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Sensor and Simulation Notes

Note 113

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by

R. W. Latham and K. S. H. Lee
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Abstract

The capacitance between two identical spherical bowls located symmetrically on a spherical surface is calculated. The bowls are perfectly conducting and infinitely thin. The equivalent area of this structure when used as an electric field sensor is also calculated.

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Contents

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I. A Statement of the Problem

In note 106 [Ref. 1], we have computed the capacitance and equivalent area of a disk in a circular aperture. The purpose of that computation was to obtain accurate information on the low-frequency characteristics of the "circular flush-plate dipole" sensor described in note 98 [Ref. 2]. In the present note, we will make similar computations for the "single-gap hollow spherical dipole" sensor described in note 91 [Ref. 3]. Because the purpose here is so similar to that in note 106, we need no further justification of the two particular boundary-value problems we will solve. Therefore, in this section, we will merely give a precise statement of these boundary-value problems and discuss them briefly.

Figure 1 shows the conductor configuration we will study. In a spherical coordinate system, (r,θ,ϕ), the infinitely thin conductors coincide with the surfaces $(r = a;\ 0 < \theta < \alpha)$ and $(r = a;\ \pi - \alpha < \theta < \pi)$. Conductors of this shape have been called "spherical bowls" by Smythe [Ref. 4, p. 204], and others. We will use that term here.

In order to determine the capacitance between the two bowls in figure 1, we will first solve the Laplace equation,

$$\nabla^2\psi_c = 0 \qquad r > a \quad \text{or} \quad r < a \tag{1}$$

for the electrostatic potential ψ_c, where ψ_c is +1 on the top bowl, -1 on the bottom bowl, and vanishes at infinity. Also, ψ_c and $\partial\psi_c/\partial r$ are continuous through the surface $(r = a;\ \alpha < \theta < \pi - \alpha)$. The charge on the top bowl may be computed from ψ_c by evaluating an integral over the surface of that bowl, that is,

$$Q_c = \varepsilon \int_{\text{bowl}} \left\{ \lim_{r\to a-} \frac{\partial\psi_c}{\partial r} - \lim_{r\to a+} \frac{\partial\psi_c}{\partial r} \right\} dS \quad . \tag{2}$$

The capacitance is then determined by

$$C = \frac{Q_c}{2} \quad . \tag{3}$$

In order to determine the equivalent area of the two-bowl sensor, we will first solve the Laplace equation for the electrostatic potential, ψ_a, of the extra field induced when the two bowls, at the same potential, are immersed in an external electrostatic field of strength 1/a parallel to the axis of symmetry of the bowls, In other words, we will solve

$$\nabla^2\psi_a = 0 \qquad r > a \quad \text{or} \quad r < a \tag{4}$$

for the electrostatic potential ψ_a, where ψ_a is equal to cos θ on the bowls and vanishes at infinity. In addition, ψ_a and $\partial\psi_a/\partial r$ are continuous through the surface ($r = a$; $\alpha < \theta < \pi - \alpha$). Once ψ_a has been obtained, the charge on the top bowl, Q_a, may be calculated by using an equation like equation (2). The equivalent area is then determined by

$$A_{eq} = \frac{aQ_a}{\varepsilon} \quad . \tag{5}$$

The two boundary-value problems stated above are three-part mixed boundary-value problems. In general, such problems are difficult to solve. Yet, for certain special shapes of the boundary on which the potential or its normal derivative is specified, methods exist for the reduction of such problems to the solution of rather simple Fredholm integral equations of the second kind [Ref. 5, Chap. 6]; it is fairly easy to solve such equations numerically. In the next section, we will follow a simplified version of Collins's method [Ref. 5, Sect. 6.5; Ref. 6] to reduce each of our particular problems to the solution of a simple integral equation. We will discuss some of the properties of this integral equation in the third section. In the fourth section, we deduce approximations for C and A_{eq} for the cases when the bowls are either very small or almost touching. The numerical method used to solve the integral equation derived in the second section is discussed in the last section.

II. A Derivation of an Integral Equation

It is clear, for both our boundary-value problems, that $\psi(r,\theta)$ is antisymmetric about the $\theta = \pi/2$ plane. This may be assured by setting

$$\psi(r,\theta) = \Phi(r,\theta) - \Phi(r,\pi - \theta) \tag{6}$$

where $\Phi(r,\theta)$ also satisfies the Laplace equation. The most general axisymmetric solution of the Laplace equation that is continuous, vanishes at infinity, and is finite everywhere may be written in the form

$$\begin{aligned} \Phi(r,\theta) &= \sum_{n=o}^{\infty} A_n (r/a)^n P_n(\cos\theta) \qquad r < a \\ \Phi(r,\theta) &= \sum_{n=o}^{\infty} A_n (a/r)^{n+1} P_n(\cos\theta) \qquad r > a \quad , \end{aligned} \tag{7}$$

where the P_n's are Legendre polynomials [Ref. 7, Sect. 5.4] and the A_n's are constants to be determined by the boundary conditions at $r = a$. For both of the problems discussed in the previous section, we may state the conditions on ψ at $r = a$ in the form

$$\psi(a,\theta) = V(\theta) \qquad 0 < \theta < \alpha$$

$$\psi(a,\theta) = - V(\pi - \theta) \qquad \pi - \alpha < \theta < \pi$$

$$\lim_{r\to a-} \frac{\partial\psi(r,\theta)}{\partial r} = \lim_{r\to a+} \frac{\partial\psi(r,\theta)}{\partial r} \qquad \alpha < \theta < \pi - \alpha \quad .$$

These conditions will be satisfied if

$$\Phi(r,\theta) - \Phi(r,\pi - \theta) = V(\theta) \qquad 0 < \theta < \alpha$$

$$\lim_{r\to a-} \frac{\partial\Phi(r,\theta)}{\partial r} = \lim_{r\to a+} \frac{\partial\Phi(r,\theta)}{\partial r} \qquad \alpha < \theta < \pi$$

i.e.

$$\sum_{n=o}^{\infty} \{1 - (-1)^n\}A_n P_n(\cos\theta) = V(\theta) \qquad 0 < \theta < \alpha \tag{8}$$

$$\sum_{n=o}^{\infty} (2n+1)A_n P_n(\cos\theta) = 0 \qquad \alpha < \theta < \pi \tag{9}$$

We may use a Mehler representation for the Legendre polynomial [Ref. 5, p. 57; Ref. 7, p. 235] to rewrite equation (8) in the form

$$\frac{\sqrt{2}}{\pi} \sum_{n=o}^{\infty} \{1 - (-1)^n\}A_n \int_o^{\theta} \frac{\cos(n+\tfrac{1}{2})u \, du}{(\cos u - \cos\theta)^{\frac{1}{2}}} = V(\theta) \qquad 0 < \theta < \alpha$$

or

$$\frac{\sqrt{2}}{\pi} \int_o^{\theta} \frac{du}{(\cos u - \cos\theta)^{\frac{1}{2}}} \left(\sum_{n=o}^{\infty} \{1 - (-1)^n\}A_n \cos(n+\tfrac{1}{2})u \right) = V(\theta) \quad . \quad 0 < \theta < \alpha$$

This may be looked upon as an integral equation for the function within the large parentheses. Its solution, from the appendix, is

$$\sum_{n=o}^{\infty} \{1 - (-1)^n\}A_n \cos(n+\tfrac{1}{2})\theta = \frac{1}{\sqrt{2}} \frac{d}{d\theta} \int_o^{\theta} \frac{\sin u \, V(u) du}{(\cos u - \cos\theta)^{\frac{1}{2}}} \qquad 0 < \theta < \alpha$$

We define the right-hand side of this equation, which is a known function when $V(u)$ is specified, as $h(\theta)$. We then can write

$$\sum_{n=o}^{\infty} A_n \cos(n+\tfrac{1}{2})\theta = h(\theta) + \sum_{n=o}^{\infty} (-1)^n A_n \cos(n+\tfrac{1}{2})\theta \qquad 0 < \theta < \alpha \; . \tag{10}$$

Equation (9) may be transformed in a manner similar to that used above for equation (8) if one uses the alternative Mehler representation

$$P_n(\cos\theta) = \frac{\sqrt{2}}{\pi} \int_{\theta}^{\pi} \frac{\sin(n+\tfrac{1}{2})u \, du}{(\cos\theta - \cos u)^{\frac{1}{2}}} \quad . \tag{11}$$

The result is

$$\sum_{n=o}^{\infty} A_n \cos(n + \tfrac{1}{2})\theta = 0 \qquad \alpha < \theta < \pi \quad . \tag{12}$$

The function defined by the series on the left-hand side of equation (12) will be denoted $\xi(\theta)$. It is then clear, from equation (12) and orthogonality, that

$$A_n = \frac{2}{\pi}\int_o^{\alpha} \xi(\theta)\cos(n + \tfrac{1}{2})\theta d\theta \quad . \tag{13}$$

If one substitutes these values for the A_n's into equation (10), he obtains

$$\xi(\theta) = h(\theta) + \sum_{n=o}^{\infty} (-1)^n \cos(n + \tfrac{1}{2})\theta \cdot \frac{2}{\pi}\int_o^{\alpha} \xi(\theta')\cos(n + \tfrac{1}{2})\theta' d\theta' \qquad 0 < \theta < \alpha$$

By interchanging the order of summation and integration in this equation, and then rewriting the summation in closed form, one obtains

$$\xi(\theta) = h(\theta) + \frac{1}{2\pi}\int_o^{\alpha} \left\{\sec\left(\frac{\theta-\theta'}{2}\right) + \sec\left(\frac{\theta+\theta'}{2}\right)\right\} \xi(\theta')d\theta' \qquad 0 < \theta < \alpha \quad . \tag{14}$$

The A_n's can be determined, once $\xi(\theta)$ is known, by using equation (13). Therefore, the solution of equation (14) immediately gives the complete solution of our boundary-value problems. But, since only Q_t, the charge on the top bowl, is needed for the determination of C and A_{eq}, we don't really need complete solutions and may proceed more simply as follows.

$$Q_t = 2\pi a^2 \int_o^{\alpha} \sigma(\theta)\sin\theta \, d\theta \tag{15}$$

where, from equations (6) and (7)

$$\frac{\sigma(\theta)}{\varepsilon} = \lim_{r\to a-} \frac{\partial\psi(r,\theta)}{\partial r} - \lim_{r\to a+} \frac{\partial\psi(r,\theta)}{\partial r}$$

$$= \lim_{r\to a-} \left\{\frac{\partial\Phi(r,\theta)}{\partial r} - \frac{\partial\Phi(r,\pi-\theta)}{\partial r}\right\} - \lim_{r\to a+} \left\{\frac{\partial\Phi(r,\theta)}{\partial r} - \frac{\partial\Phi(r,\pi-\theta)}{\partial r}\right\}$$

For $\theta < \alpha$, we may invoke equation (12) and drop the second term in each of the

Φ limits. Thus,

$$\frac{\sigma(\theta)}{\varepsilon} = \lim_{r\to a-} \frac{\partial\Phi(r,\theta)}{\partial r} - \lim_{r\to a+} \frac{\partial\Phi(r,\theta)}{\partial r} \qquad 0 < \theta < \alpha$$

i.e.

$$\frac{a\sigma(\theta)}{\varepsilon} = \sum_{n=o}^{\infty} (2n+1)A_n P_n(\cos\theta) \qquad 0 < \theta < \alpha$$

or, from equation (11),

$$\frac{a\sigma(\theta)}{\varepsilon} = \sum_{n=o}^{\infty} (2n+1)A_n \cdot \frac{\sqrt{2}}{\pi} \int_{\theta}^{\pi} \frac{\sin(n+\frac{1}{2})u\,du}{(\cos\theta - \cos u)^{\frac{1}{2}}} \qquad 0 < \theta < \alpha$$

$$= -\frac{2\sqrt{2}}{\pi} \int_{\theta}^{\pi} \frac{du}{(\cos\theta - \cos u)^{\frac{1}{2}}} \left\{ \frac{d}{du} \sum_{n=o}^{\infty} A_n \cos(n+\tfrac{1}{2})u \right\} \qquad 0 < \theta < \alpha$$

$$= -\frac{2\sqrt{2}}{\pi} \int_{\theta}^{\alpha} \frac{\xi'(u)du}{(\cos\theta - \cos u)^{\frac{1}{2}}} \qquad 0 < \theta < \alpha \quad (16)$$

where the upper limit may now be taken to be α because $\xi'(u) = 0$ if $u > \alpha$. Now, from equations (15) and (16),

$$Q_t = -r\sqrt{2}\, a\varepsilon \int_{o}^{\alpha} \sin\theta\, d\theta \int_{\theta}^{\alpha} \frac{\xi'(u)du}{(\cos\theta - \cos u)^{\frac{1}{2}}} \quad .$$

After interchanging the order of integration, performing the inner integral, and integrating the remaining integral by parts, one may rewrite the above equation as

$$Q_t = 8a\varepsilon \int_{o}^{\alpha} \xi(u)\cos(u/2)du \quad .$$

It is now evident from equation (3) that the capacitance between two bowls of half-angle α is given by

$$\frac{C}{4a\varepsilon} = \int_{o}^{\alpha} \xi_c(\theta)\cos(\theta/2)d\theta$$

where $\xi_c(\theta)$ is the solution of equation (14) when

$$h(\theta) = \frac{1}{\sqrt{2}}\frac{d}{d\theta}\int_0^{\theta}\frac{\sin u\, du}{(\cos u - \cos\theta)^{\frac{1}{2}}} = \cos(\theta/2) \quad .$$

Similarly,

$$\frac{A_{eq}}{3\pi a^2} = \frac{8}{3\pi}\int_0^{\alpha}\xi_a(\theta)\cos(\theta/2)d\theta$$

where $\xi_a(\theta)$ is the solution of equation (14) when

$$h(\theta) = \frac{1}{\sqrt{2}}\frac{d}{d\theta}\int_0^{\theta}\frac{\sin u\cos u\, du}{(\cos\theta - \cos u)^{\frac{1}{2}}} = \cos(\theta/2)\{4\cos^2(\theta/2) - 3\} \quad .$$

We will make one further transformation to simplify equation (14) by setting

$$x = \tan(\theta/2)$$

$$\cos(\theta/2)\xi(\theta) = f(x)$$

$$\cos(\theta/2)h(\theta) = g(x) \quad .$$

These substitutions reduce equation (14) to

$$f(x) - \frac{2}{\pi}\int_0^T\frac{f(x')dx'}{1 - x^2x'^2} = g(x) \ , \qquad 0 < x < T \tag{17}$$

where $T \equiv \tan(\alpha/2)$. The expressions for C and A_{eq} now become

$$\frac{C}{4\varepsilon a} = 2\int_0^T\frac{f_c(x)dx}{1 + x^2} \tag{18}$$

$$\frac{A_{eq}}{3\pi a^2} = \frac{16}{3\pi}\int_0^T\frac{f_a(x)dx}{1 + x^2} \quad , \tag{19}$$

where f_c is determined from equation (17) by setting

$$g(x) = \frac{1}{1 + x^2} \tag{20}$$

and f_a is the solution of equation (17) when

$$g(x) = \frac{1 - 3x^2}{(1 + x^2)^2} \quad . \tag{21}$$

Equations (17) through (21) are the ones on which the numerical work was performed.

III. Some Properties of the Integral Equation

Although we will give a numerical method of obtaining accurate values of the capacitance in the last section of this note, there is still some use in deriving analytical approximations to this quantity. In this section, we will show that the approximations to the capacitance that result from an application of the method of successive approximations [Ref. 8, p. 7] to equation (17) will converge to the true capacitance, but not very quickly. We will then present a variational representation for the capacitance which will be used in the next section to obtain an accurate approximation formula.

The method of successive approximations is the name given to an iterative application of the formula

$$f_n(x) = \frac{1}{1+x^2} + \frac{2}{\pi}\int_0^T \frac{f_{n-1}(x')}{1-x^2x'^2}\,dx' \tag{22}$$

where

$$f_o(x) = \frac{1}{1+x^2} \quad .$$

We will show that if we define

$$\frac{C_n}{4\varepsilon a} \equiv 2\int_0^T \frac{f_n(x)dx}{1+x^2}$$

then

$$\left|\frac{C - C_n}{C}\right| \to 0 \quad \text{as} \quad n \to \infty \quad .$$

In fact, from equations (17) and (22),

$$f(x) - f_n(x) = \frac{2}{\pi}\int_0^T \frac{f(x') - f_{n-1}(x')}{1-x^2x'^2}\,dx' \quad .$$

Thus, defining

$$\varepsilon_n \equiv \int_0^T |f(x) - f_n(x)|dx \quad ,$$

it follows that

$$\varepsilon_n \le \varepsilon_{n-1} \cdot \frac{2}{\pi} \iint_0^T \frac{dx\,dx'}{1 - x^2x'^2} \equiv \varepsilon_{n-1}\kappa \quad , \tag{23}$$

while

$$\begin{aligned}\varepsilon_o &= \frac{2}{\pi} \iint_0^T \frac{f(x')dxdx'}{1 - x^2x'^2} \\ &= \frac{2}{\pi} \iint_0^T \frac{2f(x')}{1 + x'^2} \cdot \frac{1 + x'^2}{2(1 - x^2x'^2)} \cdot dx'dx \\ &\le \frac{C}{4\varepsilon a} \cdot \kappa \quad .\end{aligned} \tag{24}$$

From equations (23) and (24) it follows that

$$\varepsilon_n \le \kappa^{n+1} \cdot \frac{C}{4\varepsilon a} \quad , \tag{25}$$

while, from the definitions of C and C_n,

$$\begin{aligned}\left|\frac{C - C_n}{4\varepsilon a}\right| &= 2\left|\int_0^T \frac{f(x) - f_n(x)}{1 + x^2}\,dx\right| \\ &\le \varepsilon_n \cdot 2\int_0^T \frac{dx}{1 + x^2} \\ &\le \varepsilon_n\alpha \quad .\end{aligned} \tag{26}$$

From equations (25) and (26), it is now clear that

$$\left|\frac{C - C_n}{C}\right| \le \alpha\kappa^{n+1} \quad . \tag{27}$$

But

$$\kappa = \frac{2}{\pi} \iint\limits_{o}^{T} \frac{dxdx'}{1 - x^2 x'^2} = \frac{1}{\pi} \int\limits_{o}^{T} \frac{dx}{x} \ln\left(\frac{1 + Tx}{1 - Tx}\right)$$

$$< \frac{1}{\pi} \int\limits_{o}^{1} \frac{dx}{x} \ln\left(\frac{1 + x}{1 - x}\right)$$

$$< \frac{\pi}{4} < 1 \;;$$

hence equation (27) shows that the approximations to the capacitance converge to the correct value. Nevertheless, since κ can be almost as large as $\pi/4$, it may take several iterations before an acceptable approximation to C is obtained. In fact, if κ is close to its maximum and the equality sign in equation (27) were to hold, it would be necessary to make about thirty successive substitutions to obtain .1% accuracy in the capacitance.

The above results indicate the need for a more efficient method of approximating the capacitance than the straightforward iteration procedure. This more efficient method can be based on a stationary representation for the capacitance which may be derived by the usual method from equations (17), (18), and (20). This representation is

$$\frac{C}{4\varepsilon a} = \left[\int\limits_{o}^{T} \frac{2f(x)dx}{1 + x^2}\right]^2 \left[\int\limits_{o}^{T} 2f^2(x)dx - \frac{4}{\pi} \iint\limits_{o}^{T} \frac{f(x)f(x')}{1 - x^2 x'^2} dxdx'\right]^{-1} \quad . \tag{28}$$

The application of this equation will be considered in the next section.

IV. Asymptotic Approximations

If one substitutes

$$f_o(x) \equiv \frac{1}{1 + x^2}$$

into equation (28), the integrations can be performed easily and the result is

$$\frac{C}{4\varepsilon a} = \frac{\pi\, c_o^2}{c_o(\pi-\alpha) - \frac{1}{2}\,\alpha \sin\alpha + \cos\alpha\, \ln(\cos\alpha)} \tag{29}$$

where

$$c_o = \frac{C_o}{4\varepsilon a} = \frac{\alpha + \sin\alpha}{2} \tag{30}$$

is the normalized zero-order approximation to the capacitance, as defined in the previous section. It is interesting that C_o is also the effective capacitance of two capacitors in series, each of which has a capacitance equal to that of an isolated spherical bowl of half-angle α [Ref. 4, p. 204]. Equation (30) is not very accurate, but equation (29) is accurate to four figures for α less than 50^o. Equation (29) is plotted in figure 2 along with the more accurate values derived from the numerical solution of equation (17).

To obtain an approximation to the capacitance for the case where α is close to $\pi/2$ we will proceed differently. For that case we may make use of previous work [Ref. 3, eqs. (105) and (110)] to say that an approximation to the admittance between the two bowls is

$$Y = Y_o(y_{int} + y_{ext})$$

$$= Y_o \sum_{n\ odd} \frac{i\pi(2n+1)}{n(n+1)} [P_n'(0)]^2\ {}_2F_1\left(-\frac{n+1}{2},\frac{n}{2};1;\beta^2\right)\left\{\frac{ka\ j_n(ka)}{[ka\ j_n(ka)]'} - \frac{ka\ h_n(ka)}{[ka\ h_n(ka)]'}\right\}, \tag{31}$$

where $\beta = \pi/2 - \alpha$, as shown in figure 1. For small k, equation (31) reduces to the admittance of a pure capacitance whose value is given by

$$\frac{C}{4\varepsilon a} = \frac{\pi}{4} \sum_{n\ \mathrm{odd}} \left[\frac{(2n+1)}{n(n+1)} P_n^1(0)\right]^2 {}_2F_1(-\tfrac{n+1}{2},\tfrac{n}{2};1;\beta^2) \quad . \tag{32}$$

But, for n odd [Ref. 10, p. 334],

$$P_n^1(0) = (-)^{\frac{n+1}{2}} \frac{n!!}{(n-1)!!}$$

$$= \frac{2}{\sqrt{\pi}} \frac{\Gamma(n/2 + 1)}{\Gamma(n/2 + .5)} \quad , \tag{33}$$

while [Ref. 7, p. 39 and p. 213],

$${}_2F_1(-\tfrac{n+1}{2},\tfrac{n}{2};1;\beta^2) = P_n^{(0,-3/2)}(1 - 2\beta^2)$$

$$= \beta^2 P_n^{(1,-3/2)}(1 - 2\beta^2) + (1 - \beta^2) P_n^{(0,-\frac{1}{2})}(1 - 2\beta^2) \quad , \tag{34}$$

where $P_n^{(\mu,\nu)}(x)$ is a Jacobi polynomial [Ref. 7, sect. 5.2]. For small β the above equation may be approximated by

$${}_2F_1(-\tfrac{n+1}{2},\tfrac{n}{2};2;\beta^2) \approx P_n^{(0,-\frac{1}{2})}(1 - 2\beta^2) \quad .$$

But [Ref. 7, p. 211 and p. 228]

$$P_n^{(0,-\frac{1}{2})}(1 - 2\beta^2) = P_n^{(0,0)}(\sqrt{1 - \beta^2})$$

$$= P_n(\sqrt{1 - \beta^2}) \tag{35}$$

where $P_n(x)$ is a Legendre polynomial [Ref. 7, sect. 5.4]. Now one can use equations (33), (34), and (35) to write equation (32) in the form

$$\frac{C}{4\varepsilon a} \approx \sum_{n\ \mathrm{odd}} \left[\frac{(2n+1)}{n(n+1)} \cdot \frac{\Gamma(n/2 + 1)}{\Gamma(n/2 + .5)}\right]^2 P_n(\sqrt{1 - \beta^2}) \quad . \tag{36}$$

With the use of the asymptotic formula [Ref. 7, p. 12],

$$\frac{\Gamma(n/2 + 1)}{\Gamma(n/2 + .5)} \approx \sqrt{\frac{n}{2}}\ (1 + \frac{1}{4n}) \quad , \tag{37}$$

equation (36) may be rewritten as

$$\frac{C}{4\varepsilon a} = \frac{1}{2} \sum_{n \text{ odd}} \left[\frac{2n+1}{n+1}\ (1 + \frac{1}{4n})\right]^2 \frac{P_n(\sqrt{1-\beta^2})}{n} + D(\beta) \quad , \tag{38}$$

where $D(\beta)$ accounts for the error introduced by using equation (37) for even the lowest values of n. We will approximate $D(\beta)$ by $D(0)$. Its value, from a numerical summation, is

$$D(0) = .0130 \quad .$$

Now equation (38) may be rewritten

$$\frac{C}{4\varepsilon a} = \sum_{n \text{ odd}} \frac{(2n+1)}{n(n+1)} P_n(\sqrt{1-\beta^2}) + \frac{1}{32} \sum_{n \text{ odd}} \frac{(10n+1)(2n+1)}{n^3(n+1)^2} P_n(\sqrt{1-\beta^2}) + D(0) \quad .$$

The first sum in the above equation may be performed analytically [Ref. 7, p. 239]; the second sum may also be reduced to well-known forms if, in that sum, we set β as zero. These manipulations give

$$\frac{C}{4\varepsilon a} = \ln\left(\frac{1+\sqrt{1-\beta^2}}{\beta}\right) + \frac{1}{32} \left\{\frac{7}{8} \pi^2 + \zeta(3) - \ln 2\right\} + .0130$$

or, neglecting terms of order β^2,

$$\frac{C}{4\varepsilon a} = .9872 - \ln \beta \tag{39}$$

This equation is correct as β approaches zero, and it is in error by less than a quarter of one percent of β is less than a tenth; if plotted on figure 4, it would be indistinguishable from the true curve.

An approximation to A_{eq} for small β may be derived by calculating the charge induced on one half of a closed conducting sphere immersed in a uniform electrostatic field. The result of this calculation is

$$A_{eq} = 3\pi a^2$$

in agreement with reference 3. Actually, if we write

$$A_{eq} = 3\pi a^2 \sin \alpha \qquad (40)$$

we are in error by less than two percent for α greater than 45°.

For small α we may use the zero-order approximate solution of equation (17) to say that

$$\frac{A_{eq}}{3\pi a^2 \sin \alpha} \rightarrow \frac{8}{3\pi} \quad .$$

In the following table we give an indication of the errors involved in the various approximations derived in this section.

Table of Relative Error

α(degrees)	Eqn. (29)	Eqn. (39)	Eqn. (40)
0	0	∞	17.809%
10	$< 10^{-3}$%	254%	12.100%
20	$< 10^{-3}$%	102%	7.841%
30	$< 10^{-3}$%	52%	4.758%
40	- .010%	29%	2.628%
50	- .050%	15%	1.261%
60	- .196%	7.6%	.481%
70	- .758%	3.0%	.120%
80	-3.166%	.7%	.010%
90	- 100%	$\rightarrow 0$	0

V. Discussion of the Numerical Technique

In order to obtain numerical data from equation (17) one can use a numerical quadrature formula for the integral. One of the most accurate of these formulae is Gauss's. Using Gauss's formula, one can write

$$\sum_{j=1}^{N} \left\{ \delta_{ij} - \frac{2}{\pi} \frac{w(x_j)}{1-x_i^2 x_j^2} \right\} f(x_j) = g(x_i) + \varepsilon_N(x_i) \quad , \quad i = 1, \ldots N \tag{41}$$

where

$$x_j = Tx_j^{(N)}$$

$$w(x_j) = Tw_j^{(N)}$$

The $x_j^{(N)}$ are the roots of the Legendre polynomial and the $w_j^{(N)}$ are the Gaussian coefficients for the interval (0,1) [Ref. 10, p. 922].

In equations (41), $\varepsilon_N(x_i)$ represents the error introduced into equation (17) by the use of the numerical integration formula; this error decreases with increasing N. If the $\varepsilon_N(x_i)$ are neglected, the resulting set of equations may be solved to obtain approximate values of the $f(x_i)$. These approximate values may be used in the Gaussian integration of equations (18) and (19) to obtain approximate values of C and A_{eq}. An extensive discussion of this technique has been given by Kantorovich and Krylov [Ref. 9, Chap. II, §1].

Kantorovich and Krylov also deduce bounds on the error involved in the above procedure. Unfortunately, their equation for the error bound involves elements of the matrix inverse to that implied by equations (41). These matrix elements are not produced naturally in any efficient method of solving equations (41), and so we must rely on the brute-force error estimation method of doubling N until the calculated C or A_{eq} remains the same to the number of significant figures desired. However, we need not make this check at every value of T. The error in approximating equation (17) by equations (41) without the $\varepsilon_N(x_i)$ decreases with decreasing T, so that if we find, for instance, an N good enough for a T of .9, that N will also be good enough for all T less than .9.

The numerical results for C and A_{eq} are given in tables 1 through 4 and in figures 2 through 5. The normalizations of C and A_{eq} for tables 1 and 2 and figures 2 and 3 have been chosen to give only a small variation in the tabulated quantities over the whole range of α. Tables 3 and 4 and figures 4 and 5 have been included to give more extensive data for the important values of α close to $\pi/2$ (β close to zero).

Table 1

This table gives the value of the capacitance between two spherical bowls located symmetrically on a sphere of radius a. The capacitance is tabulated as a function of α, the half-angle subtended at the center of the sphere by one of the bowls, and it is normalized to $4\epsilon a \sin \alpha$. The first digit of α, in degrees, is to be read from the left-hand column, while the second digit is to be read from the top row.

α	0	1	2	3	4	5	6	7	8	9
0	1.00000	1.00562	1.01134	1.01719	1.02316	1.02925	1.03547	1.04181	1.04829	1.05491
1	1.06168	1.06859	1.07564	1.08286	1.09023	1.09776	1.10546	1.11334	1.12139	1.12963
2	1.13806	1.14667	1.15550	1.16423	1.17377	1.18324	1.19294	1.20286	1.21304	1.22347
3	1.23416	1.24513	1.25639	1.26793	1.27978	1.29194	1.30443	1.31727	1.33045	1.34402
4	1.35796	1.37231	1.38707	1.40228	1.41794	1.43408	1.45072	1.46790	1.48561	1.50391
5	1.52281	1.54236	1.56258	1.58352	1.60519	1.62767	1.65100	1.67520	1.70036	1.72653
6	1.75377	1.78217	1.81181	1.84275	1.87512	1.90903	1.94459	1.98195	2.02128	2.06273
7	2.10654	2.15293	2.20219	2.25465	2.31066	2.37071	2.43534	2.50521	2.58117	2.66424
8	2.75580	2.85758	2.97200	3.10239	3.25364	3.43334	3.65418	3.93988	4.34380	5.03587

Table 2

This table gives the value of the equivalent area of an electric field sensor made up of two spherical bowls on a sphere of radius a. The equivalent area is tabulated as a function of α, the half-angle subtended at the center of the sphere by one of the bowls, and is normalized to $3\pi a^2 \sin \alpha$. The first digit of α, in degrees, is to be read from the left-hand column, while the second digit is to be read from the top row.

α	0	1	2	3	4	5	6	7	8	9
0	.848826	.853504	.858103	.862625	.867068	.871433	.875719	.879924	.884051	.888097
1	.892063	.895949	.899754	.903478	.907122	.910684	.914168	.917570	.920892	.924133
2	.927294	.930377	.933379	.936302	.939146	.941912	.944601	.947211	.949745	.952203
3	.954584	.956891	.959122	.961280	.963367	.965379	.967321	.969192	.970994	.972727
4	.974392	.975991	.977523	.978990	.980395	.981736	.983017	.984236	.985398	.986501
5	.987548	.988541	.989478	.990364	.991198	.991984	.992720	.993410	.994054	.994654
6	.995212	.995729	.996207	.996648	.997051	.997420	.997756	.998061	.998337	.998584
7	.998803	.998998	.999171	.999321	.999452	.999562	.999657	.999736	.999802	.999855
8	.999898	.999929	.999953	.999972	.999983	.999991	.999996	.999998	.999999	1.00000

Table 3

This table gives the capacitance of a single-gap hollow spherical dipole sensor whose sphere radius is a. The capacitance is normalized to $4\epsilon a$ and is tabulated as a function of the half-angle of the gap, β. The first two decimal points of β, in radians, are to be read from the left-hand column, while the third decimal point is to be read from the top row.

β	0	1	2	3	4	5	6	7	8	9
.00	∞	7.89496	7.20180	6.79632	6.50863	6.28547	6.10313	5.94897	5.81543	5.69765
.01	5.59226	5.49693	5.40988	5.32982	5.25568	5.18665	5.12208	5.06144	5.00424	4.95014
.02	4.89881	4.84997	4.80342	4.75893	4.71633	4.67545	4.63618	4.59840	4.56198	4.52684
.03	4.49289	4.46005	4.42826	4.39742	4.36752	4.33847	4.31025	4.28279	4.25606	4.23001
.04	4.20464	4.17987	4.15571	4.13213	4.10906	4.08652	4.06446	4.04289	4.02177	4.00106
.05	3.98079	3.96092	3.94141	3.92229	3.90353	3.88508	3.86699	3.84920	3.83174	3.81455
.06	3.79766	3.78104	3.76468	3.74859	3.73276	3.71716	3.70181	3.68667	3.67177	3.65706
.07	3.64258	3.62829	3.61422	3.60031	3.58660	3.57308	3.55973	3.54655	3.53355	3.52071
.08	3.50802	3.49549	3.48312	3.47088	3.45880	3.44684	3.43503	3.42336	3.41183	3.40041
.09	3.38912	3.37794	3.36691	3.35597	3.34515	3.33445	3.32385	3.31338	3.30300	3.29271

Table 4

This table gives the equivlent area of a single-gap hollow spherical dipole sensor whose sphere radius is a. This area is normalized to $3\pi a^2$ and is tabulated as a function of the half-angle of the gap, β. The first two decimal points of β, in radians, are to be read from the left-hand column, while the third decimal point is to be read from the top row. In all cases here the first two decimal points of the normalized area are .99; we tabulate the next four decimal points.

β	0	1	2	3	4	5	6	7	8	9
.00		9999	9998	9996	9992	9988	9982	9976	9968	9960
.01	9950	9940	9928	9916	9902	9888	9873	9856	9839	9819
.02	9799	9779	9758	9736	9712	9688	9661	9635	9608	9580
.03	9551	9519	9488	9456	9423	9387	9352	9316	9277	9239
.04	9199	9159	9118	9075	9032	8987	8940	8895	8847	8799
.05	8749	8698	8648	8595	8541	8486	8432	8375	8317	8285
.06	8199	8138	8076	8014	7951	7886	7819	7753	7686	7617
.07	7546	7476	7406	7332	7259	7184	7109	7031	6955	6876
.08	6796	6714	6632	6551	6467	6383	6297	6210	6123	6032
.09	5943	5853	5762	5668	5575	5479	5384	5287	5190	5090

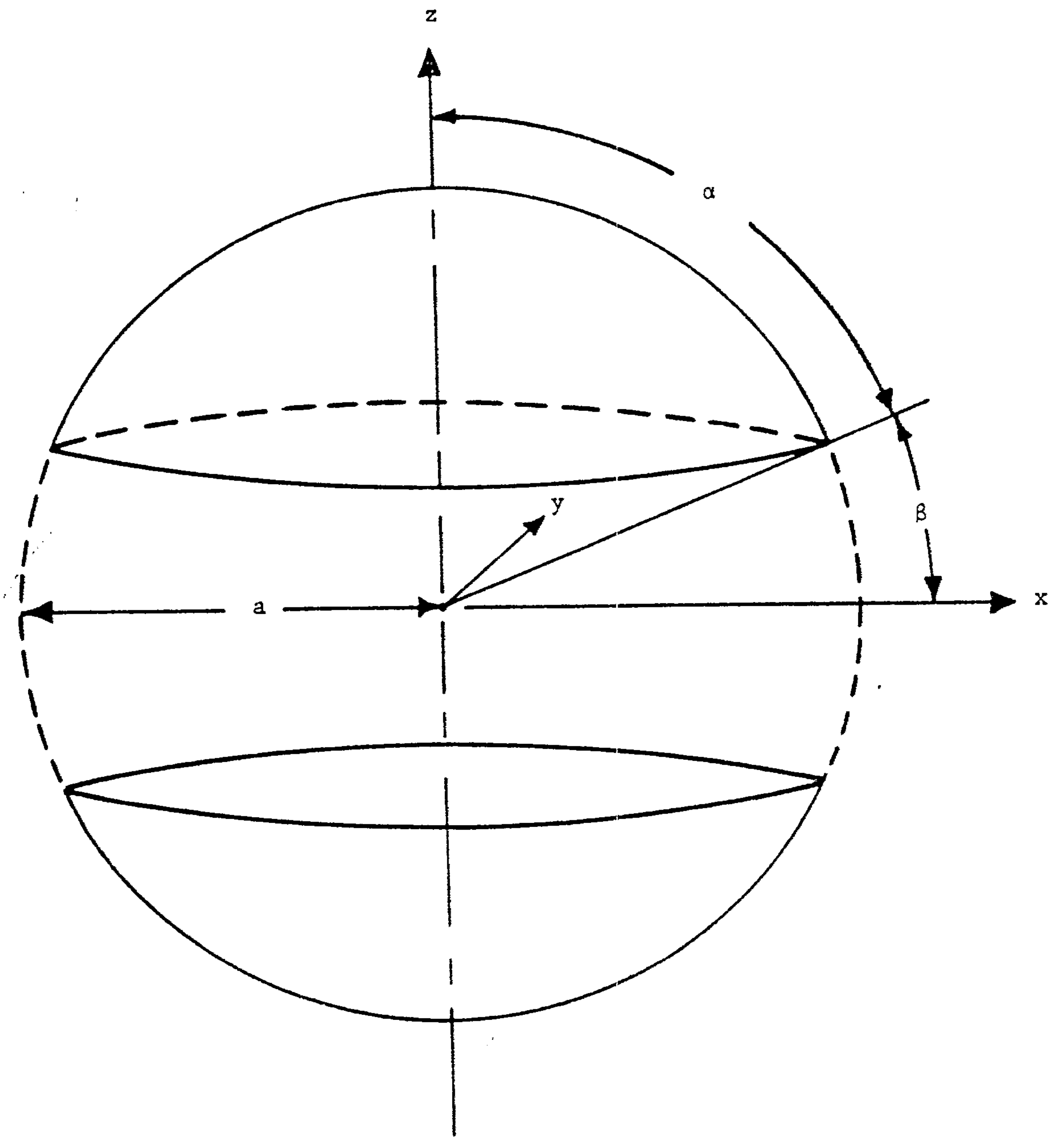

Figure 1: Two identical spherical bowls with half-angle α located symmetrically on the surface of a sphere of radius a.

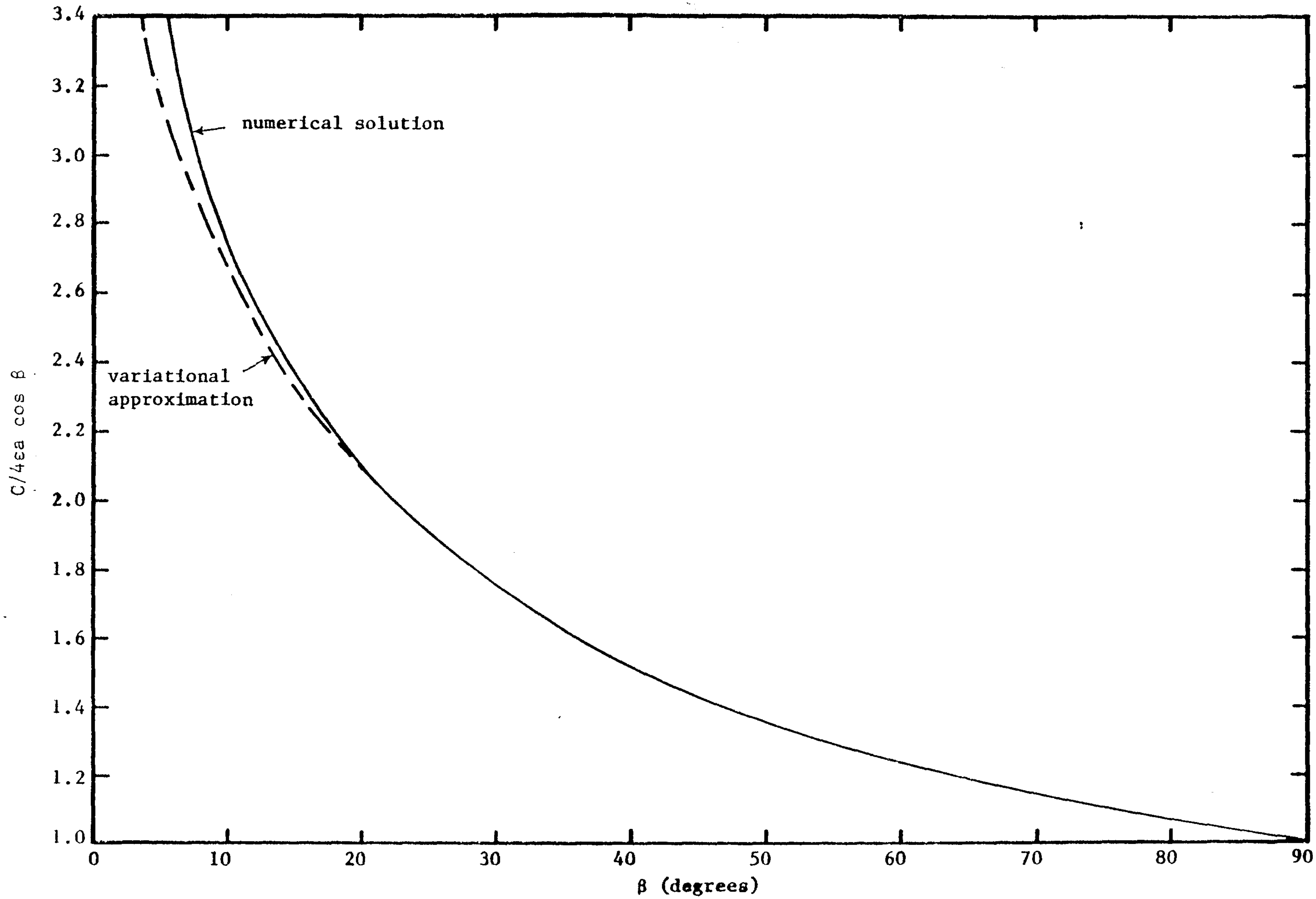

Figure 2: Capacitance between two spherical bowls versus the gap half-angle.

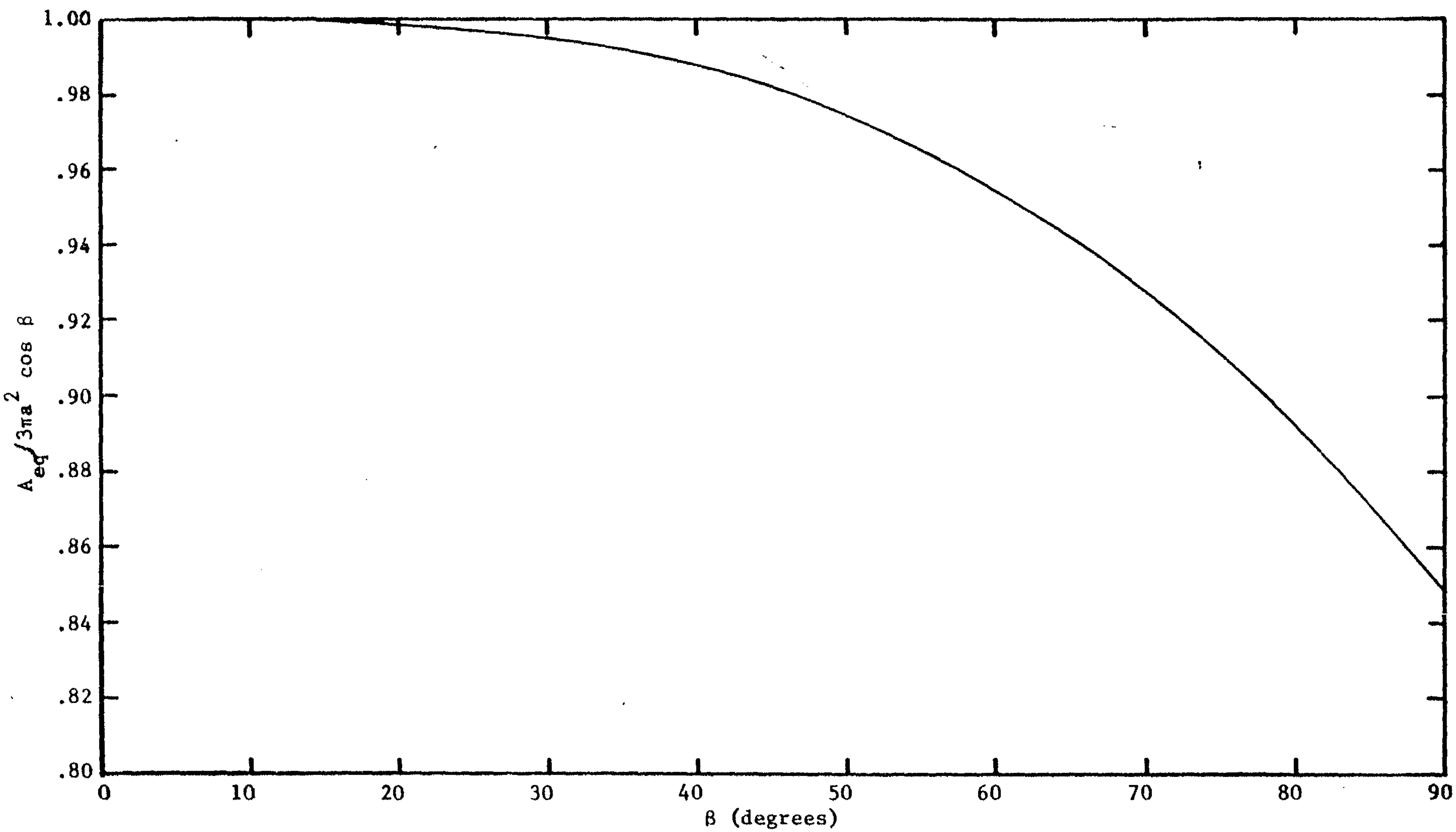

Figure 3: Equivalent area of spherical bowl sensor versus the gap half-angle.

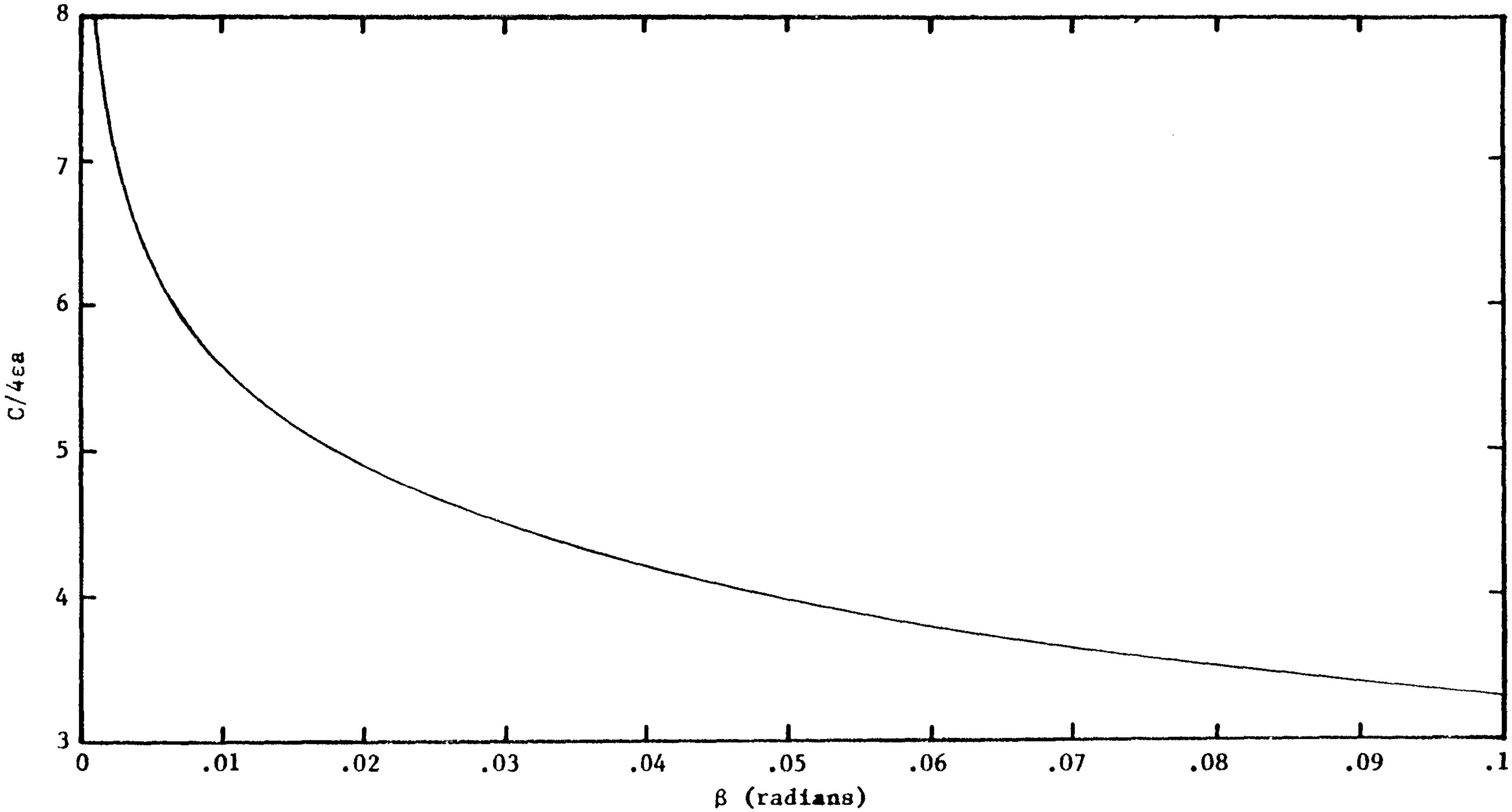

Figure 4: Capacitance of the single-gap hollow spherical dipole versus the gap half-angle.

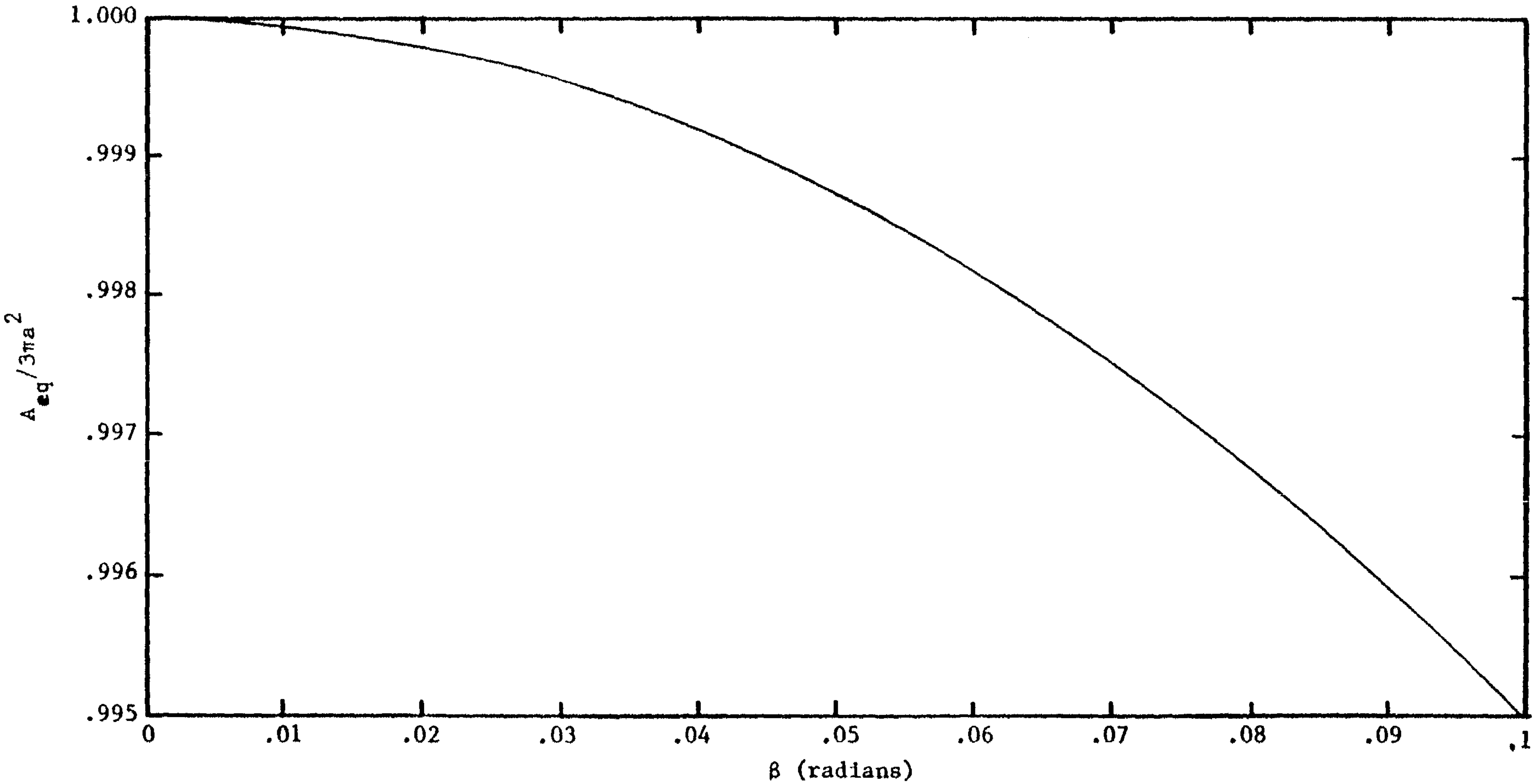

Figure 5: Equivalent area of single-gap hollow spherical dipole versus the gap half-angle.

Appendix

For completeness, we present here the well-known solutions of the two integral equations,

$$\int_0^x \frac{f(y)dy}{(\cos y - \cos x)^{\frac{1}{2}}} = g(x) \qquad 0 < x < \alpha \le \pi \tag{A.1}$$

and

$$\int_x^\pi \frac{f(y)dy}{(\cos x - \cos y)^{\frac{1}{2}}} = g(x) \qquad 0 \le \alpha < x < \pi \quad . \tag{A.2}$$

These equations are special cases of two more general integral equations that have been solved by Srivastav [Ref. 11]. Because the solutions of Srivastav's equations may be useful in other applications, we will present these solutions first. We will then specialize them in order to write the solutions of equations (A.1) and (A.2) in a simple form.

Let us first examine the integral equation,

$$\int_a^x \frac{f(y)dy}{\{w(x) - w(y)\}^\mu} = g(x) \qquad a < x < b \quad , \tag{A.3}$$

where $0 < \mu < 1$ and $w(x)$ possesses a positive derivative in (a,b). If one multiplies both sides of equation (A.3) by

$$\frac{w'(x)}{\{w(z) - w(x)\}^{1-\mu}} \quad ,$$

integrates over x from a to z, and interchanges the order of integration on the left-hand side, he obtains

$$\int_a^z f(y)dy \int_y^z \frac{w'(x)dx}{\{w(x) - w(y)\}^\mu \{w(z) - w(x)\}^{1-\mu}} = \int_a^z \frac{w'(x)g(x)dx}{\{w(z) - w(x)\}^{1-\mu}} \quad . \tag{A.4}$$

In the inner integral on the left-hand side of equation (A.4), one can change to the new independent variable,

$$t = \frac{w(x) - w(y)}{w(z) - w(y)} \quad ,$$

and obtain the integral defining the beta function [Ref. 7, p. 7],

$$\int_0^1 \frac{dt}{t^{\mu}(1 - t)^{1-\mu}} \equiv B(\mu, 1 - \mu)$$

$$= \pi \csc \pi\mu \quad .$$

If one substitutes this value in equation (A.4) and differentiates, he obtains the solution to equation (A.3) in the form

$$f(z) = \frac{\sin \pi\mu}{\pi} \cdot \frac{d}{dz} \int_a^z \frac{w'(x)g(x)dx}{\{w(z) - w(x)\}^{1-\mu}} \quad . \tag{A.5}$$

In a similar manner, Srivastav has also shown that the solution of the equation,

$$\int_x^b \frac{f(y)dy}{\{w(y) - w(x)\}^{\mu}} = g(x) \qquad a < x < b \quad , \tag{A.6}$$

where μ and $w(x)$ have the same properties as in equation (A.3), may be written in the form

$$f(z) = -\frac{\sin \pi\mu}{\pi} \cdot \frac{d}{dz} \int_z^b \frac{w'(x)g(x)dx}{\{w(x) - w(z)\}^{1-\mu}} \quad . \tag{A.7}$$

If we now choose μ to be $\frac{1}{2}$ and $w(x)$ to be $(1 - \cos x)$, it is clear from equation (A.5) that the solution of equation (A.1) is

$$f(z) = \frac{1}{\pi} \cdot \frac{d}{dz} \int_0^z \frac{\sin x \; g(x)dx}{(\cos x - \cos z)^{\frac{1}{2}}} \quad . \tag{A.8}$$

It also follows, as a special case of equation (A.7), that the solution of equation (A.2) is

$$f(z) = -\frac{1}{\pi} \cdot \frac{d}{dz} \int_z^{\pi} \frac{\sin x \; g(x)dx}{(\cos z - \cos x)^{\frac{1}{2}}} \quad . \tag{A.9}$$

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