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Optimum Spacing of N Loops in a $\dot{B}$ Sensor

by

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Abstract

In this note, the possibility of minimizing the total inductance of an N loop $\dot{B}$ sensor is considered by numerically determining the optimum spacing between the loops. This technique involves formulating a set of simultaneous non-linear equations for the loop seperations using the maxima-minima theory with constraints, and then solving these equations using a multi-dimensional Newton Raphson algorithm.

The numerical results presented here include the coil positions for up to 20 coils for various a/h values, and curves of the minimum mutual inductance of these coils shown as a function of the a/h value. In addition, difference curves showing the difference between the mutual inductance for uniform loop spacing and the minimum mutual inductance are presented.

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I. Introduction

Consider an electromagnetic sensor designed to measure the time rate of change of a magnetic flux density, $\dot{B}$. A sensor of this type might consist of N loops of wire forming a coil of length 2h and radius a, as shown in Fig. 1a. Assuming that the load connected to the sensor is purely resistive and constant in time, the Thévenin equivalent circuit of the loaded sensor can be represented by Fig. 1b. As given by Baum[1], the load voltage in the frequency domain is written as

$$V(s) = sB(s)A_{eq}R/(R + sL), \tag{1}$$

where A_{eq} is defined as the equivalent area of the coil. For frequencies ω such that

$$\omega << \omega_o = R/L, \tag{2}$$

this relation simplifies to give

$$V(s) \approx sB(s)A_{eq} \tag{3}$$

which is equivalent to the relation

$$V(t) \approx \dot{B}A_{eq} \tag{4}$$

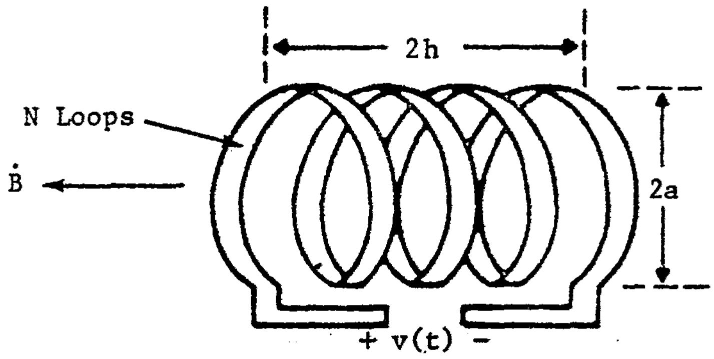
in the time domain for an initially relaxed sensor.

For the collection of N co-linear coils, the equivalent area is given approximately by

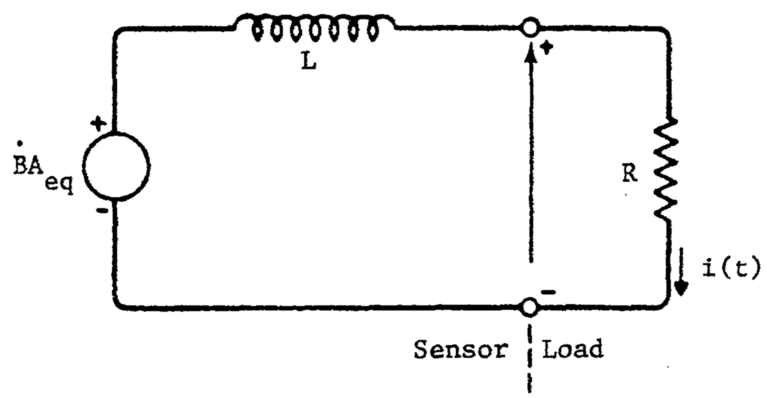
$$A_{eq} = N\pi a^2 \tag{5}$$

so that the load voltage as given by Eq. 4 can then be rewritten as

$$V(t) = N\pi a^2\dot{B}. \tag{6}$$



(a)



(b)

Figure 1. Diagram of $\dot{B}$ sensor showing the pertinent dimensions (a) and the Thévenin equivalent circuit of the sensor with a resistive load (b).

From this last relation it is obvious that by increasing the number of turns of the coil, the sensitivity of the sensor can be increased. It must be remembered, however, that the frequency range over which Eq. 6 is valid is limited by the constraint of Eq. 2. As N is increased, the inductance L is also increased, and hence the bandwidth decreases, giving a distorted time response at the output of the sensor.

In the practical design and construction of $\dot{B}$ sensors, it is necessary to limit the physical size of the sensors. In the case of N coaxial loops, the maximum seperation between any two of the loops is limited by the design constraints. Moreover, the choice of the number of loops, N, may be dictated by the sensitivity of the electronic equipment used to measure the voltage across the output. With these parameters assumed to be fixed, the problem reduces to that of obtaining the maximum bandwidth of the system. From Eq. 2 it is seen that this is equivalent to attempting to minimize the inductance of the N loops with the total length of the sensor coil specified.

Baum(1) further defines an equivalent volume of the sensor in terms of its electrical properties. In comparing this equivalent volume to the physical volume, a figure of merit is obtained which depends inversely on the inductance of the N loop sensor. Minimizing the inductance in this problem is therefore equivalent to maximizing the figure of merit of the sensor.

In this note, the determination of the optimum seperations for minimum inductance of N loops is carried out by formulating and solving a system of non-linear equations using maxima-minima theory with constraints. This same problem has been attacked in a different manner in another note by Lee and Latham(2). There, the current distribution in the azimuthal direction on a conducting cylinder of length 2h and radius a when immersed in a quasi-static magnetic field was found via an integral equation solution. This current distribution is then used to approximate the locations of the N loops. Whereas this technique is valid for very large numbers of loops, the accuracy decreases as N becomes small. The present direct approach for finding the optimal loop seperations will be more accurate for relatively small values of N.

II. Determination of the Loop Seperations for Minimum Inductance

In attempting to minimize the inductance of the N loops of Fig. 1a, the maxima-minima theory for functions of many variables is utilized[3]. The constraint that the N loops are to be positioned between the limits $x = -h$ and $x = h$ is included through a Lagrange multiplier. The solution of the resulting system of equations is then determined numerically on a digital computer.

Consider the N loops each of radius a and having positions x_i $(i = 1 \cdots N)$ to be located coaxially as shown in Fig. 2. Assuming that the loops are connected in such a way that the mutual inductance between any two of them is a positive quantity, the total inductance of the system may be written as

$$L = NL_o + \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} M_{ij} \tag{7}$$

where L_o is the self inductance of each loop and M_{ij} is the mutual inductance between the $i^{\underline{th}}$ and $j^{\underline{th}}$ loop.

The mutual inductance M_{ij} between two identical loops may be expressed as a function of the seperation of the two as[4]

$$M(d_{ij}) = \mu a[(\frac{2}{k} - k)K(k) - \frac{2}{k} E(k)] \tag{8}$$

where

$$k = \frac{2a}{\sqrt{d_{ij}^2 + 4a^2}} \tag{9}$$

and

$$d_{ij} = x_j - x_i . \tag{10}$$

The value d_{ij} is the seperation of the two loops and the functions K and E are the complete elliptic integrals defined as

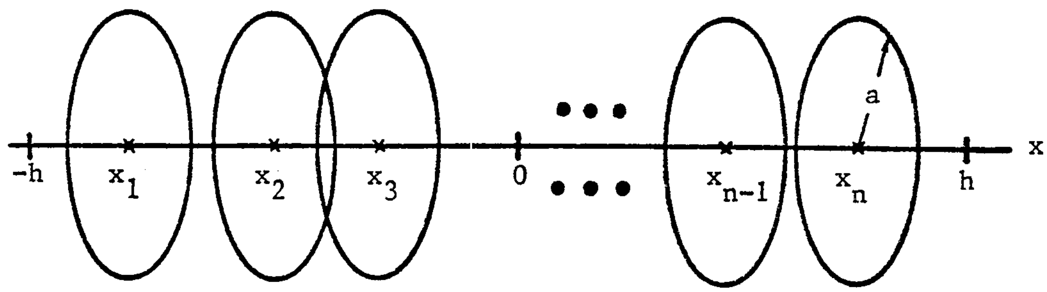


Figure 2. N identical loops of radius a which are to be spaced within the region x = -h and x = h so as to minimize the total inductance of the coil.

$$K(k) = \int_o^{\pi/2} (1 - k^2 \sin^2\phi)^{-\frac{1}{2}} d\phi$$

$$E(k) = \int_o^{\pi/2} (1 - k^2 \sin^2\phi)^{\frac{1}{2}} d\phi. \tag{11}$$

These are investigated in Refs. (5) and (6).

In order to minimize the total sensor inductance L of Eq. 7, it is sufficient to minimize the total mutual inductance, M_T, where

$$M_T = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} M(d_{ij}) \tag{12}$$

since the term NL_o is independent of the loop positions. Instead of regarding the independent variables in this problem as being the loop locations themselves, it is convenient to treat the seperations between adjacent loops as being the independent variables. By defining

$$\begin{aligned} z_1 &= x_2 - x_1 \\ z_2 &= x_3 - x_2 \\ &\vdots \\ z_{N-1} &= x_N - x_{N-1} \end{aligned} \tag{13}$$

it is seen that there are N - 1 unknown seperations to be determined. The seperation d_{ij} between the $i^{\underline{th}}$ and $j^{\underline{th}}$ loop may then be expressed in terms of the z values as

$$d_{ij} = z_i + z_{i+1} + \cdots z_{j-1}. \tag{14}$$

It is then possible to write Eq. 12 as

$$M_T(z_1, z_2, \cdots z_{N-1}) = 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} M(z_i + z_{i+1} + \cdots z_{j-1}) \tag{15}$$

where the fact that $M_{ij} = M_{ji}$ has been employed to change the j summation from $1 \rightarrow N$ to $i + 1 \rightarrow N$. This total mutual inductance M_T is then to be minimized subject to the constraint that

$$\sum_{i=1}^{N-1} z_i = 2h \tag{16}$$

which is the requirement that the total coil length be equal to 2h.

Using standard techniques[3], a Lagrange multiplier λ and a new function F are introduced where

$$F(\bar{z},\lambda) = \sum_{i=1}^{N} \sum_{j=i+1}^{N} M(z_i + \cdots z_{j-1}) + \lambda\left(\sum_{i=1}^{N-1} z_i - 2h\right). \tag{17}$$

The vector $\bar{z}$ represents the independent variables $z_1, z_2, \cdots z_{N-1}$.

Finding the unconstrained minimum of this equation is equivalent to finding the minimum of Eq. 15 subject to the constraint of Eq. 16. This is achieved by taking derivitives of F with respect to the unknowns $\bar{z}$ and λ and setting them to zero. Doing this yields the following set of N non-linear equations

$$\sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{\partial M}{\partial z_\ell} (z_i + z_{i+1} + \cdots z_\ell + \cdots z_{j-1}) + \lambda = 0, \qquad \ell = 1,2,...N-1$$

and (18)

$$\sum_{i=1}^{N-1} z_i - 2h = 0$$

It is obvious that for $\ell < i$ and $\ell > j - 1$, the derivative $\partial M/\partial z_\ell$ in Eq. 18 is zero, since M does not depend on these values of z_ℓ. Hence, the limits of the summations may be modified to yield

$$\sum_{i=1}^{\ell} \sum_{j=\ell+1}^{N} \frac{\partial M}{\partial z_\ell} (z_i + z_{i+1} + \cdots z_\ell + \cdots z_{j-1}) + \lambda = 0, \qquad \ell = 1,2,\cdots N-1$$

and (19)

$$\sum_{i=1}^{N-1} z_i - 2h = 0.$$

In attempting to determine the solution $(\overline{z},\lambda)$ of the N equations represented by Eq. 19, it is necessary to first express the derivatives $\partial M/\partial z_\ell$ in terms of the complete elliptic integrals which can be readily calculated. From Eq. 8, it is seen that $\partial M/\partial z_\ell$ may be expressed as

$$\frac{\partial M}{\partial z_\ell}(z_i + \cdots z_{j-1}) = \mu a\left[-\left(\frac{2}{k^2}+1\right)K(k) + \frac{2}{k^2}E(k) + \left(\frac{2}{k}-k\right)K'(k) - \frac{2}{k}E'(k)\right]\frac{dk}{dz_\ell} \tag{20}$$

where K' and E' represent the functions $\partial K(k)/\partial k$ and $\partial E(k)/\partial k$ respectively and $k = \left[1 + (z_i + z_{i+1} + \cdots z_\ell + \cdots z_{j-1})^2/4a^2\right]^{-\frac{1}{2}}$. Expressing these derivatives in terms of the K and E functions themselves as in Jahnke and Emde[5], and noting that from Eq. 9

$$\frac{\partial k}{\partial z_\ell} = -\frac{(z_i+z_{i+1}+\cdots z_{j-1})}{(2a)^2 k^3}, \tag{21}$$

the derivative may thus be expressed as

$$\frac{\partial M}{\partial z_\ell}(z_i + \cdots z_{j-1}) = \frac{\mu}{4a}\frac{(z_i+\cdots z_{j-1})}{k^3}\left[\frac{2K(k)}{k^2} + \frac{2-k^2}{k^2(1-k^2)}E(k)\right] \tag{22}$$

where $i \leq \ell \leq j-1$.

An alternate approach would be to use a finite difference technique to evaluate these derivatives. In this manner, the derivative may be written as

$$\frac{\partial M}{\partial z_\ell}(z_i + \cdots z_{j-1}) \approx \frac{M(z_i+z_{i+1}+\cdots(z_\ell+\varepsilon)+\cdots z_{j-1}) - M(z_i+z_{i+1}+\cdots(z_\ell-\varepsilon)+\cdots z_{j-1})}{2\varepsilon} \tag{23}$$

where $i \leq \ell \leq j-1$ and ε is some quantity much smaller than any of the seperations z_i. These derivatives, either calculated by Eq. 22 or estimated by Eq. 23, are then employed in Eq. 19 to numerically determine the solution $(\overline{z},\lambda)$.

Once the positions of the N loops have been found, the total inductance of the sensor can be determined. Assuming that each loop consists of a wire of radius b and permeability μ', the self inductance of a loop is expressed as[8]

$$L_o = \mu(2a-b)\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right] + \frac{\mu' a}{4} \tag{24}$$

with

$$k = 2\left(a(a - b)\right)^{1/2}/(2a - b) \tag{25}$$

As in previous equations, a is the loop radius, μ is the permeability of the space surrounding the sensor and K and E are elliptic entegrals. The first term in Eq. 24 correspond to the inductance of the loop, whereas the second term arises from the internal inductance of the wire of the loop.

For b/a very small, k is nearly unity so the approximations

$$\begin{aligned} K(k) &\approx \ln\left[4/(1 - k^2)^{1/2}\right] \\ E(k) &\approx 1 \end{aligned} \tag{26}$$

yield on approximate expression for L_o in the form

$$L_o \approx \mu a\left[\ln(\frac{8a}{b}) - 2\right] + \frac{\mu' a}{4} \tag{27}$$

Upon calculating the correct value of the self inductance of a single loop, as well as the total mutual inductance from Eq. 12, the total inductance of the $\dot{B}$ sensor can be determined via Eq. 7.

III. The Numerical Solution

In solving the set of N non-linear simultaneous equations [19], a multi-dimensional Newton Raphson technique is used.[3] Basically, this involves expanding the non-linear functions to first order in the independent variables about some initial trial solution, and then determining a new solution by solving the resulting N linear equations. This process continues until the solution has converged sufficiently. The computer program used for this purpose was a standard subroutine supplied by the Control Data Corporation.

The calculation of the elliptic integrals defined by Eq. 11 may be rapidly done using an eight-term approximation given by Hastings[7]. With these expansions, the errors in the calculated elliptic integrals are at all times less than 1.5×10^{-8} in magnitude.

To begin the iterative solution, it is necessary to first guess an initial solution $(\overline{z}_o, \lambda_o)$. For the initial spacing vector $\overline{z}_o$, it is convenient to consider all of the coils as being equally spaced. Hence, $z_i = 2h/(N - 1)$ for all i. For the value of the Lagrange multiplier, it was found that by choosing $\lambda_o = \mu a/z_1$, the solution would converge to the solution for minimum inductance fairly rapidly if the finite difference expression for the derivitives are used. If, however, the exact derivitives of Eq. 22 are used, the solution approaches that of a maximum value of the mutual inductance. This corresponds physically to there being a loop at each end $x = \pm h$ and the rest clustered about the point $x = 0$. Clearly this solution is not desired, so the finite difference technique was subsequently employed as it converges to the minimum solution.

It was also observed that the numerical solution does not converge for large values of a/h. In addition, the maximum a/h value for which the solution is convergent decreases as N increases. This behavior may be attributed to the use of the finite difference expression for the derivitives $\partial M/\partial z_\ell$, or perhaps due to the Newton Raphson solution itself. This lack of convergence limits somewhat the range of applicability of this technique, but as mentioned before, another method[2] is available for treating the case of many loops. As a result, the problem of extending the range of convergence of this method has not been pursued further.

IV. Computational Results

Using the numerical technique previously discussed, the optimum locations for three to twenty loops in a $\dot{B}$ sensor has been determined. The a/h ratio varies from .2 to 20. for relatively small numbers of loops and, for reasons already mentioned, the maximum value of a/h decreases with increasing N. The loop locations, given by x_i, are presented (in numerical form) in Table 1. In all cases, only the non-trivial loop locations have been reported. For example, loops 1 and N are always located at x = -h and x = h, respectively. Moreover, the individual loop locations are symmetric about the x = 0 point. This implies that less than N/2 values must be specified for determining the locations of the N loops. Note also, that if N is odd, there is a loop at x = 0. This loop, like those at x = ±h, is not included in Table 1.

From the loop position data, it is seen that for a fixed N, the loops tend to cluster about the ends x = ±h as the a/h ratio increases. This same type of behavior was noted by Lee and Latham.(2)

The minimum possible mutual inductance of the sensor is plotted in Fig. 3 as a function of a/h for various values of N. Notice that the mutual inductance is normalized with respect to the quantity $L_n = \mu\pi a^2 N^2/(2h)$ which is the total inductance of a coil of N loops as determined by assuming that the magnetic field within the coil is constant.(4) Letting M_T stand for the minimum mutual inductance, the normalized quantity plotted in Fig. 3 as a function of a/h is then $M_N = M_T/L_n = M_T(2h)/\mu\pi a^2 N^2$.

It is interesting to compare the differences between the minimum or optimum mutual inductance, M_T, and that obtained for a uniform distribution of loops in the sensor, which is denoted by M_U. Figure 4 shows the relative percent difference between these two quanties defined as $(M_U - M_T)/M_U \times 100\%$. This relative difference is plotted as a function of a/h for the various values of N. It is seen that the difference curves have a peak in them at approximately a/h = 2, and that this difference tends to increase as the number of loops increases.

Figure 5 represents the maximum possible percent difference as a function of the number of loops, N. Clearly, the data given in this curve is for integer values of the abscissa only; the curve connecting the data points has been included only to show the trend in the difference as N increases. In Ref. 2, the maximum difference was found to be about 5.6% for the limiting case of N being very large. From Figure 5 it is seen that the difference is still growing after N = 10 and it is not inconceivable that the difference might approach this value in the limit. The errors involved in using a uniform distribution of loops instead of the optimum distribution is relatively small, due to the fact that the inductance can be shown to be a variational quantity. That is, a first order change in the loop seperations produces a second order change in the inductance. This property has been studied in Ref. 2 and will not be discussed further.

Table I. Non-trivial* locations (x_i/h) for N loops in a $\dot{B}$ sensor for various a/h values.

N	i \ a/h	.2	.4	.6	.8	1.0	1.2	1.4	1.6	1.8	2.0	4.0	10.0	20.0
4	3	.339	.345	.355	.366	.377	.386	.394	.401	.406	.411	.433	.444	.446
5	4	.509	.524	.542	.560	.575	.588	.598	.606	.612	.617	.641	.652	.654
6	4	.205	.213	.223	.232	.239	.246	.251	.256	.259	.262	.276	.283	.285
	5	.613	.634	.658	.678	.694	.706	.716	.723	.729	.734	.754	.763	.764
7	5	.343	.358	.375	.390	.402	.413	.421	.427	.433	.437	.457	.466	.468
	6	.682	.709	.735	.755	.770	.782	.790	.796	.801	.805	.821	.828	.830
8	5	.148	.155	.163	.170	.176	.181	.185	.188	.191	.193	.203	.208	.209
	6	.442	.464	.486	.504	.519	.531	.540	.547	.553	.558	.579	.589	.591
	7	.733	.764	.790	.809	.822	.832	.839	.844	.848	.852	.865	.870	.871
9	6	.259	.274	.288	.300	.310	.318	.324	.330	.334	.337	.353	.361	.362
	7	.517	.544	.570	.590	.606	.618	.627	.634	.640	.645	.665	.675	.676
	8	.771	.804	.830	.847	.859	.867	.873	.877	.881	.883	.894	.899	.899
10	6	.116	.123	.129	.135	.139	.143	.146	.149	.151	.153			
	7	.347	.367	.386	.401	.414	.424	.432	.438	.443	.448			
	8	.576	.608	.635	.656	.671	.683	.692	.699	.705	.709			
	9	.802	.836	.860	.875	.885	.892	.897	.901	.904	.906			
11	7	.209	.222	.234	.243	.252	.258	.264	.268					
	8	.417	.442	.464	.482	.496	.508	.516	.523					
	9	.624	.658	.686	.707	.722	.734	.742	.749					
	10	.826	.861	.883	.897	.905	.911	.916	.919					
12	7	.095	.101	.107	.112	.115	.118							
	8	.286	.304	.320	.333	.344	.352							
	9	.475	.504	.529	.548	.563	.575							
	10	.663	.700	.728	.748	.763	.773							
	11	.847	.880	.901	.913	.921	.926							
13	8	.175	.187	.197	.205	.212								
	9	.350	.372	.391	.407	.420								
	10	.524	.556	.582	.602	.618								
	11	.696	.734	.762	.781	.795								
	12	.864	.897	.915	.926	.932								
14	8	.081	.086	.091	.095									
	9	.243	.259	.273	.284									
	10	.405	.431	.452	.470									
	11	.565	.600	.627	.648									
	12	.724	.763	.790	.809									
	13	.878	.910	.927	.936									

*Two loops are located at x/h=±1 another at x/h=0 if N is odd, and the others at the values $\pm x_i/h$ as given here.

Table I. (Cont'd.)

N	i \ a/h	.2	.4	.6
15	9	.151	.161	.170
	10	.302	.322	.338
	11	.452	.481	.505
	12	.601	.637	.665
	13	.748	.787	.814
	14	.890	.921	.936
16	9	.071	.075	.080
	10	.212	.226	.238
	11	.353	.376	.395
	12	.493	.524	.550
	13	.632	.670	.698
	14	.769	.809	.834
	15	.901	.930	.944
17	10	.133	.142	
	11	.265	.283	
	12	.397	.423	
	13	.529	.562	
	14	.659	.698	
	15	.788	.827	
	16	.910	.937	
18	10	.063	.067	
	11	.188	.200	
	12	.312	.333	
	13	.437	.466	
	14	.561	.596	
	15	.684	.723	
	16	.804	.842	
	17	.918	.944	
19	11	.118	.126	
	12	.237	.253	
	13	.355	.378	
	14	.472	.503	
	15	.589	.626	
	16	.705	.745	
	17	.818	.856	
	18	.925	.949	
20	11	.056	.060	
	12	.168	.180	
	13	.280	.300	
	14	.392	.419	
	15	.504	.536	
	16	.615	.652	
	17	.724	.764	
	18	.831	.868	
	19	.932	.954	

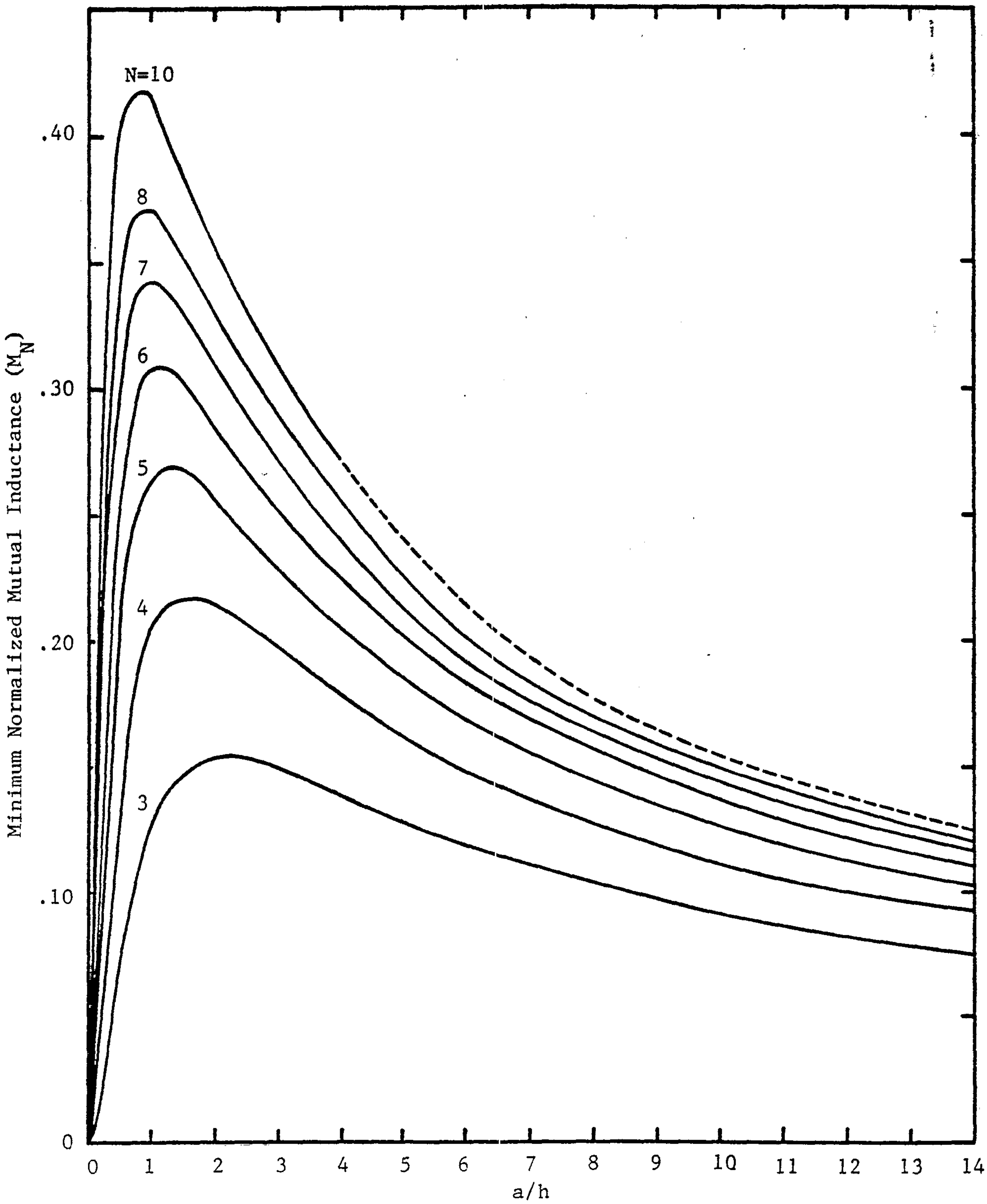


Figure 3. Plots of the optimum or minimum normalized mutual inductance of the coil shown as a function of a/h for various numbers of loops, N.

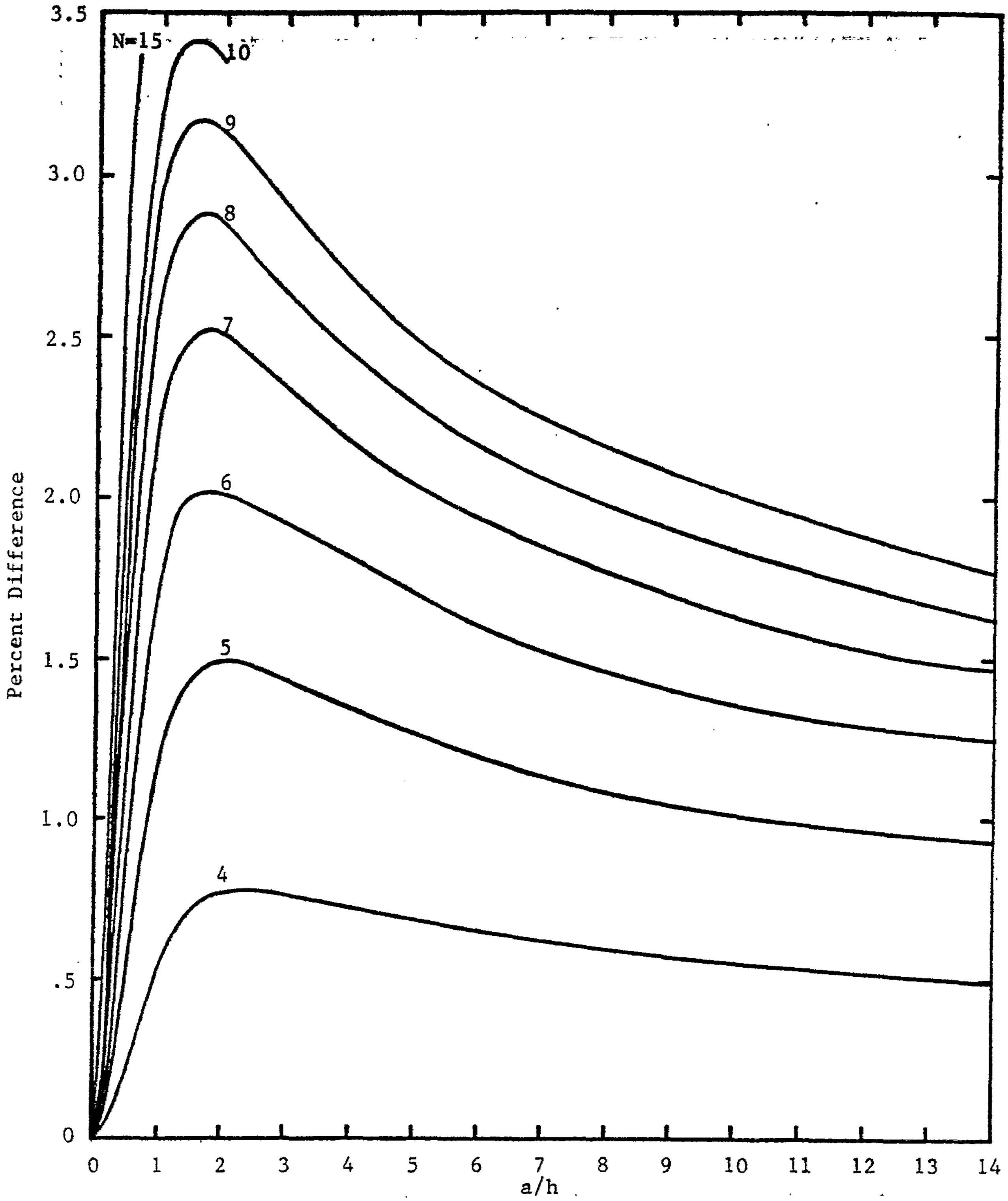


Figure 4. Plots of the percent difference between the optimum mutual inductance and the mutual inductance due to a uniform loop spacing shown as a function of a/h for various numbers of loops, N.

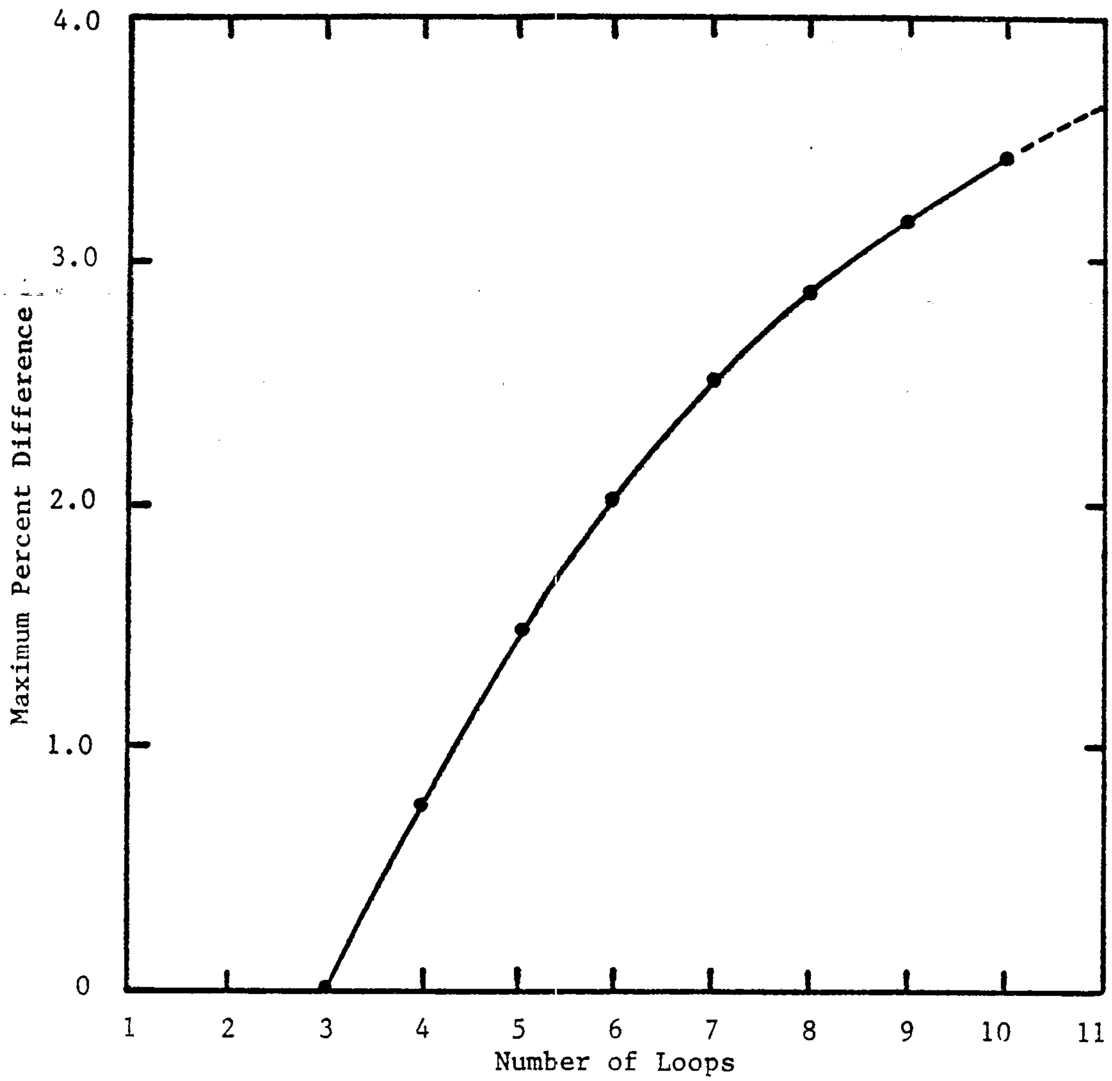


Figure 5. The maximum percent difference between the optimum mutual inductance and the mutual inductance due to a uniform loop spacing, shown as a function of the number of loops.

V. Conclusion

A technique for determining the optimum locations of N loops which make up a $\dot{B}$ sensor has been discussed. By requiring that the total mutual inductance of the coil be a minimum, a system of non-linear simultaneous equations may be formulated and solved using a multi-dimensional Newton-Raphson technique. This method was found to be convergent for small a/h values and for loops, but did not converge sufficiently for large a/h ratios with many loops.

A table showing the optimum loop locations has been presented, as well as curves of the normalized minimum mutual inductance for various N and a/h values. The percent difference in the mutual inductances that is encountered in using the uniform spacing instead of the optimum spacing is also displayed, and it is noted that this difference is less than 4% for up to ten loops.

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