Sensor and Simulation Notes

Note 124

February 1971

Magnetic Field on a Cylinder in a Loop

by

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Abstract

The magnetostatic field at the surface of an infinitely long, perfectly conducting, circular cylinder is computed. The source of the field is a circular, current-carrying loop coaxial with the cylinder.

Contents

Acknowledgment

We thank Mr. R. W. Sassman for the computer programing and Mrs. G. Peralta for the typing.

I. Introduction

An EMP simulator in the shape of a half toroid has been described in Sensor and Simulation Note 94 [Ref. 1]. The low-frequency magnetic field distribution for such a simulator was calculated in Sensor and Simulation Note 112 [Ref. 2]. The effect of a hemispherical test body on the low-frequency magnetic field was determined in Sensor and Simulation Note 120 [Ref. 3]. In this brief note we will present data on the low-frequency magnetic field when the test body may be idealized as an infinite, perfectly conducting, semicircular cylinder coaxial with the half toroid, the half toroid being in the vertical position.

In making our calculations we will make the same assumptions concerning the effect of the ground conductivity on the magnetic field that were made in Notes 112 and 120. That is to say, firstly, we will assume that the frequency is not exactly zero (which would make it necessary to determine separately the effect of the current distribution in the lower half space on the magnetic field. See, for example, Ref. 4). Secondly, we will assume the frequency is low enough so that, as far as the magnetic field distribution above the ground surface is concerned, the ground may be thought of as perfectly conducting and image theory may be applied. The above two assumptions are compatible in many cases of practical interest, as has been pointed out previously [Ref. 2, p. 2].

II. An Integral Representation

If the half toroid simulator is in the vertical position and image theory is invoked it is clear that the magnetic field at the surface of an infinite semicircular cylinder will be the same as that at the surface of an infinite circular cylinder coaxial with a current-carrying loop. Such a structure is shown schematically in figure 1. Figure 1 is presented mainly to indicate the coordinate systems we will be using.

One representation of the magnetic vector potential due to the current in the loop shown in Figure 1 is [Ref. 5]:

$$A_\phi^1(\rho,z) = \frac{\mu_o aI}{\pi}\int_0^\infty I_1(k\rho)K_1(ka)\cos(kz)dk \qquad \rho < a \quad . \tag{1}$$

We must add to this the magnetic vector potential due to the currents flowing on the infinite cylinder. This latter potential may be written in the form

$$A_\phi^2(\rho,z) = \frac{\mu_o aI}{\pi}\int_0^\infty f(k)K_1(k\rho)\cos(kz)dk \qquad \rho > b \quad . \tag{2}$$

But

$$B_\rho \propto \frac{\partial A_\phi}{\partial z}$$

and, since the cylinder is assumed to be perfectly conducting, B_ρ is zero at its surface. This condition may be satisfied by setting

$$f(k) = -\frac{I_1(kb)K_1(ka)}{K_1(kb)} \quad . \tag{3}$$

If one substitutes equation (3) in equation (2) and then computes

$$B_z(\rho,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho(A_\phi^1 + A_\phi^2) \quad , \tag{4}$$

he finds

$$B_z(\rho,z) = \frac{\mu aI}{\pi}\int_0^\infty k\left[K_1(ka)I_o(k\rho) + \frac{K_1(ka)I_1(kb)K_o(k\rho)}{K_1(kb)}\right]\cos(kz)dk \qquad \rho < a \tag{5}$$

By using the Wronskian relation for the modified Bessel functions the above expression for $B_z(\rho,z)$ at the surface of the cylinder ($\rho = b$) may be reduced to

$$B_z(b,z) = \frac{\mu a I}{\pi b} \int_0^\infty \frac{K_1(ka)}{K_1(kb)} \cos(kz)dk \quad . \tag{6}$$

The current density on the cylinder is given by

$$K_\phi(z) = B_z/\mu_o \quad .$$

We may define a parameter

$$\alpha \equiv b/a$$

and introduce a normalized distance along the cylinder,

$$d \equiv z/a$$

to express the normalized surface current density on the cylinder

$$f(z) \equiv \frac{2aK_\phi(z)}{I} \quad ,$$

in the form

$$f(d,\alpha) = \frac{2}{\alpha\pi} \int_0^\infty \frac{K_1(x)}{K_1(\alpha x)} \cos(xd)dx \tag{7}$$

As α approaches zero (i.e. as the loop recedes from the cylinder), equation (7) reduces to

$$f(d,\alpha) \xrightarrow[\alpha\to 0]{} \frac{2}{\pi} \int_0^\infty xK_1(x)\cos(xd)dx = (d^2 + 1)^{-3/2} \quad . \tag{8}$$

The central member of the above relation comes from the small argument asymptotic form of the modified Bessel function, while the equality is a special case of the equation given in Reference 6. Equation (8) is also to be expected from a consideration of the magnetic field on the axis of an isolated circular loop.

By using equations (7) and (8), one can define a "relative error" in the normalized current density, i.e. the difference between the true current density and that which would be present if the loop were extremely remote. This "relative error" will be defined by

$$\Delta(d,\alpha) \equiv (1 + d^2)^{3/2} f(d,\alpha) - 1 \tag{9}$$

Equations (7) and (9) were evaluated numerically.

III. Numerical Calculations

As stated at the end of the previous section the numerical computations were based on equation (7). Accuracy in the numerical results is assured by the fact that the integration intervals between 0 and $5/\alpha$ were kept small compared to $1/z$ and the contribution to the integral from the region $x > 5/\alpha$ was approximated, using the large argument asymptotic forms of the modified Bessel functions, by

$$\int_{5/\alpha}^{\infty} \frac{K_1(x)}{K_1(\alpha x)} \cos(xd)dx \approx \sqrt{\alpha}\, e^{-(1-\alpha)^{5/\alpha}} \cdot \frac{(1-\alpha)\cos(5d/\alpha) - d\sin(5d/\alpha)}{(1-\alpha)^2 + d^2} \quad .$$

In Table 1 are the values of $f(d,\alpha)$ for various values of d and α. The corresponding values of $\Delta(d,\alpha)$ are given in Table 2. Figure 2 contains plots of $f(d,\alpha)$ as a function of d with α as a parameter while Figure 3 contains the same information plotted as a function of α with d as a parameter. Figures 4 and 5 are plots of the $\Delta(d,\alpha)$ corresponding to Figures 2 and 3.

Table 1: $f(d,\alpha)$; normalized surface current on an infinite cylinder

α \ d	0	.2	.4	.6	.8	1.0	1.2
0	1.000	.943	.800	.631	.476	.354	.262
.02	1.002	.945	.801	.630	.476	.353	.262
.04	1.007	.948	.803	.630	.475	.352	.261
.06	1.014	.953	.805	.630	.473	.350	.259
.08	1.022	.960	.807	.630	.472	.348	.257
.10	1.032	.967	.811	.629	.470	.346	.255
.12	1.043	.976	.814	.629	.467	.343	.253
.14	1.056	.985	.818	.629	.465	.340	.250
.16	1.069	.995	.822	.628	.462	.337	.248
.18	1.083	1.007	.827	.628	.460	.334	.245
.20	1.099	1.019	.831	.627	.457	.331	.241
.22	1.116	1.032	.836	.626	.453	.327	.238
.24	1.134	1.045	.841	.625	.450	.323	.234
.26	1.154	1.060	.846	.623	.446	.318	.231
.28	1.175	1.076	.852	.622	.441	.314	.227
.30	1.198	1.092	.857	.620	.437	.309	.223
.32	1.223	1.110	.862	.617	.432	.304	.218
.34	1.249	1.128	.867	.615	.427	.299	.214
.36	1.277	1.148	.872	.612	.421	.294	.209
.38	1.308	1.168	.877	.608	.415	.288	.205
.40	1.341	1.190	.881	.603	.409	.282	.200
.42	1.376	1.212	.885	.599	.402	.276	.195
.44	1.414	1.236	.889	.593	.395	.269	.189
.46	1.455	1.261	.892	.587	.387	.262	.184
.48	1.499	1.287	.894	.580	.378	.255	.178
.50	1.547	1.315	.895	.572	.370	.248	.173
.52	1.600	1.343	.896	.563	.360	.240	.167
.54	1.657	1.373	.895	.553	.350	.232	.161
.56	1.720	1.403	.892	.543	.340	.224	.155
.58	1.788	1.435	.888	.531	.329	.215	.149
.60	1.864	1.467	.882	.518	.318	.207	.142
.62	1.948	1.500	.874	.503	.306	.198	.136
.64	2.041	1.534	.863	.488	.294	.188	.129
.66	2.147	1.567	.850	.471	.281	.179	.122
.68	2.264	1.599	.833	.453	.268	.169	.115
.70	2.398	1.629	.813	.433	.254	.160	.107
.72	2.551	1.657	.789	.412	.240	.150	.100
.74	2.727	1.680	.761	.389	.225	.140	.093
.76	2.933	1.696	.728	.365	.210	.130	.086
.78	3.176	1.703	.691	.339	.194	.120	.078
.80	3.468	1.696	.648	.312	.178	.110	.071
.82	3.824	1.673	.601	.284	.161	.099	.064
.84	4.268	1.628	.549	.255	.144	.089	.057
.86	4.840	1.555	.492	.225	.126	.078	.050
.88	5.601	1.447	.430	.193	.109	.067	.043
.90	6.665	1.301	.364	.162	.091	.056	.036
.92	8.260	1.111	.294	.129	.073	.045	.029
.94	10.916	.879	.222	.097	.054	.034	.022
.96	16.225	.608	.149	.064	.036	.023	.015
.98	32.145	.310	.074	.032	.018	.011	.007

Table 2: $\Delta(d,\alpha)$; "Relative Error" in normalized surface current

α \ d	0	.2	.4	.6	.8	1.0	1.2
0	.000	.000	.000	.000	.000	.000	.000
.02	.002	.002	.001	-.000	-.001	-.002	-.002
.04	.007	.006	.003	-.001	-.003	-.005	-.006
.06	.014	.011	.005	-.001	-.006	-.010	-.012
.08	.022	.018	.009	-.001	-.010	-.015	-.019
.10	.032	.026	.013	-.002	-.014	-.022	-.027
.12	.043	.035	.017	-.002	-.018	-.029	-.036
.14	.056	.045	.022	-.003	-.023	-.037	-.046
.16	.069	.056	.027	-.004	-.029	-.046	-.056
.18	.083	.068	.033	-.005	-.035	-.055	-.068
.20	.099	.080	.039	-.006	-.041	-.065	-.080
.22	.116	.094	.045	-.007	-.048	-.076	-.093
.24	.134	.109	.051	-.009	-.056	-.087	-.107
.26	.154	.124	.057	-.011	-.064	-.099	-.121
.28	.175	.141	.064	-.014	-.073	-.112	-.136
.30	.198	.158	.070	-.017	-.082	-.125	-.152
.32	.223	.177	.077	-.021	-.093	-.139	-.168
.34	.249	.196	.083	-.025	-.104	-.154	-.185
.36	.277	.217	.089	-.030	-.115	-.170	-.202
.38	.308	.239	.095	-.036	-.128	-.186	-.220
.40	.341	.262	.101	-.043	-.141	-.202	-.239
.42	.376	.286	.106	-.051	-.156	-.220	-.258
.44	.414	.311	.111	-.059	-.171	-.239	-.278
.46	.455	.338	.114	-.069	-.188	-.258	-.299
.48	.499	.365	.117	-.081	-.205	-.278	-.320
.50	.547	.394	.119	-.093	-.224	-.299	-.341
.52	.600	.424	.119	-.107	-.243	-.320	-.363
.54	.657	.456	.118	-.122	-.264	-.343	-.386
.56	.720	.488	.115	-.139	-.285	-.366	-.409
.58	.788	.522	.110	-.158	-.308	-.391	-.433
.60	.864	.556	.102	-.179	-.332	-.415	-.458
.62	.948	.591	.092	-.201	-.357	-.441	-.483
.64	1.042	.627	.079	-.226	-.382	-.467	-.509
.66	1.147	.662	.062	-.253	-.409	-.494	-.536
.68	1.264	.696	.041	-.282	-.437	-.521	-.563
.70	1.398	.728	.015	-.314	-.466	-.548	-.591
.72	1.551	.757	-.015	-.347	-.496	-.576	-.619
.74	1.727	.781	-.050	-.383	-.527	-.604	-.646
.76	1.933	.798	-.090	-.422	-.560	-.632	-.674
.78	2.176	.806	-.137	-.462	-.593	-.661	-.701
.80	2.468	.799	-.190	-.505	-.627	-.690	-.729
.82	2.824	.775	-.249	-.549	-.662	-.719	-.756
.84	3.268	.726	-.314	-.596	-.698	-.749	-.783
.86	3.840	.649	-.386	-.644	-.735	-.779	-.810
.88	4.601	.535	-.463	-.693	-.772	-.810	-.837
.90	5.665	.380	-.545	-.744	-.809	-.841	-.863
.92	7.260	.179	-.632	-.795	-.848	-.872	-.890
.94	9.916	-.068	-.722	-.846	-.886	-.904	-.917
.96	15.225	-.355	-.814	-.898	-.924	-.936	-.944
.98	31.145	-.671	-.908	-.949	-.962	-.968	-.972

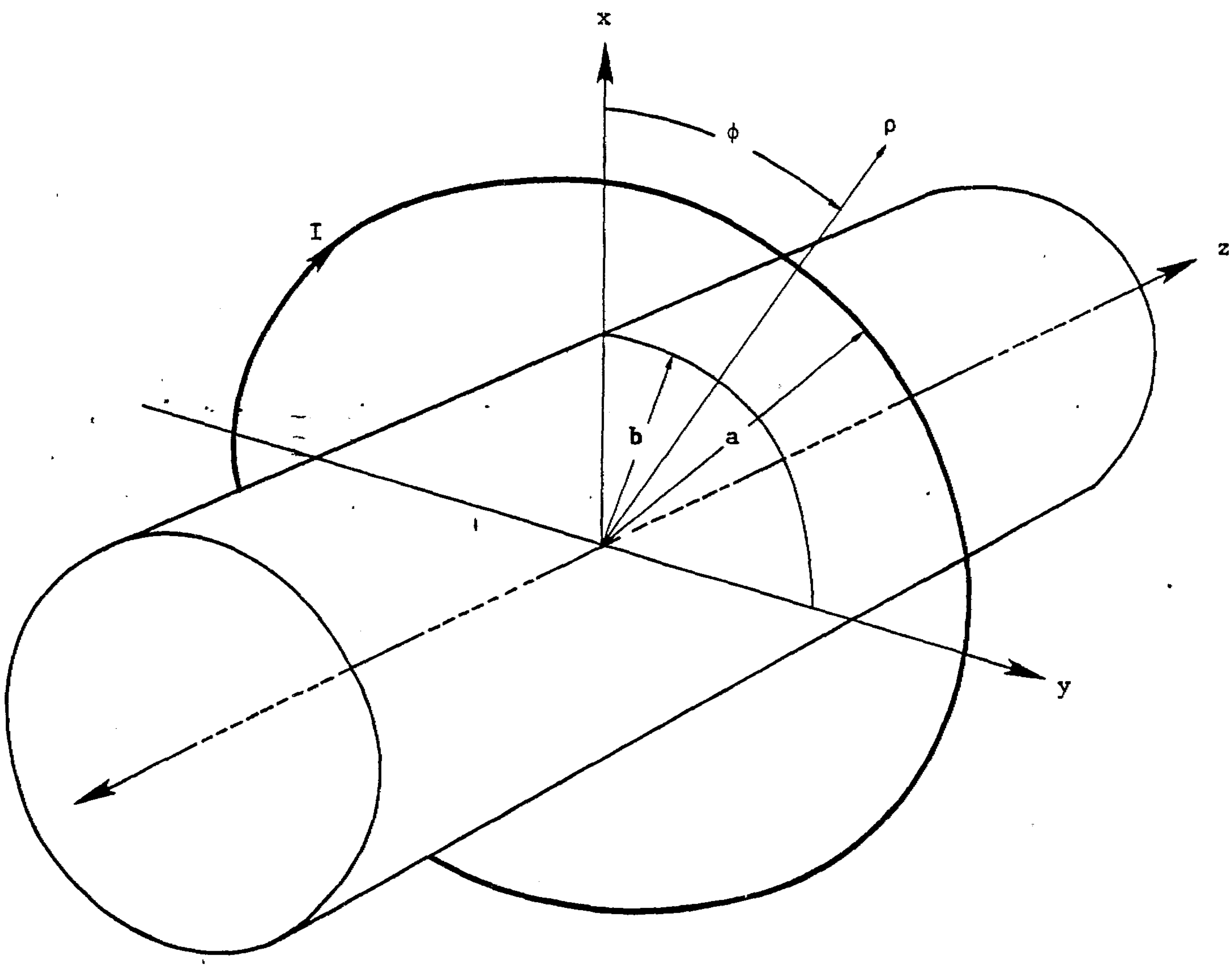

Figure 1: An infinite cylinder within a circular loop.

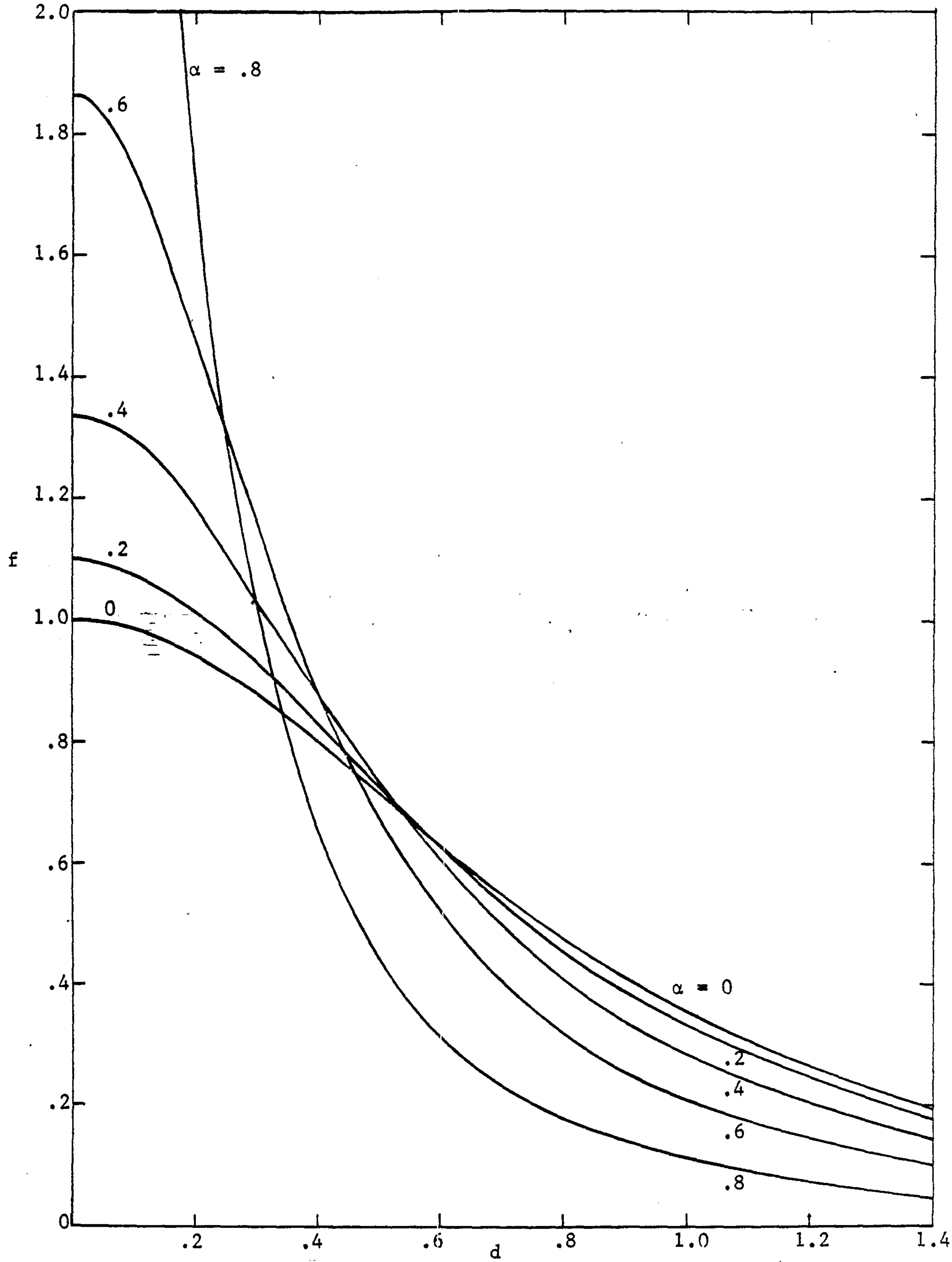

Figure 2. f(d,α) versus d with α as a parameter.

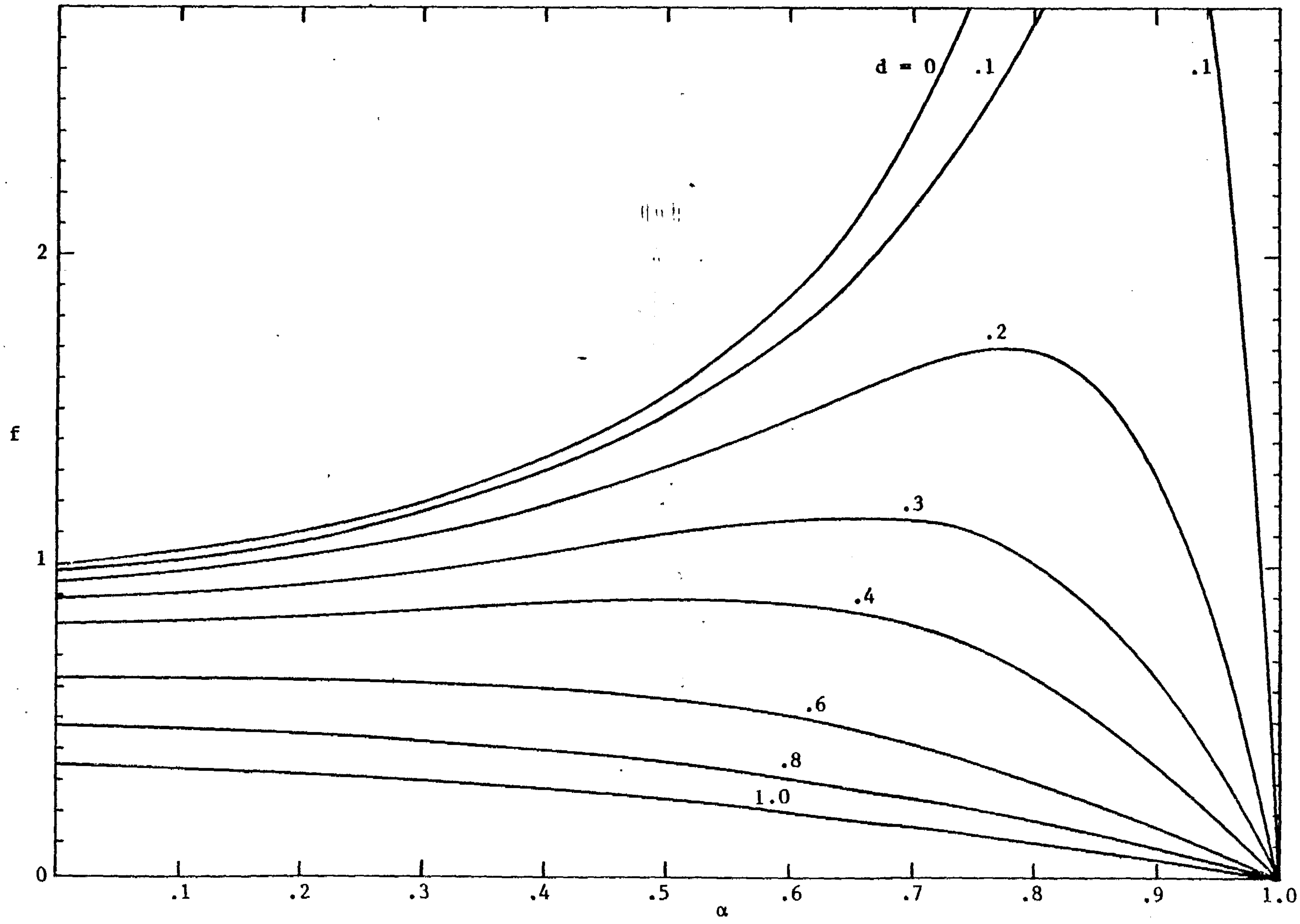

Figure 3. f(d,α) versus α with d as a parameter.

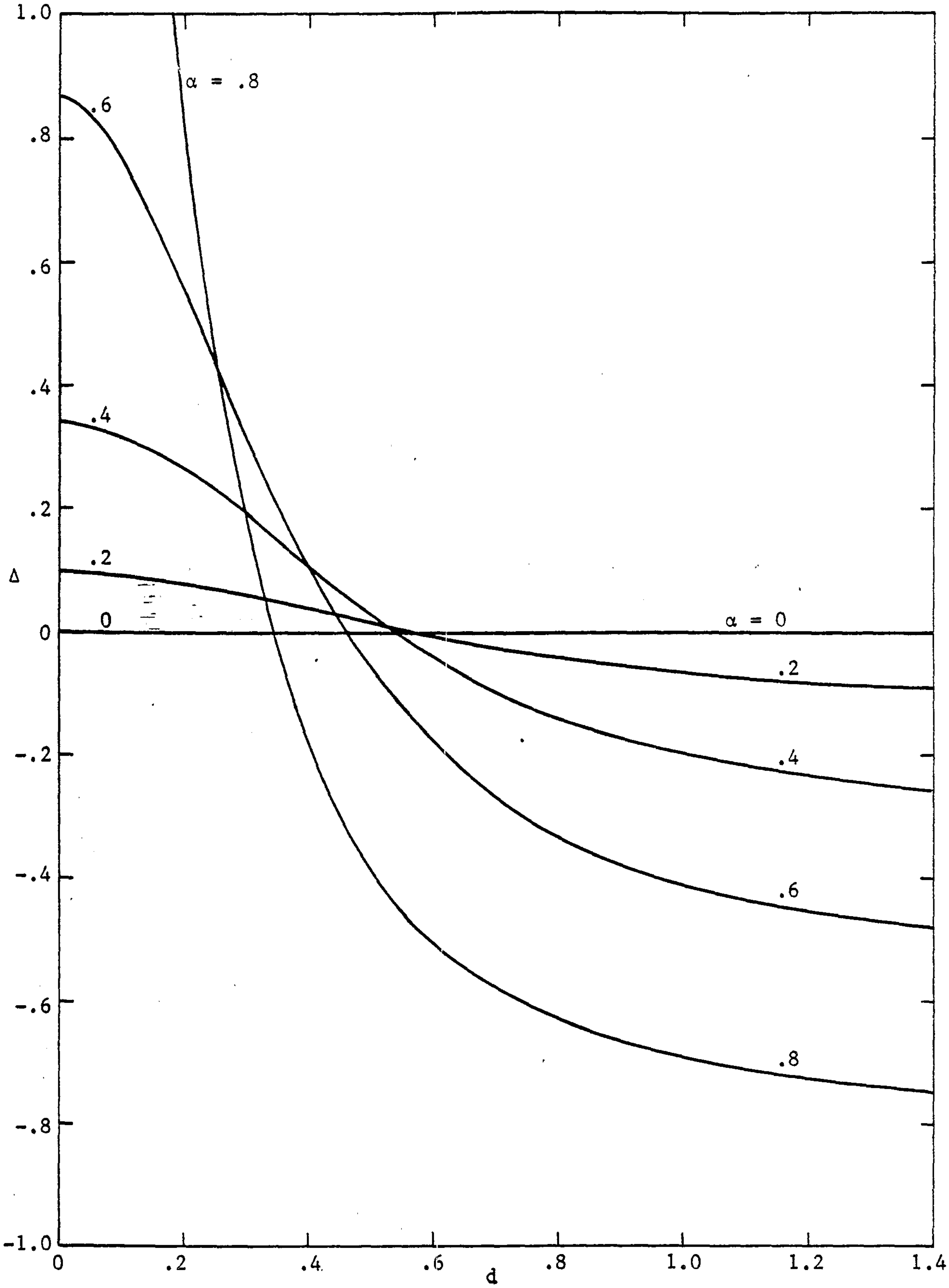

Figure 4. $\Delta(d,\alpha)$ versus d with α as a parameter.

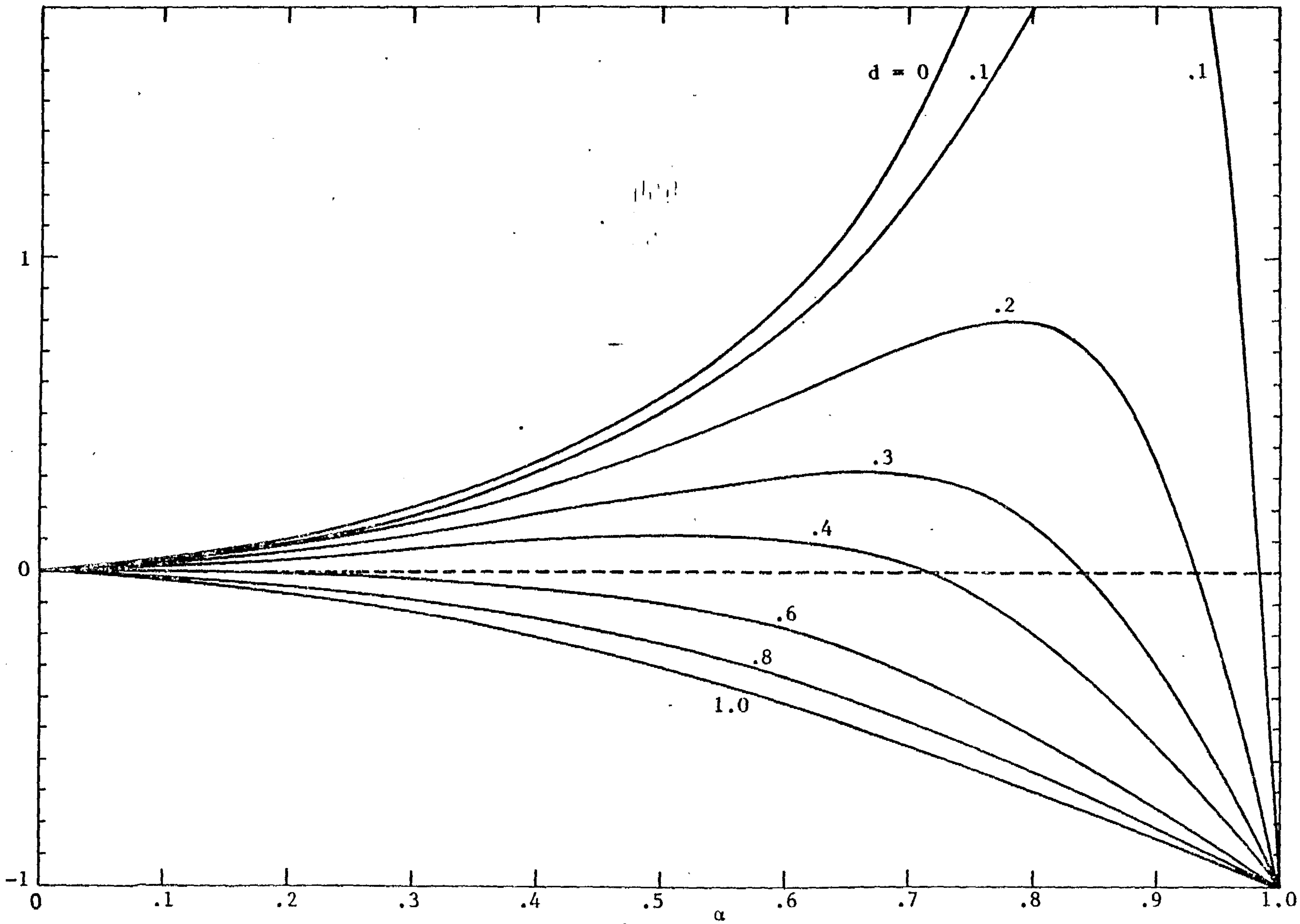

Figure 5. Δ(d,α) versus α with d as a parameter.

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