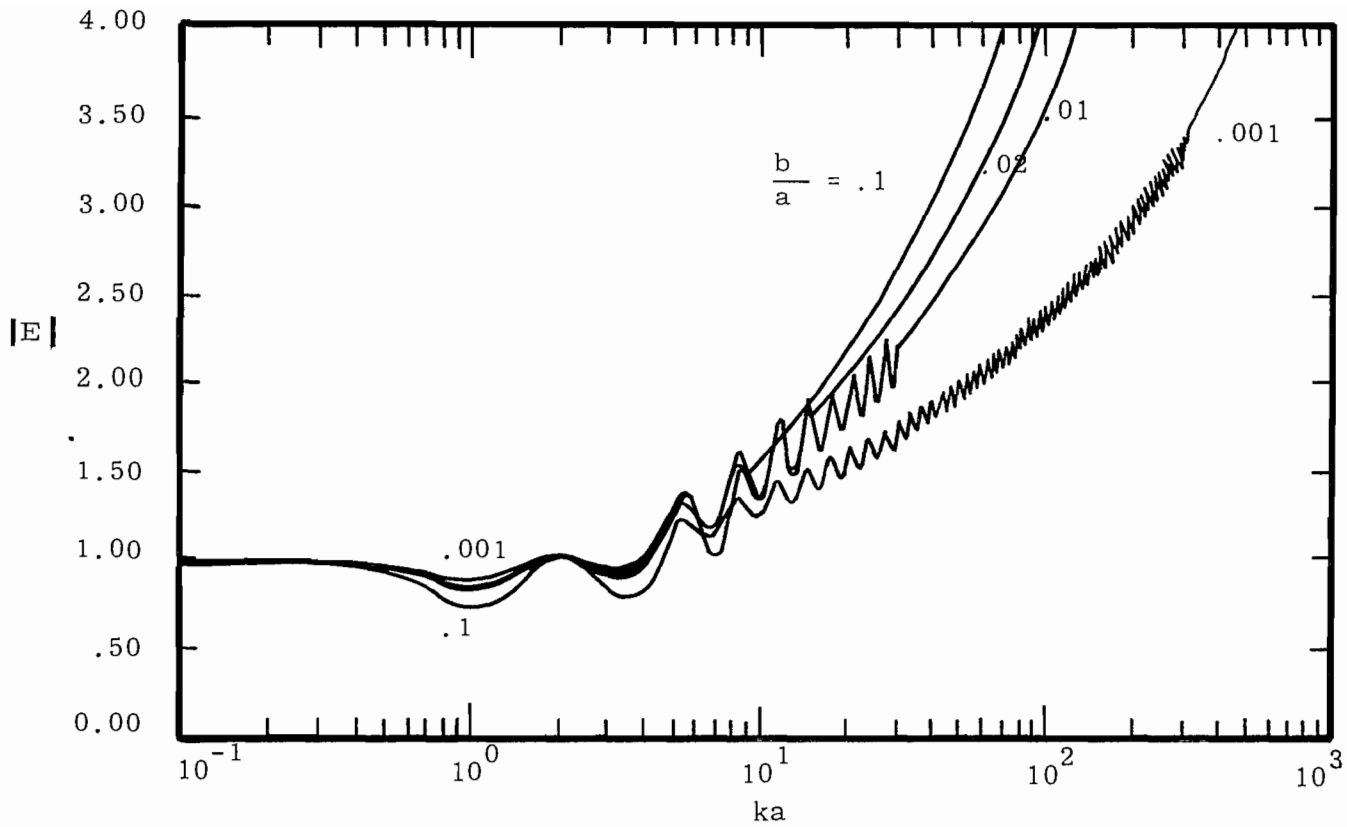
A. MAGNITUDE AND PHASE OF E



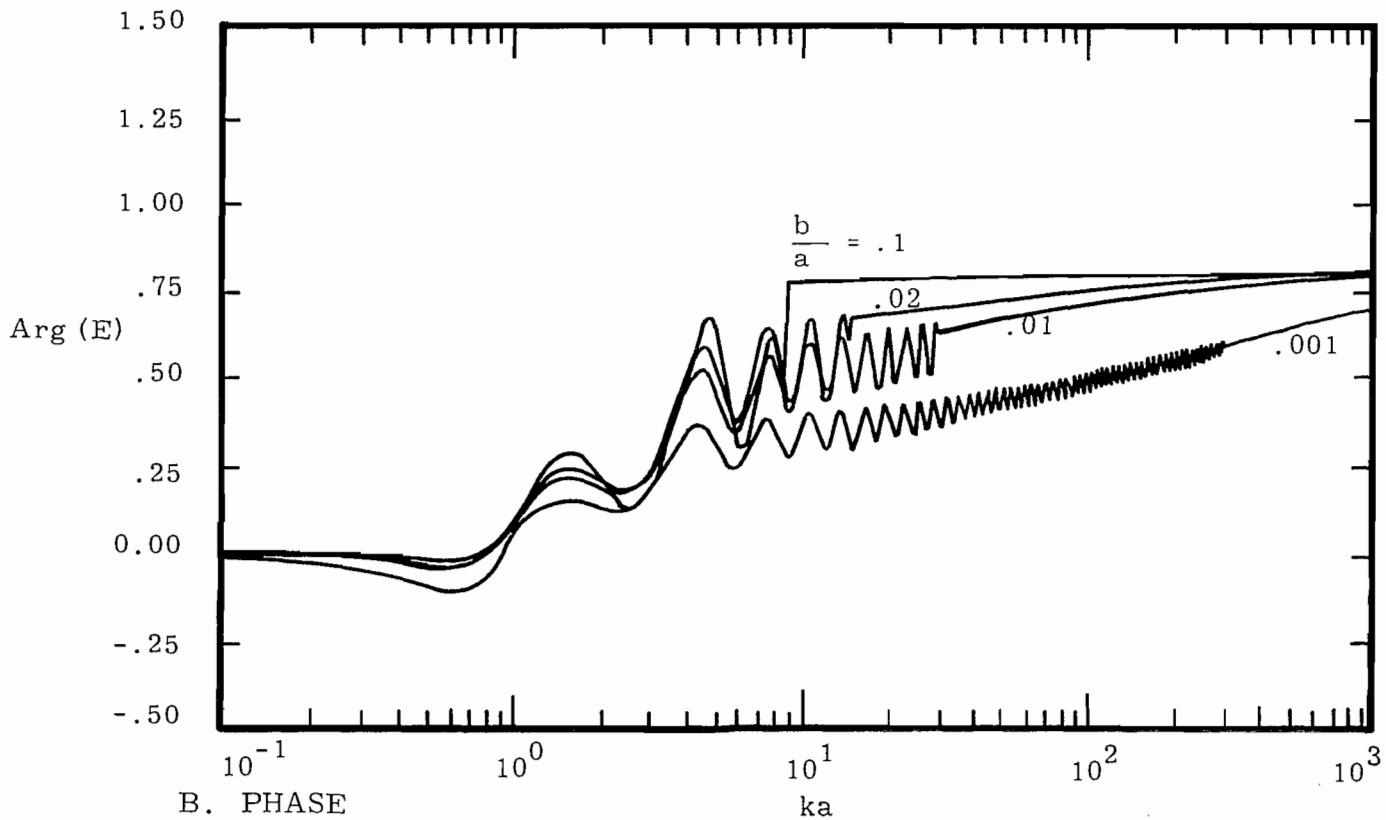
B. MAGNITUDE AND PHASE OF H

FIGURE 16. FIELDS AND ASYMPTOTIC FORMS FOR

$$Z/Z_0 = \ln(8 \frac{a}{b}) - 2 \text{ AND } \frac{b}{a} = 0.001.$$

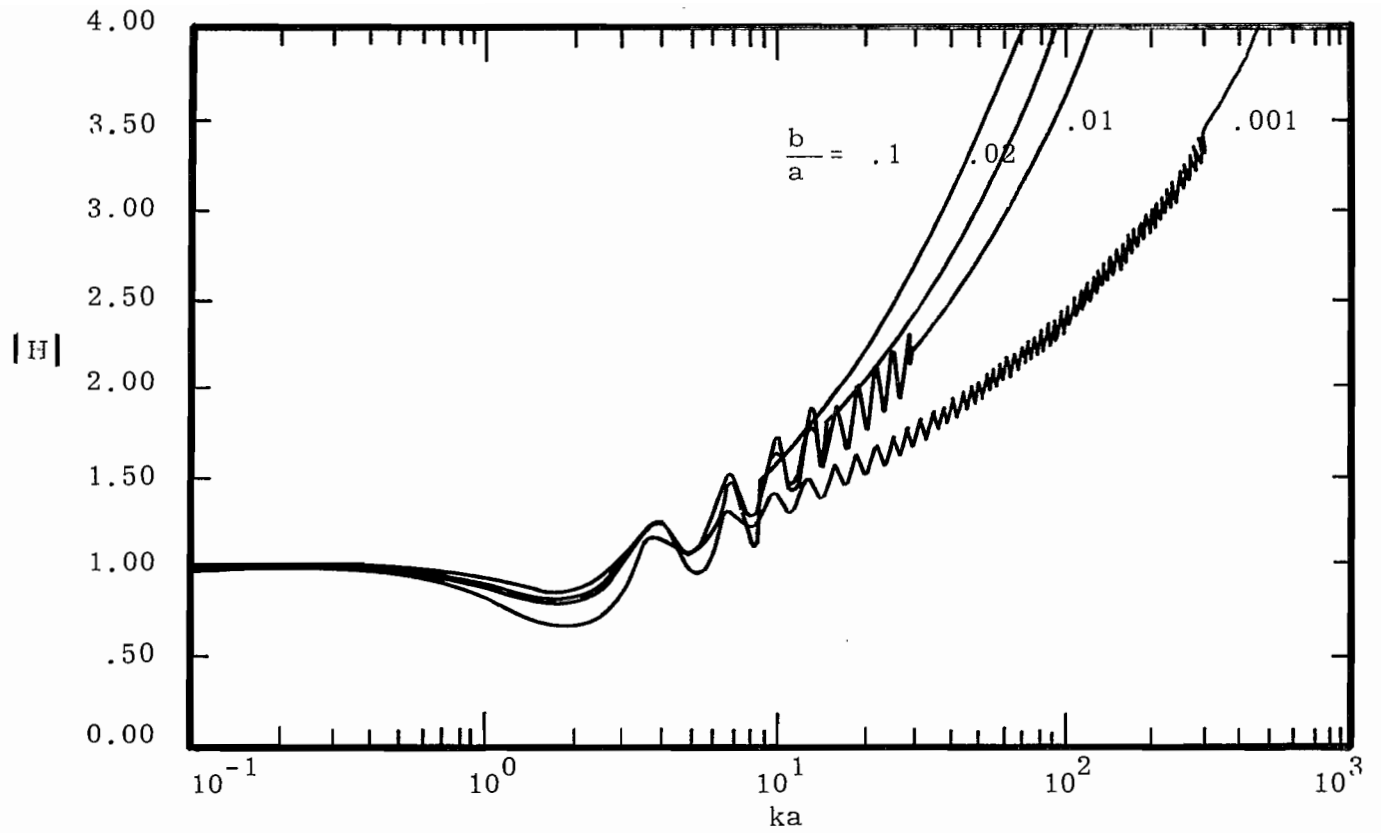


A. MAGNITUDE

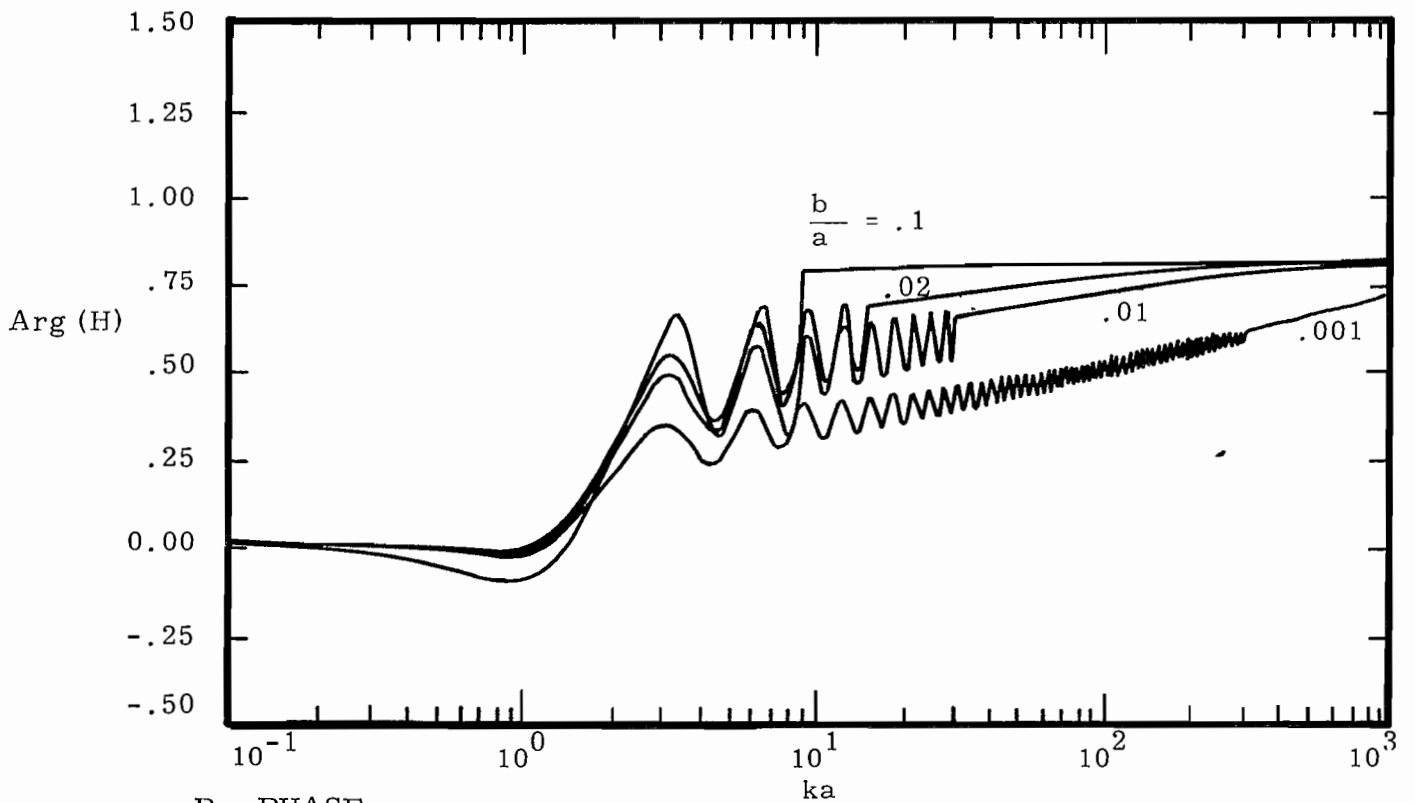


B. PHASE

FIGURE 17. E-FIELD FOR  $Z/Z_0 = \ln(8 \frac{a}{b}) - 2$  WITH  $\frac{b}{a}$  AS A PARAMETER



A. MAGNITUDE



B. PHASE

FIGURE 18. H-FIELD FOR  $Z/Z_0 = \ln(8 \frac{a}{b}) - 2$  WITH  $\frac{b}{a}$  AS A PARAMETER

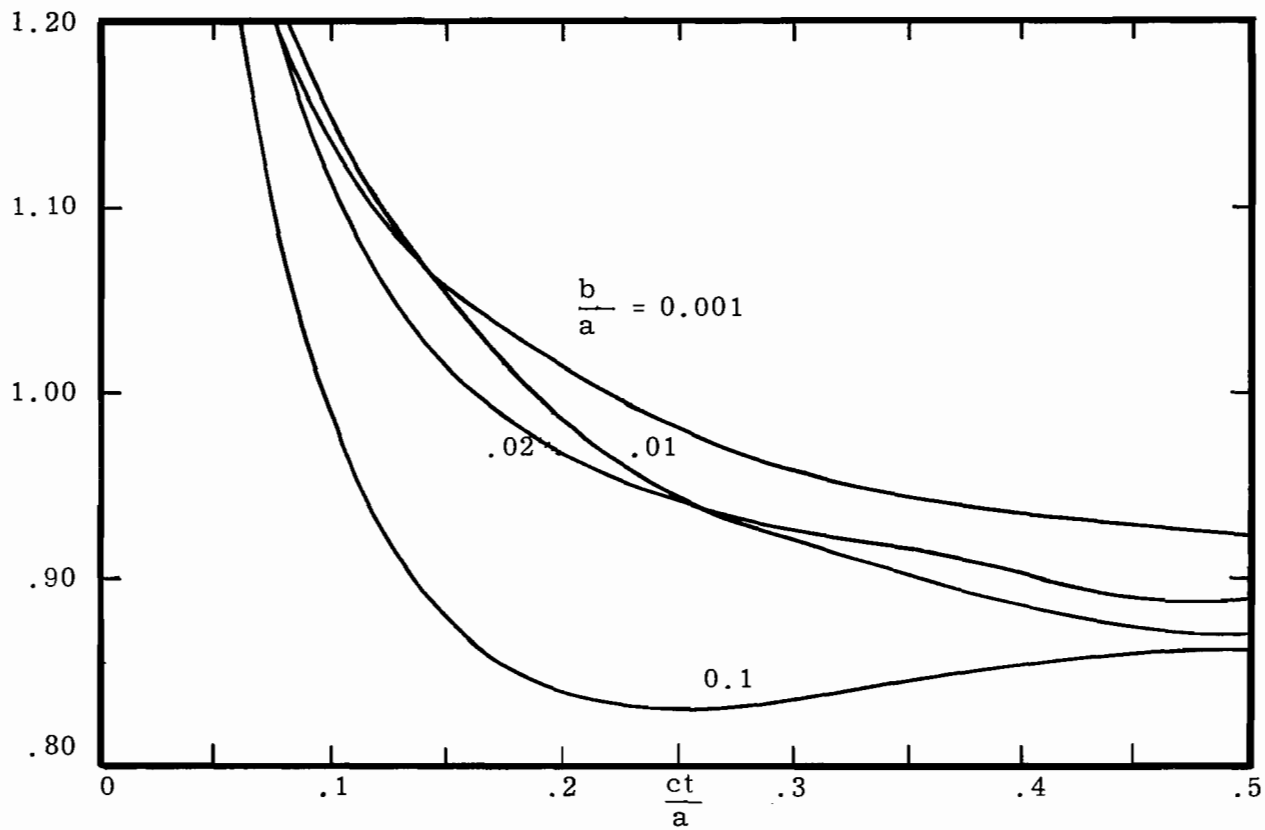
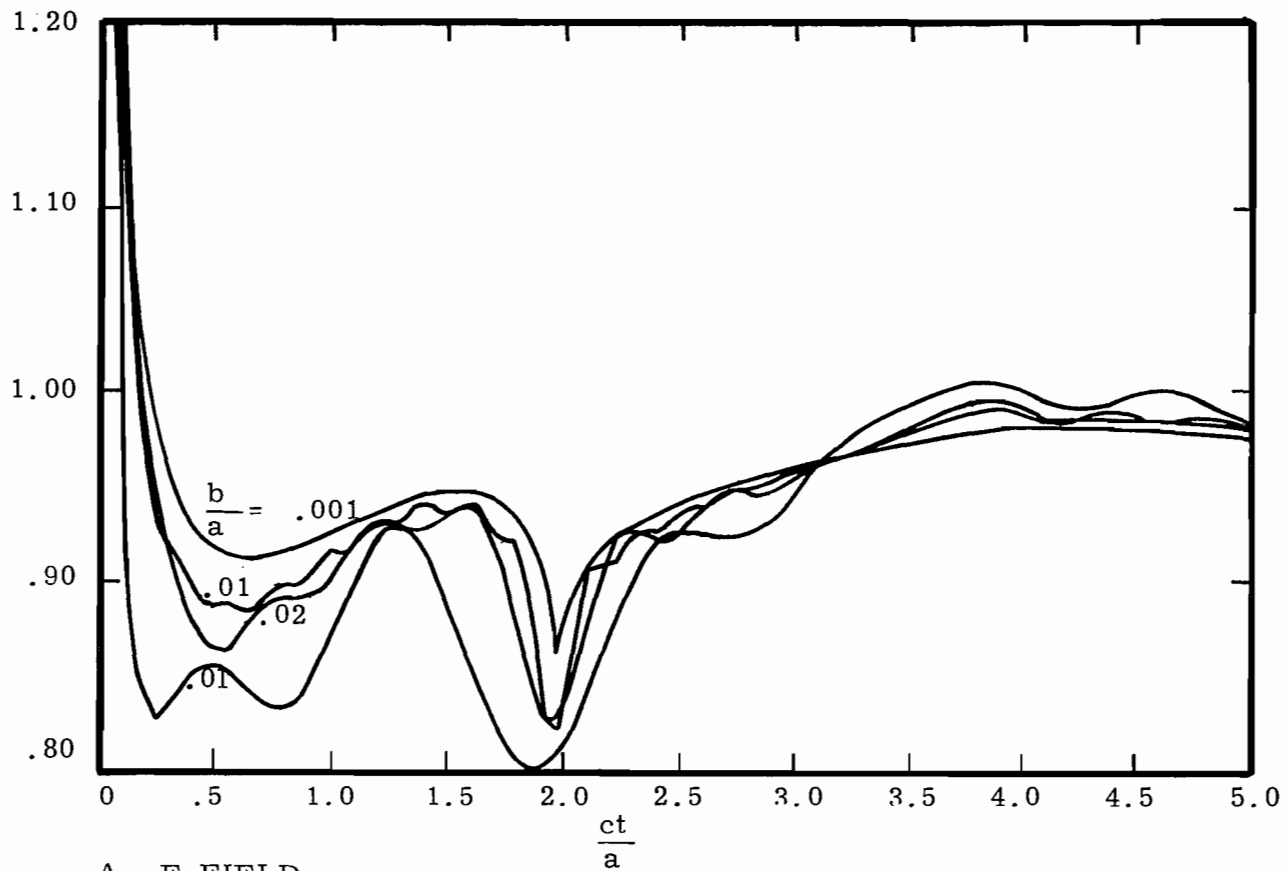


FIGURE 19. STEP-FUNCTION RESPONSE FOR THE E-FIELD IN THE TIME DOMAIN WITH  $Z/Z_0 = \ln(8 \frac{a}{b}) - 2$  AND  $\frac{b}{a}$  AS A PARAMETER

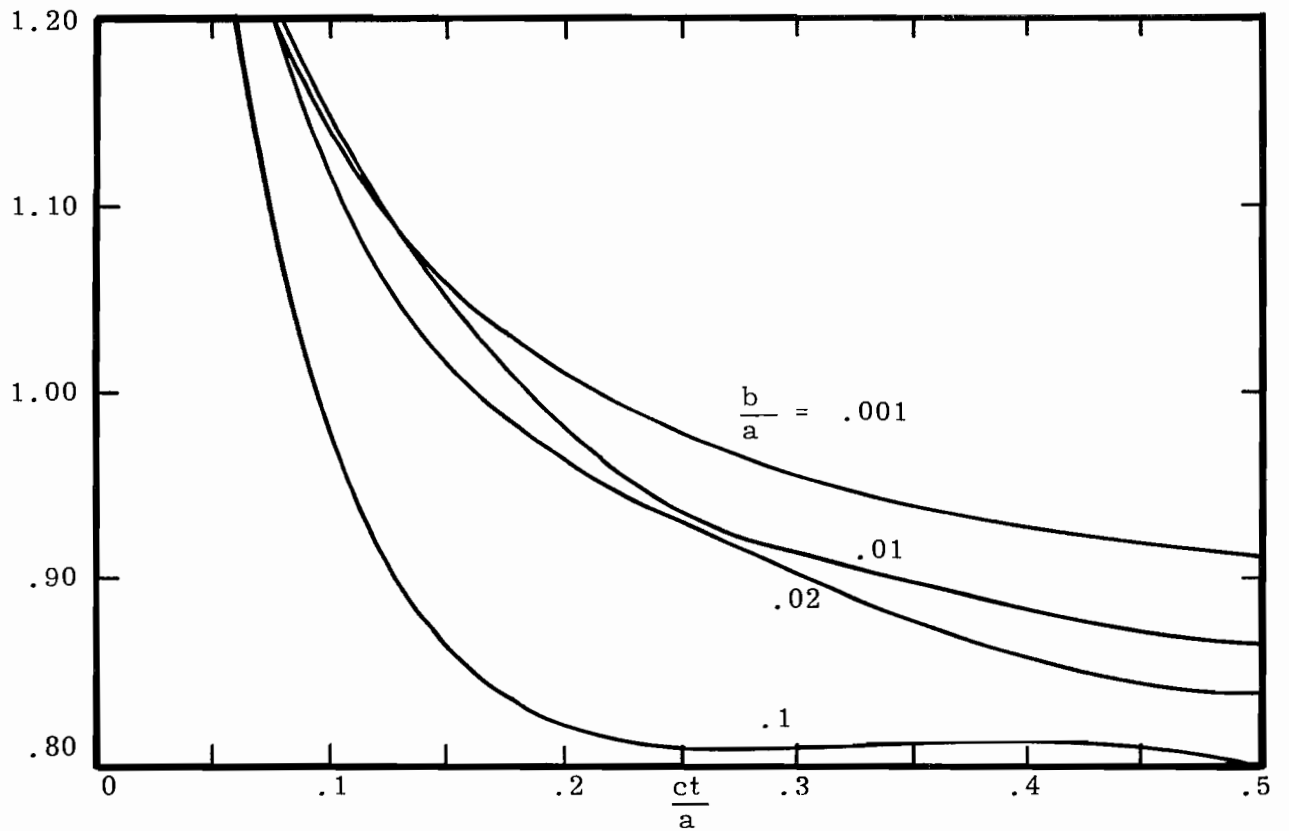
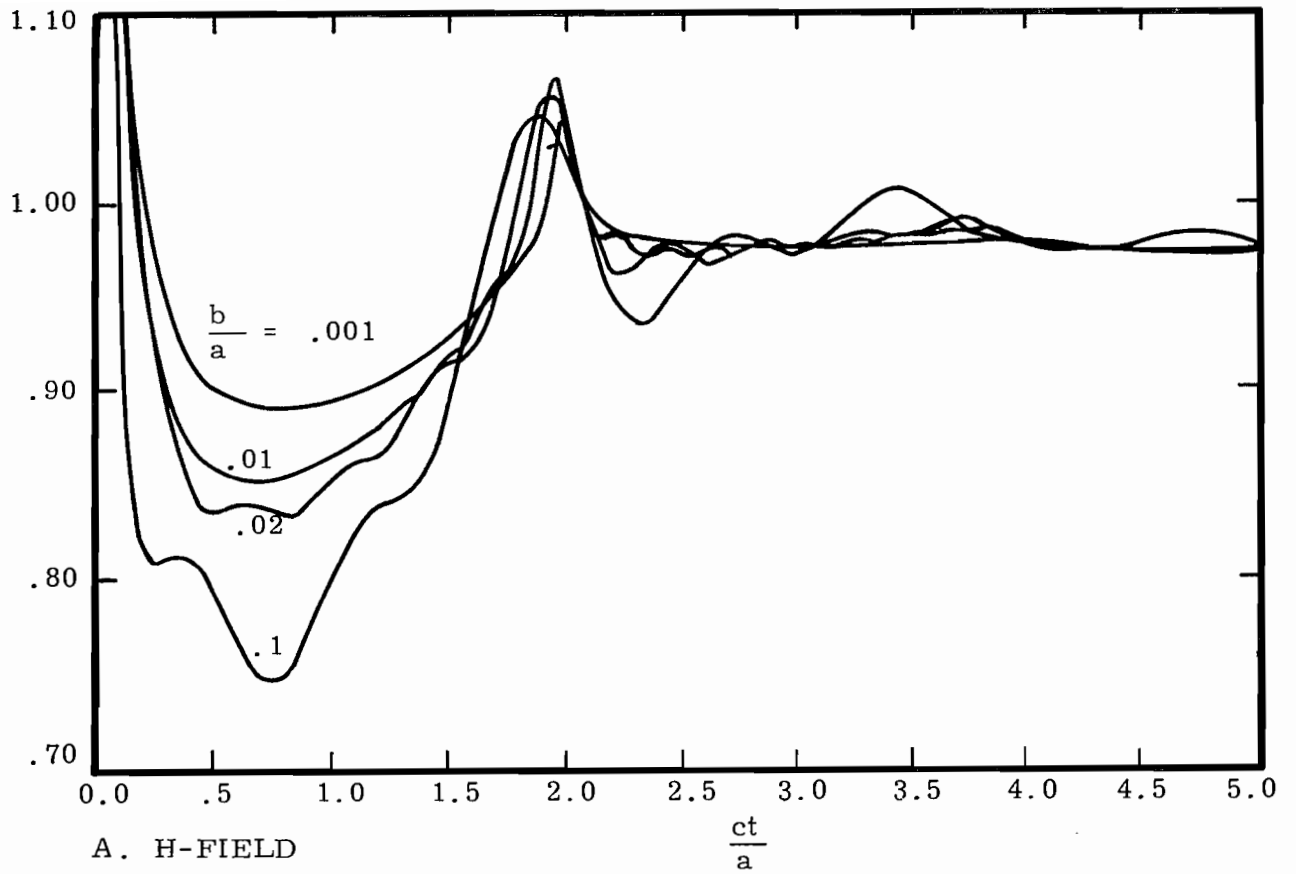
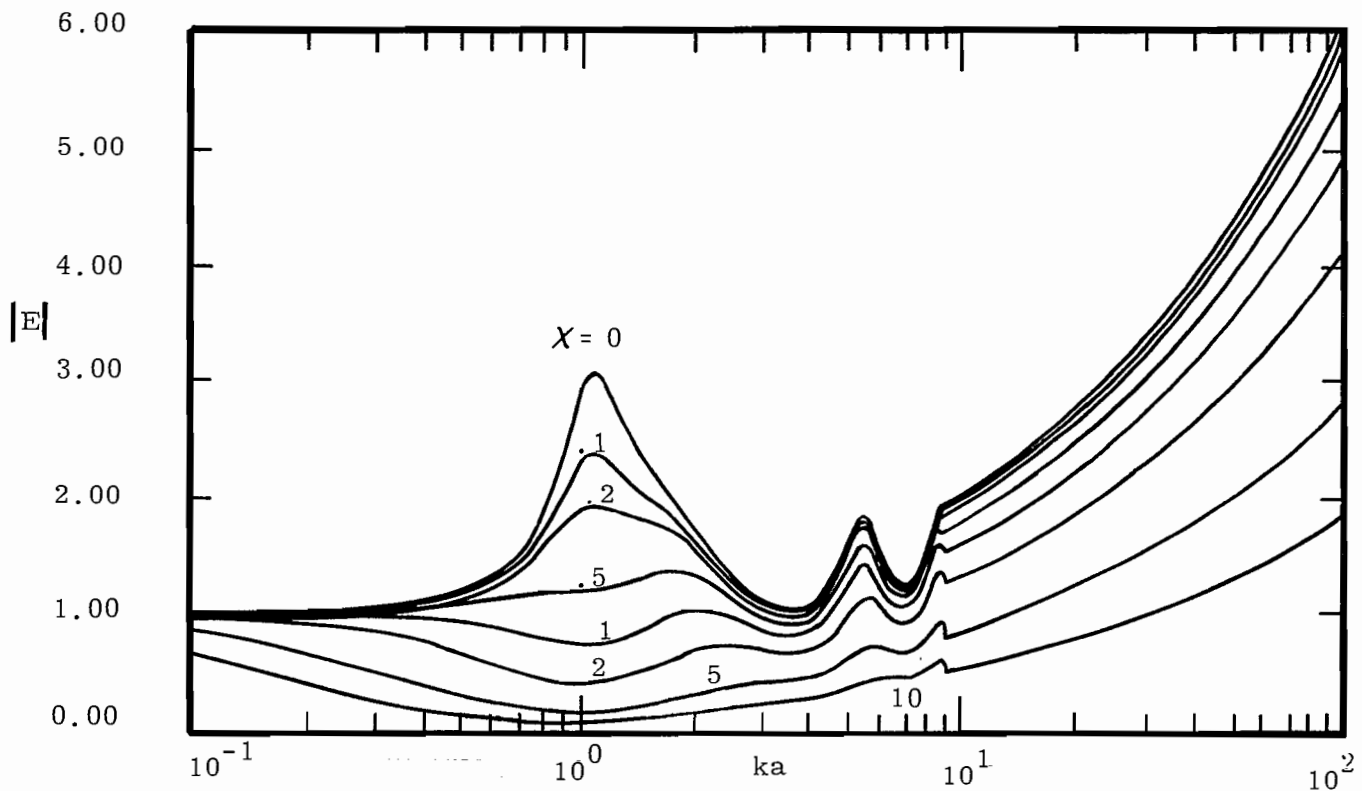
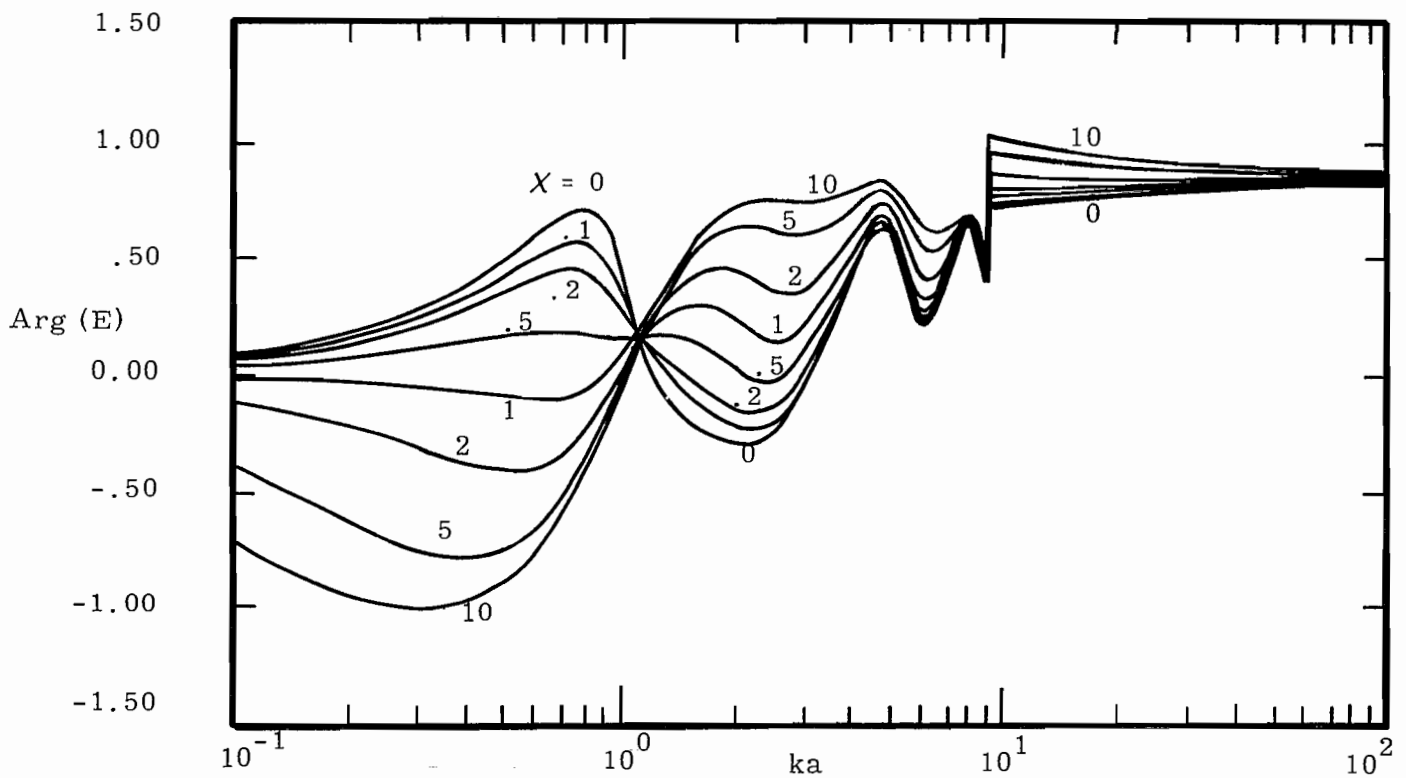


FIGURE 20. STEP-FUNCTION RESPONSE FOR THE H-FIELD IN THE TIME DOMAIN WITH  $Z/Z_0 = 1 \ln(8 \frac{a}{b}) - 2$  AND  $\frac{b}{a}$  AS A PARAMETER.



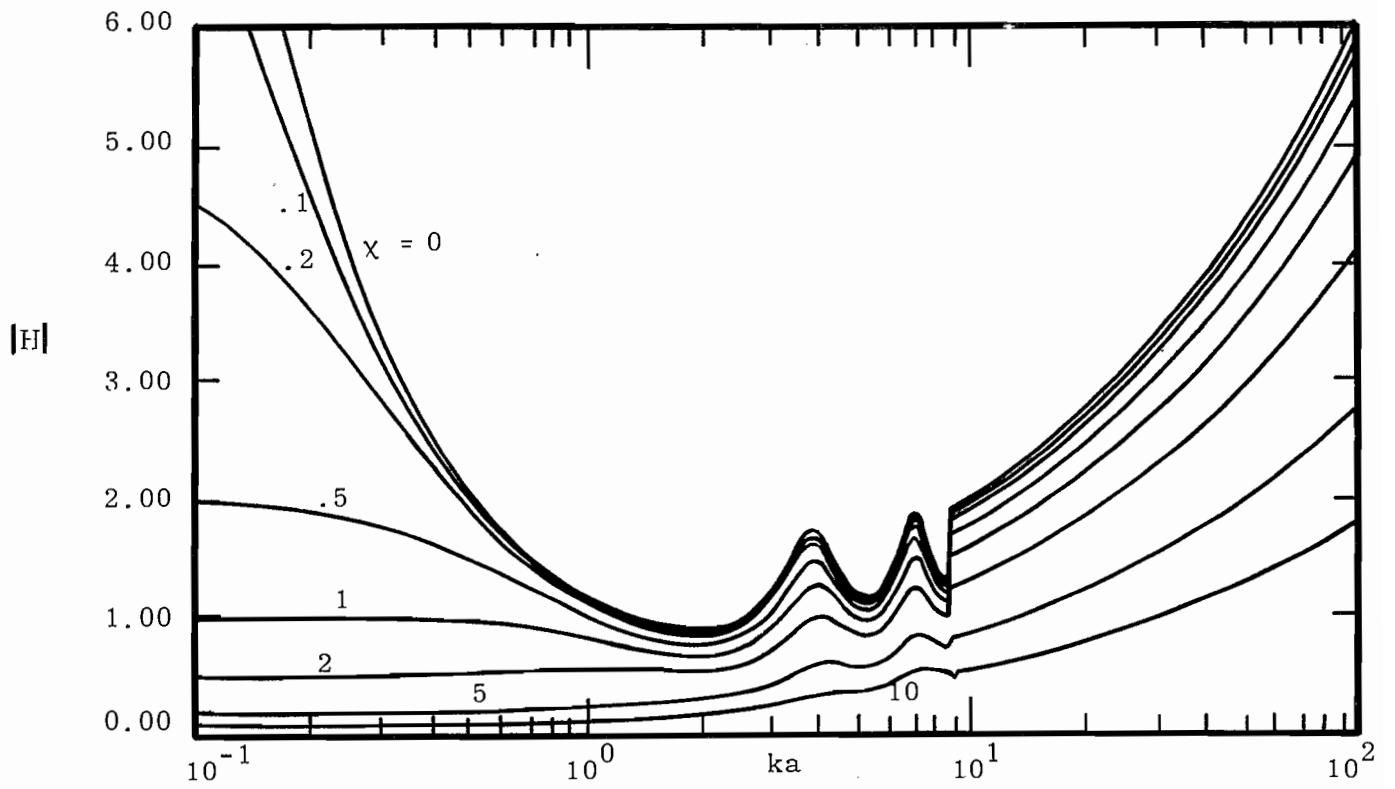


A. MAGNITUDE OF E

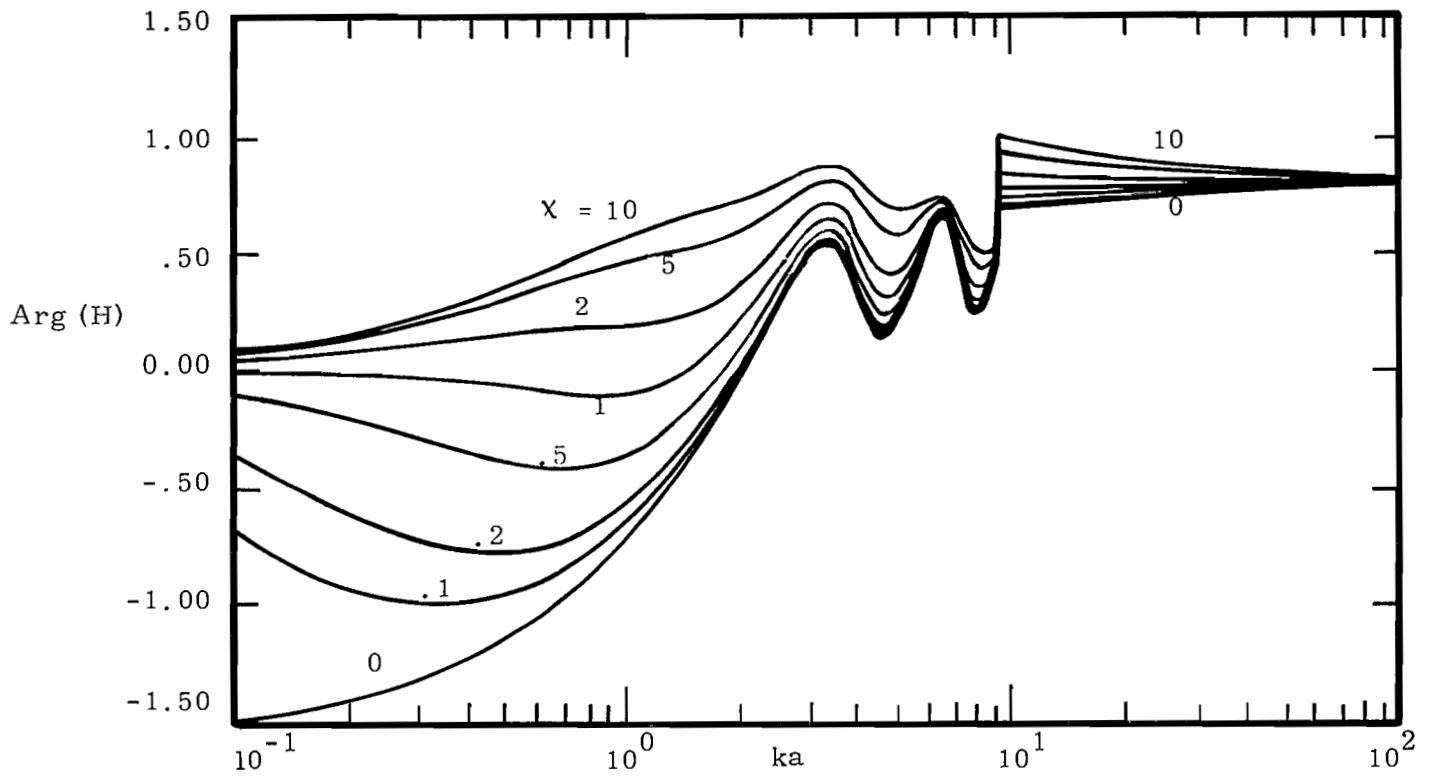


B. PHASE OF E

FIGURE 21. E-FIELD FOR  $\frac{b}{a} = 0.1$  WITH  $\chi$  AS A PARAMETER

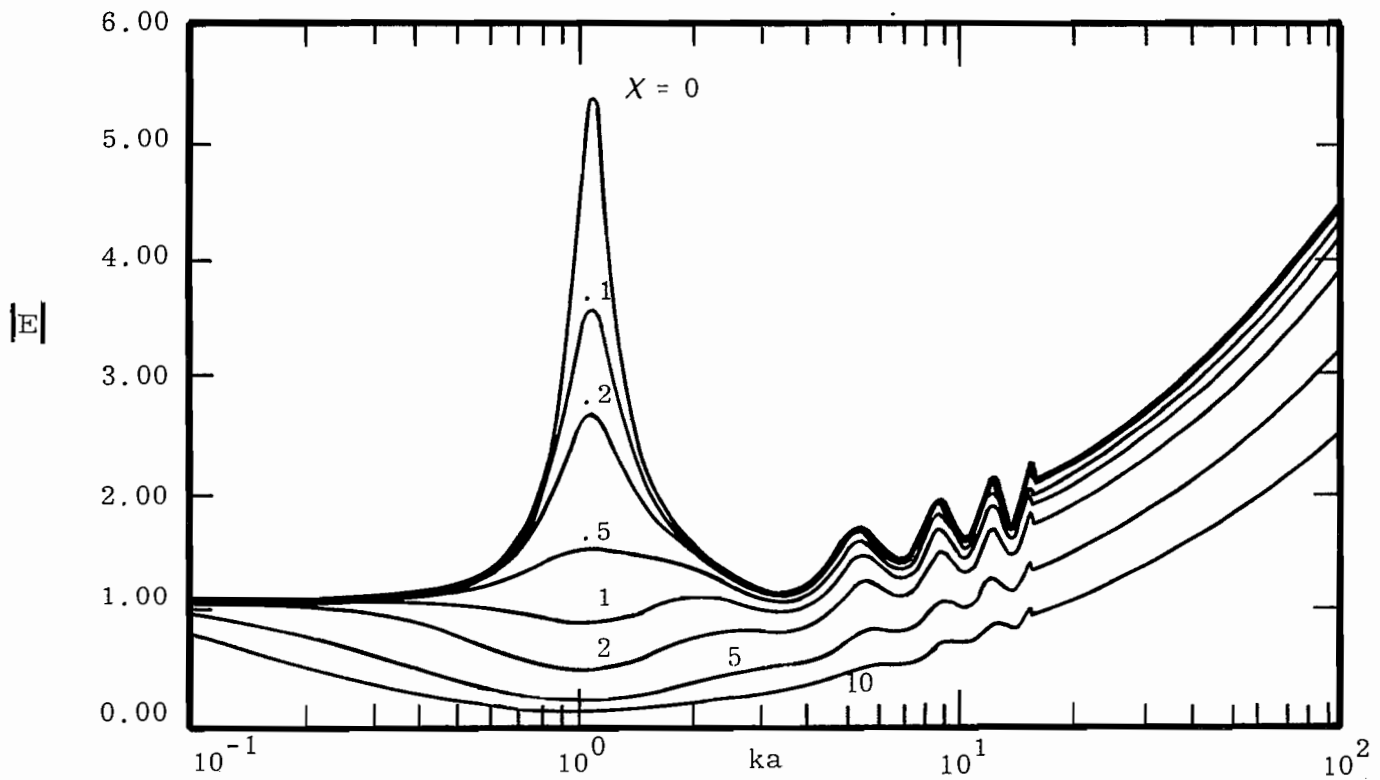


A. MAGNITUDE OF H

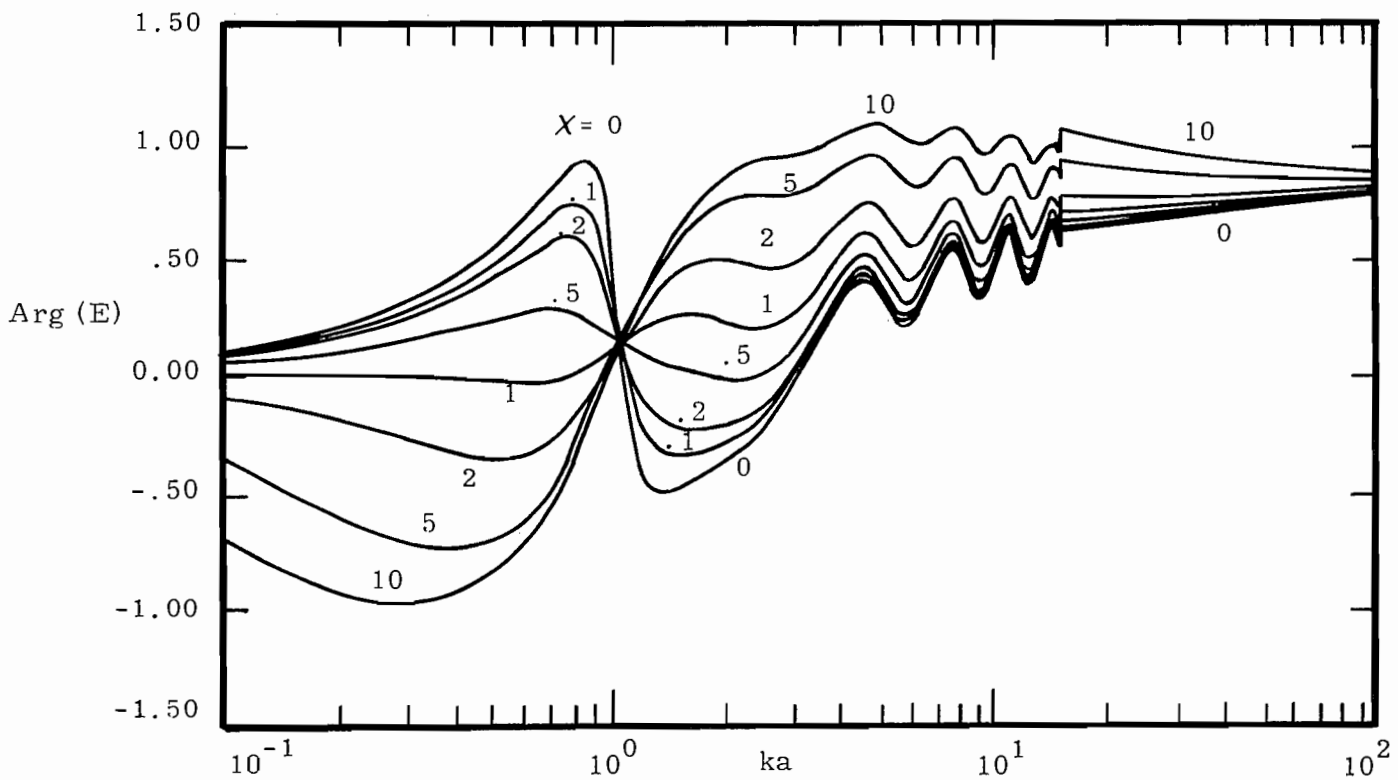


B. PHASE OF H

FIGURE 22. H-FIELD FOR  $\frac{b}{a} = 0.1$  WITH  $\chi$  AS A PARAMETER

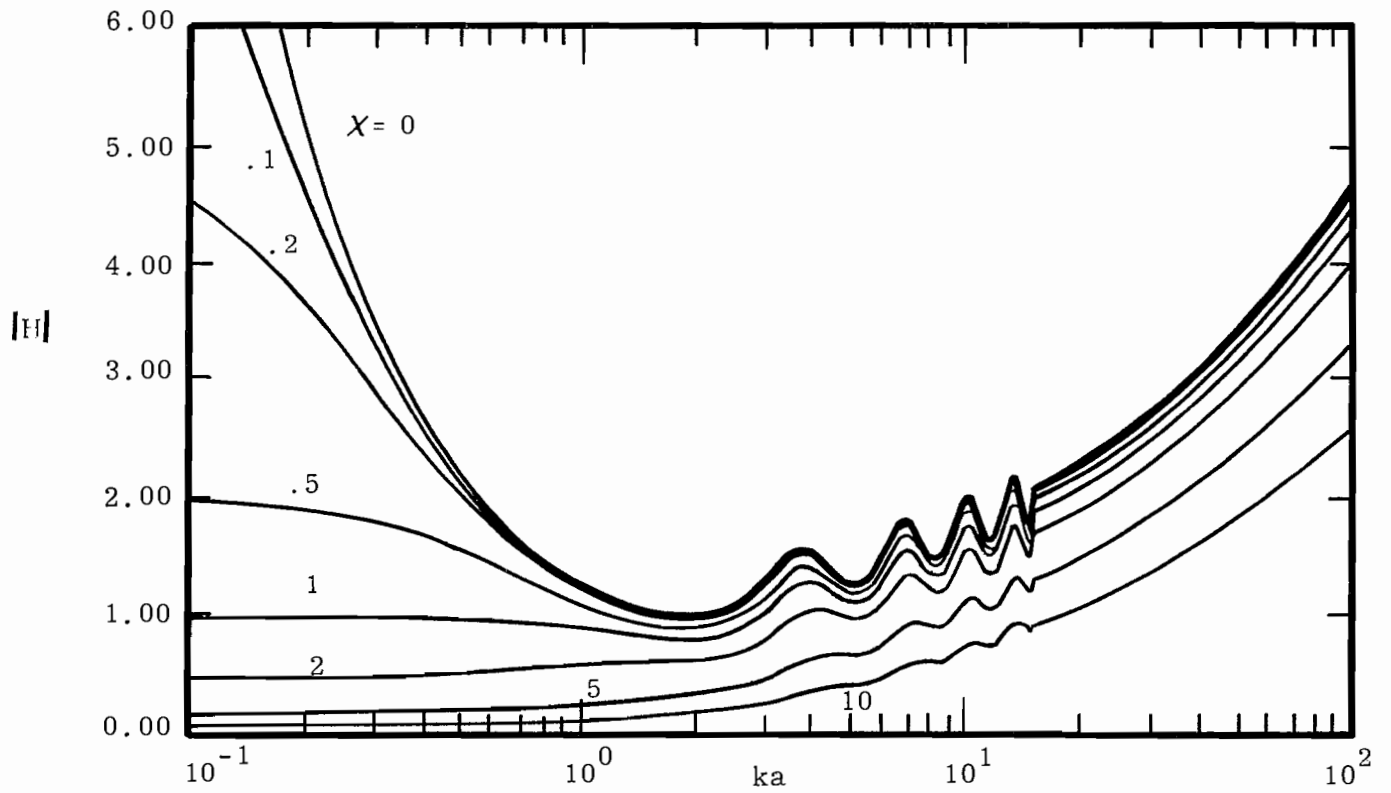


A. MAGNITUDE OF E

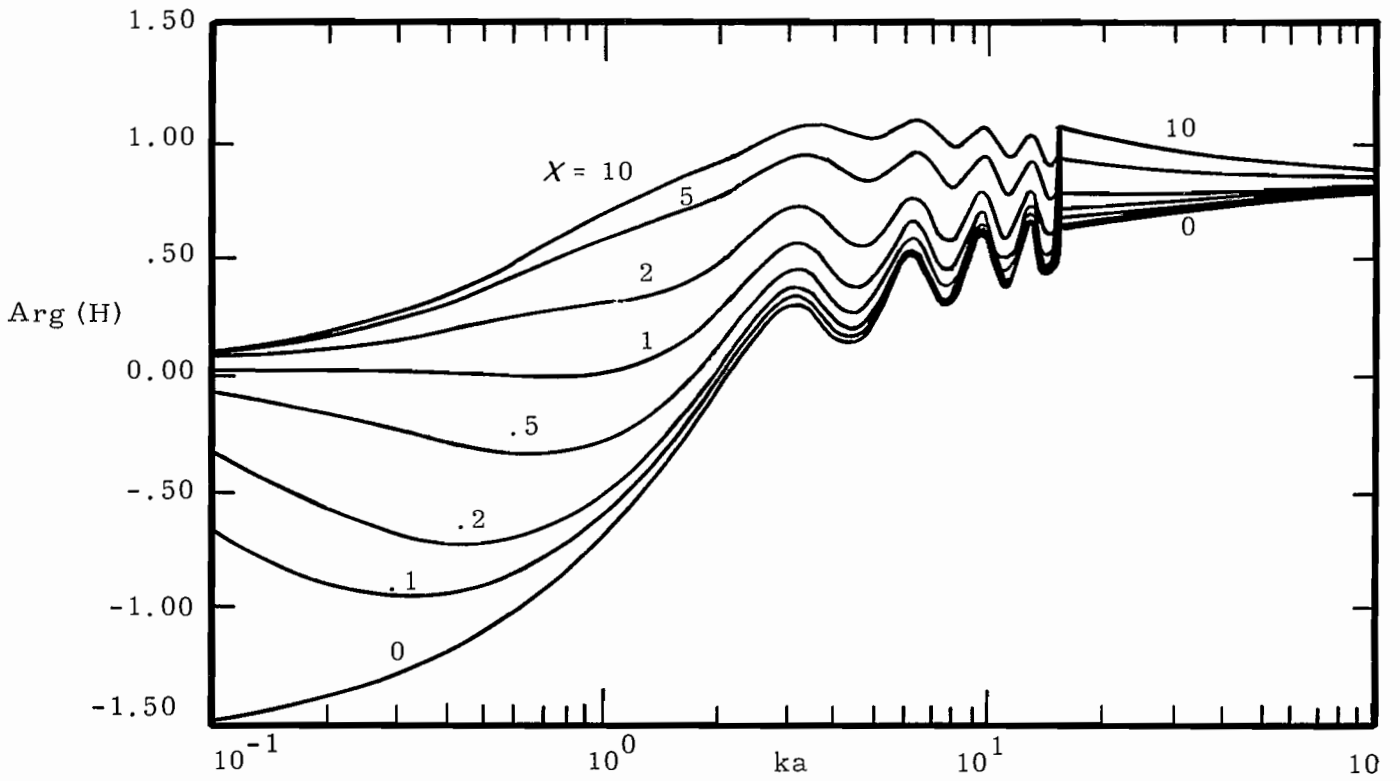


B. PHASE OF E

FIGURE 23. E-FIELD FOR  $\frac{b}{a} = 0.02$  WITH  $X$  AS A PARAMETER

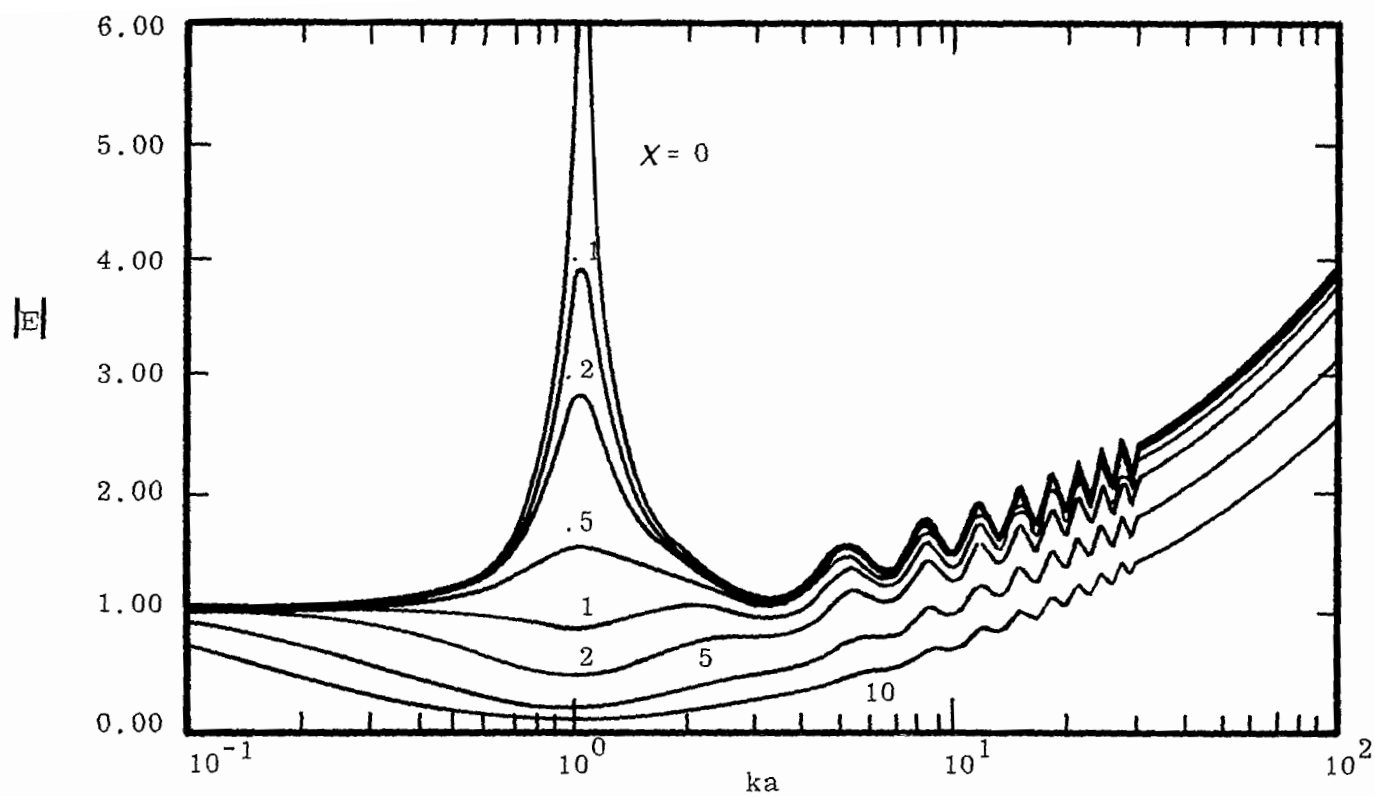


A. MAGNITUDE OF H

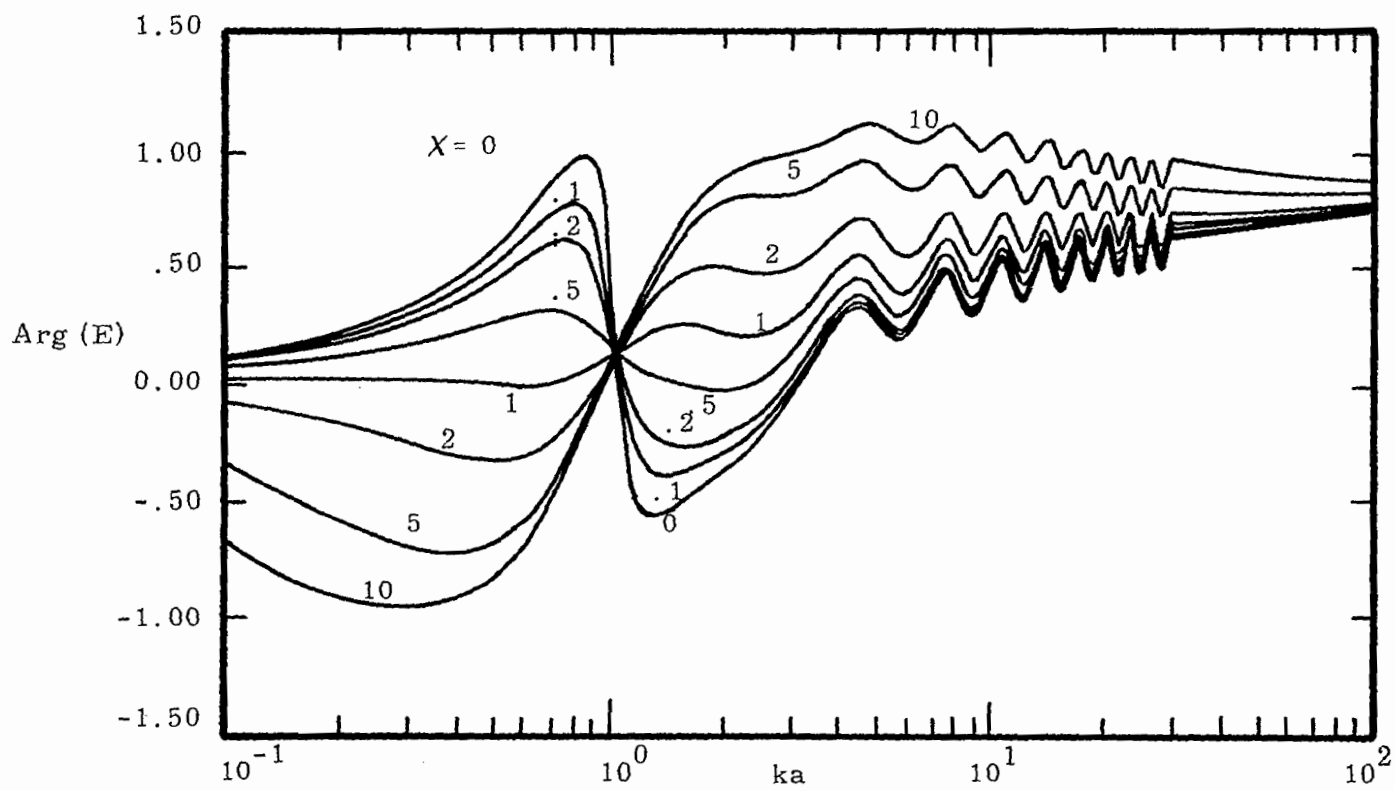


B. PHASE OF H

FIGURE 24. H-FIELD FOR  $\frac{b}{a} = 0.02$  WITH  $X$  AS A PARAMETER

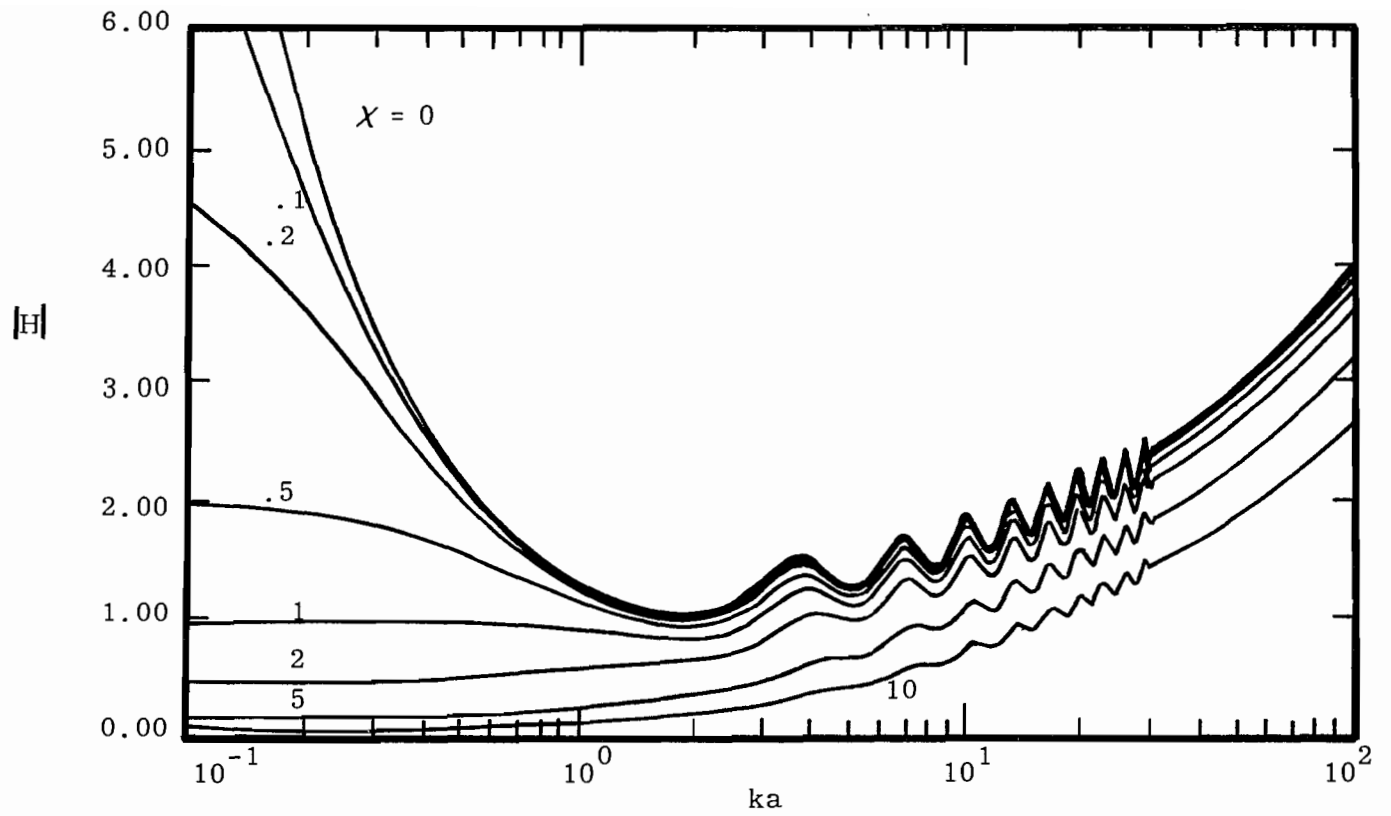


A. MAGNITUDE OF E

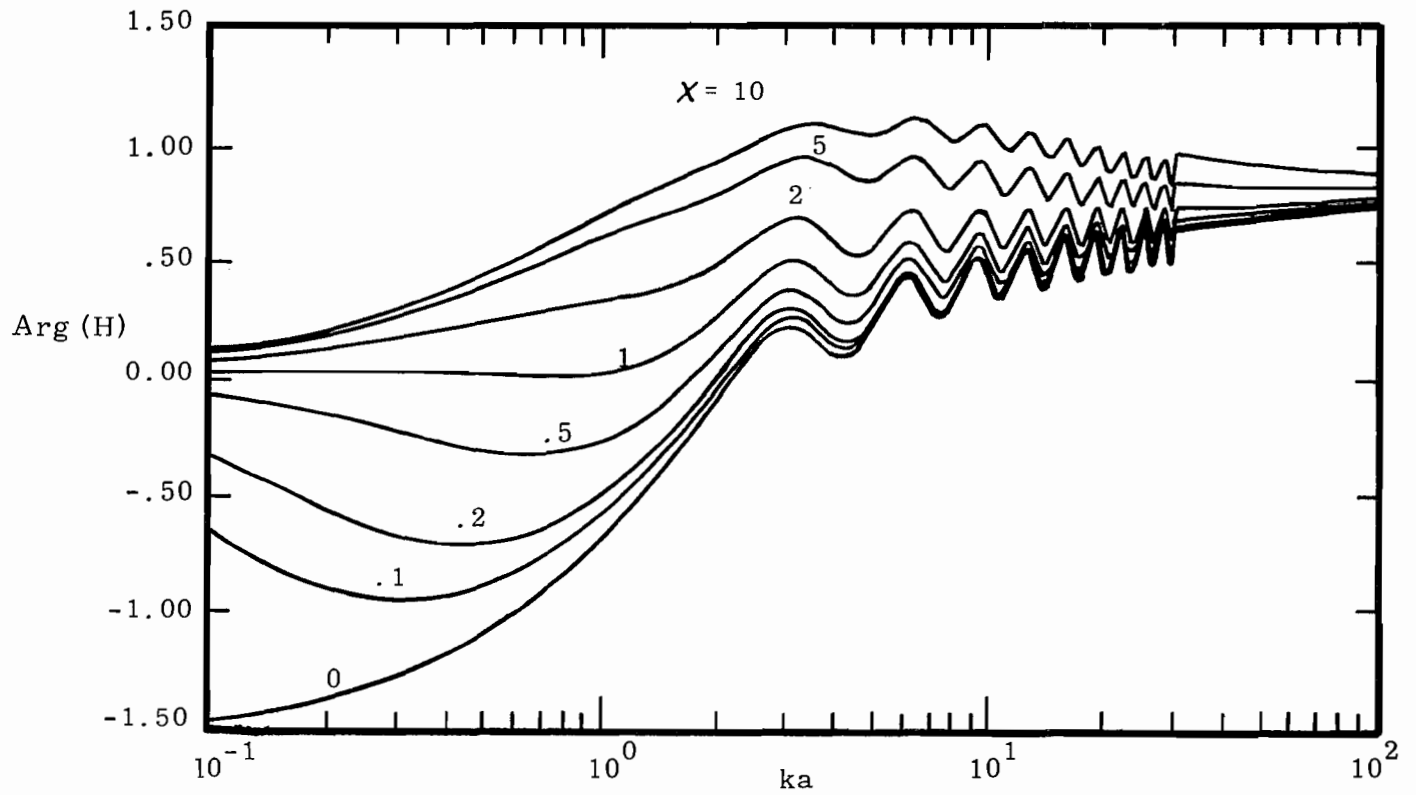


B. PHASE OF E

FIGURE 25. E-FIELD FOR  $\frac{b}{a} = 0.01$  WITH  $X$  AS A PARAMETER

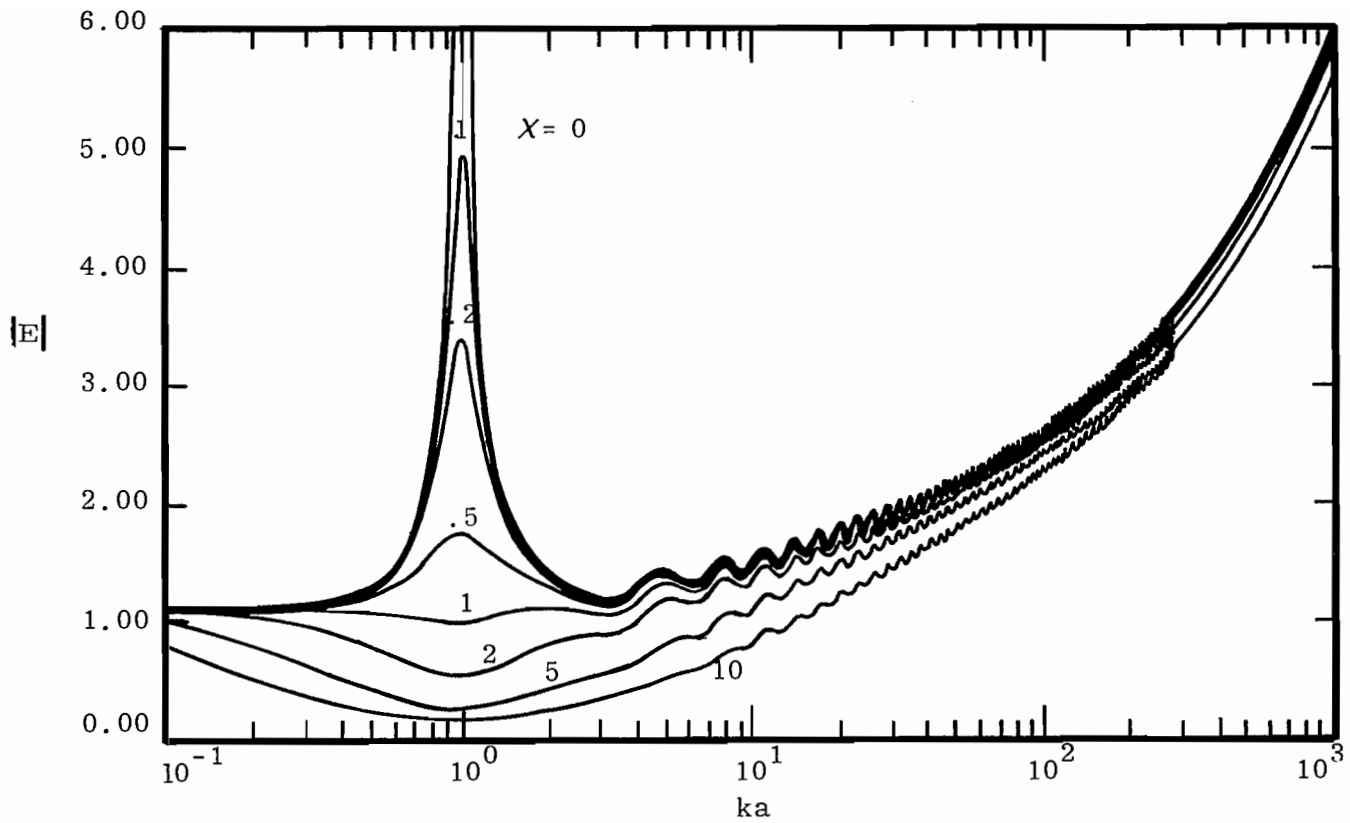


A. MAGNITUDE OF H

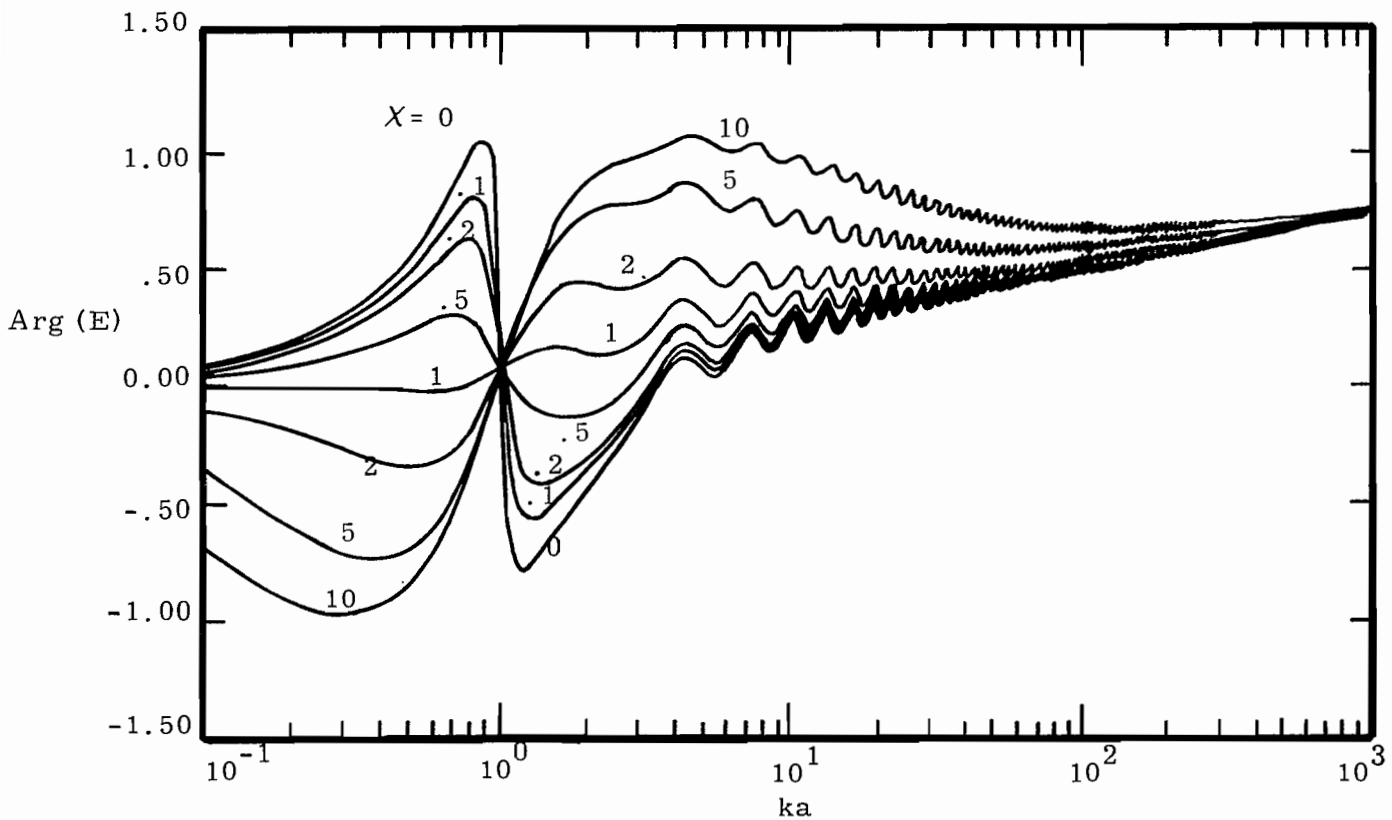


B. PHASE OF H

FIGURE 26. H-FIELD FOR  $\frac{b}{a} = 0.01$  WITH  $\chi$  AS A PARAMETER

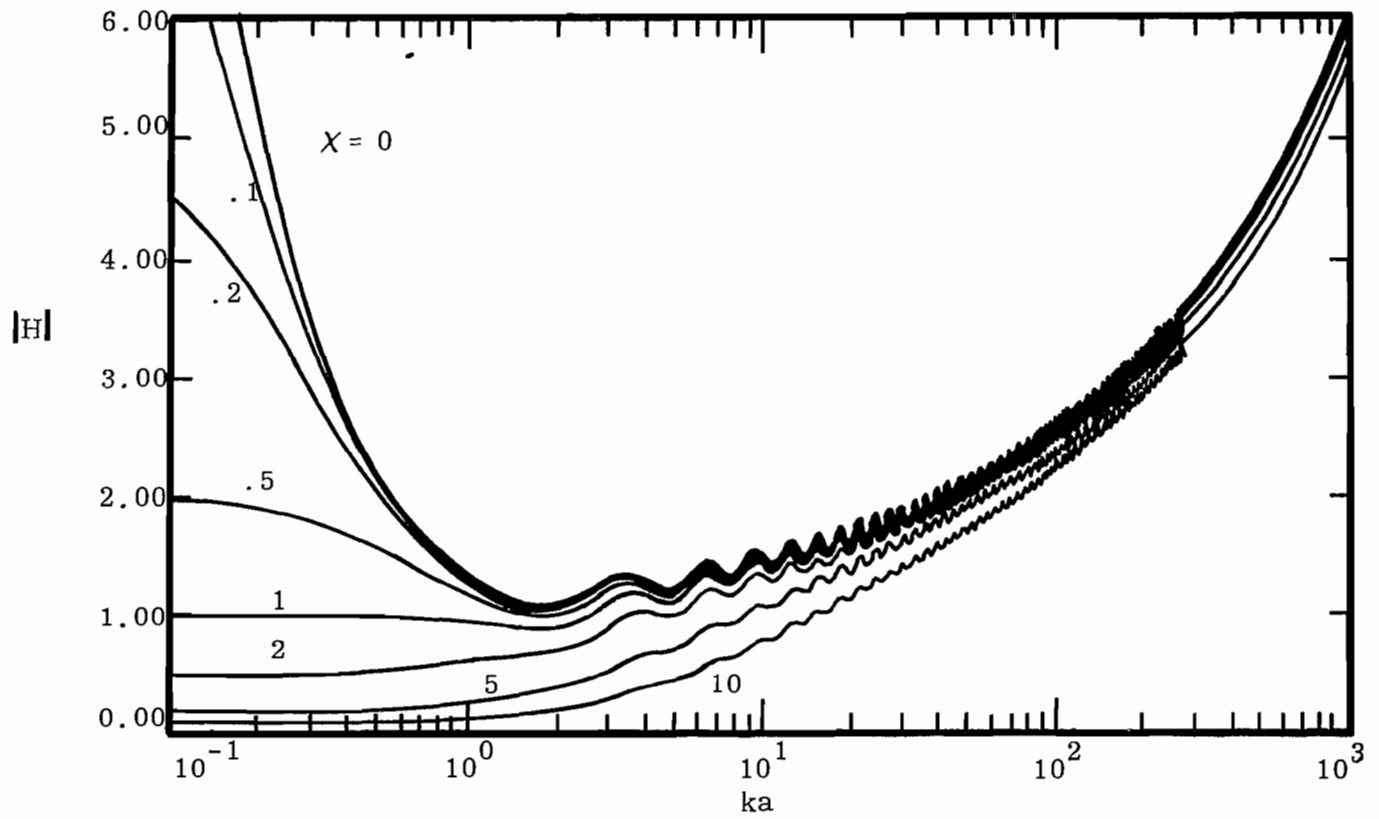


A. MAGNITUDE OF E

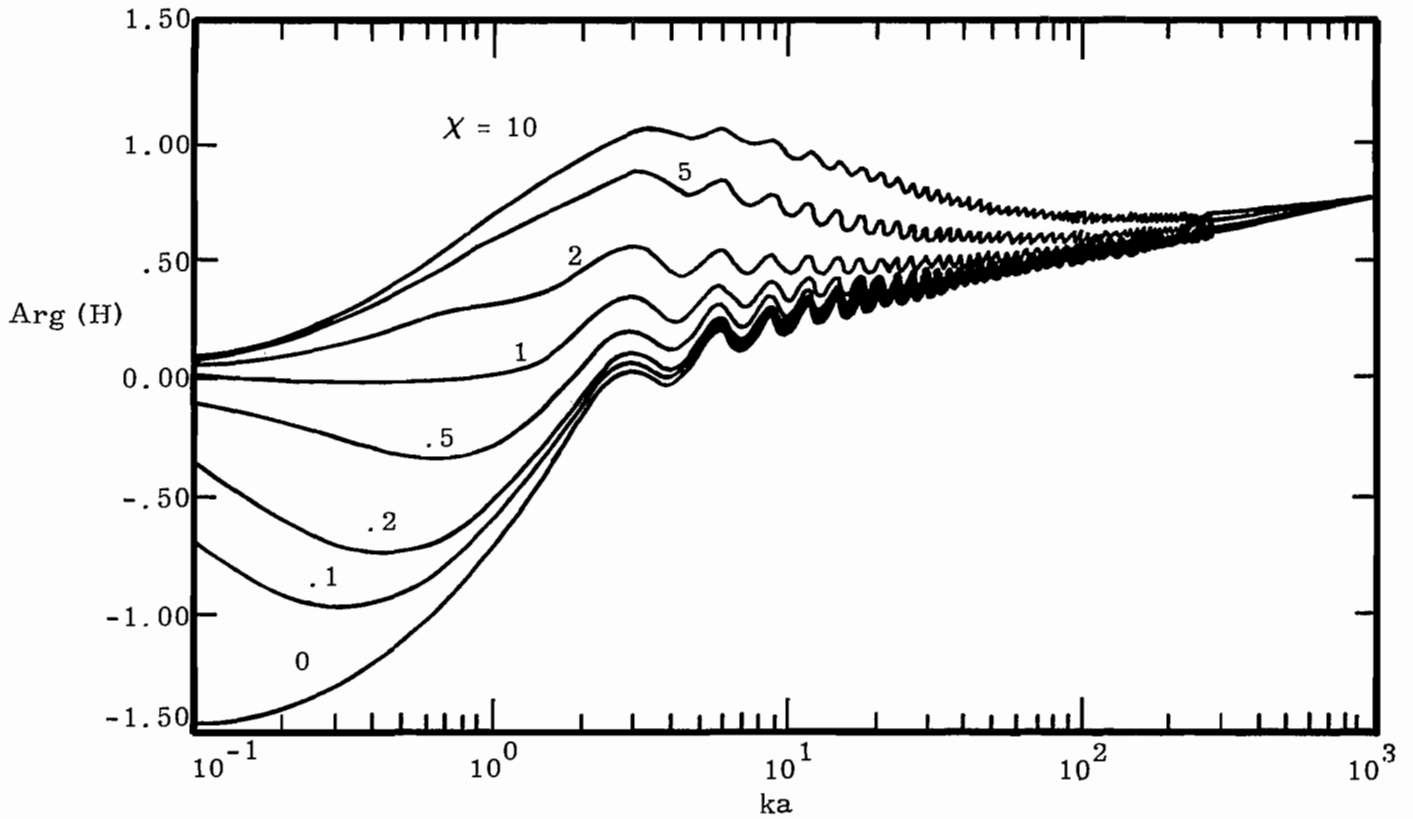


B. PHASE OF E

FIGURE 27. E-FIELD FOR  $\frac{b}{a} = 0.001$  WITH  $X$  AS A PARAMETER



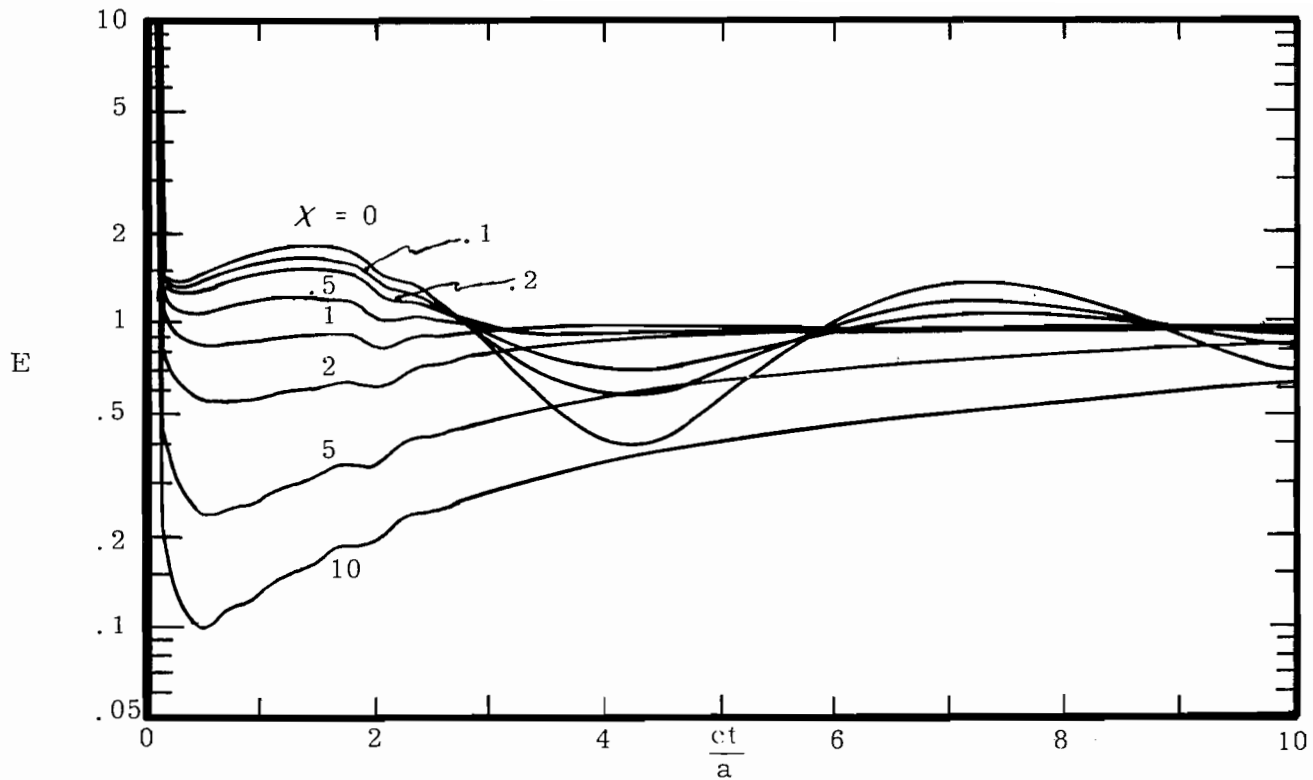
A. MAGNITUDE OF H



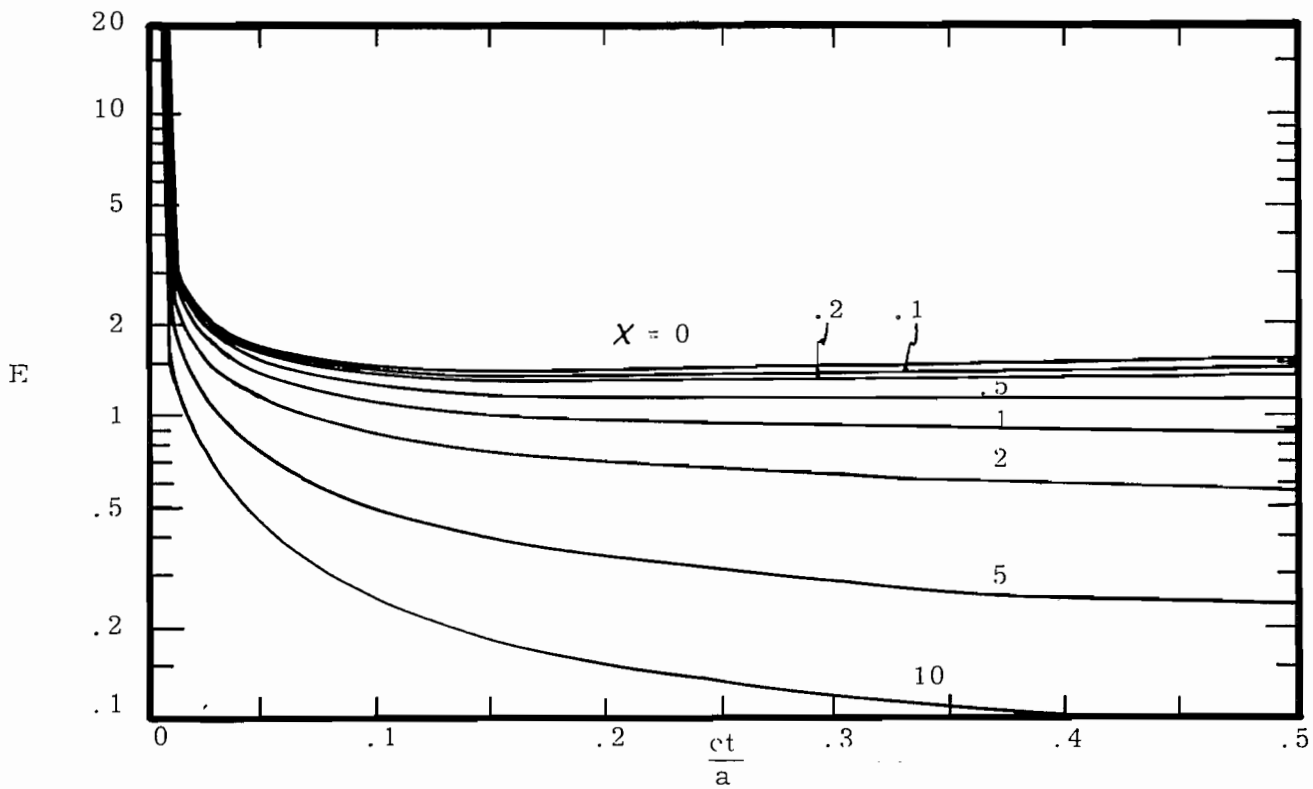
B. PHASE OF H

FIGURE 28. H-FIELD FOR  $\frac{b}{a} = 0.001$  WITH  $\chi$  AS A PARAMETER



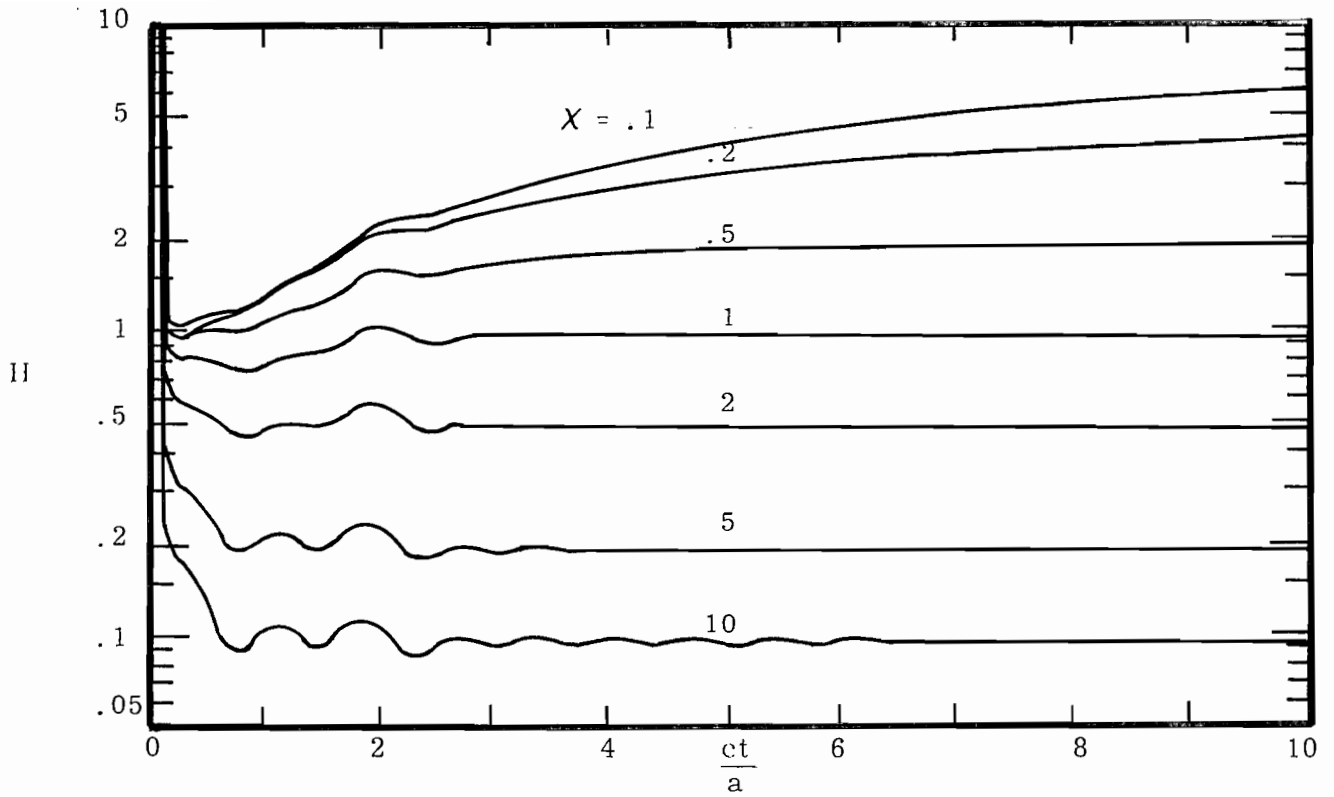


A. E-FIELD IN THE TIME DOMAIN

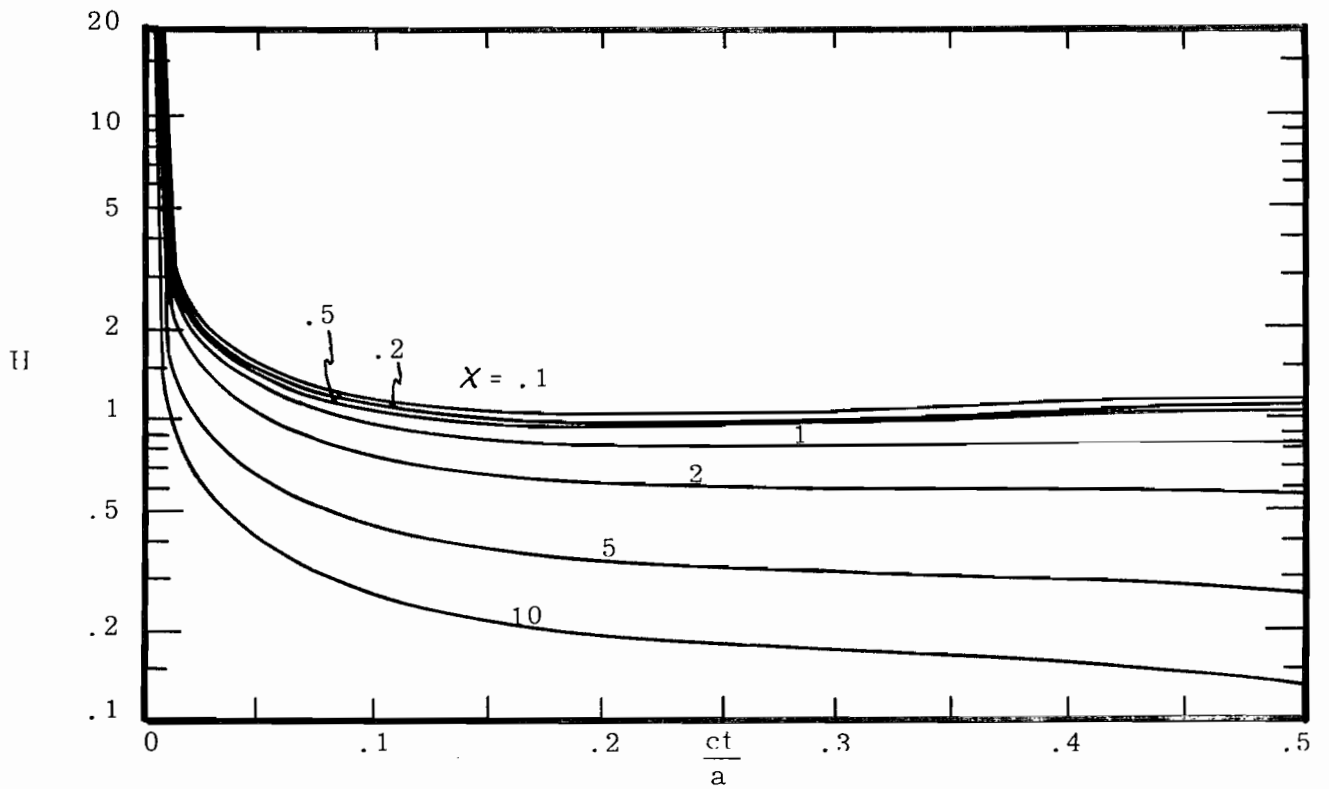


B. SCALE OF A EXPANDED

FIGURE 29. STEP-FUNCTION RESPONSE FOR THE E-FIELD FOR  $\frac{b}{a} = 0.1$  WITH  $X$  AS A PARAMETER.

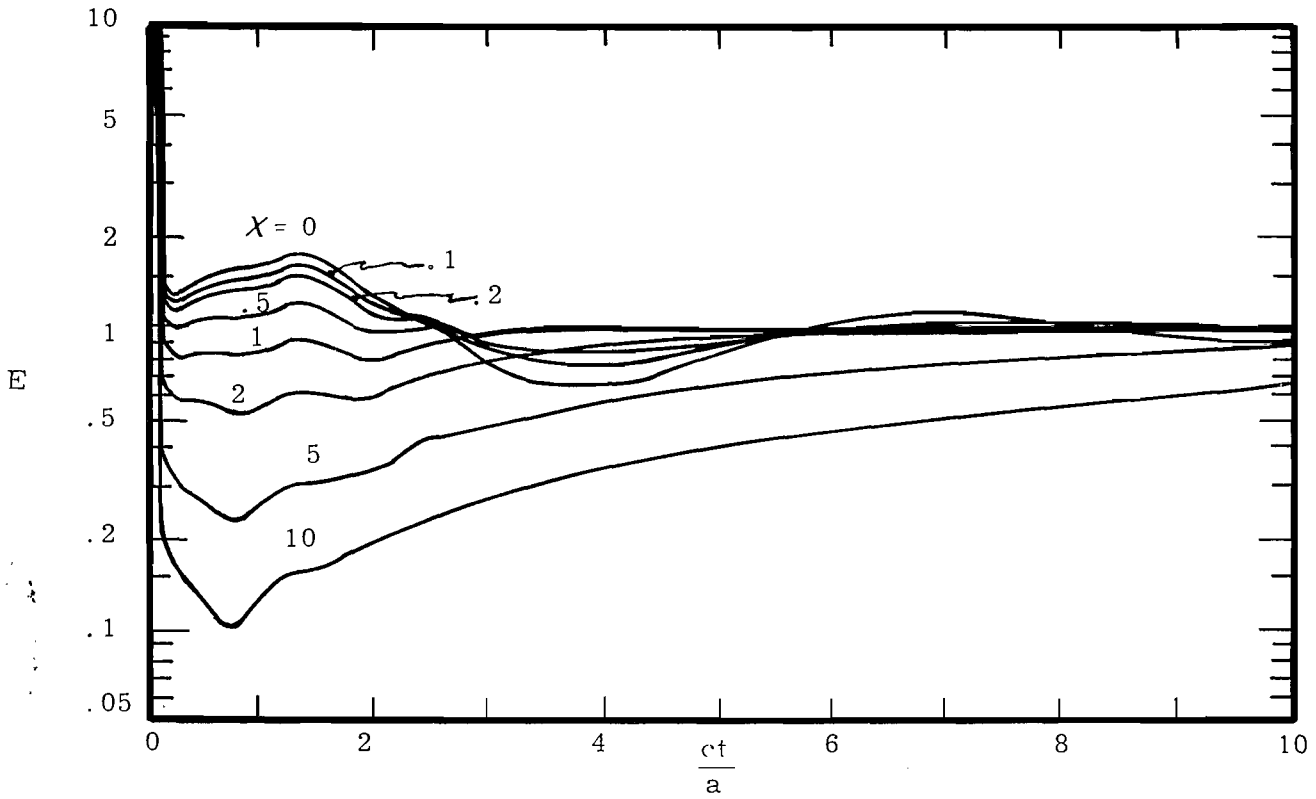


A. H-FIELD IN THE TIME DOMAIN

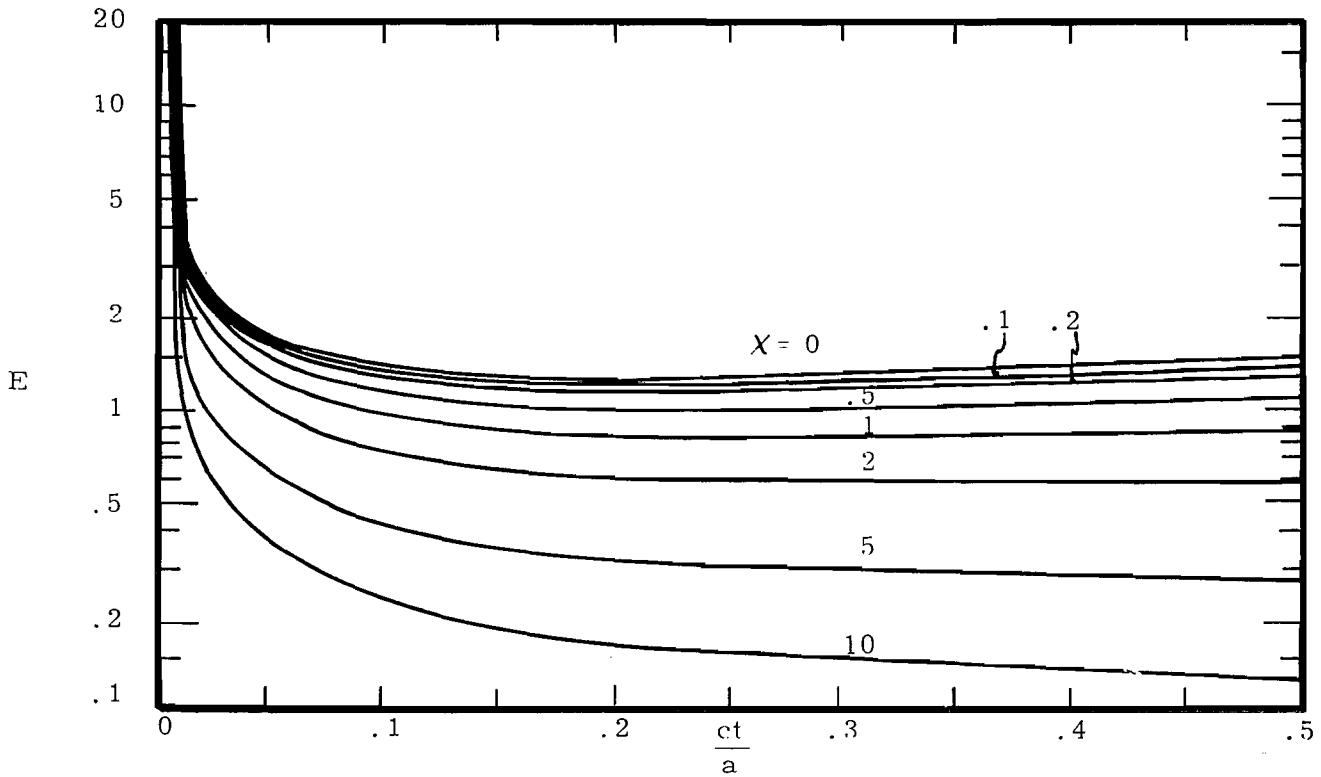


B. SCALE OF A EXPANDED

FIGURE 30. STEP-FUNCTION RESPONSE FOR THE H-FIELD FOR  $\frac{b}{a} = 0.1$  WITH  $X$  AS A PARAMETER.

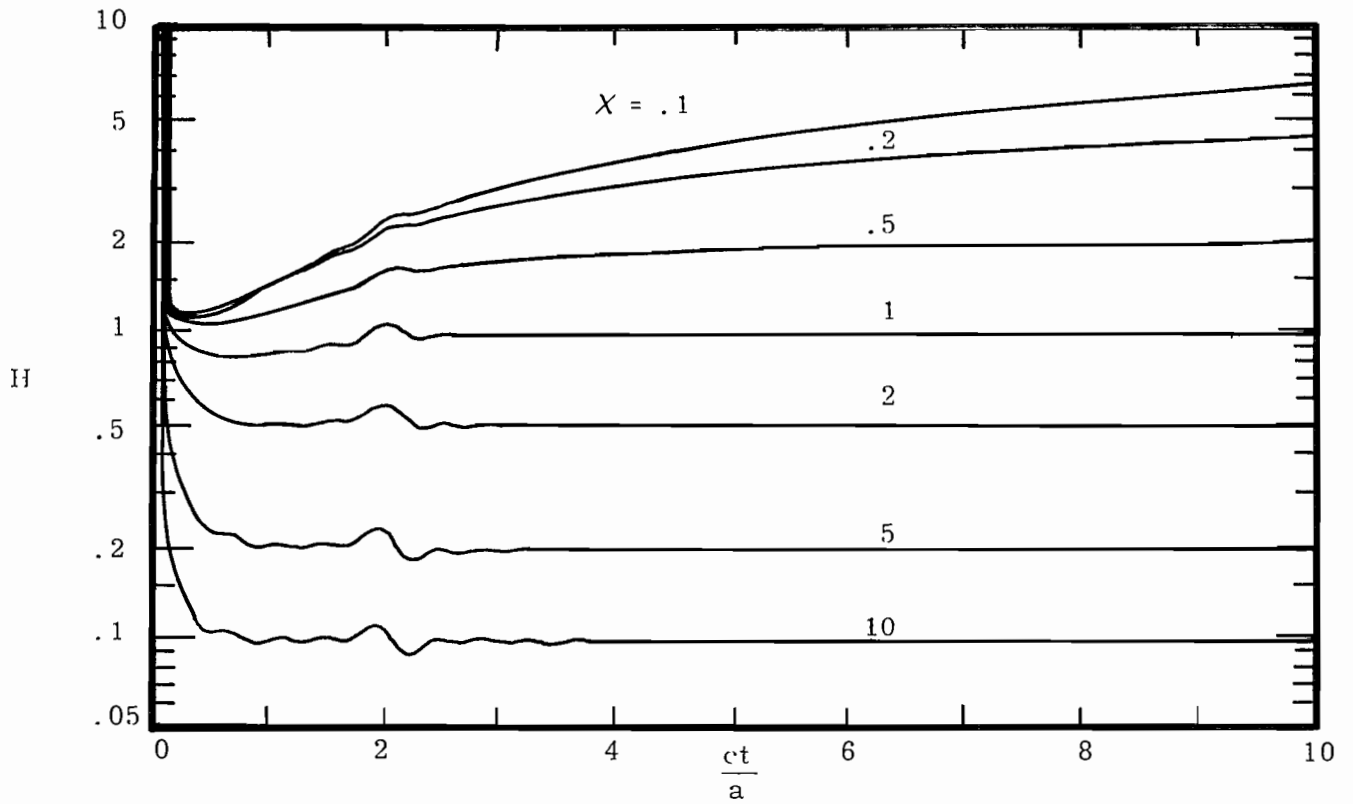


A. E-FIELD IN THE TIME DOMAIN

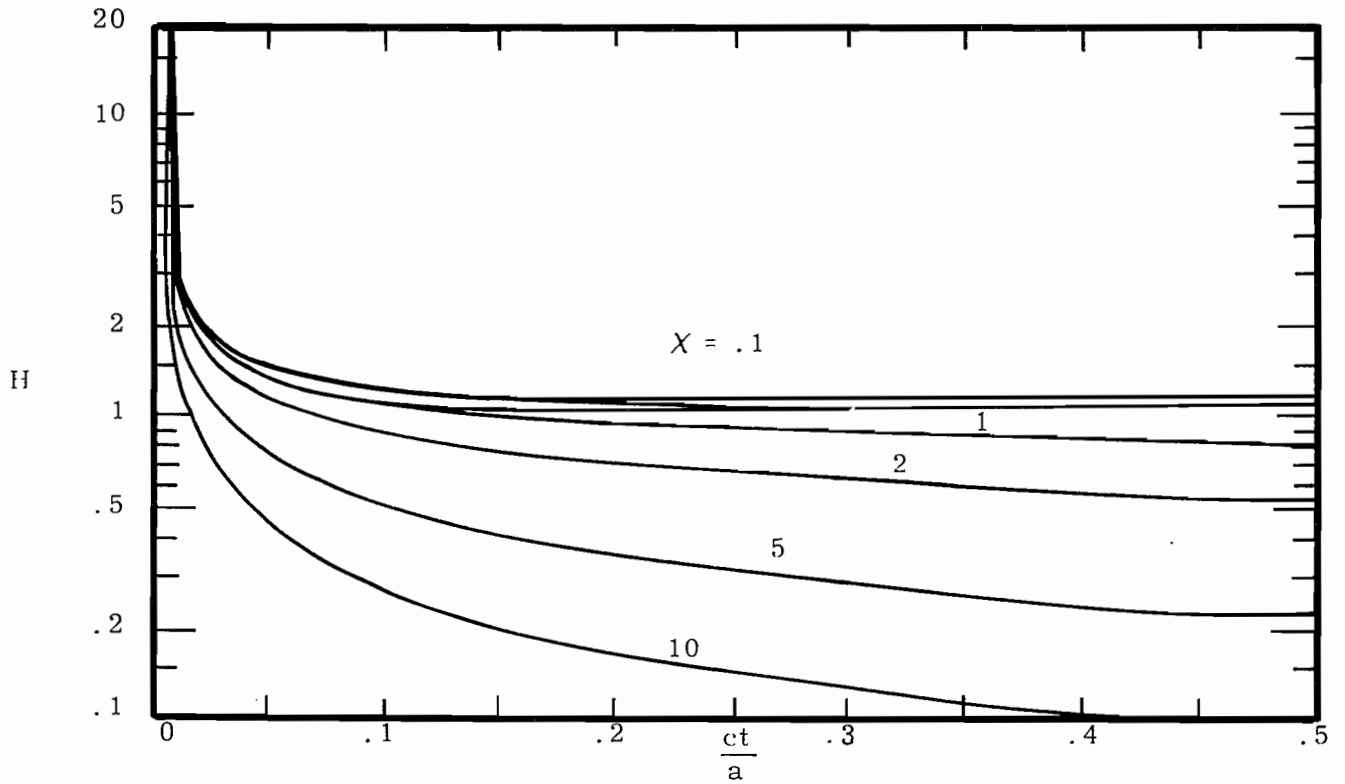


B. SCALE OF A EXPANDED

FIGURE 31. STEP-FUNCTION RESPONSE FOR THE E-FIELD FOR  $\frac{b}{a} = 0.02$  WITH  $X$  AS A PARAMETER.

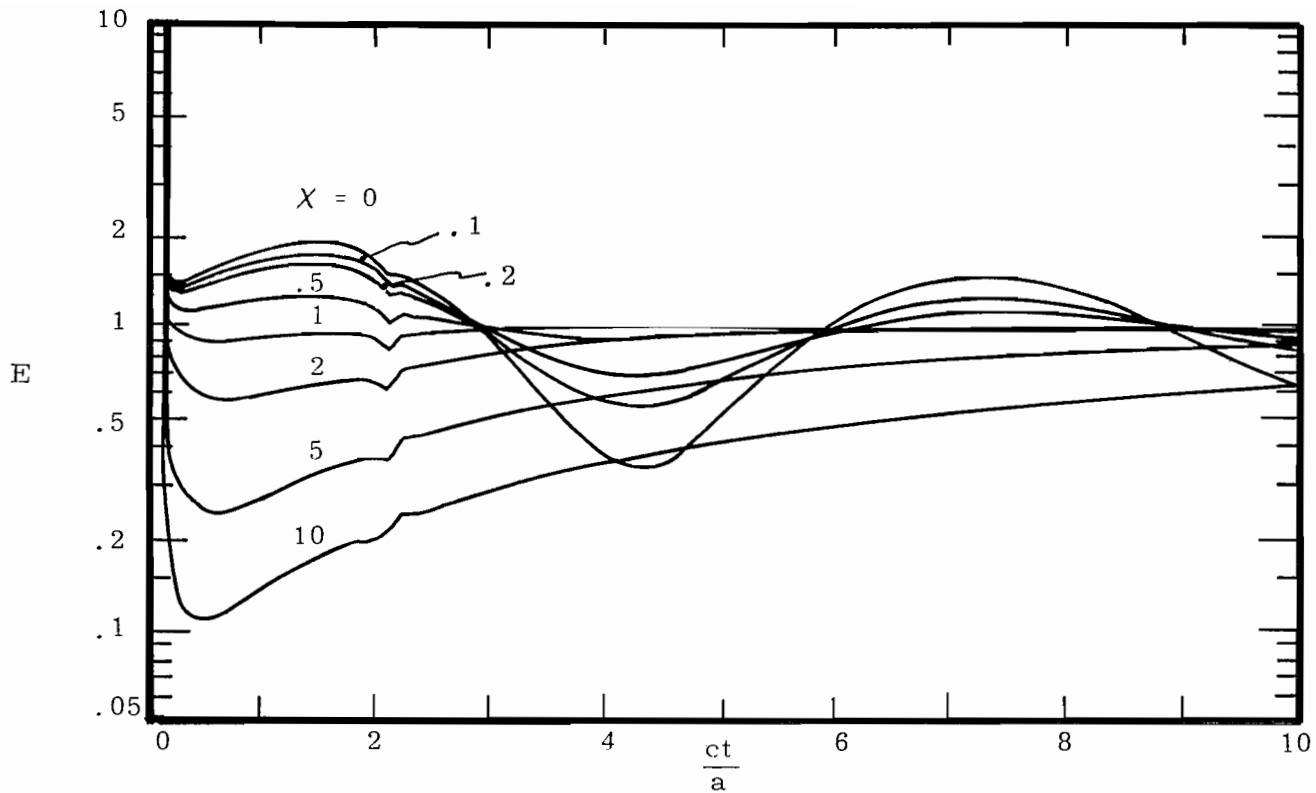


A. H-FIELD IN THE TIME DOMAIN

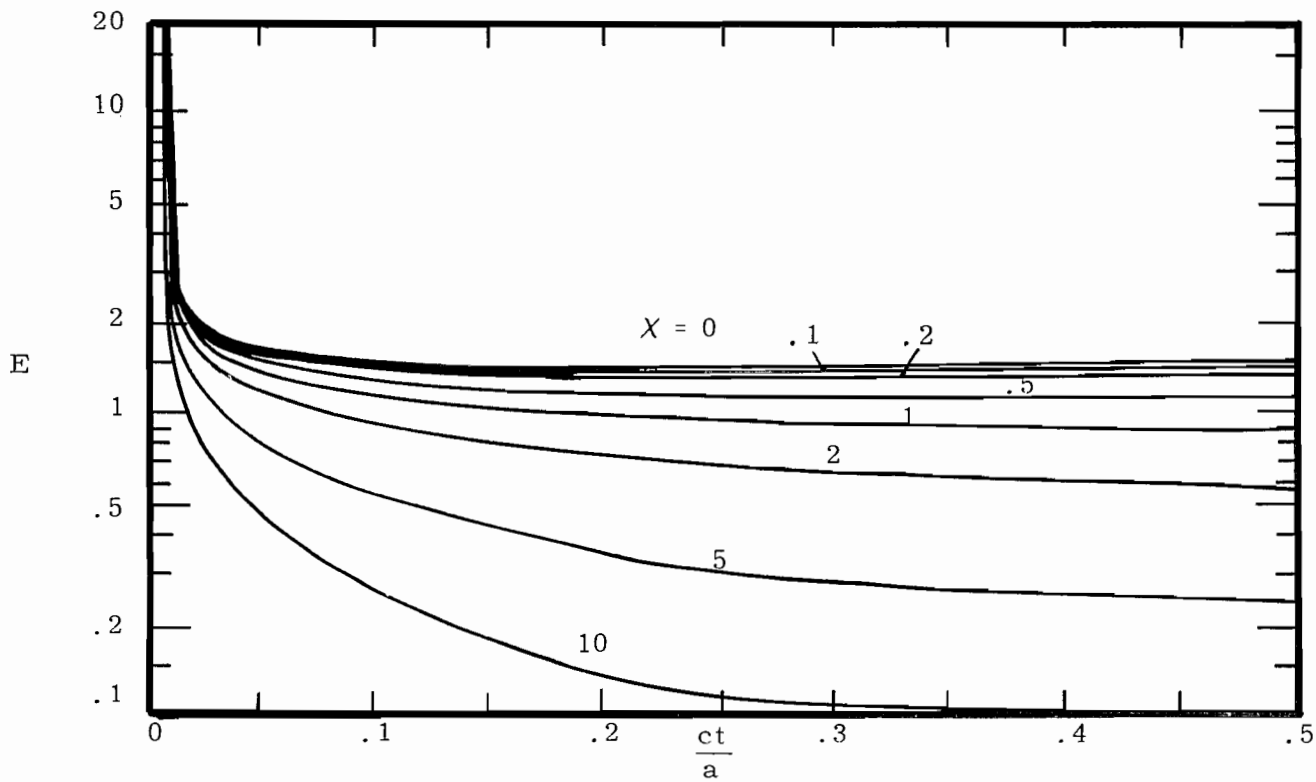


B. SCALE OF A EXPANDED

FIGURE 32. STEP-FUNCTION RESPONSE FOR THE H-FIELD FOR  $\frac{b}{a} = 0.02$  WITH  $X$  AS A PARAMETER.

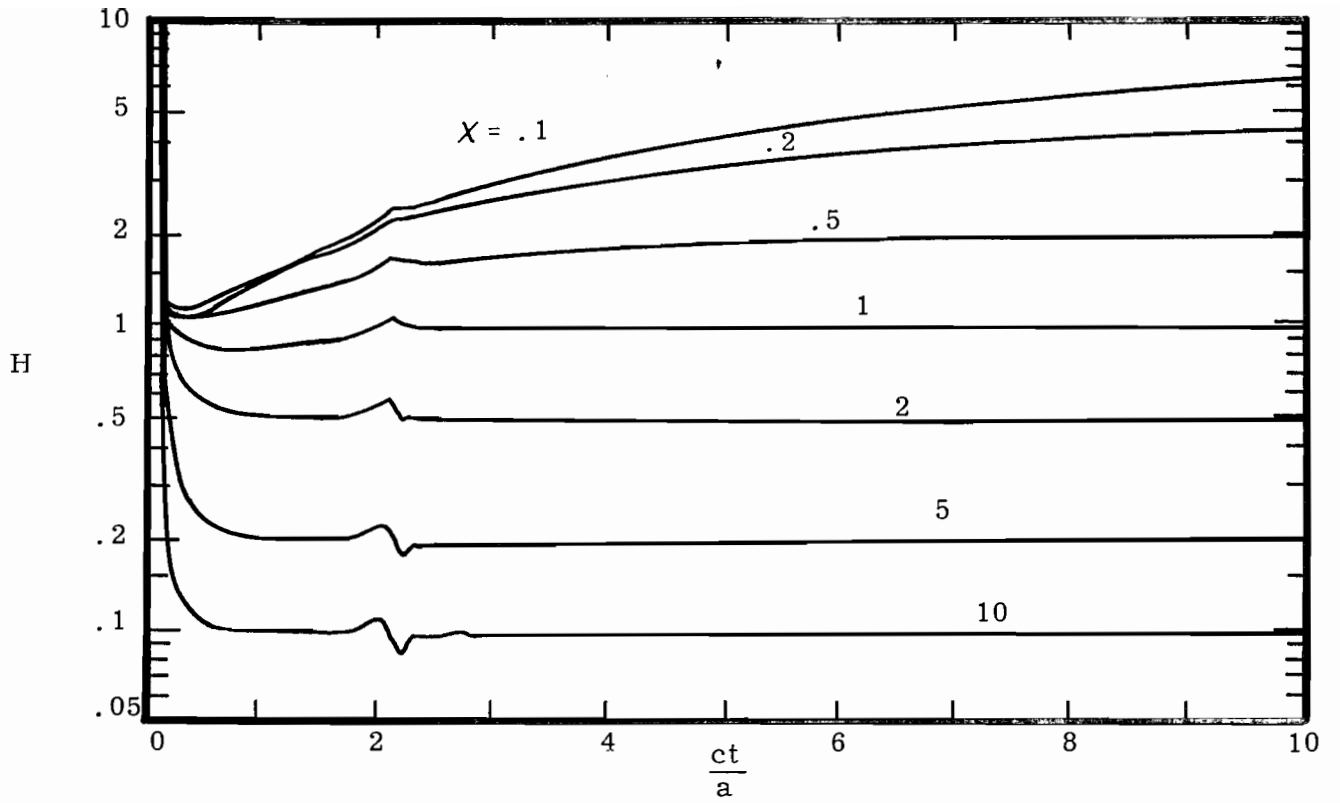


A. E-FIELD IN THE TIME DOMAIN

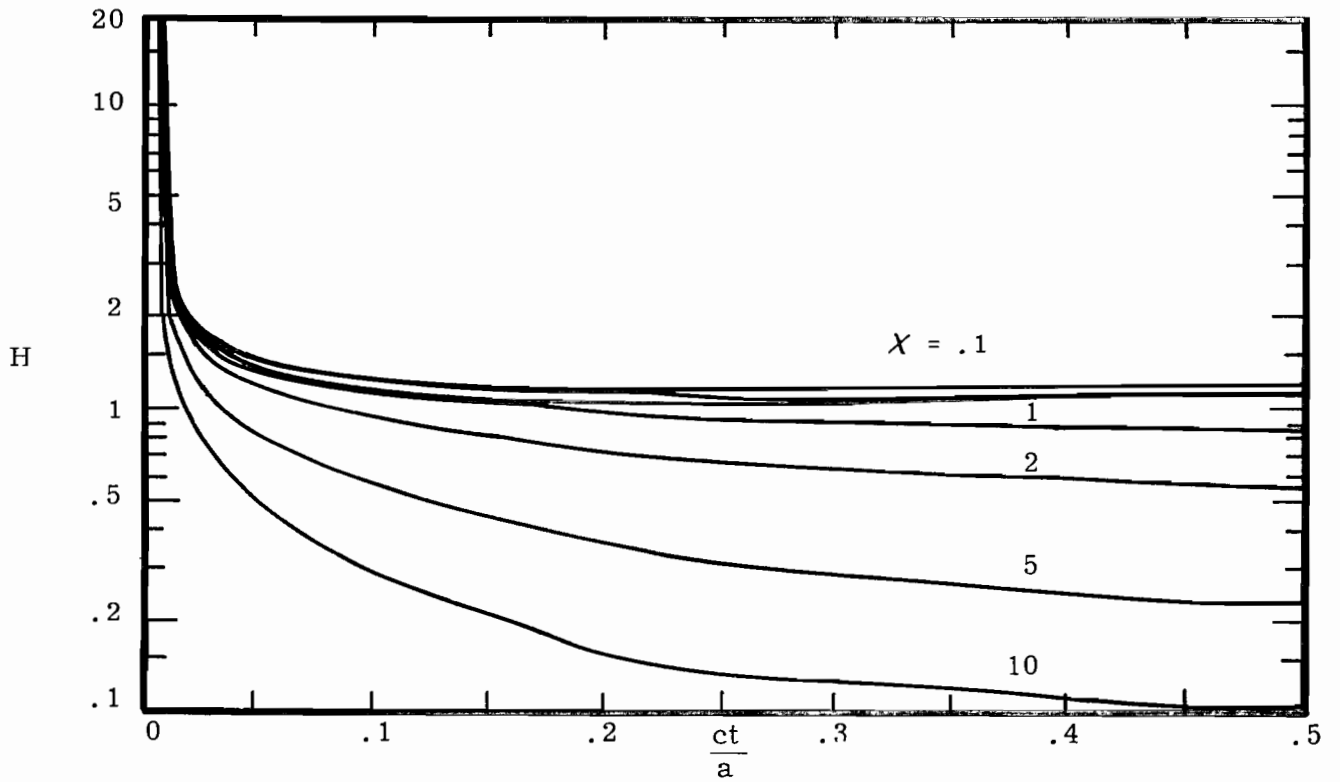


B. SCALE OF A EXPANDED

FIGURE 33. STEP-FUNCTION RESPONSE FOR THE E-FIELD FOR  $\frac{b}{a} = 0.01$  WITH  $X$  AS A PARAMETER.

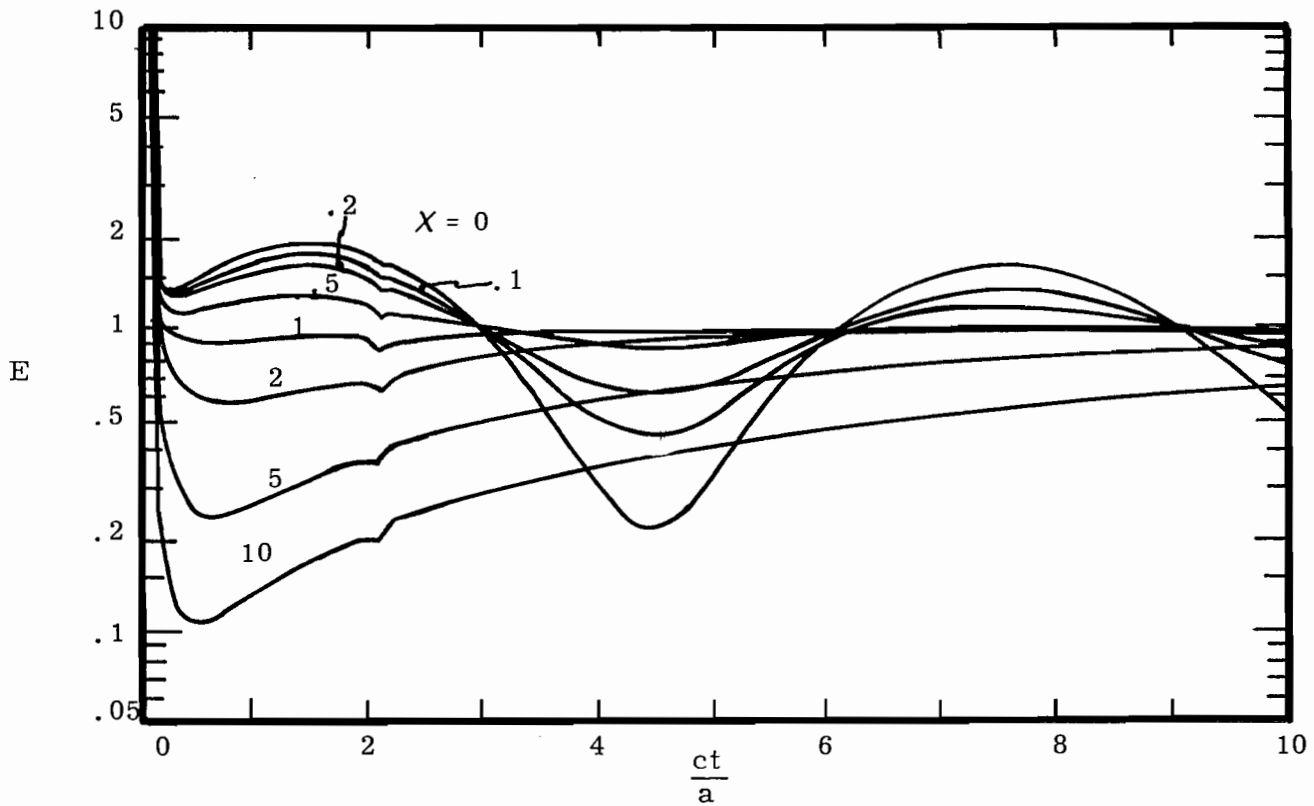


A. H-FIELD IN THE TIME DOMAIN

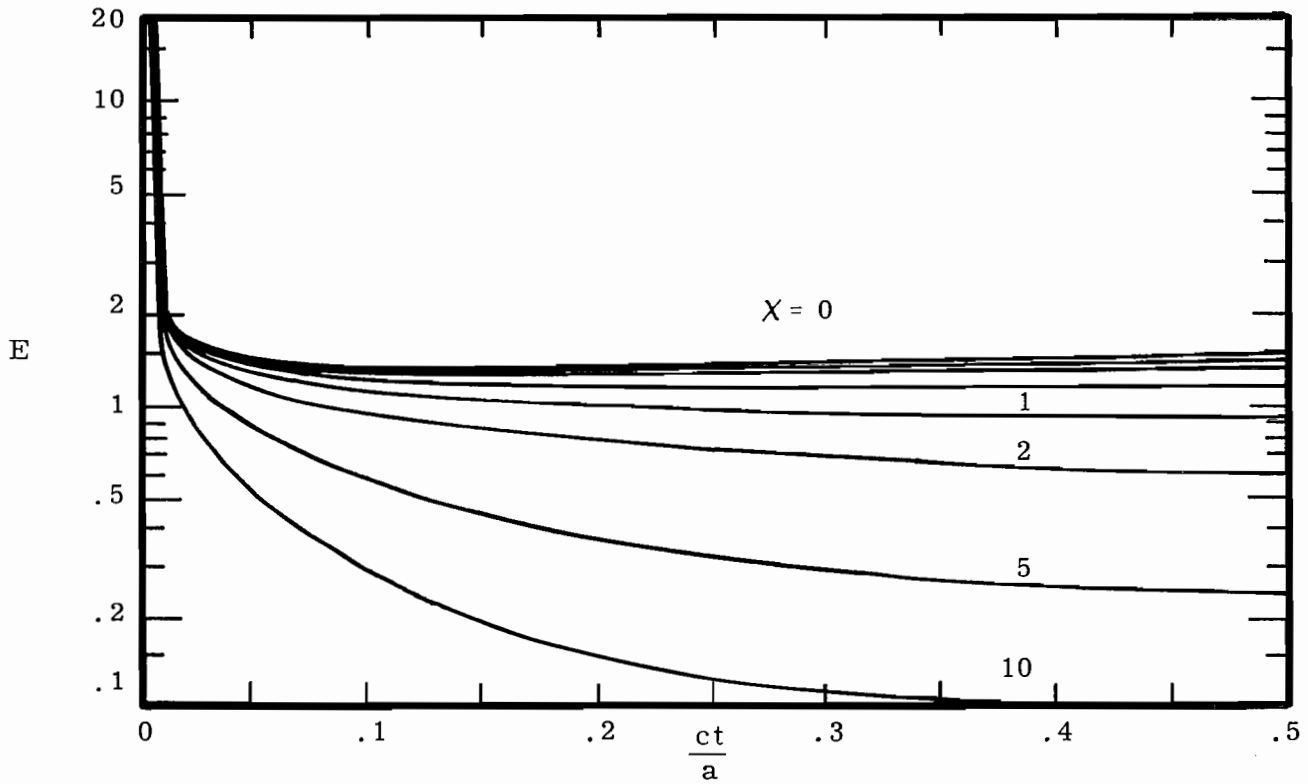


B. SCALE OF A EXPANDED

FIGURE 34. STEP-FUNCTION RESPONSE FOR THE H-FIELD FOR  $\frac{b}{a} = 0.01$  WITH  $X$  AS A PARAMETER.

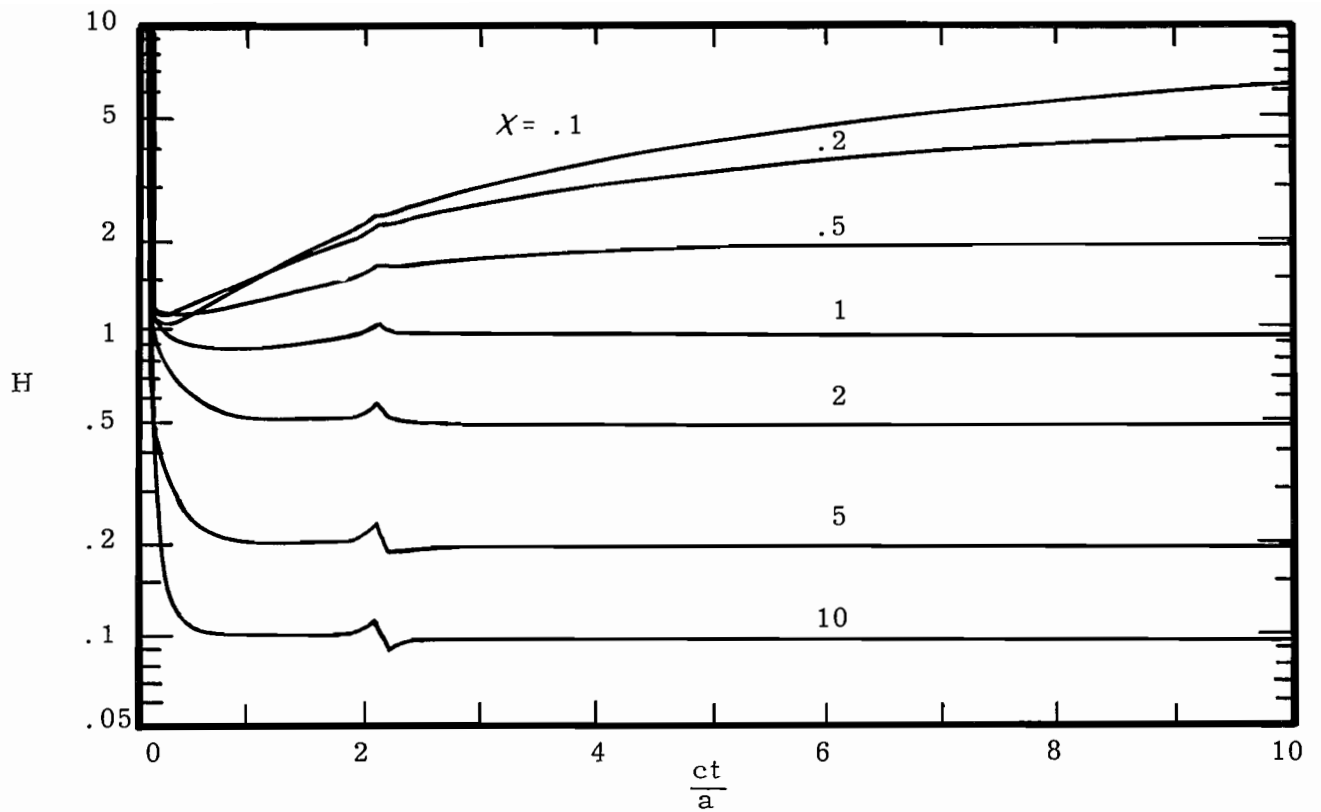


A. E-FIELD IN THE TIME DOMAIN

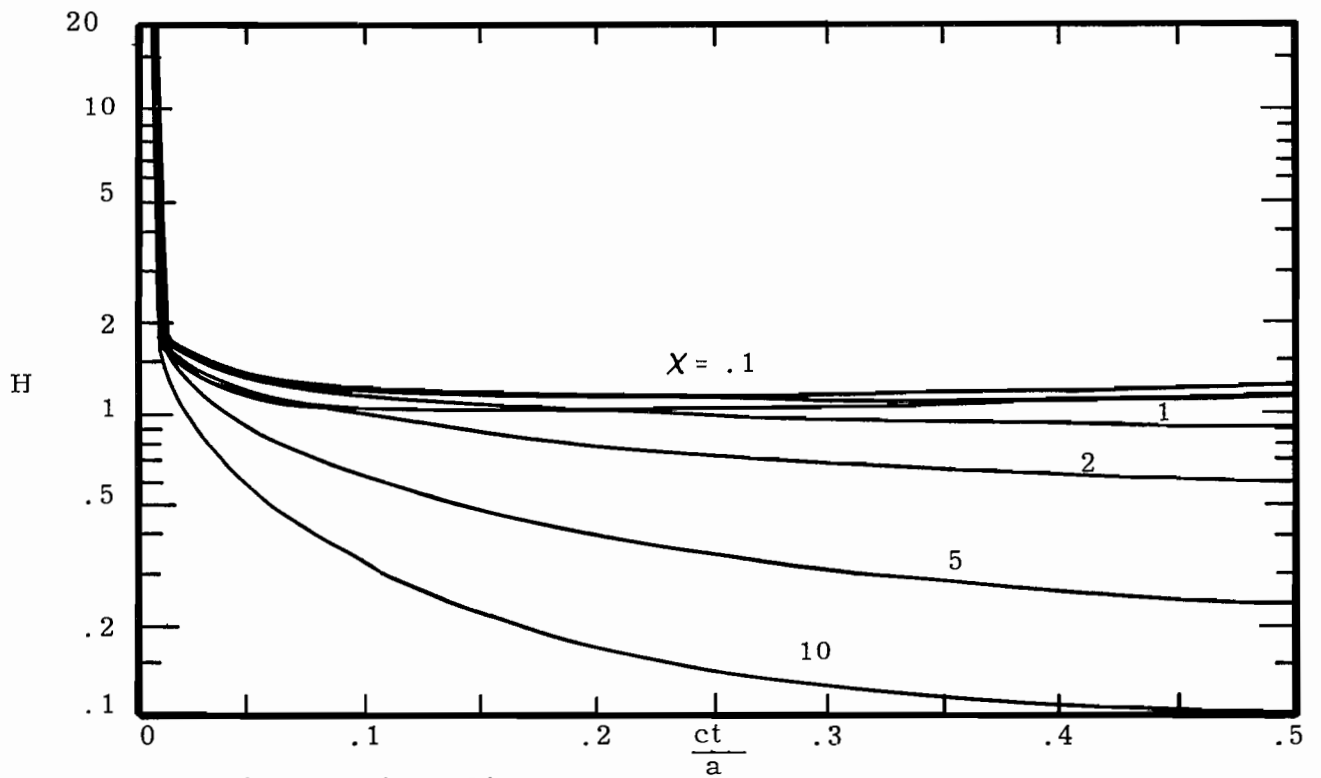


B. SCALE OF A EXPANDED

FIGURE 35. STEP-FUNCTION RESPONSE FOR THE E-FIELD FOR  $\frac{b}{a} = 0.001$  WITH  $\chi$  AS A PARAMETER.



A. H-FIELD IN THE TIME DOMAIN



B. SCALE OF A EXPANDED

FIGURE 36. STEP-FUNCTION RESPONSE FOR THE H-FIELD FOR  $\frac{b}{a} = 0.001$  WITH  $X$  AS A PARAMETER.