Sensor and Simulation Notes

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Scattering by Two Perfectly Conducting, Circular, Coaxial Disks

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Abstract

The electric field on the axis of a symmetrical parallel plate dipole consisting of two perfectly conducting, circular, coaxial disks is calculated when the incident field is a plane wave. Two quantities have special interest, namely, (1) the electric field at the center of the dipole and (2) the integral of the electric field along the axis between the two plates. Analytical expressions for the low-frequency behavior of these two quantities are derived in the case where the distance between the two disks is large. For other frequencies and for separations of the two disks not too large these two quantities of interest are calculated numerically.

I. Introduction

A common type of EMP sensor for measuring the electric field is the parallel plate dipole (PPD). The PPD consists of two thin, perfectly conducting, parallel plates with some device which picks up the electric field or voltage between the two plates.

In previous notes[1,2] some of the characteristics of a symmetrical PPD consisting of two perfectly conducting circular disks have been investigated. In this note we will continue these investigations and obtain quantitative information on the behavior of this dipole for a wide range of frequencies. To begin with, the two disks are assumed to have the same radius, to be of zero thickness, and to be exposed to an incident plane wave. Specifically we will calculate, pro primo, the electric field at the center of the dipole and, pro secundo, the integral of the electric field between the two plates along the axis of the dipole. Because of the symmetry of the problem these two quantities can be calculated, provided the appropriate factor of two is accounted for, by considering the problem of one disk above a perfectly conducting plane in the presence of one incident plane wave or, by considering the equivalent problem of two parallel disks in the presence of two symmetrically incident plane waves.

In section II we first scalarize the problem of two parallel, coaxial disks of different radii by expressing the scattered electromagnetic field in terms of suitable components of the Hertz potentials. From the boundary conditions on the disks we then derive differential equations for the different components of the Hertz potentials. Making use of the solutions of these differential equations, Green's theorem, some suitable transformation, and the edge conditions, we formulate in section III pairs of simultaneous Fredholm integral equations of the second kind. From a knowledge of the solution of these integral equations the scattered electromagnetic field can be calculated everywhere by performing simple integrations.

In section IV we express both the axial component of the scattered electric field on the axis of the two disks and the integral of this field along the axis between the two plates in terms of single integrals which involve the solution of the integral equations formulated in section III. In the special case of two equal disks each pair of simultaneous Fredholm integral equations reduces to one Fredholm integral equation of the second kind. This equation is solved iteratively for low frequencies and for large separation between the disks. These iterations are used to derive analytical expressions for both the electric field at the center of the dipole and the integral of the electric field along the axis between the two plates. For other frequencies and separations not large these two quantities are calculated numerically and graphed as a function of frequency for different size of the sensor and angles of the incident field.

Finally, in an appendix, we consider the electrostatic problem of two circular, coaxial, conducting disks immersed in a homogeneous incident field as the limit when the frequency in the dynamic problem treated in sections II and II tends to zero. A pair of two simultaneous Fredholm integral equations for the electrostatic potential of the scattered field is derived. These integral equations resemble very much that derived by Love[5] for the electrostatic potential of two, equal, circular, coaxial, conducting disks equally charged or oppositely charged.

II. Formulation of the Boundary Conditions in Terms of Differential Equations

In this section we will formulate boundary conditions in terms of a set of differential equations for the following problem. Consider two perfectly conducting, infinitely thin, coaxial, circular disks, S_+ and S_-, with radius a_+ and a_-, respectively. The distance between the two disks is 2d (see figure 1a). The two disks, S_+ and S_-, are located at

$$z = d \ , \quad 0 \le \rho \le a_+ \ , \quad 0 \le \phi < 2\pi$$

$$z = -d \ , \quad 0 \le \rho \le a_- \ , \quad 0 \le \phi < 2\pi$$

where (ρ,ϕ,z) are cylindrical coordinates. We assume that the disks are immersed in an incident electromagnetic field and the harmonic-time dependence $(e^{-i\omega t})$ will be understood throughout the note.

It is convenient to assume that the incident field can be split into two separate fields, one with the magnetic field parallel to the disks and the other with the electric field parallel to the disks. The boundary value problems for these two incident fields can be treated separately and the solution for an arbitrary incident field may then be obtained by superposition.

A. Incident Magnetic Field Parallel to the Disks

In this case we suppose that the incident field consists of two plane waves (see figure 1a) such that

$$\underline{E}^{inc} = \underline{E}_1 e^{ikz\cos\theta_1 - ikx\sin\theta_1} + \underline{E}_2 e^{ikz\cos\theta_2 - ikx\sin\theta_2}$$
$$\underline{H}^{inc} = \underline{E}_1 Z_o^{-1} e^{ikz\cos\theta_1 - ikx\sin\theta_1} + \underline{E}_2 Z_o^{-1} e^{ikz\cos\theta_2 - ikx\sin\theta_2} \qquad (1)$$

where $k = \omega/c$, c is the vacuum speed of light and Z_o is the free-space wave impedance ($Z_o \approx 377\Omega$). With $E_1 = E_2 = E_o$ and $\theta_1 = \pi - \theta_2 - \theta_o$ we have

$$\underline{E}^{inc} = 2iE_o \cos\theta_o \sin(kz\cos\theta_o)e^{-ikx\sin\theta_o}\hat{x}$$

$$+ 2E_o \sin\theta_o \cos(kz\cos\theta_o)e^{-ikx\sin\theta_o}\hat{z} \tag{2}$$

$$\underline{H}^{inc} = 2E_o Z_o^{-1} \cos(kz\cos\theta_o)e^{-ikx\sin\theta_o}\hat{y}$$

In cylindrical coordinates $\underline{E}^{inc}$ in equation (2) can be written as

$$\underline{E}^{inc} = 2E_o \cos\theta_o \sin(kz\cos\theta_o)\{-J_1(k\rho\sin\theta_o) + \sum_{m=1}^{\infty} i^m[J_{m-1}(k\rho\sin\theta_o)$$

$$- J_{m+1}(k\rho\sin\theta_o)]\cos m\phi\}\hat{\rho} - 2E_o\cos\theta_o\sin(kz\cos\theta_o)\sum_{m=1}^{\infty} i^m$$

$$[J_{m+1}(k\rho\sin\theta_o) + J_{m-1}(k\rho\sin\theta_o)]\sin m\phi\hat{\phi} + 2E_o\sin\theta_o\cos(kz\cos\theta_o)$$

$$\{J_o(k\rho\sin\theta_o) + 2\sum_{m=1}^{\infty} i^m J_m(k\rho\sin\theta_o)\cos m\phi\}\hat{z} \tag{3}$$

The scattered field can be determined from the magnetic and the electric Hertz potentials, $\underline{\pi}^{(m)}$ and $\underline{\pi}^{(e)}$:

$$\underline{E}^{sc} = i\omega \text{ curl } \underline{\pi}^{(m)} + \text{curl curl } \underline{\pi}^{(e)}$$

$$= i\omega \text{ curl } \underline{\pi}^{(m)} + \text{grad div } \underline{\pi}^{(e)} + k^2\underline{\pi}^{(e)}$$

$$\underline{B}^{sc} = \text{curl curl } \underline{\pi}^{(m)} - ikc^{-1} \text{ curl } \underline{\pi}^{(e)} \tag{4}$$

$$= \text{grad div } \underline{\pi}^{(m)} + k^2\underline{\pi}^{(m)} - ikc^{-1} \text{ curl } \underline{\pi}^{(e)}$$

where $\underline{\pi}^{(m)}$ and $\underline{\pi}^{(e)}$ satisfy the Helmholtz equation

$$\Delta\underline{\pi} + k^2\underline{\pi} = 0 \tag{5}$$

In the problem of two parallel circular coaxial disks the Hertz potentials take the form

$$\pi_x^{(m)} = E_o(i\omega)^{-1} \sum_{m=1}^{\infty} \xi_m(\rho,z)\sin m\phi$$

$$\pi_y^{(m)} = E_o(i\omega)^{-1} \sum_{m=0}^{\infty} \xi_m(\rho,z)\cos m\phi \tag{6}$$

$$\pi_z^{(m)} = E_o(i\omega)^{-1} \sum_{m=0}^{\infty} \eta_m(\rho,z)\sin m\phi$$

$$\underline{\pi}^{(e)} = E_o\psi(\rho,z)\hat{z} \tag{7}$$

In cylindrical coordinates equation (6) becomes

$$\pi_\rho^{(m)} = E_o(i\omega)^{-1} \sum_{m=1}^{\infty} \xi_{m-1}(\rho,z)\sin m\phi$$

$$\pi_\phi^{(m)} = E_o(i\omega)^{-1} \sum_{m=1}^{\infty} \xi_{m-1}(\rho,z)\cos m\phi \tag{8}$$

where ξ_m and η_m satisfy the differential equation

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} + \frac{\partial^2}{\partial z^2} + k^2\right]_{\eta_m}^{\xi_m} = 0 \tag{9}$$

Thus, we have

$$E_\rho^{sc} = E_o\left[\frac{\partial^2\psi_o}{\partial\rho\partial z} + \sum_{m=1}^{\infty} \left(\rho^{-1}m\eta_m - \frac{\partial\xi_{m-1}}{\partial z}\right)\cos m\phi\right]$$

$$E_\phi^{sc} = E_o \sum_{m=1}^{\infty} \left(\frac{\partial\xi_{m-1}}{\partial z} - \frac{\partial\eta_m}{\partial\rho}\right)\sin m\phi \tag{10}$$

The boundary conditions are

$$E_\rho = E_\rho^{inc} + E_\rho^{sc} = 0$$

$$E_\phi = E_\phi^{inc} + E_\phi^{sc} = 0 \tag{11}$$

on S_+ and S_-.

We notice here that

$$E_x^{inc} = E_y^{inc} = 0 \quad \text{for } z = d_n$$

where

$$d_n = n\pi(k \cos \theta_o)^{-1} \quad , \quad n \text{ integer.}$$

With a plane, perfectly conducting sheet (S) of arbitrary shape located at $z = d_n$ the boundary conditions

$$E_x^{inc} + E_x^{sc} = E_y^{inc} + E_y^{sc} = 0 \quad \text{on S}$$

are satisfied by

$$\underline{E}^{sc} \equiv 0$$

Thus, the sheet S will not give rise to any scattered field. For the particular case of two coaxial perfectly conducting disks located at $z = \pm d_n$ see equations (30), (37), (42), (51) and (52) in section III.

For $m \geq 1$ we have the boundary-condition equations on S_+ and S_-

$$\frac{m\eta_m}{\rho} - \frac{\partial \xi_{m-1}}{\partial z} = 2i^m \cos\theta_o \sin(kz\cos\theta_o)[J_{m+1}(k\rho\sin\theta_o) - J_{m-1}(k\rho\sin\theta_o)]$$

$$\frac{\partial \xi_{m-1}}{\partial z} - \frac{\partial \eta_m}{\partial \rho} = 4i^m \cos\theta_o \sin(kz\cos\theta_o) J_m(k\rho\sin\theta_o) \tag{12}$$

from which it follows that

$$\frac{\partial \eta_m}{\partial \rho} - \frac{m\eta_m}{\rho} + k\sin\theta_o A_m(z) J_{m+1}(k\rho\sin\theta_o) = 0 \quad , \quad \begin{cases} 0 \leq \rho \leq a_+ \ , \ z = d \\ 0 \leq \rho \leq a_- \ , \ z = -d \end{cases} \tag{13}$$

where

$$A_m(z) = 2\varepsilon_m i^m k^{-1} \cot\theta_o \sin(kz\cos\theta_o)$$

and

$$\varepsilon_m = \begin{cases} 1 \ , & m = 0 \\ 2 \ , & m \geq 1 \end{cases}$$

Solving equation (13) we have

$$\eta_m^+(\rho) = \eta_m(\rho, d) = B_m^+ \rho^m + A_m^+ J_m(k\rho \sin\theta_o) \quad , \quad 0 \leq \rho \leq a_+$$

$$\eta_m^-(\rho) = \eta_m(\rho, -d) = B_m^- \rho^m + A_m^- J_m(k\rho \sin\theta_o) \quad , \quad 0 \leq \rho \leq a_- \tag{14}$$

where

$$A_m^+ = A_m(d) \quad , \quad A_m^- = A_m(-d)$$

and the unknown constants of integration, B_m^+ and B_m^- ($m = 1, 2, 3, \ldots.$), are to be determined from the edge conditions. Moreover,

$$\frac{\partial \xi_m}{\partial z}(\rho, d) = (m+1)B_{m+1}^+ \rho^m + k \sin\theta_o A_{m+1}^+ J_m(k\rho \sin\theta_o) \quad , \quad 0 \leq \rho \leq a_+$$

$$\frac{\partial \xi_m}{\partial z}(\rho, -d) = (m+1)B_{m+1}^- \rho^m + k \sin\theta_o A_{m+1}^- J_m(k\rho \sin\theta_o) \quad , \quad 0 \leq \rho \leq a_- \tag{15}$$

For m = 0 we have the boundary condition

$$\frac{\partial^2 \psi}{\partial\rho\,\partial z} = 2\cos\theta_o \sin(kz\cos\theta_o) J_1(k\rho\sin\theta_o), \begin{cases} 0 \leq \rho \leq a_+ \ , & z = d \\ 0 \leq \rho \leq a_- \ , & z = -d \end{cases} \tag{16}$$

with the solution

$$\frac{\partial\psi}{\partial z}(\rho,d) = B_o^+ - A_o^+ J_o(k\rho \sin\theta_o)$$

(17)

$$\frac{\partial\psi}{\partial z}(\rho,-d) = B_o^- - A_o^- J_o(k\rho \sin\theta_o)$$

where B_o^+ and B_o^-, the unknown constants of integration, are to be determined from the edge conditions.

B. Incident Electric Field Parallel to the Disks

With the incident field consisting of two plane waves (see figure 1b) we have

$$\underline{E}^{inc} = 2E_o \cos(kz \cos\theta_o) e^{-ikx \sin\theta_o} \hat{y} \tag{18}$$

or in cylindrical coordinates

$$\underline{E}^{inc} = 2E_o \cos(kz\cos\theta_o) \sum_{m=1}^{\infty} i^{m-1}[J_{m+1}(k\rho \sin\theta_o) + J_{m-1}(k\rho\sin\theta_o)]\sin m\phi\hat{\rho}$$

$$+ 2E_o \cos(kz\cos\theta_o)\{iJ_1(k\rho\sin\theta_o) + \sum_{m=1}^{\infty} i^{m+1}[J_{m+1}(k\rho\sin\theta_o)$$

$$- J_{m-1}(k\rho\sin\theta_o)]\cos m\phi\}\hat{\phi} \tag{19}$$

In this case the scattered field can be described as

$$\underline{E}^{sc} = i\omega \text{ curl } \underline{\pi}^{(m)}$$

(20)

$$\underline{B}^{sc} = \text{grad div } \underline{\pi}^{(m)} + k^2\underline{\pi}^{(m)}$$

where

$$
\begin{aligned}
\pi_\rho^{(m)} &= E_o(i\omega)^{-1} \sum_{m=1}^{\infty} \chi_{m-1}(\rho,z)\cos m\phi \\
\pi_\phi^{(m)} &= - E_o(i\omega)^{-1} \sum_{m=1}^{\infty} \chi_{m-1}(\rho,z)\sin m\phi \\
\pi_z^{(m)} &= E_o(i\omega)^{-1} \sum_{m=0}^{\infty} \tau_m(\rho,z)\cos m\phi
\end{aligned}
\tag{21}
$$

The boundary conditions on S_+ and S_- give

$$
\begin{aligned}
\tau_m^+(\rho) &= \tau_m(\rho,d) = D_m^+\rho^m + C_m J_m(k\rho \sin\theta_o) \quad , \quad 0 \le \rho \le a_+ \\
\tau_m^-(\rho) &= \tau_m(\rho,d) = D_m^-\rho^m + C_m J_m(k\rho \sin\theta_o) \quad , \quad 0 \le \rho \le a_-
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\frac{\partial\chi_m}{\partial z}(\rho,d) &= (m+1)D_{m+1}^+\rho^m \quad , \quad 0 \le \rho \le a_+ \\
\frac{\partial\chi_m}{\partial z}(\rho,-d) &= (m+1)D_{m+1}^-\rho^m \quad , \quad 0 \le \rho \le a_-
\end{aligned}
\tag{23}
$$

for $m \ge 1$ and

$$
\begin{aligned}
\tau_o^+(\rho) &= \tau_o(\rho,d) = D_o^+ + C_o J_o(k\rho \sin\theta_o) \quad , \quad 0 \le \rho \le a_+ \\
\tau_o^-(\rho) &= \tau_o(\rho,-d) = D_o^- + C_o J_o(k\rho \sin\theta_o) \quad , \quad 0 \le \rho \le a_-
\end{aligned}
\tag{24}
$$

Here

$$
C_m = 2\varepsilon_m i^{m-1}(k \sin\theta_o)^{-1}\cos(kd\cos\theta_o)
$$

and D_m^+ and D_m^- are constants of integration to be determined from the edge conditions.

III. Reduction of the Differential Equations to Integral Equations

In this section we will derive integral equations from the boundary-value equations formulated in section II. When the solution of these integral equations is known the scattered field at any point can be determined from a simple integration.

A. Integral Equations for $\eta_m(\rho,z)$ and $\tau_m(\rho,z)$

From equation (14) it follows that

$$\lim_{z\to\pm d+} \eta_m(\rho,z) = \lim_{z\to\pm d-} \eta_m(\rho,z)$$

Introduce the notations

$$\frac{\partial\eta'_m}{\partial z}(\rho,z) = \lim_{\zeta\to z+} \frac{\partial\eta_m}{\partial\zeta}(\rho,\zeta)$$

$$\frac{\partial\eta''_m}{\partial z}(\rho,z) = \lim_{\zeta\to z-} \frac{\partial\eta_m}{\partial\zeta}(\rho,\zeta)$$

$$y_m^+(\rho) = \frac{\partial\eta''_m}{\partial z}(\rho,d) - \frac{\partial\eta'_m}{\partial z}(\rho,d) \qquad (25)$$

$$y_m^-(\rho) = \frac{\partial\eta''_m}{\partial z}(\rho,-d) - \frac{\partial\eta'_m}{\partial z}(\rho,-d)$$

It follows from Green's theorem that

$$\eta_m(\rho,z) = \pi^{-1}\int_o^{2\pi}\int_{S_+} G(\rho,\rho',\phi-\phi',z-d)y_m^+(\rho')\sin m\phi' \sin m\phi dS'd\phi$$

$$+\ \pi^{-1}\int_o^{2\pi}\int_{S_-} G(\rho,\rho',\phi-\phi',z+d)y_m^-(\rho')\sin m\phi' \sin m\phi dS'd\phi \qquad (26)$$

where

$$G(\rho,\rho',\psi,\zeta) = (4\pi R)^{-1} e^{ikR}$$

and

$$R = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos\psi + \zeta^2}$$

A simple manipulation of equation (26) gives

$$\eta_m(\rho,z) = \int_0^{2\pi}\int_0^{a_+} G(\rho,\rho',\psi,z-d)\rho' y_m^+(\rho')\cos m\psi d\rho' d\psi$$

$$+ \int_0^{2\pi}\int_0^{a_-} G(\rho,\rho',\psi,z+d)\rho' y_m^-(\rho')\cos m\psi d\rho' d\psi$$

$$= \frac{1}{2}\int_0^{a_+}\int_0^{\infty} p\gamma^{-1} J_m(\rho p) J_m(\rho' p) e^{-|z-d|\gamma} \rho' y_m^+(\rho') d\rho' dp$$

$$+ \frac{1}{2}\int_0^{a_+}\int_0^{\infty} p\gamma^{-1} J_m(\rho p) J_m(\rho' p) e^{-|z+d|\gamma} \rho' y_m^-(\rho') d\rho' dp \qquad (27)$$

where

$$\gamma = \begin{cases} \sqrt{p^2 - k^2}, & p > k \\ -i\sqrt{k^2 - p^2}, & p < k \end{cases}$$

The path of integration is along the real axis in the complex p-plane with downward indentation at $p = k$, as shown in figure 2. Together with equation (14) we get the following system of integral equations that $y_m^+(\rho)$ and $y_m^-(\rho)$ must satisfy

$$\int_0^{a_+} K_m(\rho,\rho',0)\rho' y_m^+(\rho')d\rho' + \int_0^{a_-} K_m(\rho,\rho',2d)\rho' y_m^-(\rho')d\rho' = B_m^+\rho^m + A_m^+ J_m(k\rho \sin\theta_o)$$

$$0 < \rho < a_+$$

$$\int_0^{a_-} K_m(\rho,\rho',0)\rho' y_m^-(\rho')d\rho' + \int_0^{a_+} K_m(\rho,\rho',2d)\rho' y_m^+(\rho')d\rho' = B_m^-\rho^m + A_m^- J_m(k\rho \sin\theta_o) \qquad (28)$$

$$0 < \rho < a_-$$

where

$$K_m(\rho,\rho',z) = \frac{1}{2}\int_0^\infty p\gamma^{-1} J_m(\rho p) J_m(\rho' p) e^{-|z|\gamma}\, dp$$

From the analysis in appendix A it follows that this system of integral equations can be reduced to

$$y_m^\pm(\rho) = -\frac{2}{\pi}\rho^{m-1}\frac{d}{d\rho}\int_\rho^{a_\pm}\frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}}\, u^{1-m}\, Y_m^\pm(u)du \qquad (29)$$

where $Y_m^+(u)$ and $Y_m^-(u)$ satisfy the system of Fredholm integral equations of the second kind

$$Y_m^+(u) + \int_0^{a_+} L_m(u,v)Y_m^+(v)dv + \int_0^{a_-} M_m(u,v,d)Y_m^-(v)dv = F_m^+(u) \quad , \quad 0 < u < a_+$$

$$Y_m^-(u) + \int_0^{a_-} L_m(u,v)Y_m^-(v)dv + \int_0^{a_+} M_m(u,v,d)Y_m^+(v)dv = F_m^-(u) \quad , \quad 0 < u < a_- \qquad (30)$$

Here

$$L_m(u,v) = \sqrt{uv}\int_0^\infty \gamma^{1-2m}[(k^2+\gamma^2)^m - \gamma^{2m}] J_{m-\frac{1}{2}}(u\gamma) J_{m-\frac{1}{2}}(v\gamma)d\gamma$$

$$+ i\sqrt{uv}\int_0^k \gamma^{1-2m}(k^2-\gamma^2)^m\, I_{m-\frac{1}{2}}(u\gamma) I_{m-\frac{1}{2}}(v\gamma)d\gamma$$

$$M_m(u,v) = \sqrt{uv} \int_0^{\infty} p^{2m+1} \gamma^{-2m} J_{m-\frac{1}{2}}(u\gamma) J_{m-\frac{1}{2}}(v\gamma) e^{-2d\gamma} \, dp$$

$$F_m^{\pm}(u) = 2u^{-m} \frac{d}{du} \int_0^u \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} \rho^{m+1} [B_m^{\pm} \rho^m + A_m^{\pm} J_m(k\rho \sin \theta_o)] d\rho$$

Defining

$$t_m^{\pm}(\rho) = \frac{\partial \tau_m'}{\partial z}(\rho, \pm d) - \frac{\partial \tau_m'}{\partial z}(\rho, \pm d)$$

one can show that the same transformations as those leading to equation (29) give

$$t_m^{\pm}(\rho) = -\frac{2}{\pi} \rho^{m-1} \frac{d}{d\rho} \int_{\rho}^{a_{\pm}} \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} u^{1-m} T_m^{\pm}(u) du \tag{31}$$

Here $T_m^+(u)$ and $T_m^-(u)$ satisfy the system of integral equations

$$\begin{aligned} T_m^+(u) + \int_0^{a_+} L_m(u,v) T_m^+(v) dv + \int_0^{a_-} M_m(u,v,d) T_m^-(v) dv &= E_m^+(u) \\ T_m^-(u) + \int_0^{a_-} L_m(u,v) T_m^-(v) dv + \int_0^{a_+} M_m(u,v,d) T_m^+(v) dv &= E_m^-(u) \end{aligned} \tag{32}$$

where

$$E_m^{\pm}(u) = 2u^{-m} \frac{d}{du} \int_0^u \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} \rho^{m+1} [D_m^{\pm} \rho^m + C_m J_m(k\rho \sin \theta_o)]$$

B. Integral Equations for $\xi_m(\rho,z)$ and $\chi_m(\rho,z)$

Introduce the notations

$$\xi'_m(\rho,z) = \lim_{\zeta\to z+} \xi_m(\rho,\zeta)$$

$$\xi''_m(\rho,z) = \lim_{\zeta\to z-} \xi_m(\rho,\zeta) \tag{33}$$

$$x_m^{\pm}(\rho) = \xi''_m(\rho,\pm d) - \xi'_m(\rho,\pm d)$$

Green's theorem then gives

$$\xi_m(\rho,z) = \frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{a_+} G(\rho,\rho',\psi,z-d)\rho' x_m^+(\rho')\cos m\psi\, d\rho' d\psi$$

$$+ \frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{a_-} G(\rho,\rho',\psi,z+d)\rho' x_m^-(\rho')\cos m\psi\, d\rho' d\psi \tag{34}$$

From equation (15) we have for $m \geq 1$

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2\right]\left[\int_0^{a_+} K_m(\rho,\rho',0)\rho' x_m^+(\rho')d\rho' + \int_0^{a_-} K_m(\rho,\rho',2d)\rho' x_m^-(\rho')d\rho'\right]$$

$$+ (m+1)B_{m+1}^+\rho^m + k\sin\theta_o A_{m+1}^+ J_o(k\rho\sin\theta_o) = 0 \quad , \quad 0 < \rho < a_+$$

$$\tag{35}$$

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2\right]\left[\int_0^{a_-} K_m(\rho,\rho',0)\rho' x_m^-(\rho')d\rho' + \int_0^{a_+} K_m(\rho,\rho',2d)\rho' x_m^+(\rho')d\rho'\right]$$

$$+ (m+1)B_{m+1}^-\rho^m + k\sin\theta_o A_{m+1}^- J_o(k\rho\sin\theta_o) = 0 \quad , \quad 0 < \rho < a_-$$

From the analysis in appendix B it follows that this system of integral equations can be reduced to

$$x_m^{\pm}(\rho) = 2\pi^{-1}\rho^m \int_\rho^{a_\pm} \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} u^{-m} X_m^{\pm}(u)du \tag{36}$$

where $X_m^+(u)$ and $X_m^-(u)$ satisfy the system of Fredholm integral equations of the second kind

$$X_m^+(u) + \int_o^{a_+} U_m(u,v)X_m^+(v)dv + \int_o^{a_-} P_m(u,v,d)X_m^-(v)dv = G_m^+(u) \quad , \quad 0 < u < a_+$$

$$X_m^-(u) + \int_o^{a_-} U_m(u,v)X_m^-(v)dv + \int_o^{a_+} P_m(u,v,d)X_m^+(v)dv = G_m^-(u) \quad , \quad 0 < u < a_- \qquad (37)$$

Here

$$U_m(u,v) = \sqrt{uv}\int_o^\infty \gamma^{1-2m}[(k^2+\gamma^2)^m - \gamma^{2m}]J_{m+\frac{1}{2}}(u\gamma)J_{m+\frac{1}{2}}(v\gamma)d\gamma$$

$$+ i\sqrt{uv}\int_o^k \gamma^{1-2m}(k^2-\gamma^2)^m\, I_{m+\frac{1}{2}}(u\gamma)I_{m+\frac{1}{2}}(v\gamma)d\gamma$$

$$P_m(u,v) = \sqrt{uv}\int_o^\infty p^{2m+1}\,\gamma^{-2m}\,J_{m+\frac{1}{2}}(u\gamma)J_{m+\frac{1}{2}}(v\gamma)e^{-2d\gamma}\,d\gamma$$

$$G_m^\pm(u) = u^{-m-1}\frac{d}{du}\int_o^u \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}}[B_{m+1}^\pm\,\rho^{2m+3} + 2A_{m+1}^\pm\,\rho^{m+2}\,J_{m+1}(k\rho\,\sin\theta_o)]d\rho$$

With

$$w_m^\pm(\rho) = \chi_m''(\rho,\pm d) - \chi_m'(\rho,\pm d)$$

we obtain, in the same way,

$$w_m^\pm(\rho) = 2\pi^{-1}\,\rho^m\int_\rho^{a_+}\frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}}\,u^{-m}\,W_m^\pm(u)du \qquad (38)$$

Here $W_m^+(u)$ and $W_m^-(u)$ satisfy the system of integral equations

$$W_m^+(u) + \int_0^{a_+} U_m(u,v)W_m^+(v)dv + \int_0^{a_-} P_m(u,v,d)W_m^-(v)dv = H_m^+(u) \quad , \quad 0 \le u \le a_+$$

$$W_m^-(u) + \int_0^{a_-} U_m(u,v)W_m^-(v)dv + \int_0^{a_+} P_m(u,v,d)W_m^+(v)dv = H_m^-(u) \quad , \quad 0 \le u \le a_- \tag{39}$$

where

$$H_m^{\pm}(u) = u^{-m-1} \frac{d}{du} \int_0^u \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} D_{m+1}^{\pm} \rho^{2m+3} d\rho$$

C. Integral Equations for $\psi(\rho,z)$

The Green's theorem combined with equation (17) gives

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + k^2\right]\left[\int_0^{a_+} K_o(\rho,\rho',0)\rho'\sigma^+(\rho')d\rho' + \int_0^{a_-} K_o(\rho,\rho',2d)\rho'\sigma^-(\rho')d\rho'\right]$$

$$+ B_o^+ - A_o^+ J_o(k\rho \sin\theta_o) = 0 \quad , \quad 0 < \rho < a_+$$

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + k^2\right]\left[\int_0^{a_-} K_o(\rho,\rho',0)\rho'\sigma^-(\rho')d\rho' + \int_0^{a_+} K_o(\rho,\rho',2d)\rho'\sigma^+(\rho')d\rho'\right]$$

$$+ B_o^- - A_o^- J_o(k\rho \sin\theta_o) = 0 \quad , \quad 0 < \rho < a_- \tag{40}$$

where

$$\sigma^{\pm}(\rho) = \psi''(\rho,\pm d) - \psi'(\rho,\pm d)$$

From the analysis in appendix B it follows that

$$\sigma^{\pm}(\rho) = 2\pi^{-1} \int_\rho^{a_\pm} \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} S^{\pm}(u)du \tag{41}$$

Here $S^+(u)$ and $S^-(u)$ satisfy the system of Fredholm integral equations of the second kind

$$S^+(u) + \int_0^{a_+} U_o(u,v)S^+(v)dv + \int_0^{a_-} P_o(u,v,d)S^-(v)dv = F_o^+(u) \quad , \quad 0 < u < a_+$$

(42)

$$S^-(u) + \int_0^{a_-} U_o(u,v)S^-(v)dv + \int_0^{a_+} P_o(u,v,d)S^+(v)dv = F_o^-(u) \quad , \quad 0 < u < a_-$$

where

$$U_o(u,v) = i\sqrt{uv}\int_0^k \gamma I_{\frac{1}{2}}(u\gamma)I_{\frac{1}{2}}(v\gamma)d\gamma = i\pi^{-1}\left[\frac{\sinh k(u+v)}{u+v} - \frac{\sinh k(u-v)}{u-v}\right]$$

$$P_o(u,v,d) = \sqrt{uv}\int_0^\infty pJ_{\frac{1}{2}}(u\gamma)J_{\frac{1}{2}}(v\gamma)e^{-2d\gamma}\,dp = \frac{2}{\pi}\int_\Gamma \sin u\gamma \sin v\gamma e^{-2d\gamma}\,d\gamma$$

and the path of integration Γ is from $-ik$ to $+\infty$ as shown in figure 3. Performing the integration we obtain

$$P_o(u,v,d) = P_o'(u,v,d) + P_o''(u,v,d) \qquad (43)$$

where

$$P_o'(u,v,d) = \pi^{-1}e^{i2kd}\left[\frac{2d\cosh k(u-v)}{4d^2+(u-v)^2} - \frac{2d\cosh k(u+v)}{4d^2+(u+v)^2}\right]$$

$$P_o''(u,v,d) = i\pi^{-1}e^{i2kd}\left[\frac{(u-v)\sinh k(u-v)}{4d^2+(u-v)^2} - \frac{(u+v)\sinh k(u+v)}{4d^2+(u+v)^2}\right]$$

In equation (42) the right hand sides are given by

$$F_o^{\pm}(u) = u^{-1}\frac{d}{du}\int_0^u \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}}\left[B_o^{\pm}\rho^3 - 2(k\sin\theta_o)^{-1}A_o^{\pm}\rho^2 J_1(k\rho\sin\theta_o)\right]d\rho$$

$$= 2k^{-1}B_o^{\pm}\sinh(ku) - 2(k\cos\theta_o)^{-1}A_o^{\pm}\sinh(ku\cos\theta_o) \qquad (44)$$

D. Determination of the Unknown Constants From the Edge Conditions

The surface currents, $\underline{i}^{+}(\rho,\phi)$ on S_{+} and $\underline{i}^{-}(\rho,\phi)$ on S_{-}, are given by

$$\underline{i}^{\pm}(\rho,\phi) = \hat{z} \times [\underline{H}'(\rho,\phi,\pm d) - \underline{H}''(\rho,\phi,\pm d)] \tag{45}$$

where

$$\underline{H}'(\rho,\phi,z) = \lim_{\zeta \to z+} \underline{H}(\rho,\phi,\zeta)$$

$$\underline{H}''(\rho,\phi,z) = \lim_{\zeta \to z-} \underline{H}(\rho,\phi,\zeta)$$

Thus,

$$\begin{aligned} i_{\rho}^{+}(\rho,\phi) &= - H_{\phi}'^{sc}(\rho,\phi,d) + H_{\phi}''^{sc}(\rho,\phi,d) \\ i_{\phi}^{+}(\rho,\phi) &= H_{\rho}'^{sc}(\rho,\phi,d) - H_{\phi}''^{sc}(\rho,\phi,d) \end{aligned} \tag{46}$$

since $\underline{H}^{inc}$ is continuous everywhere. The edge conditions are

$$\begin{aligned} i_{\rho}^{+}(\rho,\phi) &= O[(a_{+} - \rho)^{\frac{1}{2}}] \\ i_{\phi}^{+}(\rho,\phi) &= O[(a_{+} - \rho)^{-\frac{1}{2}}] \end{aligned} \tag{47}$$

as $\rho \to a_{+} - 0$.

In the case where the incident magnetic field is parallel to the disks, the expansion of $i_{\rho}^{+}(\rho,\phi)$ in a Fourier series,

$$i_{\rho}^{+}(\rho,\phi) = \sum_{m=0}^{\infty} i_{\rho m}^{+}(\rho)\cos m\phi \tag{48}$$

combined with equations (4), (6), (8), (25), (33) and (46) gives

$$i_{\rho o}^{+} = ikZ_o^{-1}E_o \frac{d\sigma^{+}}{d\rho}$$

(49)

$$i_{\rho m}^{+} = E_o(i\omega\mu_o)^{-1}\left[k^2 x_{m-1}^{+} + \frac{m}{\rho}\frac{dx_{m-1}^{+}}{d\rho} - \frac{m(m-1)}{\rho^2}x_{m-1}^{+} + \frac{m}{\rho}y_m^{+}\right]$$

which become from equations (29), (36), (41)

$$i_{\rho o}^{+}(\rho) = \frac{\sqrt{2}\ kE_o S^{+}(a_+)}{i\pi Z_o\sqrt{a_+(a_+-\rho)}} + O[(a_+ - \rho)^{\frac{1}{2}}]$$

(50)

$$i_{\rho m}^{+}(\rho) = \frac{\sqrt{2}\ mE_o\ Y_m^{+}(a_+)-X_{m-1}^{+}(a_+)}{i\omega\mu_o\pi a_+\sqrt{a_+(a_+-\rho)}} + O[(a_+ - \rho)^{\frac{1}{2}}], \quad m \geq 1$$

as $\rho \rightarrow a_+ - 0$. Thus, the edge conditions give

$$S^{+}(a_+) = 0$$

(51)

$$Y_m^{+}(a_+) = X_{m-1}^{+}(a_+) \quad , \quad m \geq 1$$

Similarly

$$S^{-}(a_-) = 0$$

(52)

$$Y_m^{-}(a_-) = X_{m-1}^{-}(a_-) \quad , \quad m \geq 1$$

from which we can determine $B_m^{\pm}$, $m \geq 0$.

In the case where the incident electric field is parallel to the disks, we have the edge conditions

$$T_m^{\pm}(a_{\pm}) = W_{m-1}^{\pm}(a_{\pm}) \quad , \quad m \geq 1 \tag{53}$$

from which we can determine $D_m^{\pm}$, $m \geq 1$.

Here we wish to add that the edge condition for $i_{\phi m}^{\pm}$ is satisfied by the conditions (51) - (53).

IV. The Electric Field on the Axis of the Disks

A. An Integral Expression for the Electric Field

The electric field on the axis of the two disks is given by

$$E_z = E_z^{inc} + E_o e \tag{54}$$

where

$$e(z) = \frac{\partial^2}{\partial z^2}\,\psi(0,z) + k^2\psi(0,z)$$

and $\psi(\rho,z)$ is given by

$$\psi(\rho,z) = \psi^+(\rho,z) + \psi^-(\rho,z) \tag{55}$$

where

$$\psi^+(\rho,z) = \frac{\partial}{\partial z}\int_{S_+} G(\rho,\rho',\phi,z-d)\rho'\sigma^+(\rho')d\rho' d\phi$$

$$\psi^-(\rho,z) = \frac{\partial}{\partial z}\int_{S_-} G(\rho,\rho',\phi,z+d)\rho'\sigma^-(\rho')d\rho' d\phi$$

We now manipulate ψ^+ (and, similarly, ψ^-) as follows.

$$\psi^+(\rho,z) = \frac{1}{4\pi}\frac{\partial}{\partial z}\int_{S_+}\frac{\rho'\sigma^+(\rho')\exp\left[ik\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi+(z-d)^2}\right]}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi+(z-d)^2}}\,d\rho' d\phi$$

$$= \frac{1}{2\pi^2}\frac{\partial}{\partial z}\int_{S_+}\int_o^{a_+}\frac{S^+(u)\cosh(k\sqrt{u^2-\rho'^2})\rho'\exp\left[ik\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi+(z-d)^2}\right]}{\sqrt{u^2-\rho'^2}\,\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi+(z-d)^2}}\,du\,d\rho' d\phi$$

$$= \pi^{-1}\frac{\partial}{\partial z}\int_o^{a_+} S^+(u)du\int_o^{\infty} p\gamma^{-1}J_o(\rho p)e^{-|z-d|\gamma}\,dp\int_o^{u}\frac{\rho' J_o(\rho' p)\cosh(k\sqrt{u^2-\rho'^2})}{\sqrt{u^2-\rho'^2}}\,d\rho' \tag{56}$$

Making use of Sonine's second integral we have

$$\psi^{+}(\rho,z) = \pi^{-1}\frac{\partial}{\partial z}\int_{o}^{a_+} S^{+}(u)du \int_{o}^{\infty} p\gamma^{-2}J_{o}(\rho p)\sin(u\gamma)e^{-|z-d|\gamma}\, dp$$

$$= \frac{\operatorname{sgn}(d-z)}{2\pi i}\int_{o}^{a_+}\left[\frac{e^{ik\sqrt{\rho^2+(|z-d|-iu)^2}}}{\sqrt{\rho^2+(|z-d|-iu)^2}} - \frac{e^{ik\sqrt{\rho^2+(|z-d|+iu)^2}}}{\sqrt{\rho^2+(|z-d|+iu)^2}}\right]S^{+}(u)du \qquad (57)$$

and the real part of the square root is positive. For $\rho = 0$ we have

$$\psi^{+}(0,z) = \pi^{-1}\operatorname{sgn}(d-z)e^{ik|z-d|}\int_{o}^{a_+}\frac{u\cosh ku - i|z-d|\sinh ku}{u^2+(z-d)^2}S^{+}(u)du \qquad (58)$$

Thus, for $-d < z < d$, we have

$$\psi(0,z) = \pi^{-1}e^{ik(d-z)}\int_{o}^{a_+}\frac{u\cosh ku + i(z-d)\sinh ku}{u^2+(z-d)^2}S^{+}(u)du$$

$$- \pi^{-1}e^{ik(d+z)}\int_{o}^{a_-}\frac{u\cosh ku - i(z+d)\sinh ku}{u^2+(z+d)^2}S^{-}(u)du \qquad (59)$$

From the analysis in appendix C it follows that $S^{+}(u) = O(u)$ as $u \to 0$ and thus the integrals in equation (59) exist for $-d \le z \le d$. We also note that the expression (55) for $\psi(\rho,z)$ satisfies the ϕ-independent wave equation off the disks S_+ and S_-. The electric field on the axis can be obtained by differentiation of equation (59). In order to obtain expressions more suitable for numerical calculations we proceed as follows: suppose $S^{+}(u)$ is differentiable twice. Integrating by parts of equation (57) and keeping in mind that $S^{+}(a_+) = S^{+}(0) = 0$ we get

$$\frac{\partial\psi^{+}}{\partial z}(\rho,z) = \frac{1}{2\pi}\int_{o}^{a_+}\left[\frac{e^{ik\sqrt{\rho^2+(|z-d|-iu)^2}}}{\sqrt{\rho^2+(|z-d|-iu)^2}} + \frac{e^{ik\sqrt{\rho^2+(|z-d|+iu)^2}}}{\sqrt{\rho^2+(|z-d|+iu)^2}}\right]V^{+}(u)du \qquad (60)$$

and

$$\frac{\partial^2\psi^+}{\partial z^2}(\rho,z) = \frac{\mathrm{sgn}(d-z)}{2\pi i}\left[\frac{e^{ik\sqrt{\rho^2+(|z-d|-ia_+)^2}}}{\sqrt{\rho^2+(|z-d|-ia_+)^2}} - \frac{e^{ik\sqrt{\rho^2+(|z-d|+ia_+)^2}}}{\sqrt{\rho^2+(|z-d|+ia_+)^2}}\right]V^+(a_+)$$

$$-\frac{\mathrm{sgn}(d-z)}{2\pi i}\int_0^{a_+}\left[\frac{e^{ik\sqrt{\rho^2+(|z-d|-iu)^2}}}{\sqrt{\rho^2+(|z-d|-iu)^2}} - \frac{e^{ik\sqrt{\rho^2+(|z-d|+iu)^2}}}{\sqrt{\rho^2+(|z-d|+iu)^2}}\right]E^+(u)du \quad (61)$$

where

$$V^+(u) = \frac{dS^+}{du}(u) \qquad (62)$$

$$E^+(u) = \frac{dV^+}{du}(u) \qquad (63)$$

Thus, for $-d < z < d$, we have, after substituting (55), (59), (60) and (61) into (54),

$$e(z) = \pi^{-1}e^{ik(d-z)}\int_0^{a_+}\frac{u\cosh ku+i(z-d)\sinh ku}{u^2+(z-d)^2}[k^2S^+(u) - E^+(u)]du$$

$$-\pi^{-1}e^{ik(d+z)}\int_0^{a_-}\frac{u\cosh ku-i(z+d)\sinh ku}{u^2+(z+d)^2}[k^2S^-(u) - E^-(u)]du$$

$$+\pi^{-1}e^{ik(d-z)}\frac{a_+\cosh ka_+ + i(z-d)\sinh ka_+}{a_+^2+(z-d)^2}V^+(a_+)$$

$$-\pi^{-1}e^{ik(d+z)}\frac{a_-\cosh ka_- - i(z+d)\sinh ka_-}{a_-^2+(z+d)^2}V^-(a_-) \qquad (64)$$

The next quantity of interest is v(d) given by

$$v(d) = E_o^{-1}\int_{-d}^{d}E_z(0,z)dz$$

which upon substitution of equation (54) becomes

$$v(d) = v_1(d) + v_2(d) + v_3(d) \tag{65}$$

where

$$v_1(d) = E_o^{-1} \int_{-d}^{d} E_z^{inc}(0,z)dz = 4k^{-1} \tan\theta_o \sin(kd\cos\theta_o)$$

$$v_2(d) = \frac{\partial\psi}{\partial z}(0,d) - \frac{\partial\psi}{\partial z}(0,-d)$$

$$v_3(d) = k^2 \int_{-d}^{d} \psi(0,z)dz$$

For $|z| \neq d$ we have

$$\frac{\partial\psi}{\partial z}(0,z) = \frac{\partial\psi^+}{\partial z}(0,z) + \frac{\partial\psi^-}{\partial z}(0,z)$$

$$= \pi^{-1} e^{ik|z-d|} \int_o^{a_+} \frac{|z-d|\cosh ku + iu\sinh ku}{(z-d)^2+u^2} V^+(u)du$$

$$+ \pi^{-1} e^{ik|z+d|} \int_o^{a_-} \frac{|z+d|\cosh ku + iu\sinh ku}{(z+d)^2+u^2} V^-(u)du \tag{66}$$

and

$$v_2(d) = \frac{\partial\psi}{\partial z}(0,d) - \frac{\partial\psi}{\partial z}(0,-d) = \frac{V^+(0)-V^-(0)}{2}$$

$$+ \frac{i}{\pi}\int_o^{a_+} \frac{\sinh ku}{u} V^+(u)du - \frac{i}{\pi}\int_o^{a_-} \frac{\sinh ku}{u} V^-(u)du$$

$$- \frac{e^{2ikd}}{\pi} \int_o^{a_+} \frac{2d\cosh ku + iu\sinh ku}{4d^2+u^2} V^+(u)du$$

$$+ \frac{e^{2ikd}}{\pi} \int_o^{a_-} \frac{2d\cosh ku + iu\sinh ku}{4d^2+u^2} V^-(u)du \tag{67}$$

From the theory of the exponential integral[11] it follows that

$$v_3(d) = k^2 \int_{-d}^{d} \psi(0,z)dz = (2\pi i)^{-1}k^2\left[\int_0^{a_+} K(u)S^+(u)du - \int_0^{a_-} K(u)S^-(u)du\right] \quad (68)$$

where

$$K(u) = E_1(ku - 2ikd) - E_1(ku) + E_1(-ku - i0) - E_1(-ku - 2ikd)$$

and $E_1(\zeta)$ is the exponential integral,

$$E_1(\zeta) = \int_\zeta^\infty t^{-1}e^{-t}dt \quad , \quad |\arg \zeta| < \pi$$

B. Low Frequency Approximation for Two Equal Disks

In the special case of two equal disks it follows from the system of integral equations (42) that

$$S^+(u) = -S^-(u) = S(u) \quad (69)$$

and $S(u)$ satisfies the Fredholm integral equation

$$S(u) + \int_0^a [U_o(u,v) - P_o(u,v,d)]S(v)dv = F_o(u) \quad , \quad 0 \le u \le a \quad (70)$$

where

$$F_o(u) = 2k^{-1}B_o \sinh ku - 2(k \cos \theta_o)^{-1}A_o^+ \sinh (ku \cos \theta_o)$$

The edge condition [c.f. equation (51)] gives

$$S(a) = 0 \quad (71)$$

which enables us to determine the unknown constant B_o.

The kernel of equation (70) is small when the normalized wave number of the incident wave is small and the distance between the two plates is sufficiently large, i.e., when $\alpha = ka << 1$ and $\eta = kd$ is of the order of unity. Thus, an iterative solution of equation (70) can be obtained in this case. This iterative solution gives

$$S(u) = \sin 2\theta_o \sin(\eta \cos \theta_o) a^2 \alpha\zeta(\zeta^2-1)\{1-\alpha^2[1/6-(\zeta^2+1)(1+\cos^2\theta_o)/20]/3\}+O(\alpha^5) \quad (72)$$

$$V(u) = \sin 2\theta_o \sin(\eta \cos \theta_o) a\alpha\{3\zeta^2-1+\alpha^2[(1+\cos^2\theta_o)\zeta^4/4-\zeta^2/2+7/60-\cos^2\theta_o/20]\}/3+O(\alpha^5) \quad (73)$$

$$E(u) = 2 \sin 2\theta_o \sin(\eta \cos \theta_o)\alpha\{1+\alpha^2[(1+\cos^2\theta_o)\zeta^3-\zeta]/6\}+O(\alpha^5) \quad (74)$$

$$E_z(0,0) = 2E_o[\sin \theta_o+2 \sin 2\theta_o \sin(\eta \cos \theta_o)e^{i\eta}\alpha^5(\eta^4+i\eta^3-\eta^2-2)/(45\pi\eta^6)]+O(\alpha^7) \quad (75)$$

$$v(d) = d\eta^{-1} \sin 2\theta_o \sin(\eta \cos \theta_o)[2 \cos^{-2}\theta_o-\alpha^2/3+\alpha^4(11+6 \cos^2\theta_o)/180]+O(\alpha^6) \quad (76)$$

where

$$\zeta = u/a$$

C. Numerical Results

The integral equation (70) was solved numerically for $0 \le ka \le 10$, $d/a = .1, .05, .02$ and $\theta_o = 18^\circ, 36^\circ, 54^\circ, 72^\circ$. With this solution available we then, by simple integrations, calculated e(0) and v(d) from equations (64) and (65), respectively. The results of these calculations are shown in figures 4-11. In the graphs we have used the normalized quantities

$$e' = 1 + e(0)/(2 \sin \theta_o)$$

$$v' = v(d)/(4d \sin \theta_o)$$

Figures 4-7 show $|e'|$ and $\arg\{e'\}$ as a function of ka with d/a as a parameter for different values of θ_o. We found that the curves for $|v'|$ and $\arg\{v'\}$ are very similar to those of $|e'|$ and $\arg\{e'\}$. Therefore, we have decided to plot $|e' - v'|$ and $\arg\{e' - v'\}$ in figures 8-11 as a function of ka with d/a = .1, .05. For d/a = .02 the difference between e' and v' is negligable and hence the corresponding curves are omitted in the figures. It is found empirically that for ka < 1 we have

$$|e'| \approx 1 - k^2a^2 \cos^2\theta_o/6$$

To conclude this section we want to point out that $e' = v' = 1$ for $\theta_o = 90^\circ$ and $E_z(0,z) = 0$ for $\theta_o = 0^\circ$, as expected.

The geometry of the problem

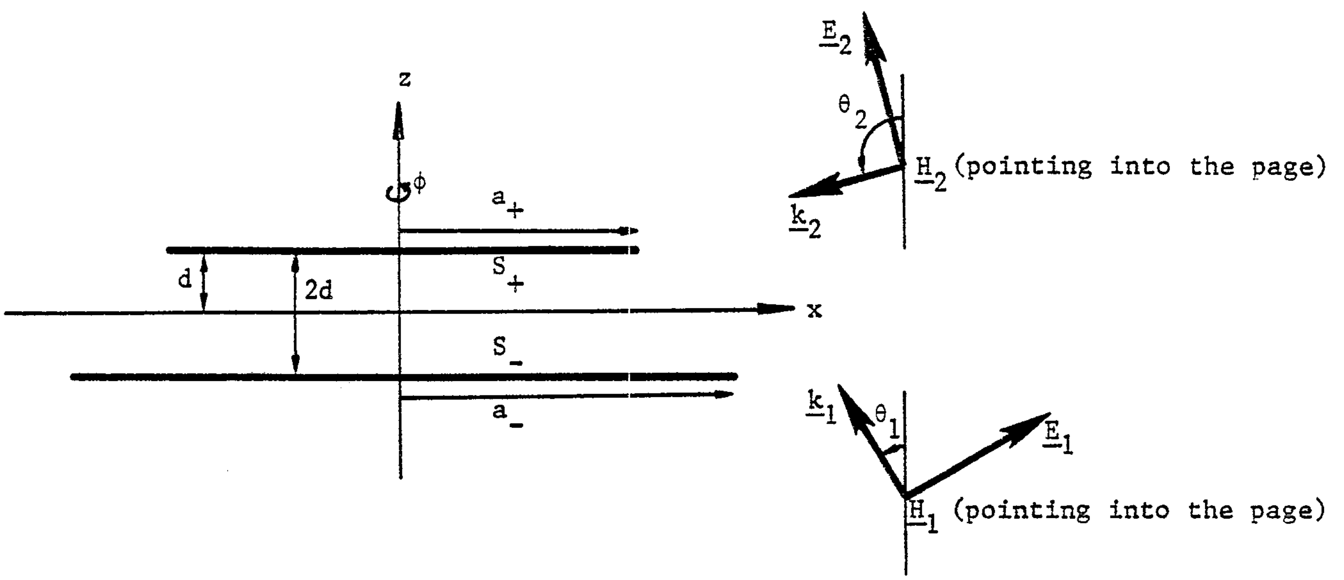


Figure 1a. Incident magnetic field parallel to the disks.

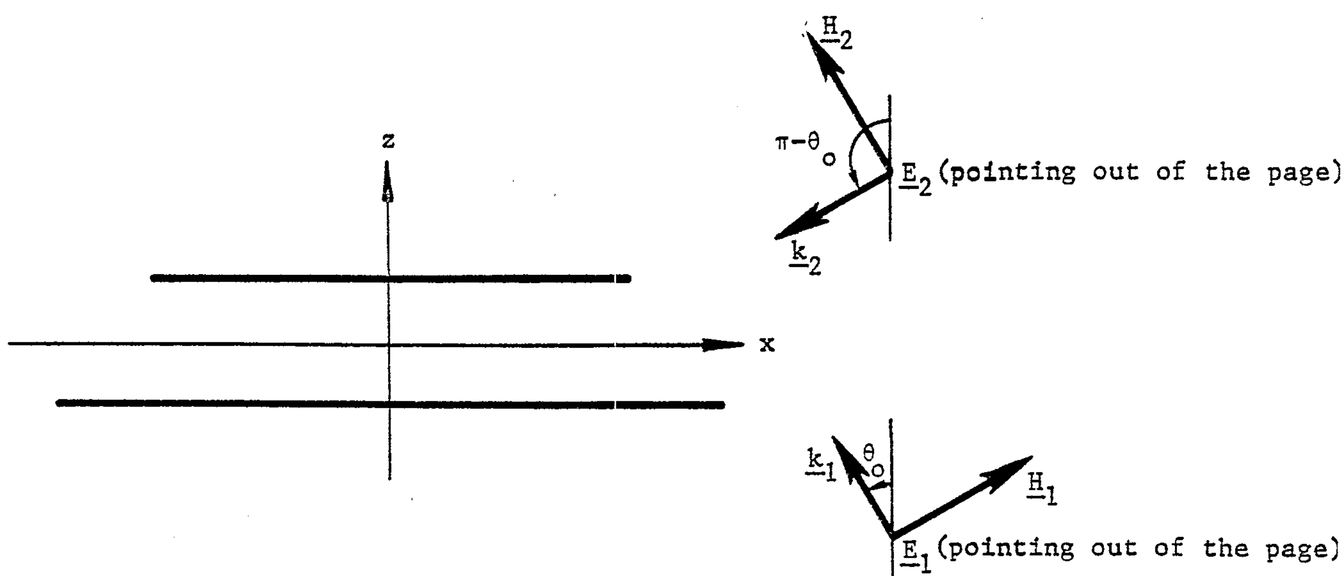


Figure 1b. Incident electric field parallel to the disks.

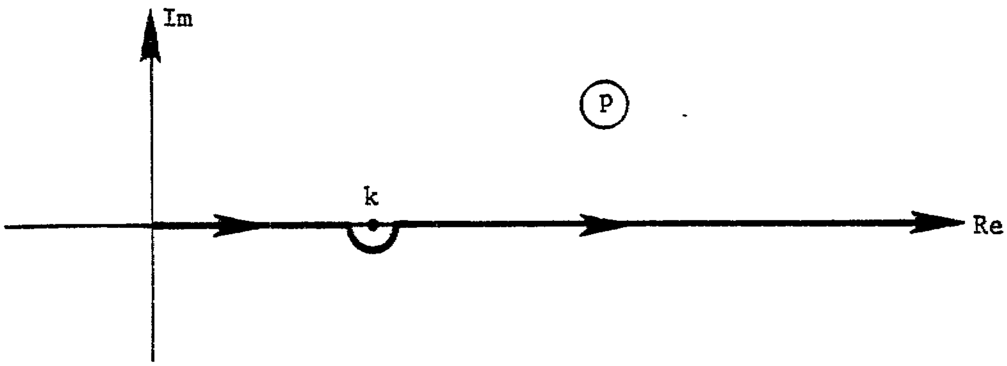


Figure 2. Path of integration in equation (27) of section III.

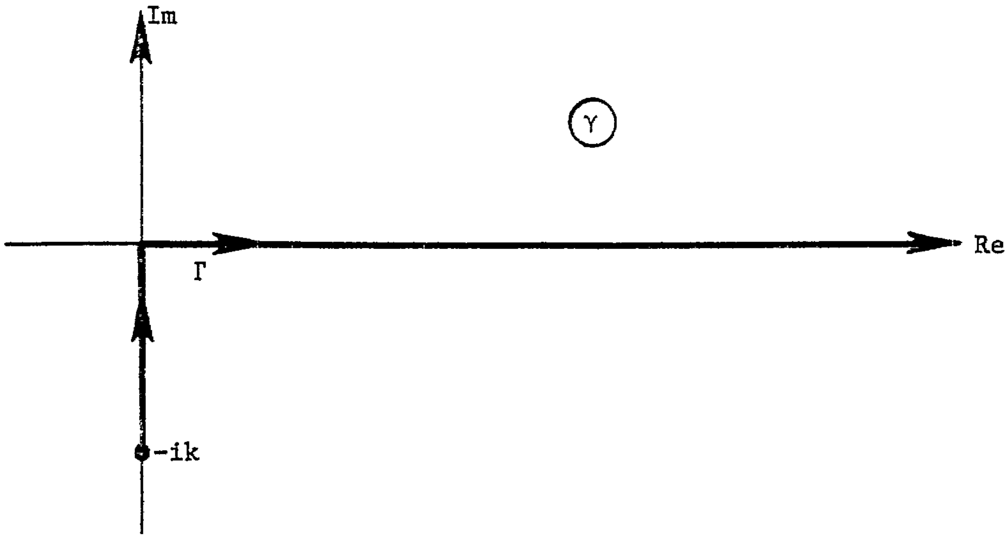
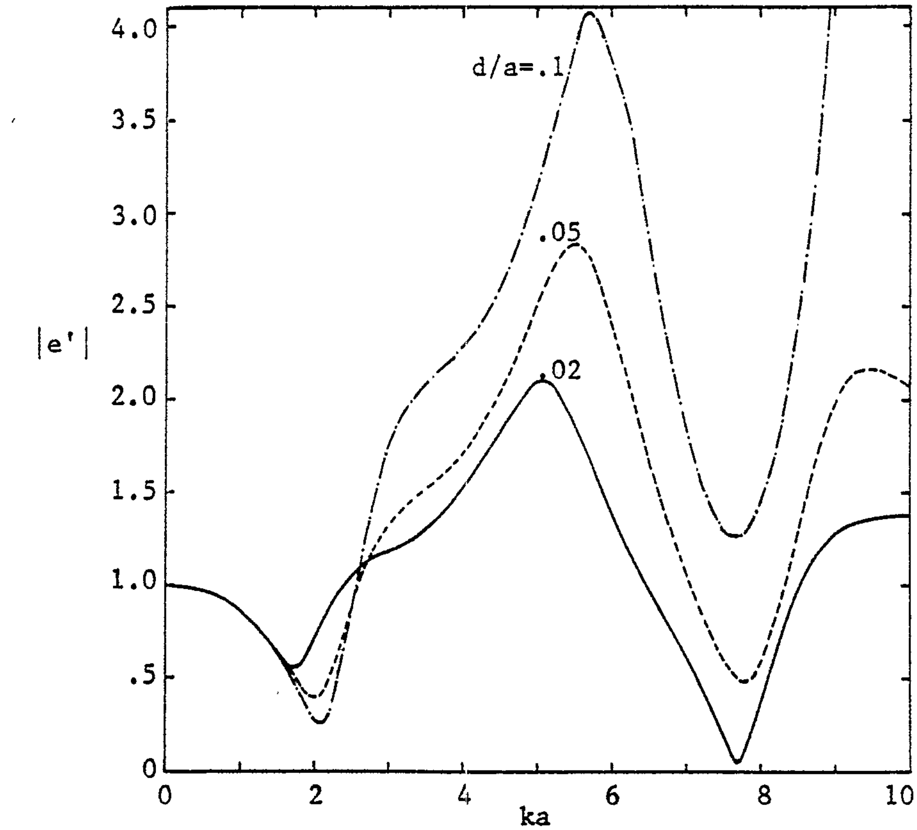


Figure 3. Path of integration in equation (42) of section III.



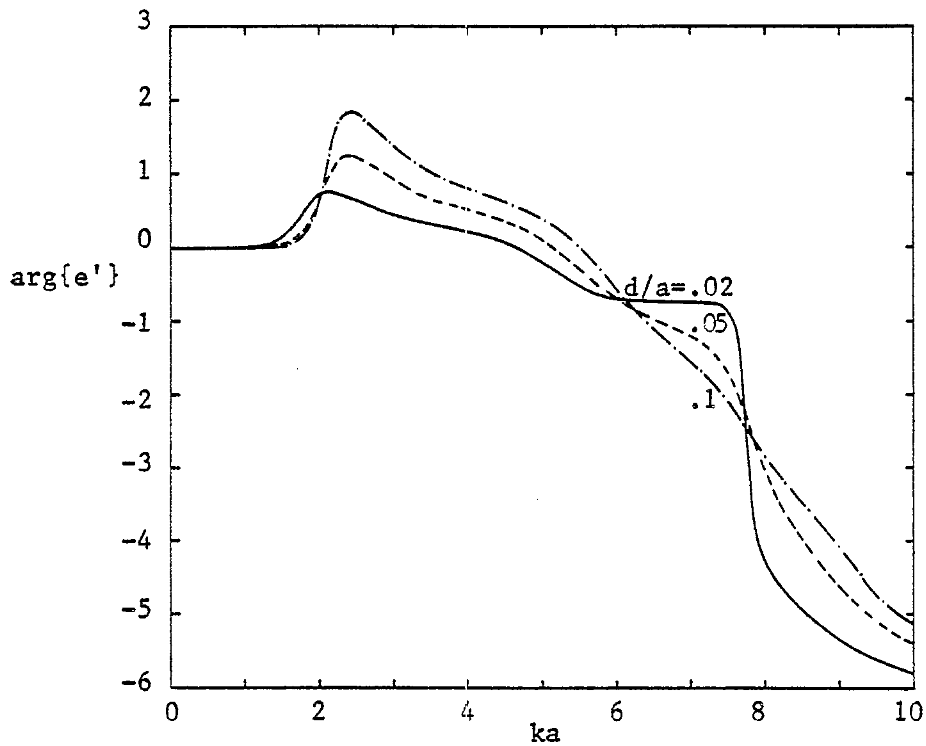


Figure 4. Normalized electric field at the center of two disks for $\theta_o = 18^o$.

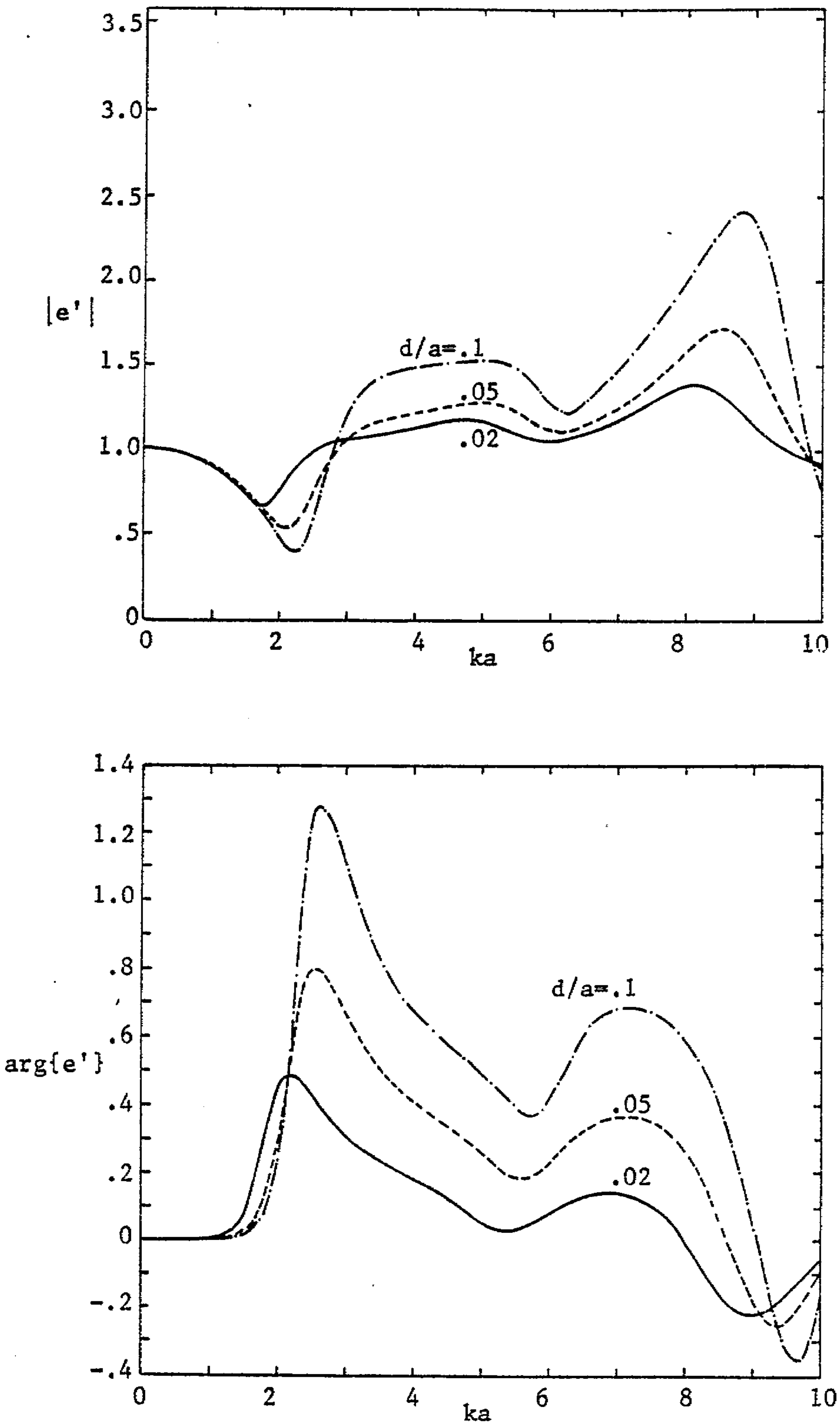


Figure 5. Normalized electric field at the center of two disks for $\theta_o = 36^o$.

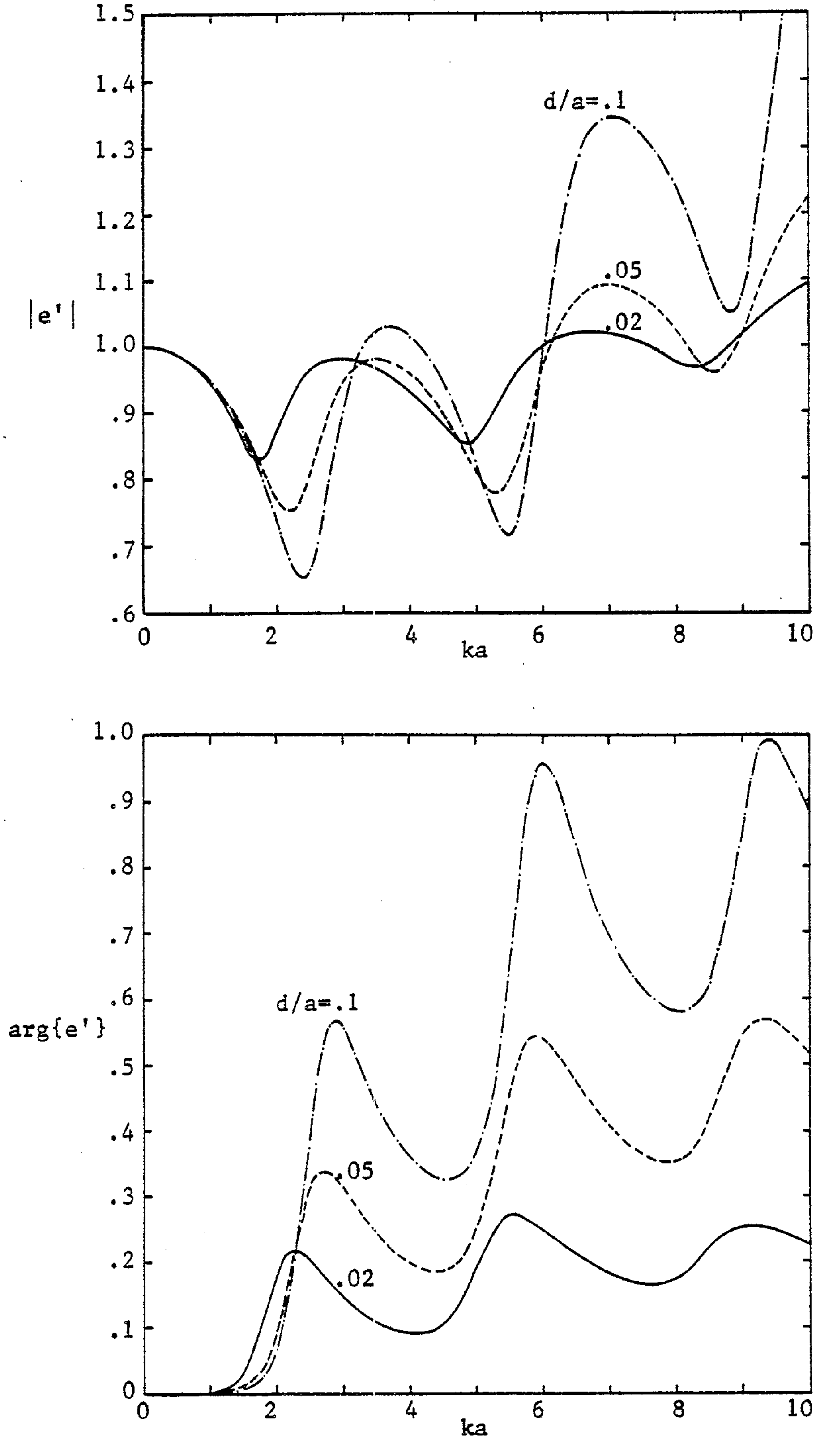


Figure 6. Normalized electric field at the center of two disks for $\theta_o = 54^\circ$.

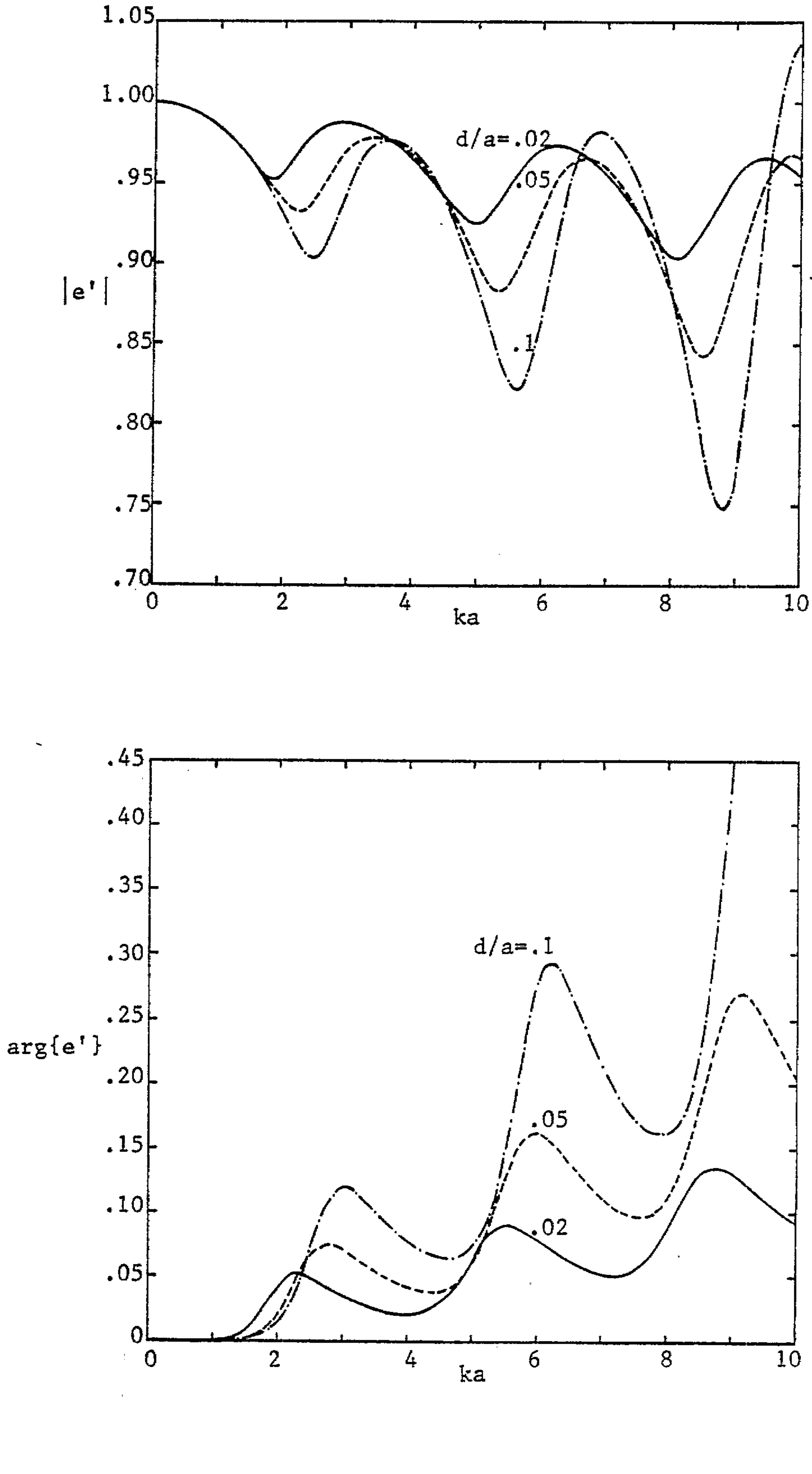
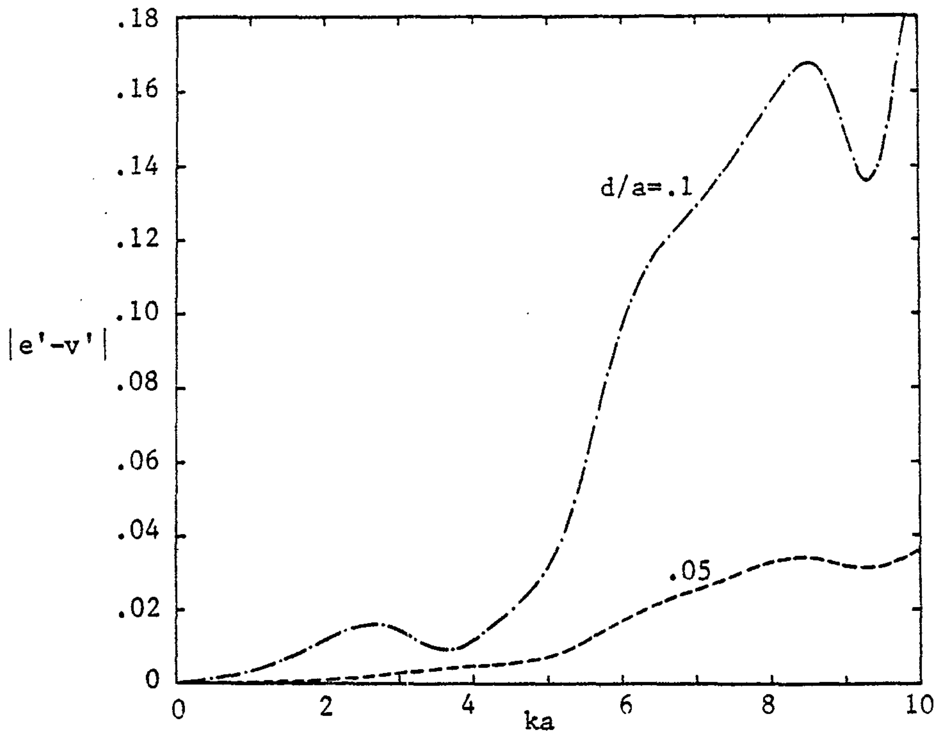


Figure 7. Normalized electric field at the center of two disks for $\theta_o = 72^o$.



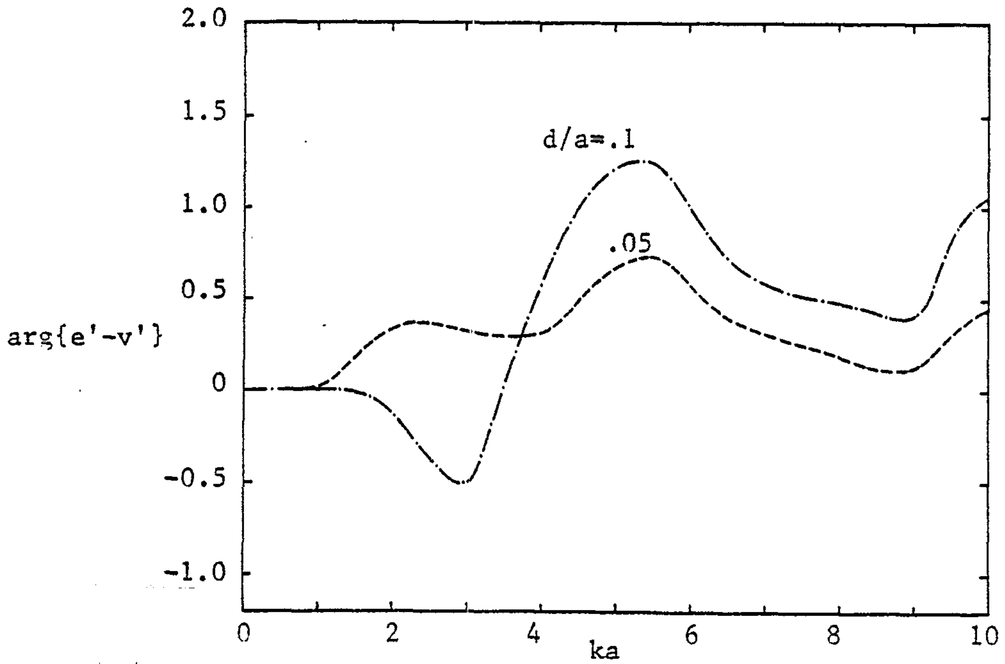


Figure 8. The difference quantity e' - v' for $\theta_o = 18^o$.

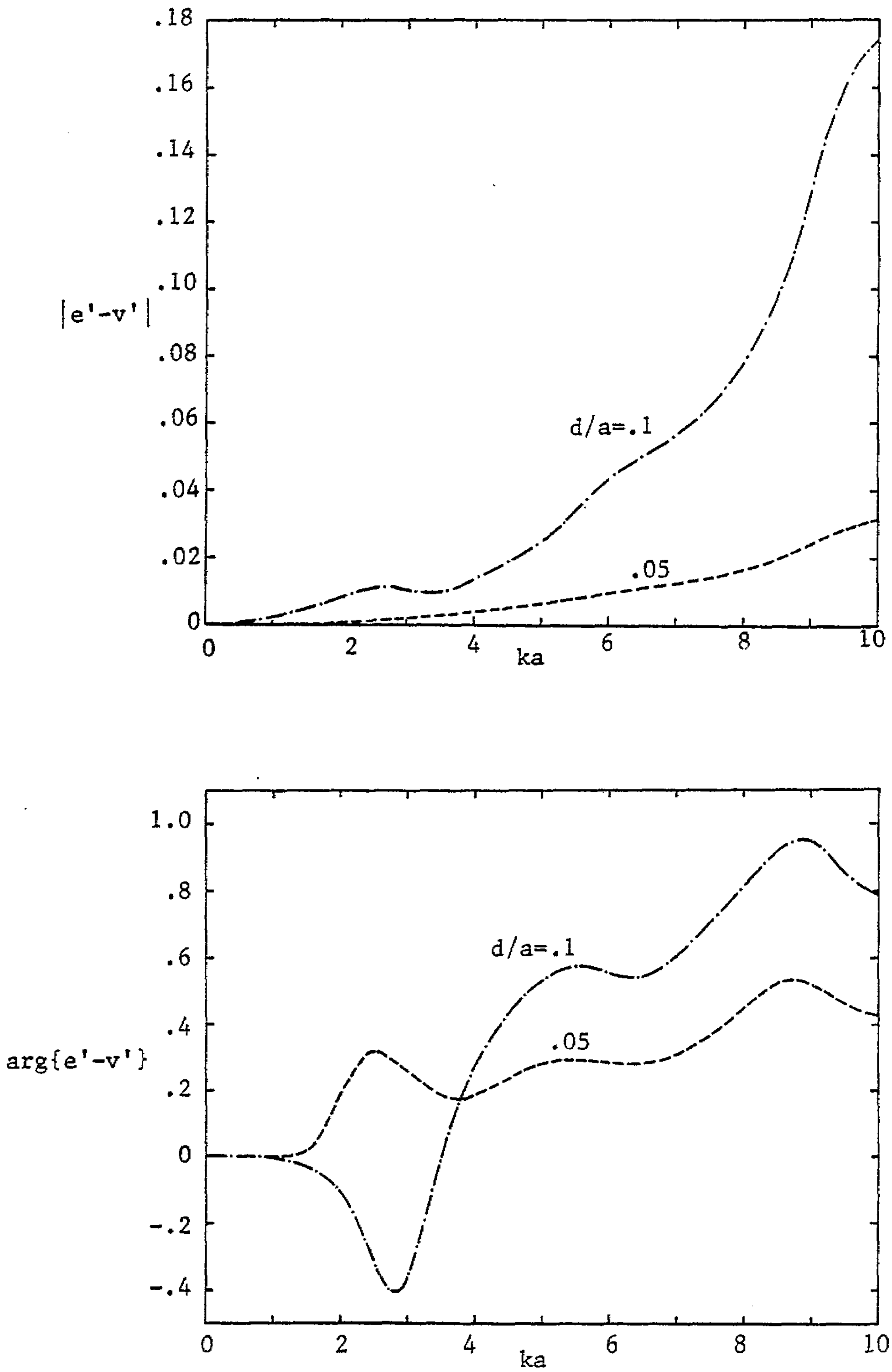


Figure 9. The difference quantity e' - v' for $\theta_o = 36^o$.

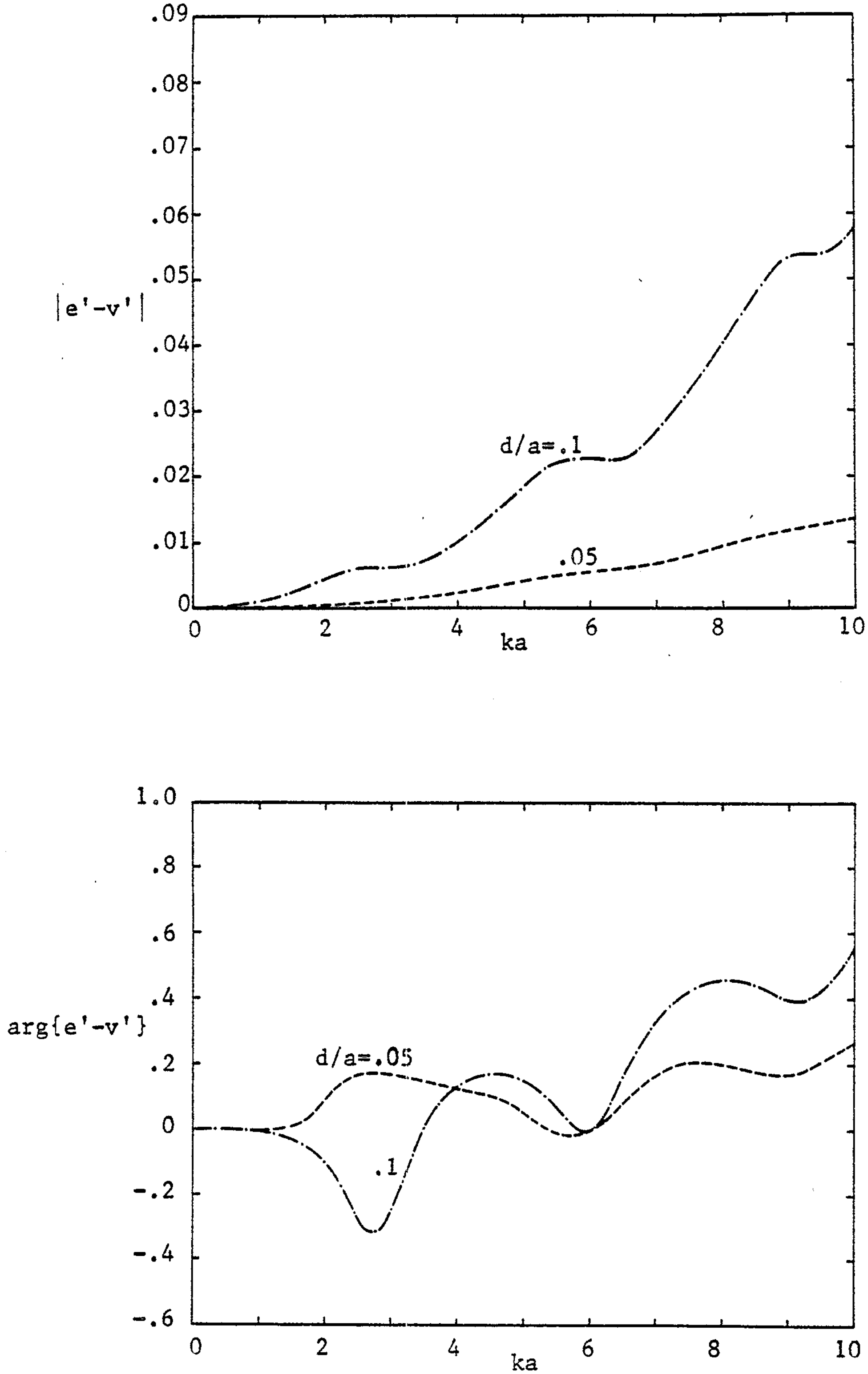


Figure 10. The difference quantity e' - v' for $\theta_o = 54^o$.

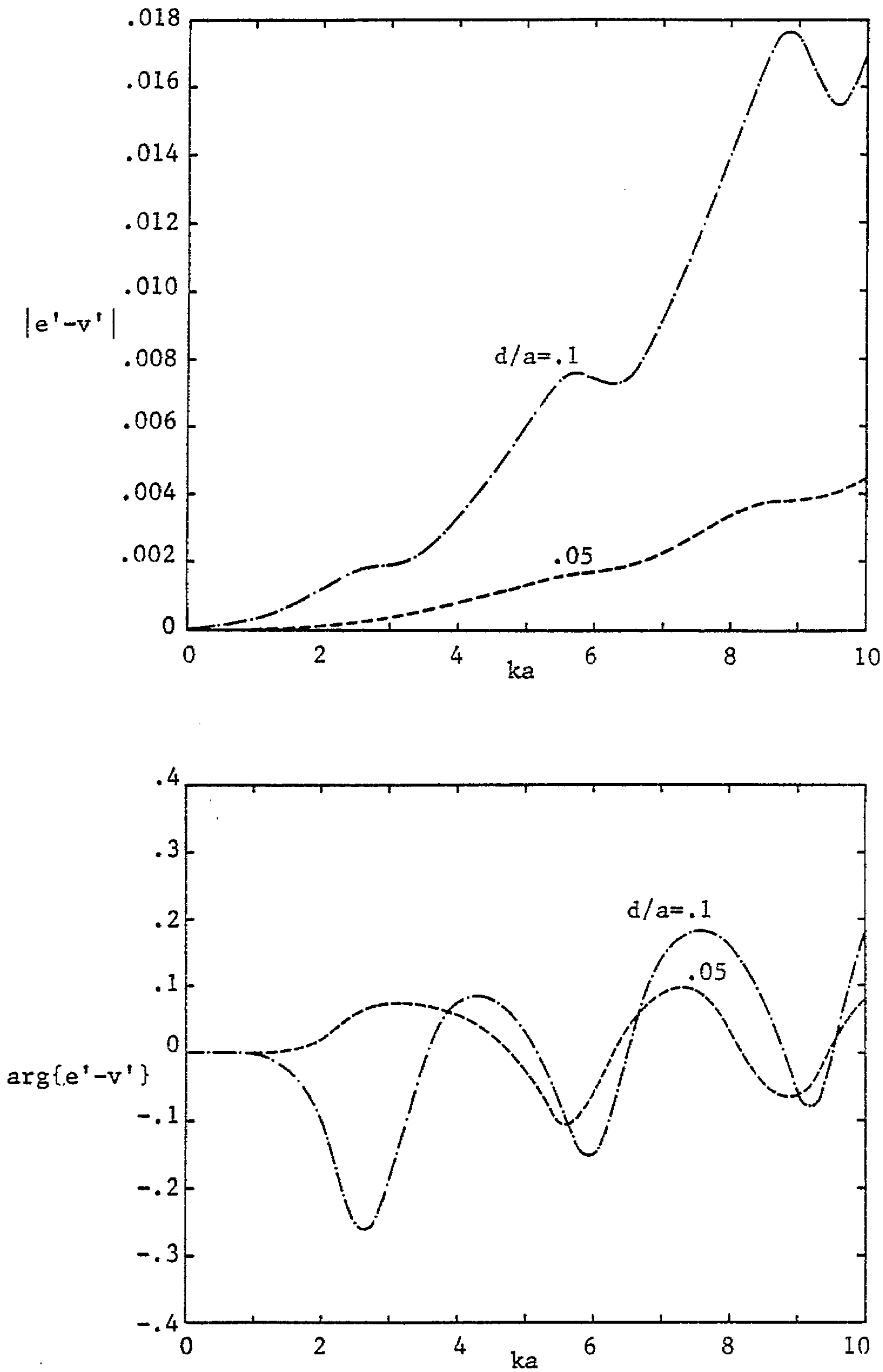


Figure 11. The difference quantity e' - v' for $\theta_o = 72^\circ$.

Appendix A

In this appendix we will transform the singular integral equation of the first kind (28) in section III to a Fredholm integral equation of the second kind.

Consider the integral equation

$$\int_0^{a_+} K_m(\rho,\rho',0)\rho' y_m^+(\rho')d\rho' + \int_0^{a_-} K_m(\rho,\rho',2d)\rho' y_m^-(\rho')d\rho' =$$

$$B_m^+\rho^m + A_m^+ J_m(k\rho \sin\theta_o) \quad , \quad 0 < \rho < a_+ \qquad \text{(A1)}$$

where

$$K_m(\rho,\rho',z) = \frac{1}{2}\int_0^\infty p\gamma^{-1} J_m(\rho p) J_m(\rho' p) e^{-z\gamma} dp$$

and $K_m(\rho,\rho',0)$ has a singularity at $\rho = \rho'$. We now split $K_m(\rho,\rho',0)$ into three parts

$$K_m(\rho,\rho',0) = K_m^{(1)}(\rho,\rho') + K_m^{(2)}(\rho,\rho') + K_m^{(3)}(\rho,\rho') \qquad \text{(A2)}$$

where

$$K_m^{(1)}(\rho,\rho') = \frac{1}{2}\int_0^\infty \frac{\gamma^{2m} J_m(\rho\sqrt{k^2+\gamma^2}) J_m(\rho'\sqrt{k^2+\gamma^2})}{(k^2+\gamma^2)^m} d\gamma$$

$$K_m^{(2)}(\rho,\rho') = \frac{1}{2}\int_0^\infty \left[1 - \frac{\gamma^{2m}}{(k^2+\gamma^2)^m}\right] J_m(\rho\sqrt{k^2+\gamma^2}) J_m(\rho'\sqrt{k^2+\gamma^2}) d\gamma$$

$$K_m^{(3)}(\rho,\rho') = \frac{i}{2}\int_0^k J_m(\rho\sqrt{k^2-\gamma^2}) J_m(\rho'\sqrt{k^2-\gamma^2}) d\gamma$$

Note that $K_m^{(1)}(\rho,\rho')$ is singular at $\rho = \rho'$. From Sonine's second integral[10] and Hankel's inversion formula it follows that

$$K_m^{(1)}(\rho,\rho') = \pi^{-1}(\rho\rho')^{-m} \int_0^{\min(\rho,\rho')} \frac{r^{2m}\cos(k\sqrt{\rho^2-r^2})\cos(k\sqrt{\rho'^2-r^2})}{\sqrt{\rho^2-r^2}\sqrt{\rho'^2-r^2}}\,dr \tag{A3}$$

Thus, equation (A1) can be transformed into

$$\pi^{-1}\rho^{-m}\int_0^{\rho} \frac{r^{2m}\cos(k\sqrt{\rho^2-r^2})}{\sqrt{\rho^2-r^2}} \int_r^{a_+} \frac{y_m^+(\rho')\rho'^{(1-m)}\cos(k\sqrt{\rho'^2-r^2})}{\sqrt{\rho'^2-r^2}}\,d\rho'\,dr$$

$$+ \int_0^{a_+} [K_m^{(2)}(\rho,\rho') + K_m^{(3)}(\rho,\rho')]\rho' y_m^+(\rho')\,d\rho'$$

$$+ \int_0^{a_-} K_m(\rho,\rho',2d)\rho' y_m^-(\rho')\,d\rho' = f_m^+(\rho) \tag{A4}$$

where

$$f_m^+(\rho) = B_m^+\rho^m + A_m^+ J_m(k\rho\sin\theta_o)$$

But this integral equation can be written in the form

$$h_1(\rho)\int_0^{\rho} N(\rho,r)h_2(r)\int_r^{a_o} N(\rho',r)n_o(\rho')g_o(\rho')\,d\rho'\,dr$$

$$+ \sum_{p=0}^{P} \int_0^{a_p} N_p(\rho,\rho')g_p(\rho')\,d\rho' = f(\rho) \tag{A5}$$

where P is a nonnegative integer and

$$\|N_p(\rho,\rho')\| < \infty \quad , \quad 0 < p < P$$

Moreover

$$\int_0^{\rho} N(\rho,\rho')g(\rho')\,d\rho' = f(\rho) \quad , \quad 0 < \rho < a \tag{A6}$$

and

$$\int_{\rho}^{a} N(\rho',\rho)g(\rho')d\rho' = f(\rho) \quad , \quad 0 < \rho < a \tag{A7}$$

have explicit solutions for any differentiable f and any $a > 0$. Define $n_p(\rho)$ and $B_p(u,v)$, $0 \le p \le P$, by

$$N_p(\rho,\rho') = h_1(\rho)n_p(\rho') \int_0^{\rho}\int_0^{\rho'} N(\rho,u)N(\rho',v)B_p(u,v)dudv \tag{A8}$$

and we arrive at the integral equation[7]

$$h_2(u)s_o(u) + \sum_{p=0}^{P} \int_o^{a_p} B_p(u,v)s_p(v)dv = f^*(u) \quad , \quad 0 < u < a_o \tag{A9}$$

where $f^*(u)$ is the solution of

$$h_1(\rho) \int_o^{\rho} N(\rho,u)f^*(u)du = f(\rho)$$

and $s_p(u)$ is defined by

$$s_p(u) = \int_{\rho}^{a_p} N(\rho,u)n_p(\rho)g_p(\rho)d\rho$$

It is now easy to show that the integral equation (A1) can be transformed to the following Fredholm integral equation of the second kind

$$Y_m^+(u) + \int_o^{a_+} L_m(u,v)Y_m^+(v)dv + \int_o^{a_-} M_m(u,v,d)Y_m^-(v)dv = F_m^+(u) \tag{A10}$$

where

$$Y_m^{\pm}(u) = u^m \int_u^{a_{\pm}} \frac{\cos(k\sqrt{\rho^2-u^2})}{\sqrt{\rho^2-u^2}} \rho^{1-m} y_m^{\pm}(\rho)d\rho$$

and

$$F_m^+(u) = 2u^{-m} \frac{d}{du} \int_0^u \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} \rho^{m+1} f_m^+(\rho)d\rho$$

Here we have made use the fact that the solution of[3]

$$\int_0^u \frac{\cos(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} f^*(\rho)d\rho = f(u)$$

is (A11)

$$f^*(\rho) = \frac{2}{\pi} \frac{d}{d\rho} \int_0^\rho \frac{\cosh(k\sqrt{\rho^2-u^2})}{\sqrt{\rho^2-u^2}} uf(u)du$$

From Sonine's second integral it follows that

$$L_m(u,v) = \sqrt{uv} \int_0^\infty \gamma^{1-2m}[(k^2+\gamma^2)^m - \gamma^{2m}]J_{m-\frac{1}{2}}(u\gamma)J_{m-\frac{1}{2}}(v\gamma)d\gamma$$

$$+ i\sqrt{uv} \int_0^k \gamma^{1-2m}(k^2-\gamma^2)^m I_{m-\frac{1}{2}}(u\gamma)I_{m-\frac{1}{2}}(v\gamma)d\gamma \tag{A12}$$

and

$$M_m(u,v,d) = \sqrt{uv} \int_0^\infty p^{2m+1}\gamma^{-2m}J_{m-\frac{1}{2}}(u\gamma)J_{m-\frac{1}{2}}(v\gamma)e^{-2d\gamma}dp \tag{A13}$$

Moreover, the solution of the integral equation[3]

$$\int_u^a \frac{\cos(k\sqrt{\rho^2-u^2})}{\sqrt{\rho^2-u^2}} f^*(\rho)d\rho = f(u) \quad , \quad 0 < u < a$$

is (A14)

$$f^*(\rho) = -\frac{2}{\pi} \frac{d}{d\rho} \int_\rho^a \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} uf(u)du \quad , \quad 0 < \rho < a$$

from which it follows that

$$y_m^\pm(\rho) = -\frac{2}{\pi} \rho^{m-1} \frac{d}{d\rho} \int_\rho^{a_\pm} \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} u^{1-m} Y_m^\pm(u)du \tag{A15}$$

Thus, we have transformed the integral equation of the first kind (A1) into the Fredholm integral equation of the second kind (A10).

Appendix B

In this appendix we will transform the differential integral equation of the first kind (35) in section III to a Fredholm integral equation of the second kind.

Consider the integral equation for $m \geq 0$

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2\right]\left[\int_o^{a_+} K_m(\rho,\rho',0)\rho' x_m^+(\rho')d\rho' + \int_o^{a_-} K_m(\rho,\rho',2d)\rho' x_m^-(\rho')d\rho'\right] + (m+1)B_{m+1}^+\rho^m$$

$$+ k \sin\theta_o A_{m+1}^+ J_m(k\rho \sin\theta_o) = 0 \tag{B1}$$

where

$$K_m(\rho,\rho',z) = \frac{1}{2}\int_o^\infty p\gamma^{-1} J_m(\rho p) J_m(\rho' p) e^{-z\gamma} dp$$

But from the theory of Bessel functions it follows that

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} + k^2\right]K_m(\rho,\rho',z) = -(\rho\rho')^{-m-1}\frac{\partial^2}{\partial\rho\partial\rho'}[\rho\rho' H_m(\rho,\rho',z)] \tag{B2}$$

where

$$H_m(\rho,\rho',z) = \frac{1}{2}(\rho\rho')^m \int_o^\infty \gamma p^{-1} J_{m+1}(\rho p) J_{m+1}(\rho' p) e^{-z\gamma} dp$$

Integration by parts of equation (B1) gives

$$\rho^{-m-1}\frac{d}{d\rho}\rho\left[\int_o^{a_+} H_m(\rho,\rho',0)\rho' q_m^+(\rho')d\rho' + \int_o^{a_-} H_m(\rho,\rho',2d)\rho' q_m^-(\rho')d\rho'\right]$$

$$+ (m+1)B_{m+1}^+\rho^m + k\sin\theta_o A_{m+1}^+ J_m(k\rho\sin\theta_o) = 0 \tag{B3}$$

where

$$q_m^{\pm}(\rho) = \frac{d}{d\rho}[\rho^{-m}x_m^{\pm}(\rho)]$$

and we assume that $x_m^{\pm}(a_{\pm}) = 0$.

Following the technique used in appendix A equation (B3) can be transformed into

$$\frac{d}{d\rho}\left[\int_o^{\rho} \frac{r^{2m+2}\cos(k\sqrt{\rho^2-r^2})}{\pi\sqrt{\rho^2-r^2}} \int_r^{a_+} \frac{q_m^+(\rho')\cos(k\sqrt{\rho'^2-r^2})}{\sqrt{\rho'^2-r^2}} d\rho' dr\right.$$

$$\left. + \rho\int_o^{a_+} \{H_m^{(2)}(\rho,\rho') + H_m^{(3)}(\rho,\rho')\}\rho' q_m^+(\rho')d\rho' + \int_o^{a_+} H_m(\rho,\rho',2d)\rho' q_m^-(\rho')d\rho'\right]$$

$$+ (m+1)B_{m+1}^+\rho^{2m+1} + k\sin\theta_o A_{m+1}^+\rho^{m+1}J_m(k\rho\sin\theta_o) = 0 \qquad \text{(B4)}$$

where

$$H_m^{(2)}(\rho,\rho') = \frac{1}{2}(\rho\rho')^m \int_o^{\infty}\left[\frac{\gamma^2}{\gamma^2+k^2} - \frac{\gamma^{2m+2}}{(\gamma^2+k^2)^{m+1}}\right]J_{m+1}(\rho\sqrt{k^2+\gamma^2})J_{m+1}(\rho'\sqrt{k^2+\gamma^2})d\gamma$$

and

$$H_m^{(3)}(\rho,\rho') = \frac{1}{2}(\rho\rho')^m \int_o^{k} \frac{\gamma^2}{\gamma^2-k^2} J_{m+1}(\rho\sqrt{k^2-\gamma^2})J_{m+1}(\rho'\sqrt{k^2-\gamma^2})d\gamma$$

Integrating equation (B4) from 0 to ρ we get for $m \geq 0$

$$\int_o^{\rho} \frac{r^{2m+2}\cos(k\sqrt{\rho^2-r^2})}{\pi\sqrt{\rho^2-r^2}} \int_r^{a_+} \frac{q_m^+(\rho')\cos(k\sqrt{\rho'^2-r^2})}{\sqrt{\rho'^2-r^2}} d\rho' dr$$

$$+ \rho\int_o^{a_+} [H_m^{(2)}(\rho,\rho') + H_m^{(3)}(\rho,\rho')]\rho' q_m(\rho')d\rho' + \rho\int_o^{a_-} H_m(\rho,\rho',2d)\rho' q_m(\rho')d\rho'$$

$$+ \frac{1}{2}B_{m+1}^+\rho^{2m+2} + A_{m+1}^+\rho^{m+1}J_{m+1}(k\rho\sin\theta_o) = 0 \qquad \text{(B5)}$$

Using the method described in appendix A this integral equation can be transformed into

$$X_m^+(u) + \int_0^{a_+} U_m(u,v)X_m^+(v)dv + \int_0^{a_-} P_m(u,v,d)X_m^-(v)dv = G_m^+(u) \qquad \text{(B6)}$$

where

$$X_m^{\pm}(u) = - u^{m+1} \int_u^{a_+} \frac{\cos(k\sqrt{\rho^2-u^2})}{\sqrt{\rho^2-u^2}} q_m^{\pm}(\rho)d\rho$$

$$G_m^+(u) = u^{-m-1} \frac{d}{du} \int_0^u \frac{\cosh(k\sqrt{u^2-\rho^2})}{\sqrt{u^2-\rho^2}} \rho g_m^+(\rho)d\rho$$

$$g_m^+(\rho) = B_{m+1}^+\rho^{2m+2} + 2A_{m+1}^+\rho^{m+1}J_{m+1}(k\rho \sin\theta_o)$$

From Sonine's second integral it follows that

$$U_m(u,v) = \sqrt{uv} \int_0^\infty \gamma^{1-2m}[(k^2+\gamma^2)^m - \gamma^{2m}]J_{m+\frac{1}{2}}(u\gamma)J_{m+\frac{1}{2}}(v\gamma)d\gamma$$

$$+ i\sqrt{uv} \int_0^k \gamma^{1-2m}(k^2-\gamma^2)^m I_{m+\frac{1}{2}}(u\gamma)I_{m+\frac{1}{2}}(v\gamma)d\gamma \qquad \text{(B7)}$$

$$P_m(u,v,d) = \sqrt{uv} \int_0^\infty p^{2m+1}\gamma^{-2m}J_{m+\frac{1}{2}}(u\gamma)J_{m+\frac{1}{2}}(v\gamma)e^{-2d\gamma}dp \qquad \text{(B8)}$$

Thus, we have succeeded in transforming the differential integral equation (B1) into the Fredholm integral equation of the second kind (B6).

Appendix C

In this appendix we will investigate $U_o(u,v)$ and $P_o(u,v,d)$, defined by equation (42) in section III, for small values of u.

Suppose that f(v) is integrable on (0,a) for any given a > 0, and

$$\|f\| = \int_o^a |f(v)|dv \tag{C1}$$

For $u < \delta/2$ where $0 < \delta < a$ we have

$$I(u) = \int_o^a U_o(u,v)f(v)dv = I_1(u,\delta) + I_2(u,\delta) \tag{C2}$$

where

$$I_1(u,\delta) = \int_o^\delta U_o(u,v)f(v)dv$$

and

$$I_2(u,\delta) = \int_\delta^a U_o(u,v)f(v)dv$$

But

$$I_1(u,\delta) = i\pi^{-1}\int_o^\delta \left[\frac{\sinh k(u+v)}{u+v} - \frac{\sinh k(u-v)}{u-v}\right] f(v)dv = \sum_{m=1}^\infty b_m(k,\delta)u^{2m-1} \tag{C3}$$

where

$$|b_m(k,\delta)| \le 2\pi^{-1}\|f\| \sum_{n=m}^\infty \frac{k^{2n+1}\ 2n-2m+2}{(2n+1)(2m-1)!(2n-2m+2)!}$$

Thus, $I_1(u,\delta) = 0(u)$ as $u \to 0$. Moreover,

$$I_2(u,\delta) = i\pi^{-1}\int_\delta^a \left[\frac{\sinh k(u+v)}{u+v} - \frac{\sinh k(u-v)}{u-v}\right]f(v)dv$$

$$= \sum_{m=0}^\infty s_m(k,\delta)u^{2m+1} - \sum_{m=0}^\infty c_m(k,\delta)u^{2m+1} \tag{C4}$$

where

$$s_m(k,\delta) = \frac{2ik^{2m}}{\pi(2m)!} \int_\delta^a \frac{\sinh kv}{u^2-v^2} f(v)dv$$

and

$$c_m(k,\delta) = \frac{2ik^{2m+1}}{\pi(2m+1)!} \int_\delta^a \frac{v \cosh kv}{u^2-v^2} f(v)dv$$

Now

$$|s_m(k,\delta)| \le \frac{4k^{2m+1} \|f\| \ln 2 \sinh ka}{\pi\delta(2m)!} \tag{C5}$$

$$|c_m(k,\delta)| \le \frac{2k^{2m+1} \|f\| \ln(2a/\delta)\cosh ka}{\pi(2m+1)!} \tag{C6}$$

Thus, $I_2(u,\delta) = O(u)$ as $u \to 0$. Moreover,

$$P_o(u,v,d) = P_o'(u,v,d) + P_o''(u,v,d) \tag{C7}$$

where

$$P_o'(u,v,d) = \frac{e^{2ikd}}{\pi}\left[\frac{2d \cosh k(u-v)}{4d^2+(u-v)^2} - \frac{2d \cosh k(u+v)}{4d^2+(u+v)^2}\right]$$

$$= \frac{8dv \cosh kve^{2ikd}}{\pi[4d^2+(u+v)^2][4d^2+(u-v)^2]} \sum_{m=0}^{\infty} \frac{u^{2m+1}k^{2m}}{2m!}$$

$$- \frac{4d(4d^2+u^2+v^2)\sinh kve^{2ikd}}{\pi[4d^2+(u+v)^2][4d^2+(u-v)^2]} \sum_{m=0}^{\infty} \frac{u^{2m+1}k^{2m+1}}{(2m+1)!} \tag{C8}$$

and

$$P_o''(u,v,d) = \frac{ie^{2ikd}}{\pi}\left[\frac{(u-v)\sinh k(u-v)}{4d^2+(u-v)^2} - \frac{(u+v)\sinh k(u+v)}{4d^2+(u+v)^2}\right]$$

$$= \frac{2v(4d^2-u^2+v^2)\cosh kve^{2ikd}}{i\pi[4d^2+(u+v)^2][4d^2+(u-v)^2]}\sum_{m=0}^{\infty}\frac{u^{2m+1}k^{2m+1}}{(2m+1)!}$$

$$+ \frac{2(4d^2+u^2-v^2)\sinh kve^{2ikd}}{i\pi[4d^2+(u+v)^2][4d^2+(u-v)^2]}\sum_{m=0}^{\infty}\frac{u^{2m+1}k^{2m}}{2m!} \qquad \text{(C9)}$$

From this it follows that

$$I_3(u) = \int_o^a P(u,v,d)f(v)dv = O(u) \qquad \text{as } u \to 0$$

Thus, we have shown that as $u \to 0$

$$I_n(u) = O(u) \quad , \quad n = 1,2,3$$

Appendix D

In this appendix we will derive integral equations for the quantities $V^{\pm}(u)$ and $E^{\pm}(u)$ defined in equations (62) and (63) of section IV.

Defining $S_1^{\pm}(u)$ as

$$S_1^{\pm}(u) = \begin{cases} S^{\pm}(u) , & 0 \le u \le a_{\pm} \\ -S^{\pm}(-u) , & -a_{\pm} \le u \le 0 \end{cases} \tag{D1}$$

we have from equation (42) in section III

$$\begin{aligned} S_1^{+}(u) + \int_{-a_+}^{a_+} N_1(u-v)S_1^{+}(v)dv + \int_{-a_-}^{a_-} N_2(u-v)S_1^{-}(v)dv &= F_o^{+}(u) , \quad 0 \le u \le a_+ \\ S_1^{-}(u) + \int_{-a_-}^{a_-} N_1(u-v)S_1^{-}(v)dv + \int_{-a_+}^{a_+} N_2(u-v)S_1^{+}(v)dv &= F_o^{-}(u) , \quad 0 \le u \le a_+ \end{aligned} \tag{D2}$$

where

$$N_1(x) = (i\pi x)^{-1}\sinh kx$$

$$N_2(x) = \pi^{-1}e^{i2kd}\left[\frac{2d \cosh kx}{4d^2+x^2} + \frac{ix \sinh kx}{4d^2+x^2}\right]$$

The left hand side of (D2) is an odd function of u. Thus, by introducing

$$F^{\pm}(u) = \begin{cases} F_o^{\pm}(u) , & 0 \le u \le a_{\pm} \\ -F_o^{\pm}(-u) , & -a_{\pm} \le u \le 0 \end{cases} \tag{D3}$$

we have

$$\begin{aligned} S_1^{+}(u) + \int_{-a_+}^{a_+} N_1(u-v)S_1^{+}(v)dv + \int_{-a_-}^{a_-} N_2(u-v)S_1^{-}(v)dv &= F^{+}(u) , \quad -a_+ \le u \le a_+ \\ S_1^{-}(u) + \int_{-a_-}^{a_-} N_1(u-v)S_1^{-}(v)dv + \int_{-a_+}^{a_+} N_2(u-v)S_1^{+}(v)dv &= F^{-}(u) , \quad -a_- \le u \le a_- \end{aligned} \tag{D4}$$

Suppose $S_1^{\pm}(u)$ is twice differentiable. Then, integration by parts gives

$$V_1^+(u) + \int_{-a_+}^{a_+} N_1(u - v)V_1^+(v)dv + \int_{-a_-}^{a_-} N_2(u - v)V_1^-(v)dv = R_1^+(u) \quad , \ -a_+ \le u \le a_+$$

$$V_1^-(u) + \int_{-a_-}^{a_-} N_1(u - v)V_1^-(v)dv + \int_{-a_+}^{a_+} N_2(u - v)V_1^+(v)dv = R_1^-(u) \quad , \ -a_- \le u \le a_- \qquad \text{(D5)}$$

where

$$V_1^{\pm}(u) = \frac{dS_1^{\pm}}{du}(u)$$

$$R_1^{\pm}(u) = \frac{dF^{\pm}}{du}(u)$$

Moreover,

$$E_1^+(u) + \int_{-a_+}^{a_+} N_1(u - v)E_1^+(v)dv + \int_{-a_-}^{a_-} N_2(u - v)E_1^-(v)dv = Q_1^+(u) \quad , \ -a_+ \le u \le a_+$$

$$E_1^-(u) + \int_{-a_-}^{a_-} N_1(u - v)E_1^-(v)dv + \int_{-a_+}^{a_+} N_2(u - v)E_1^+(v)dv = Q_1^-(u) \quad , \ -a_- \le u \le a_- \qquad \text{(D6)}$$

where

$$E_1^{\pm}(u) = \frac{dV_1^{\pm}}{du}(u)$$

$$Q_1^{\pm}(u) = \frac{d^2F^{\pm}}{du^2}(u) + [N_1(u + a_+) - N_1(u - a_+)]V_1^+(a_+)$$

$$+ [N_2(u + a_-) - N_2(u - a_-)]V_1^-(a_-)$$

The solutions of (D5) and (D6) fulfill the symmetri relations:

$$V_1^{\pm}(-u) = V_1^{\pm}(u) \tag{D7}$$

$$E_1^{\pm}(-u) = - E_1^{\pm}(u) \tag{D8}$$

Introducing

$$V^{\pm}(u) = V_1^{\pm}(u) \tag{D9}$$

$$E^{\pm}(u) = E_1^{\pm}(u) \tag{D10}$$

$$R^{\pm}(u) = R_1^{\pm}(u) \tag{D11}$$

$$Q^{\pm}(u) = Q_1^{\pm}(u) \tag{D12}$$

for $0 \leq u \leq a_{\pm}$ we have

$$\begin{aligned} V^+(u) + \int_o^{a_+} [N_1(u - v) + N_1(u + v)]V^+(v)dv + \int_o^{a_-} [N_2(u - v) + N_2(u + v)]V^-(v)dv \\ = R^+(u) \quad , \quad 0 \leq u \leq a_+ \\ V^-(u) + \int_o^{a_-} [N_1(u - v) + N_1(u + v)]V^-(v)dv + \int_o^{a_+} [N_2(u - v) + N_2(u + v)]V^+(v)dv \\ = R^-(u) \quad , \quad 0 \leq u \leq a_- \end{aligned} \tag{D13}$$

$$E^+(u) + \int_0^{a_+} [N_1(u - v) - N_1(u + v)]E^+(v)dv + \int_0^{a_-} [N_2(u - v) - N_2(u + v)]E^-(v)dv$$

$$= Q^+(u) \quad , \quad 0 \le u \le a_+$$

$$E^-(u) + \int_0^{a_-} [N_1(u - v) - N_1(u + v)]E^-(v)dv + \int_0^{a_+} [N_2(u - v) - N_2(u + v)]E^+(v)dv \qquad \text{(D14)}$$

$$= Q^-(u) \quad , \quad 0 \le u \le a_-$$

We notice here that the only difference between the integral equations for $S^{\pm}(u)$ and $E^{\pm}(u)$ is the right hand side.

Appendix E

In this appendix we will derive an integral equation for the potential around two conducting parallel coaxial disks immersed in a homogeneous electrostatic field. The geometry of the problem is described in figure 12. The incident field,

$$\underline{E}^{inc} = E_o(\hat{x} \cos \theta_o + \hat{z} \sin \theta_o) \qquad \text{(E1)}$$

can be derived from a potential, Φ^{inc}, as

$$\underline{E}^{inc} = - E_o \text{ grad } \Phi^{inc} \qquad \text{(E2)}$$

where

$$\Phi^{inc} = -x \cos \theta_o - z \sin \theta_o + \Phi_o$$

and Φ_o is a constant.

The scattered field takes the form (c.f. equations (4) and (6) in section II)

$$\underline{E}^{sc} = E_o \text{ grad}\left[\frac{\partial \psi}{\partial z} (\rho,z)\right] + E_o \text{ curl } \underline{\pi} \qquad \text{(E3)}$$

where

$$\pi_\rho = \xi_o(\rho,z)\sin \phi$$

$$\pi_\phi = \xi_o(\rho,z)\cos \phi$$

$$\pi_z = \eta_1(\rho,z)\sin \phi$$

This electrostatic problem can be considered as the limit problem when k tends to zero of the more general problem treated in section III.

Thus, we have from equations (27), (29), and (30) in section III

$$\eta_1(\rho,z) = \frac{1}{2}\int_0^{a_+}\int_0^{\infty} J_1(\rho p)J_1(\rho' p)e^{-|z-d|p}\rho' y_1^+(\rho')\,dp\,d\rho'$$

$$+ \frac{1}{2}\int_0^{a_+}\int_0^{\infty} J_1(\rho p)J_1(\rho' p)e^{-|z+d|p}\rho' y_1^-(\rho')\,dp\,d\rho' \qquad \text{(E4)}$$

where

$$y_1^{\pm}(\rho) = -\frac{2}{\pi}\frac{d}{d\rho}\int_\rho^{a_\pm} \frac{Y_1^{\pm}(u)}{\sqrt{u^2-\rho^2}}\,du$$

and $Y_1^+(u)$ and $Y_1^-(u)$ satisfy the system of integral equations

$$Y_1^+(u) + \frac{1}{\pi}\int_0^{a_-}\left[\frac{2d}{4d^2+(u-v)^2} - \frac{2d}{4d^2+(u+v)^2}\right]Y_1^-(v)\,dv = 4(B_1^+ + \cos\theta_o)u$$

$$Y_1^-(u) + \frac{1}{\pi}\int_0^{a_+}\left[\frac{2d}{4d^2+(u-v)^2} - \frac{2d}{4d^2+(u+v)^2}\right]Y_1^+(v)\,dv = 4(B_1^- + \cos\theta_o)u \qquad \text{(E5)}$$

where B_1^+ and B_1^- are constants of integration to be determined from the edge conditions. Integration by parts of equation (E4) gives

$$\eta_1(\rho,z) = \frac{1}{2}\int_0^{a_+}\int_0^{\infty} pJ_1(\rho p)J_o(\rho' p)e^{-|z-d|p}\rho' f_1^+(\rho')\,dp\,d\rho'$$

$$+ \frac{1}{2}\int_0^{a_-}\int_0^{\infty} pJ_1(\rho p)J_o(\rho' p)e^{-|z+d|p}\rho' f_1^-(\rho')\,dp\,d\rho'$$

$$= \pi^{-1}\int_0^{a_+} Y_1^+(u)\,du\int_0^{\infty} J_1(\rho p)\sin up\, e^{-|z-d|p}\,dp$$

$$+ \pi^{-1}\int_0^{a_-} Y_1^-(u)\,du\int_0^{\infty} J_1(\rho p)\sin up\, e^{-|z+d|p}\,dp \qquad \text{(E6)}$$

where

$$f_1^{\pm}(\rho) = \frac{2}{\pi} \int_{\rho}^{a_{\pm}} \frac{Y_1^{\pm}(u)}{\sqrt{u^2-\rho^2}} du$$

Moreover, from equations (34), (36) and (37) in section III it follows that

$$\xi_o(\rho,z) = \frac{1}{2}\frac{\partial}{\partial z} \int_o^{a_+}\int_o^{\infty} J_o(\rho p)J_o(\rho' p)e^{-|z-d|p}\rho' x_o^+(\rho')dpd\rho'$$

$$+ \frac{1}{2}\frac{\partial}{\partial z} \int_o^{a_-}\int_o^{\infty} J_o(\rho p)J_o(\rho' p)e^{-|z+d|p}\rho' x_o^-(\rho')dpd\rho' \qquad \text{(E7)}$$

where

$$x_o^{\pm}(\rho) = \frac{2}{\pi} \int_{\rho}^{a_{\pm}} \frac{X_o^{\pm}(u)}{\sqrt{u^2-\rho^2}} du$$

and $X_o^+(u)$ and $X_o^-(u)$ satisfy the system of integral equations

$$X_o^+(u) + \frac{1}{\pi}\int_o^{a_-} \left[\frac{2d}{4d^2+(u-v)^2} - \frac{2d}{4d^2+(u+v)^2}\right]X_o^-(v)dv = 2(B_1^+ + 2\cos\theta_o)u$$

$$X_o^-(u) + \frac{1}{\pi}\int_o^{a_+} \left[\frac{2d}{4d^2+(u-v)^2} - \frac{2d}{4d^2+(u+v)^2}\right]X_o^+(v)dv = 2(B_1^- + 2\cos\theta_o)u \qquad \text{(E8)}$$

The edge condition is, c.f. equation (51) in section III,

$$X_o^{\pm}(a_{\pm}) = Y_1^{\pm}(a_{\pm}) \qquad \text{(E9)}$$

from which it follows that

$$B_1^{\pm} = 0 \qquad \text{(E10)}$$

and thus,

$$X_o^{\pm}(u) = Y_1^{\pm}(u) \tag{E11}$$

The edge condition, although derived in terms of the current on the disk in the dynamic case, implies that the charge density, in the static case, is $O[(a_{\pm} - \rho)^{-\frac{1}{2}}]$ as $\rho \to a_{\pm}$. We have

$$\xi_o(\rho,z) = \pi^{-1} \frac{\partial}{\partial z} \int_0^{a_+} Y_1^+(u)du \int_0^{\infty} p^{-1} J_o(\rho p) \sin up e^{-|z-d|p} dp$$

$$+ \pi^{-1} \frac{\partial}{\partial z} \int_0^{a_-} Y_1^-(u)du \int_0^{\infty} p^{-1} J_o(\rho p) \sin up e^{-|z+d|p} dp \tag{E12}$$

From equations (40), (41), (42) and (44) it is easy to show that $\psi(\rho,z) = 0$. Introducing

$$\Phi(\rho,\phi,z) = \rho\left(\frac{\partial \xi_o}{\partial z} - \frac{\partial \eta_1}{\partial \rho}\right)\cos\phi = \pi^{-1} \cos\phi \int_0^{a_+} Y_1^+(u)du \int_0^{\infty} J_1(\rho p) \sin up e^{-|z-d|p} dp$$

$$+ \pi^{-1} \cos\phi \int_0^{a_-} Y_1^-(u)du \int_0^{\infty} J_1(\rho p) \sin up e^{-|z+d|p} dp \tag{E13}$$

one can see that

$$\text{curl } \underline{\pi} = - \text{grad } \Phi \tag{E14}$$

Here $E_o\Phi$ can be interpreted as the electrostatic potential for the scattered field. We have

$$\Phi(\rho,\phi,z) = (2\pi i\rho)^{-1} \cos\phi \int_0^{a_+} \left[\frac{|z-d|+iu}{\sqrt{\rho^2+(|z-d|+iu)^2}} - \frac{|z-d|-iu}{\sqrt{\rho^2+(|z-d|-iu)^2}}\right] Y_1^+(u)du$$

$$+ (2\pi i\rho)^{-1} \cos\phi \int_0^{a_-} \left[\frac{|z+d|+iu}{\sqrt{\rho^2+(|z+d|+iu)^2}} - \frac{|z-d|-iu}{\sqrt{\rho^2+(|z+d|-iu)^2}}\right] Y_1^-(u)du \tag{E15}$$

Now define

$$Y_1^{\pm}(-u) = - Y_1^{\pm}(u) \tag{E16}$$

Then,

$$\Phi(\rho,\phi,z) = (2\pi i\rho)^{-1} \cos\phi \left[\int_{-a_+}^{a_+} \frac{|z-d|+iu}{\sqrt{\rho^2+(|z-d|+iu)^2}} Y_1^+(u)du \right.$$

$$\left. + \int_{-a_-}^{a_-} \frac{|z+d|+iu}{\sqrt{\rho^2+(|z+d|+iu)^2}} Y_1^-(u)du \right] \tag{E17}$$

and $Y_1^+(u)$ and $Y_1^-(u)$ satisfy the system of integral equations

$$Y_1^+(u) + \frac{1}{\pi} \int_{-a_-}^{a_-} \frac{2d}{4d^2+(u-v)^2} Y_1^-(v)dv = 4u \cos\theta_o \quad , \quad -a_+ \le u \le a_+$$

$$Y_1^-(u) + \frac{1}{\pi} \int_{-a_+}^{a_+} \frac{2d}{4d^2+(u-v)^2} Y_1^+(v)dv = 4u \cos\theta_o \quad , \quad -a_- \le u \le a_- \tag{E18}$$

We have

1. $\Delta\Phi(\rho,\phi,z) = 0$ except on the disks $z = \pm d$, $\rho \le a_\pm$.

2. $\Phi(\rho,\phi,z) = O(r^{-1}\rho^{-1})$ as $r \to \infty$.

3. $\Phi(\rho,\phi,z)$ is continuous when approaching along the normal from either side of S_+ or S_-.

4. $\Phi^{tot} = \Phi^{inc} + \Phi$ is constant on S_+ and S_-.

To prove the continuity on S_+ suppose that $|z-d| < d$. Then,

$$\Phi^-(\rho,\phi,z) = (2\pi i\rho)^{-1} \cos\phi \int_{-a_-}^{a_-} \frac{|z+d|+iu}{\sqrt{\rho^2+(|z+d|+iu)^2}} Y_1^-(u)du \tag{E19}$$

is continuous for $z = d$. For $0 < \rho \le a_+$ we have

$$|\rho^2 + (|z-d| + iu)^2| > |z-d|^2 + \rho^2 - u^2 \ge \rho^2 - u^2 \quad \text{if } u < \rho$$

$$|\rho^2 + (|z-d| + iu)^2| > |z-d|^2 + u^2 - \rho^2 \ge u^2 - \rho^2 \quad \text{if } u > \rho$$

Thus,

$$\left| \frac{|z-d|+iu}{\sqrt{\rho^2+(|z-d|+iu)^2}} Y_1^+(u) \right| \le \frac{|Y_1^+(u)| \cdot |z-d+iu|}{\sqrt{|\rho^2-u^2|}}$$

from which it follows that

$$\Phi^+(\rho,\phi,z) = (2\pi i\rho)^{-1} \cos\phi \int_{-a_+}^{a_+} \frac{|z+d|+iu}{\sqrt{\rho^2+(|z-d|+iu)^2}} Y_1^+(u)du$$

$$\to (2\pi\rho)^{-1} \cos\phi \int_{-a_+}^{a_+} \frac{u}{\sqrt{\rho^2-u^2}} Y_1^+(u)du = \Phi^+(\rho,\phi,d) \tag{E20}$$

as $z \to d$. Here

$$\sqrt{\rho^2 - u^2} = \begin{cases} i\sqrt{u^2 - \rho^2}, & u > \rho \\ -i\sqrt{u^2 - \rho^2}, & u < -\rho \end{cases}$$

This proves the continuity of $\Phi^+(\rho,\phi,z)$ on S_+ except at $\rho = 0$. Moreover,

$$\Phi^+(\rho,\phi,d) = (2\pi)^{-1} \cos\phi \int_0^\pi \cos\theta \, Y_1^+(\rho\cos\theta)d\theta \to 0 \tag{E21}$$

as $\rho \to 0$ (c.f. the proof given in appendix C). It is easy to show that

$$\lim_{\rho\to 0} \Phi^+(\rho,\phi,z) = 0 \quad \text{for } z \ne d$$

Thus, we have

$$\lim_{\rho\to 0}\lim_{z\to d}\Phi^+(\rho,\phi,z) = \lim_{z\to d}\lim_{\rho\to 0}\Phi^+(\rho,\phi,z) = 0 \tag{E22}$$

which completes the proof of statement 3.

In order to prove statement 4 we notice that

$$\Phi^-(\rho,\phi,d) = (2\pi i\rho)^{-1}\cos\phi \int_{-a_-}^{a_-} \frac{2d+iu}{\sqrt{\rho^2+(2d+iu)^2}} Y_1^-(u)du$$

$$= (2\pi^2 i\rho)^{-1}\cos\phi \int_{-a_-}^{a_-} Y_1^-(u)du \int_0^{\pi} \frac{(2d+iu)d\theta}{2d+iu+i\rho\cos\theta}$$

$$= (2\pi^2 i\rho)^{-1}\cos\phi \int_{-a_-}^{a_-} Y_1^-(u)du \int_0^{\pi} \frac{(2d+iu)d\theta}{2d+iu-i\rho\cos\theta}$$

$$= (2\pi\rho)^{-1}\cos\phi \int_0^{\pi} d\theta\,\pi^{-1} \int_{-a_-}^{a_-} \frac{2d}{4d^2+(\rho\cos\theta-u)^2} Y_1^-(u)du \tag{E23}$$

Equations (E18), (E20) and (E23) then give

$$\Phi(\rho,\phi,d) = \Phi^+(\rho,\phi,d) + \Phi^-(\rho,\phi,d) = \rho\cos\phi\cos\theta_o = x\cos\theta_o \tag{E24}$$

for $0 \le \rho \le a_+$. Thus,

$$\Phi^{tot}(\rho,\phi,d) = \Phi^{inc}(\rho,\phi,d) + \Phi(\rho,\phi,d) = \Phi_o - d\sin\theta_o \tag{E25}$$

and from this one can easily see the validity of statement 4.

We wish to point out in passing the similarity between the equations derived here and those derived by Love in[5].

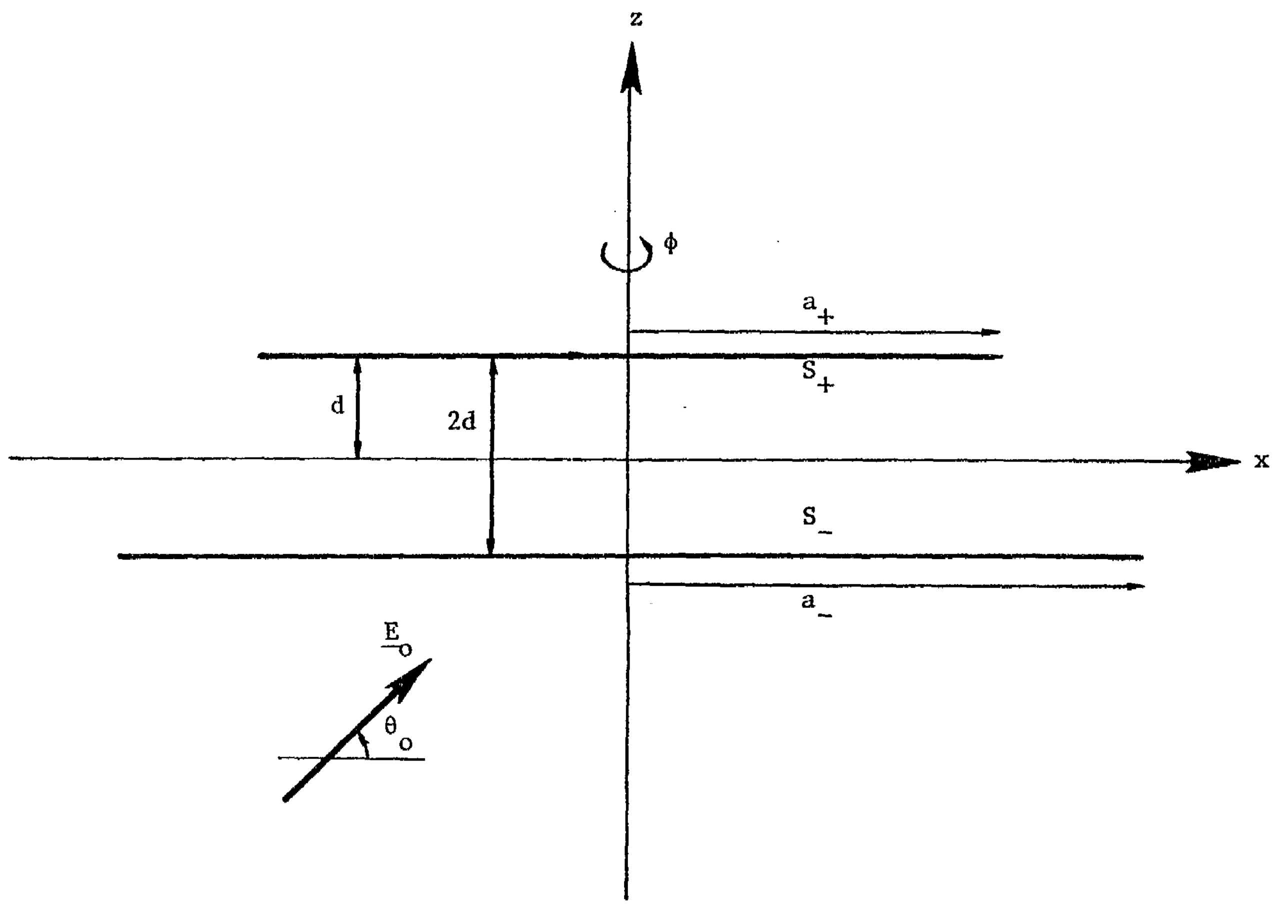


Figure 12. Two parallel coaxial disks immersed in a homogeneous electrostatic field.

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