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# Scattering of a Plane Electromagnetic Wave by a Screen With a Sinusoidal Conducting Direction

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## Abstract

In this work the scattering of a plane electromagnetic field from an infinite unidirectional conducting screen is considered. The screen models an array of sinusoidal wires whose spacing is much less than the wavelength of the incident field. First the boundary conditions imposed are shown to give a unique solution. Secondly the scattered field is calculated and it is shown to consist of TEM, TE and TM waves. It is found that an otherwise transparent screen can become significantly reflecting by introducing slight periodic curvature to the conducting direction.

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# I. Introduction

It is sometimes the case that one is interested in scattering from a structure which is an array of elements that are closely spaced compared to the wavelength of the incident electromagnetic field. When this is the case, the problem is often simplified by considering the structure to have a continuous surface on which boundary conditions are satisfied which are physically suggested by the actual array configuration. We shall refer to such a surface as a unidirectional conducting screen. In this work we examine whether the solution to the mathematical problem is unique and we show that it is. We then consider scattering by an infinite plane unidirectional conducting screen in which the conducting direction is sinusoidal. Many problems concerning unidirectional conducting screens have considered the case where the conducting direction was along parallel straight lines. For such a screen, field components alone introduce a convenient number of unknown scalars for the problem to be readily formulated. When the conducting direction has some curvature the problem is not as easily formulated. Rumsey [1] has considered the introduction of potentials for a certain class of problems involving unidirectional conducting structures, but that formulation requires a certain symmetry in the boundary conditions for the $\underline{E}$ and $\underline{H}$ fields. This symmetry is absent for the scattering problem due to the presence of the incident field. In this work we introduce the appropriate number of scalars by using certain components of the vector potential $\underline{A}$. We expand these components of $\underline{A}$ in a Fourier series with unknown coefficients. The application of boundary conditions leads to an infinite matrix which must be inverted to find these coefficients. We prove that the matrix has a unique inverse and then obtain approximate solutions using an iterative procedure that is known to converge.

## II. Uniqueness Proof

Consider the case where the conducting screen forms a closed surface S which bounds a source free volume $V_1$. The remaining part of space contains a source J and is designated as $V_2$. We denote the fields in region 1 as $\underline{E}_1$ and $\underline{H}_1$ and the constitutive parameters as $\varepsilon_1$, $\mu_1$ and $\sigma_1$. Similar notation is used for region 2. We now use Poynting's theorem and integrate over volumes 1 and 2 to obtain

$$\int_{S_\infty} \hat{a}_R \cdot (\underline{E}_2 \times \underline{H}_2^*)dS - \int_S \hat{n} \cdot (\underline{E}_2 \times \underline{H}_2^*)dS =$$

$$\int_{V_2} (i\omega\mu_2|\underline{H}_2|^2 - i\omega\varepsilon_2|\underline{E}_2|^2 - \sigma_2|\underline{E}_2|^2 - \underline{E}_2 \cdot J^*)dV \qquad (1a)$$

and

$$\int_S \hat{n} \cdot (\underline{E}_1 \times \underline{H}_1^*)dS = \int_{V_1} (i\omega\mu_1|\underline{H}_1|^2 - i\omega\varepsilon_1|\underline{E}_1|^2 - \sigma_1|\underline{E}_1|^2)dV \qquad (1b)$$

where $\hat{n}$ is the outward normal to S, $\hat{a}_R$ is a unit vector in the radial direction in a conveniently located coordinate system, and $S_\infty$ is the surface of a sphere with its center at the origin of our coordinate system and with its radius tending to infinity. The boundary conditions that we consider are such that on S,

$$\hat{n} \cdot (\underline{E}_1 \times \underline{H}_1^*) = \hat{n} \cdot (\underline{E}_2 \times \underline{H}_2^*) \qquad (2)$$

that is the energy flux is continuous through S. We also assume that $\underline{E}_2$ and

$\underline{H}_2$ decay exponentially due to $\sigma_2$. These requirements lead to a class of boundary conditions each set of which will be shown to guarantee unique solution. The choice of a particular set is dictated by the physics of the problem. When this is the case then we can combine (1a) and (1b) and use the fact that the integral over $S_\infty$ is zero to obtain

$$\int_{V_1} (i\omega\mu_1|\underline{H}_1|^2 - i\omega\varepsilon_1|\underline{E}_1|^2 - \sigma_1|\underline{E}_1|^2)dV + \int_{V_2} (i\omega\mu_2|\underline{H}_2|^2$$

$$- i\omega\varepsilon_2|\underline{E}_2|^2 - \sigma_2|\underline{E}_2|^2 - \underline{E}_2 \cdot \underline{J}^*)dV = 0 \qquad (3)$$

Let us now assume that we have another set of fields that satisfy Maxwell's equations and the same boundary conditions as $\underline{E}_1$, $\underline{H}_1$ and $\underline{E}_2$, $\underline{H}_2$ and we denote these fields as $\underline{E}_1'$, $\underline{H}_1'$ and $\underline{E}_2'$, $\underline{H}_2'$. We form the difference fields $\underline{E}_{1D} = \underline{E}_1 - \underline{E}_1'$, $\underline{H}_{1D} = \underline{H}_1 - \underline{H}_1'$, $\underline{E}_{2D} = \underline{E}_2 - \underline{E}_2'$ and $\underline{H}_{2D} = \underline{H}_2 - \underline{H}_2'$ and assume that the boundary conditions that allowed (2) to be satisfied also allow (2) to be satisfied for the difference field. This point will later be explicitly examined. When these conditions are satisfied, then we can arrive at the equation

$$\int_{V_1} (i\omega\mu_1|\underline{H}_{1D}|^2 - i\omega\varepsilon_1|\underline{E}_{1D}|^2 - \sigma_1|\underline{E}_{1D}|^2)dV$$

$$+ \int_{V_2} (i\omega\mu_2|\underline{H}_{2D}|^2 - i\omega\varepsilon_2|\underline{E}_{2D}|^2 - \sigma_2|\underline{E}_{2D}|^2)dV = 0 \qquad (4)$$

in the same manner that (3) was obtained. We note that the term containing

$\underline{J}$ is absent. This is the case because the primed and unprimed solutions satisfy the same inhomogeneous Maxwell's equations so the difference solution satisfies Maxwell's equations with the source terms cancelling each other. Equating the real part of (4) equal to zero we obtain

$$\int_{V_1} \sigma_1 |\underline{E}_{1D}|^2 dV + \int_{V_2} \sigma_2 |\underline{E}_{2D}|^2 dV = 0 \tag{5}$$

From (5) we conclude the $\underline{E}_1 = \underline{E}_1'$ and $\underline{E}_2 = \underline{E}_2'$. Setting the imaginary part of (4) equal to zero and using the results just established we conclude $\underline{H}_1 = \underline{H}_1'$ and $\underline{H}_2 = \underline{H}_2'$.

We will now show that the boundary conditions used for the screen problem allow (2) to be satisfied. Consider a right handed orthonormal set of vectors to be defined at every point on the screen $(\hat{t}, \hat{n}, \hat{s})$. The unit vector $\hat{t}$ is chosen along the direction of conduction on the screen, $\hat{n}$ is the outward normal to the surface, while $\hat{s} = \hat{t} \times \hat{n}$ and $\hat{s}$ is the direction along which no conduction takes place. For the unidirectional conducting screen we impose the following boundary conditions

$$\hat{t} \cdot \underline{E}_1 = \hat{t} \cdot \underline{E}_2 = 0 \tag{6a}$$

$$\hat{s} \cdot \underline{E}_1 = \hat{s} \cdot \underline{E}_2 \tag{6b}$$

$$\hat{t} \cdot \underline{H}_1 = \hat{t} \cdot \underline{H}_2 \tag{6c}$$

Conditions (6a) and (6b) are imposed because we assume that the screen acts like a perfect conductor in the $\hat{t}$ direction and is transparent in the $\hat{s}$

direction. Condition (6c) is imposed because the induced electric current on the sheet is $n \times (\underline{H}_2 - \underline{H}_1)$ and we want this current to flow only in the $\hat{t}$ direction. These boundary conditions have been applied in previous problems. See for example Collin [2]. In general

$$\hat{n} \cdot (\underline{E} \times \underline{H}^*) = (\hat{t} \cdot \underline{H}^*)(\hat{s} \cdot \underline{E}) - (\hat{s} \cdot \underline{H}^*)(\hat{t} \cdot \underline{E}) \tag{7}$$

Using (7) we see that if (6) is satisfied, then (2) is satisfied. We also observe that (6) is composed of linear relationships so that if $\underline{E}_1'$, $\underline{H}_1'$ and $\underline{E}_2'$, $\underline{H}_2'$ satisfied (6), the difference between the primed and unprimed solutions would also satisfy (6). Because the difference solution satisfied (6) it would also satisfy (2) (for the difference field) and the preceding uniqueness proof is applicable. We conclude that the solution to the uniconducting screen problem which satisfies (6) and appropriate conditions at infinity is unique.

## III. Solution of the Scattering Problem

The configuration that our screen models is an infinite planar array of sinusoidally shaped wires. The spacing between the wires is assumed to be much smaller than the wavelength of the incident field. The screen lies in the $z = 0$ plane and the equation for the wire passing through the origin of our coordinate system is $y = m \sin ax$. Vectors tangent and normal to the wire and lying in the plane of the screen are given by

$$\underline{t} = \hat{a}_x + ma \cos ax\, \hat{a}_y \tag{8}$$

and

$$\underline{s} = -\, ma \cos ax\, \hat{a}_x + \hat{a}_y \tag{9}$$

Notice that $\underline{t}$ and $\underline{s}$ have not been normalized. We will consider an incident plane wave of the form

$$\underline{E}_i = \hat{a}_y e^{ik_o z} \tag{10}$$

The source region, $z < 0$, is region 2 and region 1 will correspond to $z > 0$. The scattered field in region 1 is denoted as $\underline{E}^{(1)}, \underline{H}^{(1)}$ and the scattered field in region 2 is denoted as $\underline{E}^{(2)}, \underline{H}^{(2)}$. The boundary conditions that will be satisfied on the screen are

$$\underline{t} \cdot (\underline{E}_i + \underline{E}^{(1)}) = 0 \tag{11}$$

$$\underline{t} \cdot (\underline{E}_i + \underline{E}^{(2)}) = 0 \tag{12}$$

$$\underline{s} \cdot (\underline{E}_i + \underline{E}^{(1)}) = \underline{s} \cdot (\underline{E}_i + \underline{E}^{(2)}) \tag{13}$$

$$\underline{t} \cdot (\underline{H}_i + \underline{H}^{(1)}) = \underline{t} \cdot (\underline{H}_i + \underline{H}^{(2)}) \tag{14}$$

From symmetry considerations one can deduce $\underline{t} \cdot \underline{H}^{(1)} = -\, \underline{t} \cdot \underline{H}^{(2)}$ and consequently from (14) one can conclude $\underline{t} \cdot \underline{H}^{(1)} = \underline{t} \cdot \underline{H}^{(2)} = 0$. We would like to solve for $\underline{E}^{(1)}, \underline{H}^{(1)}$ and $\underline{E}^{(2)}, \underline{H}^{(2)}$ by introducing as few unknown scalars as possible. The introduction of potentials for problems involving uni-directional conducting structures was considered by Rumsey [1]. His formulation concerned propagation and radiation, but not scattering. The two former classes of problems contained a symmetry of boundary conditions for the

$\underline{E}$ and $\underline{H}$ fields that is destroyed by the presence of an incident field in the scattering problem; thus his method is not directly applicable to our problem. Our formulation is based on the introduction of the electric vector potentials $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$ for the scattered field. It is argued that these potentials have no z components, because their source, the difference in $\hat{n} \times \underline{H}$ across the screen has no z component. The difference in $\hat{n} \times \underline{E}$ across the screen is zero due to our boundary donditions so no magnetic vector potentials need be considered. That is

$$\underline{A}^{(1)} = A_x^{(1)}(x,z)\hat{a}_x + A_y^{(1)}(x,z)\hat{a}_y \quad (15)$$

$$\underline{A}^{(2)} = A_x^{(2)}(x,z)\hat{a}_x + A_y^{(2)}(x,z)\hat{a}_y \quad (16)$$

The components of the $\underline{A}$'s have no y dependence due to the y symmetry of the screen. We now introduce the following expansions for these components

$$A_x^{(1)} = \sum_{n=-\infty}^{\infty} a_n e^{inax} e^{i\gamma_n z} \quad (17)$$

$$A_y^{(1)} = \sum_{n=-\infty}^{\infty} b_n e^{inax} e^{i\gamma_n z} \quad (18)$$

$$A_x^{(2)} = \sum_{n=-\infty}^{\infty} a_n e^{inax} e^{-i\gamma_n z} \quad (19)$$

$$A_y^{(2)} = \sum_{n=-\infty}^{\infty} b_n e^{inax} e^{-i\gamma_n z} \quad (20)$$

and in order to satisfy Maxwell's equations $\gamma_n = \sqrt{k_o^2 - (na)^2}$ and the square root is defined to give outward propagating or evanescent waves. The expansion coefficients in (17) and (19) as well as in (18) and (20) are the same because of the symmetry of the scattered field. Using the standard relationships between E, H and A that is

$$E_x = -\frac{i\omega}{k_o^2}\frac{\partial^2 A_x}{\partial z^2} \tag{21}$$

$$E_y = i\omega A_y \tag{22}$$

$$H_x = -\frac{i}{\mu_o}\frac{\partial A_z}{\partial z} \tag{23}$$

$$H_y = \frac{1}{\mu_o}\frac{\partial A_x}{\partial z} \tag{24}$$

the boundary conditions (11) and (14) lead to

$$\frac{\gamma_n^2}{k_o^2} a_n + \frac{ma}{2}(b_{n-1} + b_{n+1}) = \frac{ima}{2\omega}(\delta_{n,-1} + \delta_{n,1}) \tag{25}$$

$$\gamma_n b_n - \frac{ma}{2}(\gamma_{n-1}a_{n-1} + a_{n+1}\gamma_{n+1}) = 0 \tag{26}$$

Boundary conditions (12) and (13) are automatically satisfied because of the relationship of the expansion coefficients for $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$ exhibited in (17 - 20). We can examine the infinite subset of equations (25) and (26) which corresponds to $a_{2n}$ and $b_{2n+1}$ for all n. This set of equations can be viewed as an infinite homogeneous matrix equation. This same infinite matrix will later be shown to have a unique inverse. From this fact we conclude that the homogeneous matrix equation can have only the null solution and $a_{2n}$ and $b_{2n+1}$ are zero for all n. We now make use of the symmetry of the system to derive further relationships between the expansion coefficients. From considering the direction that current would be induced on the "wires" of our screen we can see that on the screen $A_x(x) = -A_x(\pi/a - x)$ and $A_y(x) = A_y(\pi/a - x)$. Using these equalities we can easily see that $a_n = (-1)^{n+1}a_{-n}$ and $b_n = (-1)^n b_{-n}$. These conclusions could be arrived at by using matrix arguments.

We now return to (25) and (26) rewritten in the following manner

$$a_{2n+1} = -\frac{k_o^2}{\gamma_{2n+1}^2}\frac{ma}{2}(b_{2n} + b_{2n+2}) + \frac{ima}{2\omega}\frac{k_o^2}{\gamma_{2n+1}^2}\delta_{2n+1,1} \qquad n \geq 0 \qquad (27)$$

$$b_{2n} = \frac{ma}{2\gamma_{2n}}(\gamma_{2n-1}a_{2n-1} + \gamma_{2n+1}a_{2n+1}) \qquad n \geq 0 . \qquad (28)$$

This set of equations has the general form of an infinite matrix equation which can always be rewritten as

$$x_i = \sum_{\substack{j=o \\ j\neq i}}^{\infty} c_{ij}x_j + f_i \qquad (29)$$

A sufficient condition that (29) has a unique solution is

$$\sum_{\substack{j=o \\ j\neq i}}^{\infty} |c_{ij}| < 1 - \theta \ , \ \theta > 0 \ , \quad \text{all } i \qquad (30)$$

When condition (30) is satisfied the matrix associated with (29) is said to be fully regular. A fully regular matrix equation can be solved by iteration with the guarantee of convergence to a unique solution. A detailed discussion of fully regular matrices can be found in the book by Kantorovich and Krylov [3]. Condition (30) when applied to (27) and (28) leads to

$$ma \frac{k_o^2}{|\gamma_{2n+1}^2|} < 1 - \theta \qquad n \geq 0 \tag{31}$$

$$\frac{ma}{2} \left( \left| \frac{\gamma_{2n-1}}{\gamma_{2n}} \right| + \left| \frac{\gamma_{2n+1}}{\gamma_{2n}} \right| \right) < 1 - \theta \qquad n \geq 0 \tag{32}$$

As long as we are not operating at a resonant frequency, $\gamma_n = 0$, then an ma exists such that both (31) and (32) are satisfied for all n. When this is the case then we can solve (27) and (28) by iteration as described in Kantorovich and Krylov. Performing this calculation we obtain after iterating

$$b_o = \frac{i2M^2k_o}{\omega\gamma_1} \{1 - \frac{M^2k_o^2}{\gamma_1} (\frac{2}{k_o} + \frac{1}{\gamma_2}) + [\frac{M^2k_o^2}{\gamma_1} (\frac{2}{k_o} + \frac{1}{\gamma_2})]^2 + \frac{M^4k_o^4}{\gamma_1\gamma_2^2\gamma_3} - (\frac{M^2k_o^2}{\gamma_1})^3[(\frac{2}{k_o} + \frac{1}{\gamma_2})^2 + \frac{\gamma_1}{\gamma_3\gamma_2^2}][\frac{2}{k_o} + \frac{1}{\gamma_2} + \frac{\gamma_1}{\gamma_2\gamma_3}] + 0(M^8)\} \tag{33}$$

where $M = \frac{ma}{2}$. All coefficients in the expansion of $A^{(1)}$ and $A^{(2)}$ could be computed in terms of $b_o$. Once $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$ are known, then all field components can be determined using standard relationships.

Only a finite number of modes are non-evanescent depending on the frequency of the incident field. That is for $n^2a^2 < k_o^2 < (n + 1)^2a^2$ the first n scattered modes exist far from the screen while the higher scattered modes decay exponentially with the distance from the screen. For n = 0 we obtain no depolarization for this scattered mode and it is a TEM mode $(E_y, H_x)$. The remaining even numbered scattered modes are $TE(E_y, H_x, H_z)$ whereas the odd numbered scattered modes are $TM(E_x, E_z, H_y)$. We note that the scattering structure is infinite and consequently we can not discuss the behavior of the fields in the far zone; however, an interesting aspect of the problem is that solutions can exist which have scattered fields directed along the propagation direction which do not decrease with increasing distance from the scatterer.

When $\frac{a}{k_o} > 1$, then only the zero-order TEM scattered mode is non-evanescent. The magnitude of the reflection coefficient for this mode is

$$|R| = \omega |b_o| \tag{34}$$

It is interesting to note that $|R|$ can be relatively large for a small curvature of the wires which are modeled by the conducting screen. If the wires were straight, then $|R|$ would be zero. By introducing slight curvature so that the ratio of the amplitude of the sinusoidally bent wire is 15 percent of the period, we can use all of the terms through $O(M^6)$ in (33) and (34) to show that $|R| = .42 \pm .04$. In order to obtain this value of $|R|$ we chose $(\frac{a}{k_o}) = 1.405$. We could not choose $\frac{a}{k_o}$ closer to the resonant value unity because of the validity conditions (31) and (32) imposed by our perturbation method of solution. This calculation shows that an otherwise transparent screen can become significantly reflecting by introducing slight periodic curvature to the conducting direction.

## References

1. V. H. Rumsey, "A new way of solving Maxwell's equations," I.R.E. Trans. Antennas Propagation, 9, p. 461, 1961.

2. R. E. Collin, Field Theory of Guided Waves, New York: McGraw-Hill, p. 402, 1960.

3. L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis, New York: John Wiley and Sons, p. 26, 1964.