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Low-Frequency Magnetic Field Distribution for a Simulator
with the Geometry of a Half Toroid Joined to the Surface
of a Medium with Infinite Conductivity

Capt Carl E. Baum
Air Force Weapons Laboratory

Abstract

This note considers the low-frequency magnetic field distribution in a simulator with the geometry of a half toroid connected to a perfectly conducting ground or water surface. The toroid minor radius is restricted to be small compared to its major radius so that for calculating the low-frequency magnetic field the current can be considered as localized on a semicircular current path centered inside the half toroid. Using image theory to give a bent circular current path the magnetic field is found from the vector potential from such a current. The asymptotic forms of the field distribution are found near the center of the simulator and for distances far from the simulator. The field distribution is graphed for a wide range of parameters.

Foreword

The figures are placed after the summary and before the appendices. We would like to thank Mr. Terry L. Brown and Mr. Joe P. Martinez of Dikewood, and Sgt. Richard T. Clark and ALC Robert N. Marks of AFWL for the numerical calculations and graphs.

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I. Introduction

In a previous note¹ we discussed some of the features of an electromagnetic pulse simulator for simulating the case of a pulsed electromagnetic plane wave incident on the earth's surface at some angle of incidence and polarization and with a finite, nonzero (except for special cases) low-frequency content of the appropriate field components. This simulator has the geometry of a half toroid connected to the ground or water surface as shown in figure 1. A pulse generator is located at a particular position in the half toroid and the toroidal structure has impedance loading for shaping the waveform. One varies the angle ξ_1 of the plane of the half toroid with respect to the x axis (vertical), varies the angle ξ_2 of the generator from the topmost position around the half toroid, and varies the angle ξ_3 which positions the connections of the half toroid to the ground or water surface with respect to the system under test; by so doing various angles of incidence and polarization are achieved.

The ground or water conductivity and permittivity influence the fields, both in the case of an incident uniform electromagnetic plane wave and in the simulator geometry being considered. If the ground conductivity is sufficiently high the low-frequency magnetic field near the center of the simulator on the ground surface $((x, y, z) = (0, 0, 0))$ closely approximates the desired low-frequency magnetic field resulting from the incident plane wave as discussed in reference 1. In this note we consider the low-frequency magnetic field throughout the vicinity of the simulator structure for the case of an infinite ground or water conductivity.

Strictly speaking, if the ground or water conductivity is finite then in the low-frequency limit the magnetic field penetrates the lower medium, assuming a distribution dependent on the nonuniform current distribution in the lower medium plus the current in the half toroid above. Such a distribution for the low-frequency magnetic field will not in general be the same as in the case for an infinite ground or water conductivity which we consider in this note. Provided that the lower-medium conductivity is large enough such that there are frequencies of interest with both wavelengths in air large compared to a (the simulator major radius) and skin depths in the lower medium small compared to $a \cos(\xi_1)$, then the case of infinite conductivity for the lower medium gives accurate results for this case for the magnetic field in the upper medium; possible exceptions include positions where the fields are significantly changing

1. Capt Carl E. Baum, Sensor and Simulation Note 94, Some Considerations Concerning a Simulator with the Geometry of a Half Toroid Joined to a Ground or Water Surface, November 1969.

near the lower medium over distances less than or of the order of the skin depth in the lower medium. Perhaps the case of finite ground or water conductivity can be considered in future notes.

Since in this note we consider the case of infinite conductivity for the lower medium and the surface of this medium a plane, then we can use an image of the half toroid below the y, z plane as shown in figure 2 for calculating the fields. Note the choice of coordinates, the same as in reference 1, where the ground or water surface is chosen as the y, z plane with the y axis meeting the half toroid at two positions where the image also meets the half toroid. In this note we are only considering the low-frequency magnetic field which we call \vec{H} for simplicity because the complete waveform is not being considered. Let I be the low-frequency current in the toroid and the image with direction around the complete bent toroid (upper half toroid plus image) as indicated. This low-frequency current is considered as localized on two semicircles of radius a as shown in figures 2 and 3 where a is the major radius of the half toroid; the semicircular current filaments are located inside the half toroid and image of minor radius b where the minor radius is measured perpendicularly from the semicircular curve to the toroidal surface. Since we are considering the limiting case of low frequency (with infinite lower-medium conductivity), then I is uniform around the complete bent toroid and for purposes of calculating the low-frequency magnetic field distribution the generator position is then insignificant. Note that in calculating the fields in this note the presence of any system under test (as shown in figure 1) is not included. Such systems will distort the fields and interact with the simulator structure. Perhaps idealized geometries of such systems can be considered in future notes.

Besides the cartesian coordinates (x, y, z) we define a cylindrical coordinate system (Ψ, ϕ, z) with the relations²

$$x = \Psi \cos(\phi) , \quad y = \Psi \sin(\phi) \quad (1)$$

We also have spherical coordinates (r, θ, ϕ) with

$$z = r \cos(\theta) , \quad \Psi = r \sin(\theta) \quad (2)$$

For unit vectors we use the symbol \vec{e} with the particular coordinate added as a subscript.

2. All units are rationalized MKSA.

Figure 3A shows a full perspective view of the line current taken inside the toroid plus image on $r = a$ on two planes each containing the y axis and canted at angles ξ_1 with respect to the positive and negative x axes respectively. For positive ξ_1 this current path is only in the half space $z > 0$. Note the convention for positive I in the figure; for the special case of $\xi_1 = 0$ then I is taken positive in the direction of \hat{e}_ϕ .

Figure 3B shows one half of the bent circular current path with its own coordinate systems. There are cartesian coordinates (u_1, u_2, u_3) and cylindrical coordinates (λ, β, u_3) related by

$$u_1 = \lambda \cos(\beta) , \quad u_2 = \lambda \sin(\beta) \quad (3)$$

There is also the coordinate β_1 which replaces β as an integration variable when integrating over the current I on $(\lambda, u_3) = (a, 0)$. Note that all of these coordinate systems are right handed. This particular set of coordinates shown in figure 3B is used in section III to find the magnetic field from a semi-circular current filament which is then applied by superposition to both halves of the bent circular current filament to give the total low-frequency magnetic field.

For convenience define a normalized low-frequency magnetic field as

$$\vec{h} \equiv \frac{2a}{I} \vec{H} \quad (4)$$

Defining a position vector as

$$\vec{r} \equiv r \vec{e}_r = x \vec{e}_x + y \vec{e}_y + z \vec{e}_z = \psi \vec{e}_\psi + z \vec{e}_z \quad (5)$$

we have at $\vec{r} = \vec{0}$ from reference 1 the result that in the low-frequency limit for infinite lower-medium conductivity

$$\vec{H} \Big|_{\vec{r}=\vec{0}} = \frac{I}{2a} \cos(\xi_1) \vec{e}_z \quad (6)$$

so that we also have

$$\vec{h}|_{\vec{r}=\vec{0}} = \cos(\xi_1)\vec{e}_z \quad (7)$$

Thus for $\xi_1 = 0$ we have a unit normalized magnetic field at the coordinate origin. Since a uniform plane wave incident on the surface of a perfect conductor would produce a uniform resulting magnetic field at low frequencies over restricted dimensions near the conducting plane one is then interested in the deviation of the field in the simulator from a uniform field at low frequencies. For this purpose define a normalized difference field as

$$\begin{aligned} \Delta\vec{h} &\equiv \vec{h} - \cos(\xi_1)\vec{e}_z \\ &= h_x\vec{e}_x + h_y\vec{e}_y + [h_z - \cos(\xi_1)]\vec{e}_z \end{aligned} \quad (8)$$

so that at the origin

$$\Delta\vec{h}|_{\vec{r}=\vec{0}} = \vec{0} \quad (9)$$

and the origin is used as the reference point for the magnetic field distribution. For the plots we also divide the various components of \vec{h} as well as scalars such as $|\Delta\vec{h}|$ by $\cos(\xi_1)$ in order to show the size of the various quantities relative to the field at the origin; thereby one obtains fractional deviations of the field components and magnitude of the difference field from the reference field in equation 7.

II. Low-Frequency Magnetic Field in Flat Toroid

First consider the somewhat simpler special case of a flat toroid specified by $\xi_1 = 0$. This case is worked out in Smythe³ and is included for comparison to and as a special case of the more general result in the next section.

First we go through some preliminary considerations. The magnetic field is related to a vector potential as

$$\vec{B} = \mu \vec{H} = \nabla \times \vec{A} \quad (10)$$

where μ is the permeability of the medium being considered (μ_0 in our case). For our static problem the vector potential can be written as a volume integral over the current density in the form

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \int_V \frac{\vec{J}(\vec{r}_i) dV_i}{|\vec{r} - \vec{r}_i|} \quad (11)$$

where V is the volume of integration. A subscript i with coordinates (like \vec{r}_i , dV_i) is used to indicate the position of the quantity over which the integration is being performed, as distinguished from the observer position where the appropriate electromagnetic quantity is being calculated. If the current density is localized to give a current I on a contour with a unit tangent vector \vec{e}_s (with I taken positive in this direction) then equation 11 becomes

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \int_C \frac{I(\vec{r}_i) \vec{e}_s(\vec{r}_i)}{|\vec{r} - \vec{r}_i|} ds \quad (12)$$

where C is the integration contour for the line integral and s is the arc length parameter. Thus I and \vec{e}_s as well as \vec{r}_i can be considered as functions of s . The magnetic field components in cylindrical coordinates can be calculated from

3. W. R. Smythe, *Static and Dynamic Electricity*, 3rd ed., McGraw-Hill, 1968, pp. 290-291.

$$\begin{aligned}
B_\psi &= \mu H_\psi = \frac{1}{\psi} \frac{\partial}{\partial \phi} A_z - \frac{\partial}{\partial z} A_\phi \\
B_\phi &= \mu H_\phi = \frac{\partial}{\partial z} A_\psi - \frac{\partial}{\partial \psi} A_z \\
B_z &= \mu H_z = \frac{1}{\psi} \frac{\partial}{\partial \psi} (\psi A_\phi) - \frac{1}{\psi} \frac{\partial}{\partial \phi} A_\psi
\end{aligned} \tag{13}$$

Now we consider Smythe's derivation for the case of $\xi_1 = 0$. In this case the bent circular current path (shown in figure 3A) lies on the circle $(\psi, z) = (a, 0)$. The unit tangent on the contour in equation 12 becomes

$$\vec{e}_s = \vec{e}_\phi \tag{14}$$

so that the current I is taken positive in the $+\phi$ direction. Since we have a single closed contour for the line integral in equation 12 and since we are considering a static problem then I is independent of ϕ and we have by symmetry

$$A_\psi = A_z = 0 \tag{15}$$

$$A_\phi = \frac{\mu I a}{2\pi} \int_0^\pi \frac{\cos(\phi_i) d\phi_i}{|\vec{r} - \vec{r}_i|}$$

with A_ϕ independent of ϕ and where

$$|\vec{r} - \vec{r}_i| = [a^2 + \psi^2 + z^2 - 2a\psi \cos(\phi_i)]^{1/2} \tag{16}$$

The result is

$$A_\phi = \frac{\mu I}{\pi} \left(\frac{a}{\psi}\right)^{1/2} \left\{ \left[1 - \frac{m}{2}\right] K(m) - E(m) \right\} \tag{17}$$

where the parameter is taken as

$$m \equiv \frac{4a\psi}{(a+\psi)^2+z^2} = \frac{\frac{4\psi}{a}}{\left(1+\frac{\psi}{a}\right)^2 + \left(\frac{z}{a}\right)^2} \quad (18)$$

The elliptic integrals (K and E) are discussed in appendix A. The resulting magnetic field has no z component and has

$$\begin{aligned} B_\psi &= \mu H_\psi = -\frac{\partial}{\partial z} A_\phi \\ &= \frac{\mu I}{2\pi} \frac{z}{\psi} [(a+\psi)^2+z^2]^{-1/2} \left\{ -K(m) + \frac{a^2+\psi^2+z^2}{(a-\psi)^2+z^2} E(m) \right\} \\ &= \frac{\mu I}{2\pi} \frac{z}{\psi} [(a+\psi)^2+z^2]^{-1/2} \left\{ -K(m) + \frac{2-m}{2m_1} E(m) \right\} \end{aligned} \quad (19)$$

$$\begin{aligned} B_z &= \mu H_z = \frac{1}{\psi} \frac{\partial}{\partial \psi} (\psi A_\phi) \\ &= \frac{\mu I}{2\pi} [(a+\psi)^2+z^2]^{-1/2} \left\{ K(m) + \frac{a^2-\psi^2-z^2}{(a-\psi)^2+z^2} E(m) \right\} \\ &= \frac{\mu I}{2\pi} [(a+\psi)^2+z^2]^{-1/2} \left\{ K(m) - \frac{1}{2m_1} \left[2 - \left(1+\frac{a}{\psi}\right)m \right] E(m) \right\} \end{aligned}$$

For convenience we have defined a normalized magnetic field \vec{h} by equation 4. For our present case of $\xi_1 = 0$ we then have only ψ and z components given by

$$\begin{aligned}
h_{\psi} &= \frac{2a}{I} H_{\psi} \\
&= \frac{1}{\pi} \frac{z}{a} \frac{a}{\psi} \left[\left(1 + \frac{\psi}{a} \right)^2 + \left(\frac{z}{a} \right)^2 \right]^{-1/2} \left\{ -K(m) + \frac{2-m}{2m_1} E(m) \right\}
\end{aligned} \tag{20}$$

$$\begin{aligned}
h_z &= \frac{2a}{I} H_z \\
&= \frac{1}{\pi} \left[\left(1 + \frac{\psi}{a} \right)^2 + \left(\frac{z}{a} \right)^2 \right]^{-1/2} \left\{ K(m) - \frac{1}{2m_1} \left[2 - \left(1 + \frac{a}{\psi} \right) m \right] E(m) \right\}
\end{aligned}$$

There are some special cases for which the results simplify considerably. On the z axis we have only a z component given by

$$\vec{h} \Big|_{\psi=0} = \left[1 + \left(\frac{z}{a} \right)^2 \right]^{-3/2} \vec{e}_z \tag{21}$$

On the $z = 0$ plane we have

$$\begin{aligned}
m \Big|_{z=0} &= 4 \frac{\psi}{a} \left[1 + \frac{\psi}{a} \right]^{-2} \\
\vec{h} \Big|_{z=0} &= \frac{1}{\pi} \left[1 + \frac{\psi}{a} \right]^{-1} \left\{ K(m) + \frac{1 + \frac{\psi}{a}}{1 - \frac{\psi}{a}} E(m) \right\} \vec{e}_z
\end{aligned} \tag{22}$$

Note for $0 < \psi < a$ that h_z is positive while for $\psi > a$ that h_z is negative on the $z = 0$ plane.

This section has considered the special case of $\xi_1 = 0$ as an introduction to the more general case considered in the next section. The graphs for the low-frequency magnetic field distribution are considered in section VI.

III. Low-Frequency Magnetic Field in Bent Toroid

Now consider the low-frequency magnetic field distribution associated with a uniform current I around a bent circular path lying in two planes. To do this we consider the vector potential and magnetic field associated with a semicircular current path carrying a uniform current I as in figure 3B. By taking two such semicircular current paths and appropriately orienting them in space and adding the fields from the two semicircular paths the fields from the bent circular path are obtained. Note that one cannot have a semicircular current path carrying a uniform static current because of the charge buildup at the two ends of the path. However the calculation of the magnetic field by a superposition of the fields for two semicircular paths is still legitimate. The fields for one semicircular current path are calculated as an intermediate result. Note that for the semicircular current path in general $\nabla \times \vec{h} \neq \vec{0}$ because the current has a divergence, while for the full bent current path the curl of \vec{h} is zero.

In section III A we first consider the magnetic field associated with a uniform semicircular current path. Second the transformations required to apply the results for a semicircular path to a bent circular path are developed in section III B.

A. Low-frequency magnetic field associated with a uniform semicircular current filament

Consider the geometry in figure 3B which is used for both halves of the bent circular current path. Here we consider the static magnetic field associated with one half of the bent circular current path without specifying which half. The results are used later for a superposition of the fields associated with both halves. As shown in figure 3B we have cartesian coordinates (u_1, u_2, u_3) and cylindrical coordinates (Ψ, β, u_3) related as in equations 3. There is a coordinate β_i which is used as an integration variable in place of β . For convenience define

$$\beta_{\Delta} \equiv \beta_i - \beta \quad (23)$$

The vector potential and magnetic field calculated here (section III A) are related to these special coordinate systems. The vector components are designated by subscripts 1, 2, and 3 for the cartesian case and by subscripts λ , β , and 3 for the cylindrical case. These are not the total vector potential or magnetic field and the special coordinate reference identifies this fact. From the vector potential we obtain the magnetic field by using

$$B_\lambda = \mu H_\lambda = \frac{1}{\lambda} \frac{\partial}{\partial \phi} A_3 - \frac{\partial}{\partial u_3} A_\beta$$

$$B_\beta = \mu H_\beta = \frac{\partial}{\partial z} A_\lambda - \frac{\partial}{\partial \lambda} A_3 \quad (24)$$

$$B_3 = \mu H_3 = \frac{1}{\lambda} \frac{\partial}{\partial \lambda} (\lambda A_\beta) - \frac{1}{\lambda} \frac{\partial}{\partial \beta} A_\lambda$$

Now since no current is parallel to the u_3 axis then from equation 11 we have

$$A_3 = 0 \quad (25)$$

In terms of a normalized magnetic field as in equation 4 we then have

$$h_\lambda = - \frac{2a}{\mu I} \frac{\partial}{\partial u_3} A_\beta$$

$$h_\beta = \frac{2a}{\mu I} \frac{\partial}{\partial u_3} A_\lambda \quad (26)$$

$$h_3 = \frac{2a}{\mu I} \left[\frac{1}{\lambda} \frac{\partial}{\partial \lambda} (\lambda A_\beta) - \frac{1}{\lambda} \frac{\partial}{\partial \beta} A_\lambda \right]$$

Converting to cartesian form we use

$$h_1 = \cos(\beta) h_\lambda - \sin(\beta) h_\beta \quad (27)$$

$$h_2 = \sin(\beta) h_\lambda + \cos(\beta) h_\beta$$

The parameter for use with the elliptic integrals in this case has the form

$$m \equiv \frac{4a\lambda}{(a+\lambda)^2 + u_3^2} = \frac{\frac{4\lambda}{a}}{\left(1 + \frac{\lambda}{a}\right)^2 + \left(\frac{u_3}{a}\right)^2} \quad (28)$$

Now consider the vector potential which can be calculated from

$$\vec{A} = \frac{\mu I a}{4\pi} \int_0^\pi \frac{\vec{e}_{\beta_i} d\beta_i}{|\vec{r} - \vec{r}_i|} \quad (29)$$

where

$$|\vec{r} - \vec{r}_i| = \left[a^2 + \lambda^2 + u_3^2 - 2a\lambda \cos(\beta - \beta_i) \right]^{1/2} \quad (30)$$

Define

$$\psi \equiv \frac{\pi + (\beta - \beta_i)}{2}, \quad d\psi = -\frac{1}{2} d\beta_i$$

giving the relations

$$\begin{aligned} \cos(\beta - \beta_i) &= \cos(2\psi - \pi) = -\cos(2\psi) \\ &= 2 \sin^2(\psi) - 1 \\ \sin(\beta - \beta_i) &= \sin(2\psi - \pi) = -\sin(2\psi) \\ &= -2 \sin(\psi) \cos(\psi) \end{aligned} \quad (31)$$

Then we have

$$\begin{aligned}
|\vec{r} - \vec{r}_i|^2 &= (a+\lambda)^2 + u_3^2 - 4a\lambda \sin^2(\psi) \\
&= [(a+\lambda)^2 + u_3^2] [1 - m \sin^2(\psi)] \\
&= a^2 \left[\left(1 + \frac{\lambda}{a}\right)^2 + \frac{u_3^2}{a^2} \right] [1 - m \sin^2(\psi)]
\end{aligned} \tag{32}$$

The components of the vector potential are then

$$\begin{aligned}
A_\lambda &= - \frac{\mu I a}{4\pi} \int_0^\pi \frac{\sin(\beta_i - \beta) d\beta_i}{|\vec{r} - \vec{r}_i|} \\
&= - \frac{\mu I a}{2\pi} [(a+\lambda)^2 + u_3^2]^{-1/2} T_\lambda \\
&= - \frac{\mu I}{2\pi} \left[\left(1 + \frac{\lambda}{a}\right)^2 + \left(\frac{u_3}{a}\right)^2 \right]^{-1/2} T_\lambda \\
&= - \frac{\mu I}{4\pi} \left(\frac{a}{\lambda}\right)^{1/2} m^{1/2} T_\lambda \\
A_\beta &= \frac{\mu I a}{4\pi} \int_0^\pi \frac{\cos(\beta_i - \beta) d\beta_i}{|\vec{r} - \vec{r}_i|} \\
&= \frac{\mu I}{2\pi} a [(a+\lambda)^2 + u_3^2]^{-1/2} T_\beta \\
&= \frac{\mu I}{2\pi} \left[\left(1 + \frac{\lambda}{a}\right)^2 + \left(\frac{u_3}{a}\right)^2 \right]^{-1/2} T_\beta \\
&= \frac{\mu I}{4\pi} \left(\frac{a}{\lambda}\right)^{1/2} m^{1/2} T_\beta
\end{aligned} \tag{33}$$

$$A_3 = 0$$

where we have defined two integrals T_λ and T_β which can be written, using equations 30 through 32, as

$$\begin{aligned}
 T_\lambda &= \frac{1}{2} \int_0^\pi \frac{\sin(\beta_i - \beta) d\beta_i}{[1 - m \sin^2(\psi)]^{1/2}} \\
 &= 2 \int_{\frac{\beta}{2}}^{\frac{\pi + \beta}{2}} \frac{\sin(\psi) \cos(\psi)}{[1 - m \sin^2(\psi)]^{1/2}} d\psi \\
 T_\beta &= \frac{1}{2} \int_0^\pi \frac{\cos(\beta_i - \beta) d\beta_i}{[1 - m \sin^2(\psi)]^{1/2}} \\
 &= \int_{\frac{\beta}{2}}^{\frac{\pi + \beta}{2}} \frac{2 \sin^2(\psi) - 1}{[1 - m \sin^2(\psi)]^{1/2}} d\psi
 \end{aligned} \tag{34}$$

Appendix A considers some properties of the elliptic integrals and their derivatives. In appendix B (equations B4 and B6) the integrals in equations 34 are solved giving

$$\begin{aligned}
 T_\lambda &= \frac{2}{m} \left\{ \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{1/2} - \left[1 - m \cos^2\left(\frac{\beta}{2}\right) \right]^{1/2} \right\} \\
 T_\beta &= -\frac{2}{m} \left[E\left(\frac{\pi + \beta}{2} \middle| m\right) - E\left(\frac{\beta}{2} \middle| m\right) \right] \\
 &\quad + \left(\frac{2}{m} - 1\right) \left[F\left(\frac{\pi + \beta}{2} \middle| m\right) - F\left(\frac{\beta}{2} \middle| m\right) \right]
 \end{aligned} \tag{35}$$

where the elliptic integral notation (E, F, and K of one or two arguments as appropriate) is explained in appendix A. Using these results with equations 28 and 33 gives the vector potential for the semicircular current path.

In order to calculate \vec{h} for the semicircular current path we first calculate some derivatives of m from equations 33; these are

$$\begin{aligned}\frac{\partial m}{\partial \lambda} &= \frac{4a \left[(a+\lambda)^2 + u_3^2 \right] - 8a\lambda(a+\lambda)}{\left[(a+\lambda)^2 + u_3^2 \right]^2} \\ &= \frac{m}{\lambda} - \frac{1}{2\lambda} \left(1 + \frac{\lambda}{a} \right) m^2\end{aligned}\quad (36)$$

$$\frac{\partial m}{\partial u_3} = \frac{-8a\lambda u_3}{\left[(a+\lambda)^2 + u_3^2 \right]^2} = -\frac{u_3}{2a\lambda} m^2$$

There are also the useful derivatives

$$\begin{aligned}\frac{\partial}{\partial \lambda} \left[(a+\lambda)^2 + u_3^2 \right]^{-1/2} &= -(a+\lambda) \left[(a+\lambda)^2 + u_3^2 \right]^{-3/2} \\ &= -\frac{1}{8} a^{-1/2} \left(1 + \frac{\lambda}{a} \right) \left(\frac{m}{\lambda} \right)^{3/2}\end{aligned}\quad (37)$$

$$\begin{aligned}\frac{\partial}{\partial u_3} \left[(a+\lambda)^2 + u_3^2 \right]^{-1/2} &= -u_3 \left[(a+\lambda)^2 + u_3^2 \right]^{-3/2} \\ &= -\frac{u_3}{8} \left(\frac{m}{a\lambda} \right)^{3/2}\end{aligned}$$

Now we calculate the components of \vec{h} from equations 26 and 33; these are

$$\begin{aligned}h_\lambda &= -\frac{a^2}{\pi} \left\{ -\frac{u_3}{8} \left(\frac{m}{a\lambda} \right)^{3/2} T_\beta + \frac{1}{2} \left(\frac{m}{a\lambda} \right)^{1/2} \left[-\frac{u_3}{2\lambda a} m^2 \right] \frac{\partial}{\partial m} T_\beta \right\} \\ &= \frac{1}{8\pi} u_3 a^{1/2} \left(\frac{m}{\lambda} \right)^{3/2} \left\{ T_\beta + 2m \frac{\partial}{\partial m} T_\beta \right\}\end{aligned}$$

$$\begin{aligned}
h_\beta &= -\frac{a^2}{\pi} \left\{ -\frac{u_3}{8} \left(\frac{m}{a\lambda}\right)^{3/2} T_\lambda + \frac{1}{2} \left(\frac{m}{a\lambda}\right)^{1/2} \left[-\frac{u_3}{2\lambda a} m^2 \right] \frac{\partial}{\partial m} T_\lambda \right\} \\
&= \frac{1}{8\pi} u_3 a^{1/2} \left(\frac{m}{\lambda}\right)^{3/2} \left\{ T_\lambda + 2m \frac{\partial}{\partial m} T_\lambda \right\}
\end{aligned} \tag{38}$$

$$\begin{aligned}
h_3 &= \frac{a^2}{\pi\lambda} \left\{ \frac{1}{2} \left(\frac{m}{a\lambda}\right)^{1/2} T_\beta - \frac{1}{8} (a\lambda)^{-1/2} \left(1 + \frac{\lambda}{a}\right) m^{3/2} T_\beta \right. \\
&\quad \left. + \frac{1}{2} \left(\frac{\lambda m}{a}\right)^{1/2} \left[\frac{m}{\lambda} - \frac{1}{2\lambda} \left(1 + \frac{\lambda}{a}\right) m^2 \right] \frac{\partial}{\partial m} T_\beta + \frac{1}{2} \left(\frac{m}{a\lambda}\right)^{1/2} \frac{\partial}{\partial \beta} T_\lambda \right\} \\
&= \frac{1}{2\pi} \left(\frac{a}{\lambda}\right)^{3/2} m^{1/2} \left\{ \left[1 - \frac{1}{4} \left(1 + \frac{\lambda}{a}\right) m \right] T_\beta \right. \\
&\quad \left. + \left[m - \frac{1}{2} \left(1 + \frac{\lambda}{a}\right) m^2 \right] \frac{\partial}{\partial m} T_\beta + \frac{\partial}{\partial \beta} T_\lambda \right\}
\end{aligned}$$

In appendix B we have the derivatives

$$\begin{aligned}
\frac{\partial}{\partial \beta} T_\lambda &= -\sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \left\{ \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} + \left[1 - m \cos^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right\} \\
\frac{\partial}{\partial m} T_\lambda &= \frac{1}{m^2} \left\{ -\left[2 - m \sin^2\left(\frac{\beta}{2}\right) \right] \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right. \\
&\quad \left. + \left[2 - m \cos^2\left(\frac{\beta}{2}\right) \right] \left[1 - m \cos^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right\} \\
\frac{\partial}{\partial m} T_\beta &= \frac{1}{m^2} \frac{4-3m}{2m_1} \left\{ E\left(\frac{\pi+\beta}{2} \middle| m\right) - E\left(\frac{\beta}{2} \middle| m\right) \right\} \\
&\quad - \frac{1}{m^2} \frac{4-m}{2} \left\{ F\left(\frac{\pi+\beta}{2} \middle| m\right) - F\left(\frac{\beta}{2} \middle| m\right) \right\}
\end{aligned} \tag{39}$$

$$\begin{aligned}
& + \frac{1}{m} \frac{2-m}{2m_1} \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \left\{ \left[1-m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right. \\
& \left. + \left[1-m \cos^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right\}
\end{aligned}$$

found in equations B7, B9, and B11. Substituting the results of equations 35 and 39 into equations 38 gives

$$\begin{aligned}
h_\lambda &= \frac{1}{8} \frac{u_3}{\lambda} \left(\frac{a}{\lambda} m\right)^{1/2} \left\{ \frac{2-m}{m_1} \left[E\left(\frac{\pi+\beta}{2} \middle| m\right) - E\left(\frac{\beta}{2} \middle| m\right) \right] \right. \\
& \quad \left. - 2 \left[F\left(\frac{\pi+\beta}{2} \middle| m\right) - F\left(\frac{\beta}{2} \middle| m\right) \right] \right. \\
& \quad \left. + \frac{m}{m_1} (2-m) \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \left\{ \left[1-m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right. \right. \\
& \quad \left. \left. + \left[1-m \cos^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right\} \right\} \\
h_\beta &= \frac{1}{4\pi} \frac{u_3}{\lambda} \left(\frac{a}{\lambda} m\right)^{1/2} \left\{ - \left[1-m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} + \left[1-m \cos^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right\} \quad (40) \\
h_3 &= \frac{1}{2\pi} \left(\frac{a}{\lambda}\right)^{3/2} m^{1/2} \left\{ \frac{1}{4m_1} \left[\left(1+\frac{\lambda}{a}\right)m - 2\frac{\lambda}{a} \right] \left[E\left(\frac{\pi+\beta}{2} \middle| m\right) - E\left(\frac{\beta}{2} \middle| m\right) \right] \right. \\
& \quad \left. + \frac{1}{2} \frac{\lambda}{a} \left[F\left(\frac{\pi+\beta}{2} \middle| m\right) - F\left(\frac{\beta}{2} \middle| m\right) \right] \right. \\
& \quad \left. + \frac{m}{4m_1} \left[\left(1+\frac{\lambda}{a}\right)m - 2\frac{\lambda}{a} \right] \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \left\{ \left[1-m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right. \right. \\
& \quad \left. \left. + \left[1-m \cos^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right\} \right\}
\end{aligned}$$

Using equations 3 and 27 these results for the cylindrical coordinate system (λ, β, u_3) can be converted to the cartesian coordinate system (u_1, u_2, u_3) . Thus we have the normalized magnetic field for the semicircular current path.

B. Low-frequency magnetic field for bent circular current path

With equations 3, 27, 28, and 40 we have the normalized magnetic field for the case of a uniform current I on a semicircular current path as illustrated in figure 3B. Now we apply these results to the two halves of the bent circular current path with uniform current I as illustrated in figure 3A. To do this we consider two cases designated by a prime and a double prime for fields and coordinates. The normalized field \vec{h} is written as

$$\vec{h} = \vec{h}' + \vec{h}'' \quad (41)$$

where \vec{h} , \vec{h}' , and \vec{h}'' are all considered with respect to the coordinate systems (x, y, z) , (Ψ, ϕ, z) , and (r, θ, ϕ) shown in figure 3A. The prime case \vec{h}' is for the normalized magnetic field associated with the semicircular current path in the half space $x > 0$ (with the restriction $|\xi_1| < \pi/2$); the double prime case \vec{h}'' is for the normalized magnetic field associated with the semicircular current path in the half space $x \leq 0$.

Note that any field components written in terms of the coordinate systems (u_1, u_2, u_3) and (λ, β, u_3) are taken from the solution for the semicircular current path in equations 3, 27, 28, and 40. This solution is used twice with different definitions of the coordinates and field components when expressed in the (x, y, z) , (Ψ, ϕ, z) , and/or (r, θ, ϕ) coordinate systems.

Case 1: Prime

Consider \vec{h}' in (x, y, z) coordinates. To do this move the (u_1, u_2, u_3) coordinate system such that the semicircular current path (including current direction) is aligned with that half of the bent circular current path situated in $x > 0$ as shown in figure 3A. This requires the coordinate transformation

$$\begin{aligned} u_1' &= -y \\ u_2' &= \cos(\xi_1)x + \sin(\xi_1)z \\ u_3' &= -\sin(\xi_1)x + \cos(\xi_1)z \end{aligned} \quad (42)$$

with

$$\lambda' \cos(\beta') = u_1' , \quad \lambda' \sin(\beta') = u_2' \quad (43)$$

or in another form

$$\lambda' = \left[u_1'^2 + u_2'^2 \right]^{\frac{1}{2}} = \left[\cos^2(\xi_1) x^2 + y^2 + \sin^2(\xi_1) z^2 + 2 \sin(\xi_1) \cos(\xi_1) xz \right]^{\frac{1}{2}} \quad (44)$$

$$\cos(\beta') = \frac{u_1'}{\lambda'} , \quad \sin(\beta') = \frac{u_2'}{\lambda'}$$

The parameter for the elliptic functions is then calculated from

$$m' = \frac{4 \frac{\lambda'}{a}}{\left(1 + \frac{\lambda'}{a}\right)^2 + \left(\frac{u_3'}{a}\right)^2} = 1 - m_1' \quad (45)$$

where λ' and u_3' are taken from equations 42 and 44. Substitute m' and β' for m and β respectively in equations 40 and call the results h_{λ}' , h_{β}' , and h_3' . Then calculate h_1' and h_2' as

$$h_1' = \cos(\beta') h_{\lambda}' - \sin(\beta') h_{\beta}' \quad (46)$$

$$h_2' = \sin(\beta') h_{\lambda}' + \cos(\beta') h_{\beta}'$$

Having h_1' , h_2' , and h_3' convert to (x, y, z) coordinates from

$$h_x' = \cos(\xi_1) h_2' - \sin(\xi_1) h_3'$$

$$h_y' = -h_1' \quad (47)$$

$$h_z' = \sin(\xi_1) h_2' + \cos(\xi_1) h_3'$$

giving \vec{h}' in (x, y, z) coordinates.

Note for this case that all coordinates in figure 3B have had primes added to them. These transformations only apply to the prime case.

Case 2: Double Prime

Now consider \vec{h}'' . Use double primes with the coordinates in figure 3B. Orient the (u_1'', u_2'', u_3'') system such that the semi-circular current path (including current direction) is aligned with the half of the bent circular current path situated in $x \leq 0$. This requires the coordinate transformation

$$\begin{aligned} u_1'' &= y \\ u_2'' &= -\cos(\xi_1)x + \sin(\xi_1)z \\ u_3'' &= \sin(\xi_1)x + \cos(\xi_1)z \end{aligned} \tag{48}$$

with

$$\lambda'' \cos(\beta'') = u_1'' , \quad \lambda'' \sin(\beta'') = u_2'' \tag{49}$$

and

$$\lambda'' = \left[u_1''^2 + u_2''^2 \right]^{\frac{1}{2}} = \left[\cos^2(\xi_1)x^2 + y^2 + \sin^2(\xi_1)z^2 - 2\sin(\xi_1)\cos(\xi_1)xz \right]^{\frac{1}{2}} \tag{50}$$

$$\cos(\beta'') = \frac{u_1''}{\lambda''} , \quad \sin(\beta'') = \frac{u_2''}{\lambda''}$$

The parameter is

$$m'' = \frac{4\frac{\lambda''}{a}}{\left(1 + \frac{\lambda''}{a}\right)^2 + \left(\frac{u_3''}{a}\right)^2} = 1 - m_1'' \tag{51}$$

Substitute m'' and β'' for m and β respectively in equations 40, and call the results h''_λ , h''_β , and h''_3 . Then calculate h''_1 and h''_2 from

$$\begin{aligned} h''_1 &= \cos(\beta'')h''_\lambda - \sin(\beta'')h''_\beta \\ h''_2 &= \sin(\beta'')h''_\lambda + \cos(\beta'')h''_\beta \end{aligned} \tag{52}$$

and finally convert to (\bar{x}, y, z) reference using

$$\begin{aligned} h''_x &= -\cos(\xi_1)h''_2 + \sin(\xi_1)h''_3 \\ h''_y &= h''_1 \\ h''_z &= \sin(\xi_1)h''_2 + \cos(\xi_1)h''_3 \end{aligned} \tag{53}$$

giving \vec{h}'' in (x, y, z) coordinates.

Adding this result for \vec{h}'' to the result for \vec{h}' in case 1 gives the total normalized magnetic field as in equation 41.

IV. Field Distribution for Small r/a

Having developed the complete expressions for the low-frequency magnetic field resulting from the bent circular current path we now go on to consider the form of the magnetic-field distribution for small r/a. This results in simpler expressions which apply to the field near the center of the simulator (i.e. $\vec{r} = 0$) so that the spatial variation of the fields in this part of the simulator can be easily seen.

To approach this problem of the field distribution for small r/a we first consider the magnetic field near $(\lambda, \beta, u_3) = (0, 0, 0)$ for the semicircular current path as discussed in section III A. As a first step consider the two integrals T_λ and T_β which enter into the magnetic field solution in equations 38. We have the power series expansion

$$[1 - m \sin^2(\psi)]^{-\frac{1}{2}} = \sum_{\ell=0}^{\infty} \alpha_n \sin^{2\ell}(\psi) m^\ell \quad (54)$$

where

$$\alpha_\ell = (-1)^n \binom{-\frac{1}{2}}{\ell} = \frac{(-1)^n}{\ell!} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \left(-\frac{5}{2}\right) \cdots \left(\frac{1}{2} - \ell\right) \quad (55)$$

and where two numbers in a single pair of parentheses is the notation for a binomial coefficient. This series is absolutely convergent for $|\sin^2(\psi) m| < 1$. Since we are concerned with $0 \leq m < 1$ and with real ψ then we can use this series to represent the function. The first several coefficients are

$$\alpha_0 = 1, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{3}{8}, \quad \alpha_3 = \frac{5}{16} \quad (56)$$

For our present purposes we only use up through $\ell = 3$ in the series.

From equations 34 write T_λ as

$$\begin{aligned}
T_\lambda &= \int_{\frac{\beta}{2}}^{\frac{\pi+\beta}{2}} \frac{2 \sin(\psi) \cos(\psi) d\psi}{[1-m \sin^2(\psi)]^{1/2}} \\
&= \int_{\frac{\beta}{2}}^{\frac{\pi+\beta}{2}} \left\{ \sum_{\ell=0}^{\infty} \alpha_\ell 2 \sin^{2\ell+1}(\psi) \cos(\psi) m^\ell \right\} d\psi
\end{aligned} \tag{57}$$

Then we have a representation for T_λ as

$$T_\lambda = \sum_{\ell=0}^{\infty} a_\ell m^\ell \tag{58}$$

where

$$a_\ell = 2\alpha_\ell \int_{\frac{\beta}{2}}^{\frac{\pi+\beta}{2}} \sin^{2\ell+1}(\psi) \cos(\psi) d\psi \tag{59}$$

Note that a_ℓ is bounded as $\ell \rightarrow \infty$ because α_ℓ and the limits and integrand in equation 58 are bounded. Thus the series in equation 58 is absolutely convergent for $|m| < 1$. Integrating, we have

$$\begin{aligned}
a_\ell &= \frac{\alpha_\ell}{\ell+1} \left[\sin^{2\ell+2} \left(\frac{\pi+\beta}{2} \right) - \sin^{2\ell+2} \left(\frac{\beta}{2} \right) \right] \\
&= \frac{\alpha_\ell}{\ell+1} \left[\cos^{2\ell+2} \left(\frac{\beta}{2} \right) - \sin^{2\ell+2} \left(\frac{\beta}{2} \right) \right]
\end{aligned} \tag{60}$$

Using the half angle formulas

$$\sin^2\left(\frac{\beta}{2}\right) = \frac{1-\cos(\beta)}{2}, \quad \cos^2\left(\frac{\beta}{2}\right) = \frac{1+\cos(\beta)}{2} \quad (61)$$

we have

$$a_\ell = \frac{\alpha_\ell}{\ell+1} 2^{-\ell-1} \left\{ [1+\cos(\beta)]^{\ell+1} - [1-\cos(\beta)]^{\ell+1} \right\} \quad (62)$$

The first several coefficients are

$$a_0 = \cos(\beta)$$

$$a_1 = \frac{1}{4} \cos(\beta)$$

(63)

$$a_2 = \frac{1}{32} [3\cos(\beta) + \cos^3(\beta)]$$

$$a_3 = \frac{5}{8 \cdot 16} [\cos(\beta) + \cos^3(\beta)]$$

Next from equations 34 write T_β as

$$\begin{aligned} T_\beta &= \int_{\frac{\beta}{2}}^{\frac{\pi+\beta}{2}} \frac{2\sin^2(\psi)-1}{[1-m\sin^2(\psi)]^{1/2}} d\psi \\ &= \int_{\frac{\beta}{2}}^{\frac{\pi+\beta}{2}} \left\{ \sum_{\ell=0}^{\infty} \alpha_\ell [2\sin^2(\psi)-1] \sin^{2\ell}(\psi) m^\ell \right\} d\psi \end{aligned} \quad (64)$$

Then T_β can be represented as

$$T_{\beta} = \sum_{\ell=0}^{\infty} b_{\ell} m^{\ell} \quad (65)$$

where

$$b_{\ell} = \alpha_{\ell} \int_{\frac{\beta}{2}}^{\frac{\pi+\beta}{2}} [2 \sin^2(\psi) - 1] \sin^2(\psi) d\psi \quad (66)$$

Since the limits, integrand, and α_{ℓ} are bounded as $\ell \rightarrow \infty$ then b_{ℓ} is bounded and the series in equation 65 is absolutely convergent for $|m| < 1$. Letting

$$\zeta \equiv 2\psi \quad (67)$$

and using equations 62 gives

$$b_{\ell} = -\alpha_{\ell} 2^{-\ell-1} \int_{\beta}^{\pi+\beta} \cos(\zeta) [1 - \cos(\zeta)]^{\ell} d\zeta \quad (68)$$

Expanding the integrand and integrating the first several coefficients are

$$b_0 = -\frac{\alpha_0}{2} \sin(\zeta) \Big|_{\beta}^{\pi+\beta} = \sin(\beta)$$

$$b_1 = -\frac{\alpha_1}{4} \left[\sin(\zeta) - \frac{\zeta}{2} - \frac{\sin(2\zeta)}{4} \right] \Big|_{\beta}^{\pi+\beta} = \frac{\pi}{16} + \frac{1}{4} \sin(\beta)$$

$$\begin{aligned} b_2 &= -\frac{\alpha_2}{8} \left[\sin(\zeta) - \zeta - \frac{\sin(2\zeta)}{2} + \sin(\zeta) - \frac{\sin^3(\zeta)}{3} \right] \Big|_{\beta}^{\pi+\beta} \\ &= \frac{3}{64} \left[\pi + 4 \sin(\beta) - \frac{2}{3} \sin^3(\beta) \right] \end{aligned} \quad (69)$$

$$\begin{aligned}
b_3 &= -\frac{\alpha_3}{16} \left[\sin(\zeta) - \frac{3\zeta}{2} - \frac{3 \sin(2\zeta)}{4} + 3 \sin(\zeta) - \sin^3(\zeta) \right. \\
&\quad \left. - \frac{3\zeta}{8} - \frac{\sin(2\zeta)}{4} - \frac{\sin(4\zeta)}{32} \right] \Big|_{\beta}^{\pi+\beta} \\
&= \frac{5}{16 \cdot 16} \left[\frac{15}{8} \pi + 8 \sin(\beta) - 2 \sin^3(\beta) \right]
\end{aligned}$$

where one can find the integral of $\cos^{\ell+1}(\zeta)$ for small ℓ in standard reference tables.⁴

In calculating the normalized magnetic field certain derivatives of T_λ and T_β are used. From equations 58 and 65 we have

$$\frac{\partial}{\partial m} T_\lambda = \sum_{\ell=1}^{\infty} \ell a_\ell m^{\ell-1} \tag{70}$$

$$\frac{\partial}{\partial m} T_\beta = \sum_{\ell=1}^{\infty} \ell b_\ell m^{\ell-1}$$

where these series are also absolutely convergent for $|m| < 1$. Another derivative of interest is

$$\frac{\partial}{\partial \beta} T_\lambda = \sum_{\ell=0}^{\infty} c_\ell m^\ell \tag{71}$$

where

$$c_\ell = \frac{\partial}{\partial \beta} a_\ell \tag{72}$$

4. H. B. Dwight, Tables of Integrals and Other Mathematical Data, 4th ed., Macmillan, 1961, chapter 2.

Note that this last derivative can also be obtained directly from equations 34 and then using the expansion in equation 54 to obtain the absolutely convergent power series in m for $|m| < 1$. The first several coefficients are

$$\begin{aligned}
 c_0 &= -\sin(\beta) \\
 c_1 &= -\frac{1}{4} \sin(\beta) \\
 c_2 &= \frac{3}{32}[-2 \sin(\beta) + \sin^3(\beta)] \\
 c_4 &= \frac{5}{8 \cdot 16}[-4 \sin(\beta) + 3 \sin^3(\beta)]
 \end{aligned}
 \tag{73}$$

With these preliminaries now write the normalized magnetic field from equations 38 for the semicircular current path for small m as

$$\begin{aligned}
 h_\lambda &= \frac{1}{8\pi} \frac{u_3}{a} \left(\frac{am}{\lambda}\right)^{\frac{3}{2}} \left\{ b_0 + 3b_1m + 5b_2m^2 + 7b_3m^3 + O(m^4) \right\} \\
 h_\beta &= \frac{1}{8\pi} \frac{u_3}{a} \left(\frac{am}{\lambda}\right)^{\frac{3}{2}} \left\{ a_0 + 3a_1m + 5a_2m^2 + 7a_3m^3 + O(m^4) \right\} \\
 h_3 &= \frac{1}{2\pi} \left(\frac{a}{\lambda}\right)^{\frac{3}{2}} m^{\frac{1}{2}} \left\{ (b_0+c_0) + (2b_1+c_1)m + (3b_2+c_2)m^2 + (4b_3+c_3)m^3 \right. \\
 &\quad \left. + O(m^4) - \frac{1}{4} \left(1+\frac{\lambda}{a}\right) \left[b_0m + 3b_1m^2 + 5b_2m^3 + 7b_3m^4 + O(m^5) \right] \right\} \\
 &= \frac{1}{8\pi} \left(\frac{am}{\lambda}\right)^{\frac{3}{2}} \left\{ 8b_1 + 4c_1 + (12b_2+4c_2)m + (16b_3+4c_3)m^2 + O(m^3) \right. \\
 &\quad \left. - \left(1+\frac{\lambda}{a}\right) \left[b_0 + 3b_1m + 5b_2m^2 + 7b_3m^3 + O(m^4) \right] \right\}
 \end{aligned}
 \tag{74}$$

where the last result has used $b_0 + c_0 = 0$. Equations 74 have included only the first several terms in an asymptotic expansion as $m \rightarrow 0$. The remainders are included in the order symbols since an absolutely convergent power series is also an asymptotic series.

Having the magnetic field for small m now convert this result to small r/a . Note first that

$$\begin{aligned} \left(\frac{r}{a}\right)^2 &= \left(\frac{u_1}{a}\right)^2 + \left(\frac{u_2}{a}\right)^2 + \left(\frac{u_3}{a}\right)^2 \\ &= \left(\frac{\lambda}{a}\right)^2 + \left(\frac{u_3}{a}\right)^2 \end{aligned} \quad (75)$$

This result is independent of which of the two semicircular current paths are being considered since the origin of the (u_1, u_2, u_3) coordinate system is the same position as the origin of the (x, y, z) coordinate system in both cases. Then as $r/a \rightarrow 0$ we can write any cartesian coordinate divided by a as $O(r/a)$, including u_1, u_2, u_3, x, y, z ; the same applies for λ and Ψ . Then from equation 28 expand m as $r/a \rightarrow 0$ in the form

$$\begin{aligned} m &= 4 \frac{\lambda}{a} \left[\left(1 + \frac{\lambda}{a}\right)^2 + \left(\frac{u_3}{a}\right)^2 \right]^{-1} = 4 \frac{\lambda}{a} \left[1 + 2 \frac{\lambda}{a} + \left(\frac{r}{a}\right)^2 \right]^{-1} \\ &= 4 \frac{\lambda}{a} \left[1 - 2 \frac{\lambda}{a} - \left(\frac{r}{a}\right)^2 + 4 \left(\frac{\lambda}{a}\right)^2 + o\left(\left(\frac{r}{a}\right)^3\right) \right] \end{aligned} \quad (76)$$

Thus we can also write as $r/a \rightarrow 0$

$$m = o\left(\frac{r}{a}\right) \quad (77)$$

One of the term combinations in equations 74 can be written for $r/a \rightarrow 0$ as

$$\begin{aligned} \frac{1}{8} \left(\frac{am}{\lambda} \right)^{\frac{3}{2}} &= \left[\left(1 + \frac{\lambda}{a} \right)^2 + \left(\frac{u_3}{a} \right)^2 \right]^{-\frac{3}{2}} = \left[1 + 2 \frac{\lambda}{a} + \left(\frac{r}{a} \right)^2 \right]^{-\frac{3}{2}} \\ &= 1 - 3 \frac{\lambda}{a} - \frac{3}{2} \left(\frac{r}{a} \right)^2 + \frac{15}{2} \left(\frac{\lambda}{a} \right)^2 + o \left(\left(\frac{r}{a} \right)^3 \right) \end{aligned} \quad (78)$$

Next find the magnetic field for the semicircular current path for $r/a \rightarrow 0$ by substituting equations 76 through 78 into equations 74. Equations 63, 69, and 73 are needed for the coefficients. Then as $r/a \rightarrow 0$ the low-frequency normalized magnetic field for the semicircular current path has the form

$$\begin{aligned} h_\lambda &= \frac{1}{\pi} \frac{u_3}{a} \left[1 - 3 \frac{\lambda}{a} + o \left(\left(\frac{r}{a} \right)^2 \right) \right] \left\{ \sin(\beta) + \left[\frac{3\pi}{4} + 3 \sin(\beta) \right] \frac{\lambda}{a} + o \left(\left(\frac{r}{a} \right)^2 \right) \right\} \\ &= \frac{1}{\pi} \sin(\beta) \frac{u_3}{a} + \frac{3}{4} \frac{\lambda}{a} \frac{u_3}{a} + o \left(\left(\frac{r}{a} \right)^3 \right) \end{aligned}$$

$$\begin{aligned} h_\beta &= \frac{1}{\pi} \frac{u_3}{a} \left[1 - 3 \frac{\lambda}{a} + o \left(\left(\frac{r}{a} \right)^2 \right) \right] \left\{ \cos(\beta) + 3 \cos(\beta) \frac{\lambda}{a} + o \left(\left(\frac{r}{a} \right)^2 \right) \right\} \\ &= \frac{1}{\pi} \cos(\beta) \frac{u_3}{a} + o \left(\left(\frac{r}{a} \right)^3 \right) \end{aligned}$$

$$h_3 = \frac{1}{\pi} \left[1 - 3 \frac{\lambda}{a} - \frac{3}{2} \left(\frac{r}{a} \right)^2 + \frac{15}{2} \frac{\lambda}{a}^2 + o \left(\left(\frac{r}{a} \right)^3 \right) \right] \quad (79)$$

$$\begin{aligned} &\cdot \left\{ \frac{\pi}{2} + \sin(\beta) + \left[\frac{9\pi}{4} + 6 \sin(\beta) \right] \frac{\lambda}{a} \left[1 - 2 \frac{\lambda}{a} \right] + \left[\frac{75\pi}{8} + 30 \sin(\beta) - \frac{5}{2} \sin^3(\beta) \right] \left(\frac{\lambda}{a} \right)^2 \right. \\ &- \left(1 + \frac{\lambda}{a} \right) \left\{ \sin(\beta) + \left[\frac{3\pi}{4} + 3 \sin(\beta) \right] \frac{\lambda}{a} \left[1 - 2 \frac{\lambda}{a} \right] + \left[\frac{15\pi}{4} + 15 \sin(\beta) - \frac{5}{2} \sin^3(\beta) \right] \right. \\ &\left. \left. \cdot \left(\frac{\lambda}{a} \right)^2 \right\} + o \left(\left(\frac{r}{a} \right)^3 \right) \right\} \end{aligned}$$

$$= \frac{1}{2} + \frac{2}{\pi} \sin(\beta) \frac{\lambda}{a} + \frac{3}{8} \left(\frac{\lambda}{a}\right)^2 - \frac{3}{4} \left(\frac{u_3}{a}\right)^2 + O\left(\left(\frac{r}{a}\right)^3\right)$$

where up through quadratic terms in λ/a and u_3/a have been retained. Using equations 27 convert h_λ and h_β to h_1 and h_2 for $r/a \rightarrow 0$ giving for h_1 , h_2 , and h_3 the results

$$h_1 = \frac{3}{4} \cos(\beta) \frac{\lambda}{a} \frac{u_3}{a} + O\left(\left(\frac{r}{a}\right)^3\right)$$

$$= \frac{3}{4} \frac{u_1}{a} \frac{u_3}{a} + O\left(\left(\frac{r}{a}\right)^3\right)$$

$$h_2 = \frac{1}{\pi} \frac{u_3}{a} + \frac{3}{4} \sin(\beta) \frac{\lambda}{a} \frac{u_3}{a} + O\left(\left(\frac{r}{a}\right)^3\right)$$

(80)

$$= \frac{1}{\pi} \frac{u_3}{a} + \frac{3}{4} \frac{u_2}{a} \frac{u_3}{a} + O\left(\left(\frac{r}{a}\right)^3\right)$$

$$h_3 = \frac{1}{2} + \frac{2}{\pi} \sin(\beta) \frac{\lambda}{a} + \frac{3}{8} \left(\frac{\lambda}{a}\right)^2 - \frac{3}{4} \left(\frac{u_3}{a}\right)^2 + O\left(\left(\frac{r}{a}\right)^3\right)$$

$$= \frac{1}{2} + \frac{2}{\pi} \frac{u_2}{a} + \frac{3}{8} \left(\frac{u_1}{a}\right)^2 + \frac{3}{8} \left(\frac{u_2}{a}\right)^2 - \frac{3}{4} \left(\frac{u_3}{a}\right)^2 + O\left(\left(\frac{r}{a}\right)^3\right)$$

Now that we have the solution for small r/a for the semi-circular current path we can apply it to the full bent current path by superposition of the results for the prime and double prime cases as discussed in section III B. First apply the results of equations 80 to the prime case. From equations 42 the coordinate transformations are

$$u'_1 = -y$$

$$u'_2 = \cos(\xi_1)x + \sin(\xi_1)z \quad (81)$$

$$u'_3 = -\sin(\xi_1)x + \cos(\xi_1)z$$

Substituting these results into equations 80, applying the transformations for h' from equations 47, and collecting terms gives (for $r/a \rightarrow 0$)

$$\begin{aligned}
 h'_x &= \cos(\xi_1)h'_2 - \sin(\xi_1)h'_3 \\
 &= -\frac{1}{2}\sin(\xi_1) - \frac{3}{\pi}\sin(\xi_1)\cos(\xi_1)\frac{x}{a} + \frac{1}{\pi}[\cos^2(\xi_1) - 2\sin^2(\xi_1)]\frac{z}{a} \\
 &\quad + \frac{3}{8}\left\{-\sin(\xi_1)\left(\frac{y}{a}\right)^2 + [-3\cos^2(\xi_1)\sin(\xi_1) + 2\sin^3(\xi_1)]\left(\frac{x}{a}\right)^2\right. \\
 &\quad \left.+ [2\cos^3(\xi_1) - 8\sin^2(\xi_1)\cos(\xi_1)]\frac{x}{a}\frac{z}{a}\right. \\
 &\quad \left.+ [4\sin(\xi_1)\cos^2(\xi_1) - \sin^3(\xi_1)]\left(\frac{z}{a}\right)^2\right\} \\
 &\quad + o\left(\left(\frac{r}{a}\right)^3\right)
 \end{aligned}$$

$$\begin{aligned}
 h'_y &= -h'_1 \\
 &= -\frac{3}{4}\sin(\xi_1)\frac{x}{a}\frac{y}{a} + \frac{3}{4}\cos(\xi_1)\frac{y}{a}\frac{z}{a} + o\left(\left(\frac{r}{a}\right)^3\right)
 \end{aligned}$$

$$\begin{aligned}
 h'_z &= \sin(\xi_1)h'_2 + \cos(\xi_1)h'_3 \\
 &= \frac{1}{2}\cos(\xi_1) + \frac{1}{\pi}[2\cos^2(\xi_1) - \sin^2(\xi_1)]\frac{x}{a} + \frac{3}{\pi}\sin(\xi_1)\cos(\xi_1)\frac{z}{a} \\
 &\quad + \frac{3}{8}\left\{\cos(\xi_1)\left(\frac{y}{a}\right)^2 + [-4\sin^2(\xi_1)\cos(\xi_1) + \cos^3(\xi_1)]\left(\frac{x}{a}\right)^2\right. \\
 &\quad \left.+ [8\cos^2(\xi_1)\sin(\xi_1) - 2\sin^3(\xi_1)]\frac{x}{a}\frac{z}{a}\right\}
 \end{aligned} \tag{82}$$

$$+ \left[3\sin^2(\xi_1)\cos(\xi_1) - 2\cos^3(\xi_1) \right] \left(\frac{z}{a} \right)^2 \left\{ \right.$$

$$+ o\left(\left(\frac{r}{a} \right)^3 \right)$$

Next apply the results of equations 80 to the double prime case. From equations 48 the coordinate transformations are

$$u_1'' = y$$

$$u_2'' = -\cos(\xi_1)x + \sin(\xi_1)z \quad (83)$$

$$u_3'' = \sin(\xi_1)x + \cos(\xi_1)z$$

Substituting these results into equations 80, applying the transformations for \bar{h}'' from equations 53, and collecting terms gives (for $r/a \rightarrow 0$)

$$h_x'' = -\cos(\xi_1)h_2'' + \sin(\xi_1)h_3''$$

$$= \frac{1}{2}\sin(\xi_1) - \frac{3}{\pi}\sin(\xi_1)\cos(\xi_1)\frac{x}{a} + \frac{1}{\pi}\left[2\sin^2(\xi_1) - \cos^2(\xi_1) \right]\frac{z}{a}$$

$$+ \frac{3}{8}\left\{ \sin(\xi_1)\left(\frac{y}{a} \right)^2 + \left[3\cos^2(\xi_1)\sin(\xi_1) - 2\sin^3(\xi_1) \right] \left(\frac{x}{a} \right)^2 \right.$$

$$+ \left[2\cos^3(\xi_1) - 8\sin^2(\xi_1)\cos(\xi_1) \right] \frac{x}{a} \frac{z}{a}$$

$$\left. + \left[-4\sin(\xi_1)\cos^2(\xi_1) + \sin^3(\xi_1) \right] \left(\frac{z}{a} \right)^2 \right\}$$

$$+ o\left(\left(\frac{r}{a} \right)^3 \right)$$

(84)

$$h_y'' = h_1''$$

$$= \frac{3}{4} \sin(\xi_1) \frac{x}{a} \frac{y}{a} + \frac{3}{4} \cos(\xi_1) \frac{y}{a} \frac{z}{a} + o\left(\left(\frac{r}{a}\right)^3\right)$$

$$h_z'' = \sin(\xi_1) h_2'' + \cos(\xi_1) h_3''$$

$$= \frac{1}{2} \cos(\xi_1) + \frac{1}{\pi} \left[\sin^2(\xi_1) - 2\cos^2(\xi_1) \right] \frac{x}{a} + \frac{3}{\pi} \sin(\xi_1) \cos(\xi_1) \frac{z}{a}$$

$$+ \frac{3}{8} \left\{ \cos(\xi_1) \left(\frac{y}{a}\right)^2 + \left[-4\sin^2(\xi_1) \cos(\xi_1) + \cos^3(\xi_1) \right] \left(\frac{x}{a}\right)^2 \right.$$

$$\left. + \left[-8\cos^2(\xi_1) \sin(\xi_1) + 2\sin^3(\xi_1) \right] \frac{x}{a} \frac{z}{a} \right.$$

$$\left. + \left[3\sin^2(\xi_1) \cos(\xi_1) - 2\cos^3(\xi_1) \right] \left(\frac{z}{a}\right)^2 \right\}$$

$$+ o\left(\left(\frac{r}{a}\right)^3\right)$$

Having both \vec{h}' and \vec{h}'' now add the results to give (for $r/a \rightarrow 0$)

$$h_x = h_x' + h_x''$$

$$= \cos(\xi_1) \left\{ -\frac{6}{\pi} \sin(\xi_1) \frac{x}{a} + \frac{3}{2} \left[1 - 5\sin^2(\xi_1) \right] \frac{x}{a} \frac{z}{a} \right\} + o\left(\left(\frac{r}{a}\right)^3\right)$$

$$h_y = h_y' + h_y''$$

$$= \cos(\xi_1) \left\{ \frac{3}{2} \frac{y}{a} \frac{z}{a} \right\} + o\left(\left(\frac{r}{a}\right)^3\right)$$

(85)

$$h_z = h'_z + h''_z$$

$$= \cos(\xi_1) \left\{ 1 + \frac{6}{\pi} \sin(\xi_1) \frac{z}{a} + \frac{3}{4} [1 - 5 \sin^2(\xi_1)] \left(\frac{x}{a} \right)^2 + \frac{3}{4} \left(\frac{y}{a} \right)^2 - \frac{3}{4} [2 - 5 \sin^2(\xi_1)] \left(\frac{z}{a} \right)^2 \right\} + O\left(\left(\frac{r}{a} \right)^3 \right)$$

Note that $\cos(\xi_1)$ is factored out of the asymptotic expressions for each of the components. Also note that some of the terms vanish for the special case of $\xi_1 = 0$ which introduces symmetry with respect to both the z axis and the x, y plane.

V. Field Distribution for Large r/a

Another limiting form for the low-frequency magnetic field distribution which we consider is that for large r/a. Again consider the results for the semicircular current path which are represented for small m by the asymptotic form in equations 74. For r/a $\rightarrow \infty$ we can write m from equation 28 as

$$\begin{aligned}
 m &= 4\frac{\lambda}{a} \left[\left(1 + \frac{\lambda}{a}\right)^2 + \left(\frac{u_3}{a}\right)^2 \right]^{-1} = 4\frac{\lambda}{a} \left[1 + 2\frac{\lambda}{a} + \left(\frac{r}{a}\right)^2 \right]^{-1} \\
 &= 4\frac{\lambda}{r} \frac{a}{r} \left[1 + 2\frac{\lambda}{r} \frac{a}{r} + \left(\frac{a}{r}\right)^2 \right]^{-1} \\
 &= 4\frac{\lambda}{r} \frac{a}{r} \left[1 - 2\frac{\lambda}{r} \frac{a}{r} - \left(\frac{a}{r}\right)^2 + 4\left(\frac{\lambda}{r} \frac{a}{r}\right)^2 + o\left(\left(\frac{a}{r}\right)^3\right) \right] \quad (86)
 \end{aligned}$$

Note in considering the asymptotic form as r/a $\rightarrow \infty$ that λ/r , u_3/r , and β are all considered as fixed numbers. One might define a polar angle γ by

$$\lambda \equiv r \sin(\gamma) , \quad u_3 = r \cos(\gamma) \quad (87)$$

so that (r, γ , β) is a spherical coordinate system for the semicircular-current-path problem. Then as r/a $\rightarrow \infty$ one keeps γ and β fixed which implies fixed λ/r and u_3/r as well as fixed u_1/r and u_2/r . Later, when considering the full bent-circular-current-path problem, this implies fixed x/r , y/r , z/r , ψ/r , θ , and ϕ as r/a $\rightarrow \infty$. From equation 86 there is the interesting result as r/a $\rightarrow \infty$ that

$$m = o\left(\frac{a}{r}\right) \quad (88)$$

Thus a/r $\rightarrow 0$ implies m $\rightarrow 0$ and the results of equation 74 directly apply. For r/a $\rightarrow \infty$ one of the term combinations can be written as

$$\begin{aligned}
\frac{1}{8} \left(\frac{am}{\lambda} \right)^{\frac{3}{2}} &= \left[\left(1 + \frac{\lambda}{a} \right)^2 + \left(\frac{u_3}{a} \right)^2 \right]^{-\frac{3}{2}} = \left[1 + 2 \frac{\lambda}{a} + \left(\frac{r}{a} \right)^2 \right]^{-\frac{3}{2}} \\
&= \left(\frac{a}{r} \right)^3 \left[1 + 2 \frac{\lambda}{r} \frac{a}{r} + \left(\frac{a}{r} \right)^2 \right]^{-\frac{3}{2}} \\
&= \left(\frac{a}{r} \right)^3 \left[1 - 3 \frac{\lambda}{r} \frac{a}{r} - \frac{3}{2} \left(\frac{a}{r} \right)^2 + \frac{15}{2} \left(\frac{\lambda}{r} \frac{a}{r} \right)^2 + o \left(\left(\frac{a}{r} \right)^3 \right) \right] \quad (89)
\end{aligned}$$

Substituting the results of these equations into equations 74 gives for the low-frequency normalized magnetic field as $r/a \rightarrow \infty$

$$\begin{aligned}
h_\lambda &= \frac{1}{\pi} \frac{u_3}{r} \left(\frac{a}{r} \right)^2 \left[1 - 3 \frac{\lambda}{r} \frac{a}{r} - \frac{3}{2} \left(\frac{a}{r} \right)^2 + \frac{15}{2} \left(\frac{\lambda}{r} \frac{a}{r} \right)^2 + o \left(\left(\frac{a}{r} \right)^3 \right) \right] \\
&\quad \cdot \left\{ \sin(\beta) + \left[\frac{3\pi}{4} + 3 \sin(\beta) \right] \frac{\lambda}{r} \frac{a}{r} \left[1 - 2 \frac{\lambda}{r} \frac{a}{r} \right] \right. \\
&\quad \left. + \left[\frac{15\pi}{4} + 15 \sin(\beta) - \frac{5}{2} \sin^3(\beta) \right] \left(\frac{\lambda}{r} \frac{a}{r} \right)^2 + o \left(\left(\frac{a}{r} \right)^3 \right) \right\} \\
&= \frac{1}{\pi} \sin(\beta) \frac{u_3}{r} \left(\frac{a}{r} \right)^2 + \frac{3}{4} \frac{\lambda}{r} \frac{u_3}{r} \left(\frac{a}{r} \right)^3 \\
&\quad + \frac{1}{\pi} \sin(\beta) \frac{u_3}{r} \left\{ -\frac{3}{2} + \left[\frac{15}{2} - \frac{5}{2} \sin^2(\beta) \right] \left(\frac{\lambda}{r} \right)^2 \right\} \left(\frac{a}{r} \right)^4 + o \left(\left(\frac{a}{r} \right)^5 \right) \\
h_\beta &= \frac{1}{\pi} \frac{u_3}{r} \left(\frac{a}{r} \right)^2 \left[1 - 3 \frac{\lambda}{r} \frac{a}{r} - \frac{3}{2} \left(\frac{a}{r} \right)^2 + \frac{15}{2} \left(\frac{\lambda}{r} \frac{a}{r} \right)^2 + o \left(\left(\frac{a}{r} \right)^3 \right) \right] \\
&\quad \cdot \left\{ \cos(\beta) + 3 \cos(\beta) \frac{\lambda}{r} \frac{a}{r} \left[1 - 2 \frac{\lambda}{r} \frac{a}{r} \right] + \left[15 \cos(\beta) + \frac{5}{2} \cos^3(\beta) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& \cdot \left(\frac{\lambda}{r} \frac{a}{r} \right)^2 + O \left(\left(\frac{a}{r} \right)^3 \right) \Big\} \\
& = \frac{1}{\pi} \cos(\beta) \frac{u_3}{r} \left(\frac{a}{r} \right)^2 + \frac{1}{\pi} \cos(\beta) \frac{u_3}{r} \left[-\frac{3}{2} + \frac{5}{2} \cos^2(\beta) \left(\frac{\lambda}{r} \right)^2 \right] \left(\frac{a}{r} \right)^4 + O \left(\left(\frac{a}{r} \right)^5 \right) \\
h_3 & = \frac{1}{\pi} \left(\frac{a}{r} \right)^3 \left[1 - 3 \frac{\lambda}{r} \frac{a}{r} - \frac{3}{2} \left(\frac{a}{r} \right)^2 + \frac{15}{2} \left(\frac{\lambda}{r} \frac{a}{r} \right)^2 + O \left(\left(\frac{a}{r} \right)^3 \right) \right] \quad (90) \\
& \cdot \left\{ \frac{\pi}{2} + \sin(\beta) + \left[\frac{9\pi}{4} + 6\sin(\beta) \right] \frac{\lambda}{r} \frac{a}{r} + O \left(\left(\frac{a}{r} \right)^2 \right) \right. \\
& - \left(1 + \frac{\lambda}{r} \frac{a}{r} \right) \left\{ \sin(\beta) + \left[\frac{3\pi}{4} + 3\sin(\beta) \right] \frac{\lambda}{r} \frac{a}{r} \left[1 - 2 \frac{\lambda}{r} \frac{a}{r} \right] \right. \\
& \quad \left. \left. + \left[\frac{15\pi}{4} + 15\sin(\beta) - \frac{5}{2} \sin^3(\beta) \right] \left(\frac{\lambda}{r} \frac{a}{r} \right)^2 + O \left(\left(\frac{a}{r} \right)^3 \right) \right\} \right\} \\
& = - \frac{1}{\pi} \sin(\beta) \frac{\lambda}{r} \left(\frac{a}{r} \right)^2 + \left[\frac{1}{2} - \frac{3}{4} \left(\frac{\lambda}{r} \right)^2 \right] \left(\frac{a}{r} \right)^3 \\
& + \frac{1}{\pi} \left\{ \frac{9}{2} \sin(\beta) \frac{\lambda}{r} + \frac{5}{2} \left[\sin^3(\beta) - 3\sin(\beta) \right] \left(\frac{\lambda}{r} \right)^3 \right\} \left(\frac{a}{r} \right)^4 + O \left(\left(\frac{a}{r} \right)^5 \right)
\end{aligned}$$

Using equations 27 convert h_λ and h_β to h_1 and h_2 for $r/a \rightarrow \infty$ giving for h_1 , h_2 , and h_3 the results

$$\begin{aligned}
h_1 & = \cos(\beta) h_\lambda - \sin(\beta) h_\beta \\
& = \frac{3}{4} \cos(\beta) \frac{\lambda}{r} \frac{u_3}{r} \left(\frac{a}{r} \right)^3 + \frac{5}{\pi} \sin(\beta) \cos(\beta) \left(\frac{\lambda}{r} \right)^2 \frac{u_3}{r} \left(\frac{a}{r} \right)^4 + O \left(\left(\frac{a}{r} \right)^5 \right) \\
& = \frac{3}{4} \frac{u_1}{r} \frac{u_3}{r} \left(\frac{a}{r} \right)^3 + \frac{5}{\pi} \frac{u_1}{r} \frac{u_2}{r} \frac{u_3}{r} \left(\frac{a}{r} \right)^4 + O \left(\left(\frac{a}{r} \right)^5 \right)
\end{aligned}$$

$$\begin{aligned}
h_2 &= \sin(\beta)h_\lambda + \cos(\beta)h_\beta \\
&= \frac{1}{\pi} \frac{u_3}{r} \left(\frac{a}{r}\right)^2 + \frac{3}{4} \sin(\beta) \frac{\lambda}{r} \frac{u_3}{r} \left(\frac{a}{r}\right)^3 + \frac{1}{\pi} \frac{u_3}{r} \left\{ -\frac{3}{2} + \frac{5}{2} [1 + \sin^2(\beta)] \left(\frac{\lambda}{r}\right)^2 \right\} \\
&\quad \cdot \left(\frac{a}{r}\right)^4 + O\left(\left(\frac{a}{r}\right)^5\right) \\
&= \frac{1}{\pi} \frac{u_3}{r} \left(\frac{a}{r}\right)^2 + \frac{3}{4} \frac{u_2}{r} \frac{u_3}{r} \left(\frac{a}{r}\right)^3 + \frac{1}{\pi} \frac{u_3}{r} \left[-\frac{3}{2} + \frac{5}{2} \left(\frac{u_1}{r}\right)^2 + 5 \left(\frac{u_2}{r}\right)^2 \right] \left(\frac{a}{r}\right)^4 + O\left(\left(\frac{a}{r}\right)^5\right) \\
h_3 &= -\frac{1}{\pi} \frac{u_2}{r} \left(\frac{a}{r}\right)^2 + \left[\frac{1}{2} - \frac{3}{4} \left(\frac{u_1}{r}\right)^2 - \frac{3}{4} \left(\frac{u_2}{r}\right)^2 \right] \left(\frac{a}{r}\right)^3 \\
&\quad + \frac{1}{\pi} \left[\frac{9}{2} \frac{u_2}{r} - \frac{15}{2} \left(\frac{u_1}{r}\right)^2 \frac{u_2}{r} - 5 \left(\frac{u_2}{r}\right)^3 \right] \left(\frac{a}{r}\right)^4 + O\left(\left(\frac{a}{r}\right)^5\right)
\end{aligned} \tag{91}$$

With the solution for $r/a \rightarrow \infty$ for the semicircular current path we go to the solution for the full bent circular current path by the superposition procedure outlined in section III B. First apply the results of equations 91 to the prime case. The coordinate transformations are (from equations 42)

$$\begin{aligned}
u_1' &= -y \\
u_2' &= \cos(\xi_1)x + \sin(\xi_1)z \\
u_3' &= -\sin(\xi_1)x + \cos(\xi_1)z
\end{aligned} \tag{92}$$

Substituting the coordinate transformations into equations 91, applying the transformations for h' from equations 47, and collecting terms gives (for $r/a \rightarrow \infty$)

$$h'_x = \cos(\xi_1)h'_2 - \sin(\xi_1)h'_3$$

$$\begin{aligned}
 &= \frac{1}{\pi} \frac{z}{r} \left(\frac{a}{r} \right)^2 \\
 &+ \left\{ -\frac{1}{2} \sin(\xi_1) + \frac{3}{4} \sin(\xi_1) \left(\frac{y}{r} \right)^2 + \frac{3}{4} \sin(\xi_1) \left(\frac{z}{r} \right)^2 + \frac{3}{4} \cos(\xi_1) \frac{x}{r} \frac{z}{r} \right\} \left(\frac{a}{r} \right)^3 \\
 &+ \left\{ -\frac{3}{\pi} \sin(\xi_1) \cos(\xi_1) \frac{x}{r} \frac{z}{r} - \frac{3}{2\pi} [1 + 2\sin^2(\xi_1)] \frac{z}{r} \right. \\
 &\quad \left. + \frac{5}{2\pi} [2\sin(\xi_1) \cos(\xi_1) \frac{x}{r} + [1 + 2\sin^2(\xi_1)] \frac{z}{r}] \left(\frac{y}{r} \right)^2 \right. \\
 &\quad \left. + \frac{5}{\pi} \frac{z}{r} \left[\cos^2(\xi_1) \left(\frac{x}{r} \right)^2 + 2\sin(\xi_1) \cos(\xi_1) \frac{x}{r} \frac{z}{r} + \sin^2(\xi_1) \left(\frac{z}{r} \right)^2 \right] \right\} \left(\frac{a}{r} \right)^4 \\
 &+ o\left(\left(\frac{a}{r} \right)^5 \right)
 \end{aligned}$$

$$h'_y = -h'_1$$

$$\begin{aligned}
 &= \left\{ -\frac{3}{4} \sin(\xi_1) \frac{x}{r} \frac{y}{r} + \frac{3}{4} \cos(\xi_1) \frac{y}{r} \frac{z}{r} \right\} \left(\frac{a}{r} \right)^3 \\
 &+ \frac{5}{\pi} \frac{y}{r} \left\{ -\sin(\xi_1) \cos(\xi_1) \left(\frac{x}{r} \right)^2 + [\cos^2(\xi_1) - \sin^2(\xi_1)] \frac{x}{r} \frac{z}{r} \right. \\
 &\quad \left. + \sin(\xi_1) \cos(\xi_1) \left(\frac{z}{r} \right)^2 \right\} \left(\frac{a}{r} \right)^4 \\
 &+ o\left(\left(\frac{a}{r} \right)^5 \right)
 \end{aligned} \tag{93}$$

$$h_2^1 = \sin(\xi_1)h_2^1 + \cos(\xi_1)h_3^1$$

$$\begin{aligned}
 &= -\frac{1}{\pi} \frac{x}{r} \left(\frac{a}{r}\right)^2 \\
 &+ \left\{ \frac{1}{2} \cos(\xi_1) - \frac{3}{4} \cos(\xi_1) \left(\frac{x}{r}\right)^2 - \frac{3}{4} \cos(\xi_1) \left(\frac{y}{r}\right)^2 - \frac{3}{4} \sin(\xi_1) \frac{x}{r} \frac{z}{r} \right\} \left(\frac{a}{r}\right)^3 \\
 &+ \left\{ \frac{3}{2\pi} \left[1 + 2\cos^2(\xi_1) \right] \frac{x}{r} + \frac{3}{\pi} \sin(\xi_1) \cos(\xi_1) \frac{z}{r} \right. \\
 &\quad + \frac{5}{2\pi} \left[- \left[1 + 2\cos^2(\xi_1) \right] \frac{x}{r} - 2\sin(\xi_1) \cos(\xi_1) \frac{z}{r} \right] \left(\frac{y}{r}\right)^2 \\
 &\quad \left. + \frac{5}{\pi} \frac{x}{r} \left[-\cos^2(\xi_1) \left(\frac{x}{r}\right)^2 - 2\sin(\xi_1) \cos(\xi_1) \frac{x}{r} \frac{z}{r} - \sin^2(\xi_1) \left(\frac{z}{r}\right)^2 \right] \right\} \left(\frac{a}{r}\right)^4 \\
 &+ o\left(\left(\frac{a}{r}\right)^5\right)
 \end{aligned}$$

Next apply the results of equations 91 to the double prime case. From equations 48 the coordinate transformations are

$$u_1'' = y$$

$$u_2'' = -\cos(\xi_1)x + \sin(\xi_1)z \quad (94)$$

$$u_3'' = \sin(\xi_1)x + \cos(\xi_1)z$$

Substitute these transformations into equations 91, apply the transformations for \tilde{h}'' from equations 53, and collect terms to give (for $r/a \rightarrow \infty$)

$$\begin{aligned}
h''_x &= -\cos(\xi_1)h''_2 + \sin(\xi_1)h''_3 \\
&= -\frac{1}{\pi} \frac{z}{r} \left(\frac{a}{r}\right)^2 \\
&\quad + \left\{ \frac{1}{2} \sin(\xi_1) - \frac{3}{4} \sin(\xi_1) \left(\frac{y}{r}\right)^2 - \frac{3}{4} \sin(\xi_1) \left(\frac{z}{r}\right)^2 + \frac{3}{4} \cos(\xi_1) \frac{x}{r} \frac{z}{r} \right\} \left(\frac{a}{r}\right)^3 \\
&\quad + \left\{ -\frac{3}{\pi} \sin(\xi_1) \cos(\xi_1) \frac{x}{r} + \frac{3}{2\pi} [1 + 2\sin^2(\xi_1)] \frac{z}{r} \right. \\
&\quad \quad \left. + \frac{5}{2\pi} \left[2\sin(\xi_1) \cos(\xi_1) \frac{x}{r} - [1 + 2\sin^2(\xi_1)] \frac{z}{r} \right] \left(\frac{y}{r}\right)^2 \right. \\
&\quad \quad \left. + \frac{5}{\pi} \frac{z}{r} \left[-\cos^2(\xi_1) \left(\frac{x}{r}\right)^2 + 2\sin(\xi_1) \cos(\xi_1) \frac{x}{r} \frac{z}{r} - \sin^2(\xi_1) \left(\frac{z}{r}\right)^2 \right] \right\} \left(\frac{a}{r}\right)^4 \\
&\quad + o\left(\left(\frac{a}{r}\right)^5\right)
\end{aligned}$$

$$h''_y = h''_1$$

(95)

$$\begin{aligned}
&= \left\{ \frac{3}{4} \sin(\xi_1) \frac{x}{r} \frac{y}{r} + \frac{3}{4} \cos(\xi_1) \frac{y}{r} \frac{z}{r} \right\} \left(\frac{a}{r}\right)^3 \\
&\quad + \frac{5}{\pi} \frac{y}{r} \left\{ -\sin(\xi_1) \cos(\xi_1) \left(\frac{x}{r}\right)^2 + [\sin^2(\xi_1) - \cos^2(\xi_1)] \frac{x}{r} \frac{z}{r} \right. \\
&\quad \quad \left. + \sin(\xi_1) \cos(\xi_1) \left(\frac{z}{r}\right)^2 \right\} \left(\frac{a}{r}\right)^4 \\
&\quad + o\left(\left(\frac{a}{r}\right)^5\right)
\end{aligned}$$

$$\begin{aligned}
h_z'' &= \sin(\xi_1) h_2'' + \cos(\xi_1) h_3'' \\
&= \frac{1}{\pi} \frac{x}{r} \left(\frac{a}{r}\right)^2 \\
&\quad + \left\{ \frac{1}{2} \cos(\xi_1) - \frac{3}{4} \cos(\xi_1) \left(\frac{x}{r}\right)^2 - \frac{3}{4} \cos(\xi_1) \left(\frac{y}{r}\right)^2 + \frac{3}{4} \sin(\xi_1) \frac{x}{r} \frac{z}{r} \right\} \left(\frac{a}{r}\right)^3 \\
&\quad + \left\{ -\frac{3}{2\pi} [1+2\cos^2(\xi_1)] \frac{x}{r} + \frac{3}{\pi} \sin(\xi_1) \cos(\xi_1) \frac{z}{r} \right. \\
&\quad \left. + \frac{5}{2\pi} [1+2\cos^2(\xi_1)] \frac{x}{r} - 2\sin(\xi_1) \cos(\xi_1) \frac{z}{r} \right\} \left(\frac{y}{r}\right)^2 \\
&\quad + \frac{5}{\pi} \frac{x}{r} \left[\cos^2(\xi_1) \left(\frac{x}{r}\right)^2 - 2\sin(\xi_1) \cos(\xi_1) \frac{x}{r} \frac{z}{r} + \sin^2(\xi_1) \left(\frac{z}{r}\right)^2 \right] \left\{ \left(\frac{a}{r}\right)^4 \right. \\
&\quad \left. + o\left(\left(\frac{a}{r}\right)^5\right) \right\}
\end{aligned}$$

Now add the results for \vec{h}' and \vec{h}'' to obtain the results for the fields from the full bent circular current path for $r/a \rightarrow \infty$ as

$$\begin{aligned}
h_x &= h_x' + h_x'' \\
&= \cos(\xi_1) \left\{ \frac{3}{2} \frac{x}{r} \frac{z}{r} \left(\frac{a}{r}\right)^3 + \sin(\xi_1) \left[-\frac{6}{\pi} \frac{x}{r} + \frac{10}{\pi} \frac{x}{r} \left(\frac{y}{r}\right)^2 + \frac{20}{\pi} \frac{x}{r} \left(\frac{z}{r}\right)^2 \right] \left(\frac{a}{r}\right)^4 \right\} \\
&\quad + o\left(\left(\frac{a}{r}\right)^5\right)
\end{aligned}$$

$$\begin{aligned}
h_y &= h_y' + h_y'' \\
&= \cos(\xi_1) \left\{ \frac{3}{2} \frac{y}{r} \frac{z}{r} \left(\frac{a}{r}\right)^3 + \sin(\xi_1) \frac{10}{\pi} \frac{y}{r} \left[-\left(\frac{x}{r}\right)^2 + \left(\frac{z}{r}\right)^2 \right] \left(\frac{a}{r}\right)^4 \right\} + o\left(\left(\frac{a}{r}\right)^5\right)
\end{aligned} \tag{96}$$

$$h_z = h'_z + h''_z$$

$$= \cos(\xi_1) \left\{ \left[1 - \frac{3}{2} \left(\frac{x}{r} \right)^2 - \frac{3}{2} \left(\frac{y}{r} \right)^2 \right] \left(\frac{a}{r} \right)^3 + \sin(\xi_1) \left[\frac{6}{\pi} \frac{z}{r} - \frac{10}{\pi} \frac{z}{r} \left(\frac{y}{r} \right)^2 - \frac{20}{\pi} \left(\frac{x}{r} \right)^2 \frac{z}{r} \right] \left(\frac{a}{r} \right)^4 \right\} + o\left(\left(\frac{a}{r} \right)^5 \right)$$

Here $\cos(\xi_1)$ is factored out of the asymptotic expressions for convenience. Note that for $\xi_1 = 0$ the terms involving $(a/r)^4$ drop out.

For convenience these results can be converted to cylindrical and spherical coordinate representations. In cylindrical coordinates (ψ, ϕ, z) we have the relations

$$x = \psi \cos(\phi), \quad y = \psi \sin(\phi) \quad (97)$$

The field components can then be written

$$h_\psi = \cos(\phi) h_x + \sin(\phi) h_y$$

$$= \cos(\xi_1) \left\{ \frac{3}{2} \frac{\psi}{r} \frac{z}{r} \left(\frac{a}{r} \right)^3 + \sin(\xi_1) \left[-\frac{6}{\pi} \cos^2(\phi) \frac{\psi}{r} + \frac{10}{\pi} [1 + \cos^2(\phi)] \cdot \frac{\psi}{r} \left(\frac{z}{r} \right)^2 \right] \left(\frac{a}{r} \right)^4 \right\} + o\left(\left(\frac{a}{r} \right)^5 \right)$$

(98)

$$\begin{aligned}
h_\phi &= -\sin(\phi)h_x + \cos(\phi)h_y \\
&= -\cos(\xi_1)\sin(\xi_1)\frac{4}{\pi}\sin(\phi)\cos(\phi)\frac{\psi}{r}\left(\frac{a}{r}\right)^4 + O\left(\left(\frac{a}{r}\right)^5\right) \\
h_z &= \cos(\xi_1)\left\{\left[1 - \frac{3}{2}\left(\frac{\psi}{r}\right)^2\right]\left(\frac{a}{r}\right)^3 + \sin(\xi_1)\left[\frac{6}{\pi}\frac{z}{r} - \frac{10}{\pi}[1 + \cos^2(\phi)]\right.\right. \\
&\quad \left.\left.\cdot \left(\frac{\psi}{r}\right)^2\frac{z}{r}\right]\left(\frac{a}{r}\right)^4\right\} + O\left(\left(\frac{a}{r}\right)^5\right)
\end{aligned}$$

In spherical coordinates we have

$$\psi = r \sin(\theta), \quad z = r \cos(\theta) \quad (99)$$

The field components are then

$$\begin{aligned}
h_r &= \sin(\theta)h_\psi + \cos(\theta)h_z \\
&= \cos(\xi_1)\left\{\cos(\theta)\left(\frac{a}{r}\right)^3 + \sin(\xi_1)\frac{6}{\pi}[\cos^2(\theta) - \sin^2(\theta)\cos^2(\phi)]\left(\frac{a}{r}\right)^4\right\} \\
&\quad + O\left(\left(\frac{a}{r}\right)^5\right) \\
h_\theta &= \cos(\theta)h_\psi - \sin(\theta)h_z \\
&= \cos(\xi_1)\left\{\frac{1}{2}\sin(\theta)\left(\frac{a}{r}\right)^3 + \sin(\xi_1)\frac{4}{\pi}\sin(\theta)\cos(\theta)[1 + \cos^2(\phi)]\left(\frac{a}{r}\right)^4\right\} \\
&\quad + O\left(\left(\frac{a}{r}\right)^5\right) \quad (100)
\end{aligned}$$

$$h_\phi = -\cos(\xi_1)\sin(\xi_1)\frac{4}{\pi}\sin(\theta)\sin(\phi)\cos(\phi)\left(\frac{a}{r}\right)^4 + O\left(\left(\frac{a}{r}\right)^5\right)$$

VI. Graphical Results

With the general results and the asymptotic form for small r/a worked out in the previous sections we next consider the quantitative distribution of the normalized low-frequency magnetic field in the immediate neighborhood of the simulator. These results are graphed in figures 4 through 29. Note that $|\Delta\vec{h}|$ and the components of \vec{h} are divided by $\cos(\xi_1)$ for the graphs so that $h_z/\cos(\xi_1) \rightarrow 1$ as $r/a \rightarrow 0$ and to give a convenient reference for the relative size of the other field components and of the magnitude of the difference field. The equations for the computer programs are in section III B with $\Delta\vec{h}$ taken from equation 8. The special simpler case for $\xi_1 = 0$ is found in section II.

The figures come in groups. There are five groups associated with specific values of ξ_1 . The first group (for $\xi_1 = 0$) has only two figures since the results can be expressed with only two normalized field components (h_ψ and h_z) and two coordinates (ψ and z). Figure 4 plots the field components as a function of z/a for various ψ/a ; figure 5 is a contour plot of $|\Delta\vec{h}|$ on a z, ψ plane. Note because of symmetry we only consider $z > 0, \psi > 0$. The second through fifth groups have 5 figures for each group with fixed ξ_1 . The first three figures take fixed values of y/a and plot the three cartesian normalized field components as functions of z/a for various x/a . The fourth figure in the group is a contour plot of $|\Delta\vec{h}|$ on x, y planes for 3 values of y/a . The fifth figure in the group considers only the $x = 0$ plane (the ground or water surface) where there are only two nonzero normalized field components, h_y and h_z ; these two components are plotted as functions of z/a for various y/a , and there is a contour plot of $|\Delta\vec{h}|$ on the $x = 0$ plane. Note that by symmetry we only need consider $x > 0$ and $y > 0$, but both positive and negative z are needed. After the five groups of figures for fixed ξ_1 values there is a sixth group. This sixth group comprises two summary figures, numbers 26 and 27. Figure 26 plots the maximum value of $|\Delta\vec{h}|/\cos(\xi_1)$ as a function of r/a for various values of ξ_1 between 0 and $\pi/2$; the maximum is taken over all θ, ϕ on the surface of a sphere of constant r/a . Figure 27 specializes this result by constraining $x = 0$ (the ground or water surface); the maximum value of $|\Delta\vec{h}|/\cos(\xi_1)$ is taken over all θ on a circle of fixed r/a to obtain the maximum on the $x = 0$ plane; the maximum is plotted as a function of r/a for various values of ξ_1 . These maxima are determined numerically. Note that the maximum over the sphere is in general only a little larger than the maximum over the circle on the $x = 0$ plane for the same value of r/a . Figures 28 and 29 repeat the plots in figures 26 and 27 respectively, except with expanded scales.

For convenience in the contour plots a star * is used to indicate the position of the current path where it intersects the plane being considered.

VII. Summary

This note has considered the low-frequency magnetic field distribution produced by a simulator with the geometry of a half toroid connected to a perfectly conducting ground or water surface. The minor radius of the half toroid has been assumed small compared to the major radius so that the current could be localized to a semicircular current path. Using image theory then the magnetic field has been found by integrating over the uniform current around a bent circular current path. After having obtained expressions for the normalized field in terms of elliptic integrals, the asymptotic forms for positions near the "center" of the simulator ($r = 0$) and for far away positions ($r \rightarrow \infty$) have been obtained. Graphs show the magnetic field distribution in the vicinity of the simulator.

The results in this note apply to the case that low frequencies of interest have wavelengths in the air large compared to a (the major radius) but skin depths in the ground or water small compared to a . Perhaps another note can consider the limiting low-frequency magnetic field distribution for the case of a uniform finite ground or water conductivity. Furthermore the two positions where the half toroid meets the ground or water surface need not be the positions of electrical contact to the lower medium. Conductors (perhaps including various impedance elements as well) could run from the two ends of the half toroid along or just above the ground or water surface to two other positions where electrical contact to the lower medium would be made. With this flexibility and perhaps even more general conductor distributions the low-frequency current density distribution in the lower medium can be tailored somewhat so as to make the low-frequency magnetic field distribution better approximate a uniform field in the z direction near the center of the simulator ($\vec{r} = \vec{0}$).

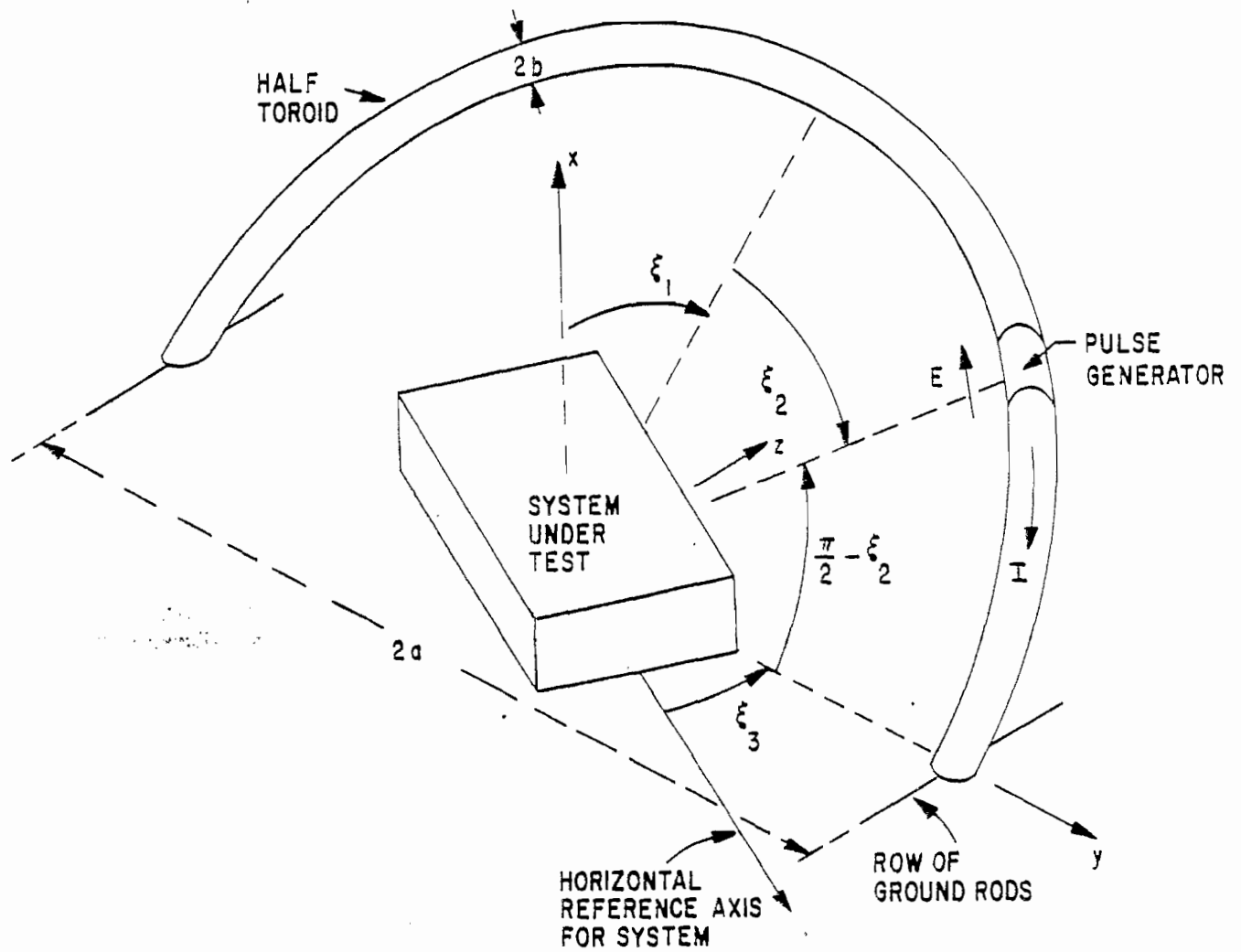
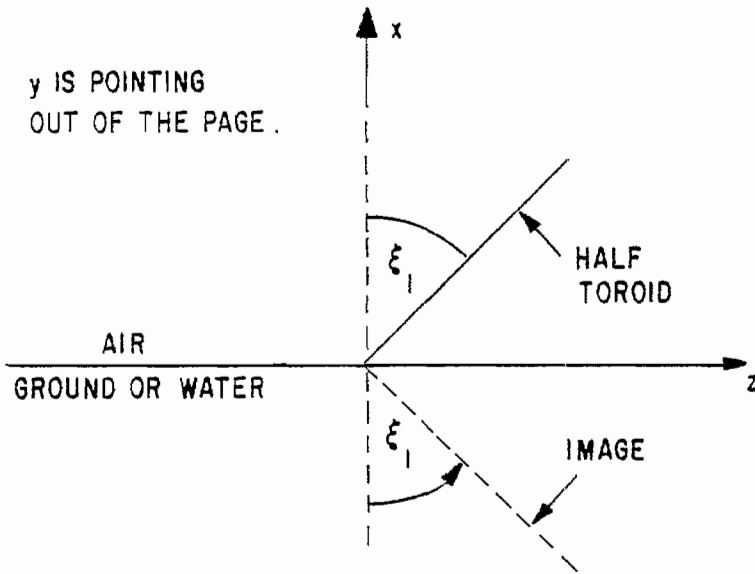
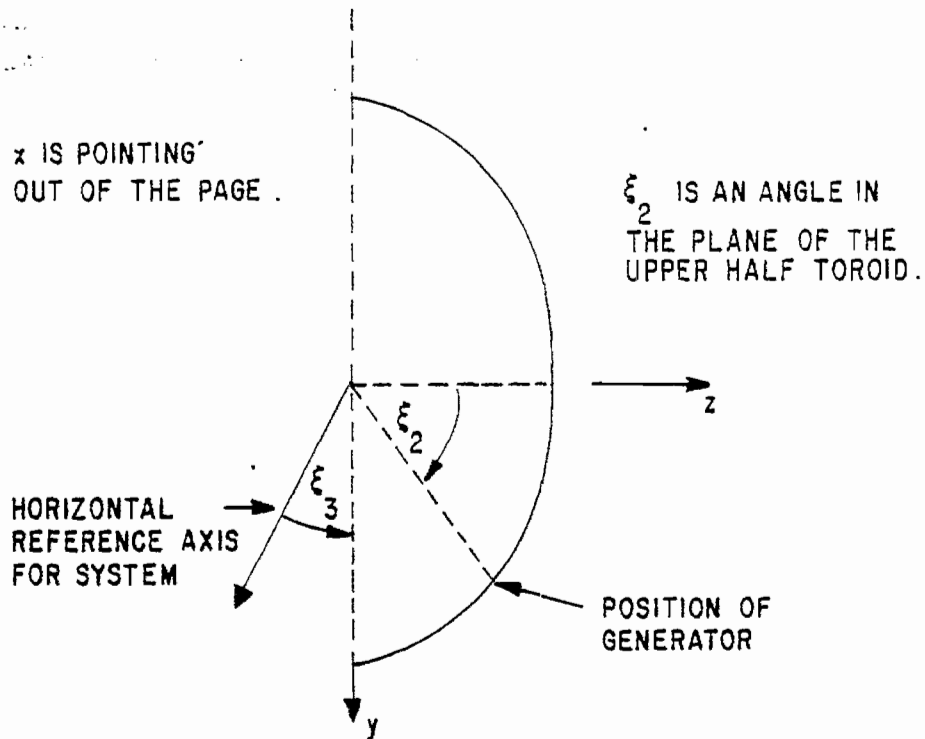


FIGURE 1. GEOMETRY OF SIMULATOR

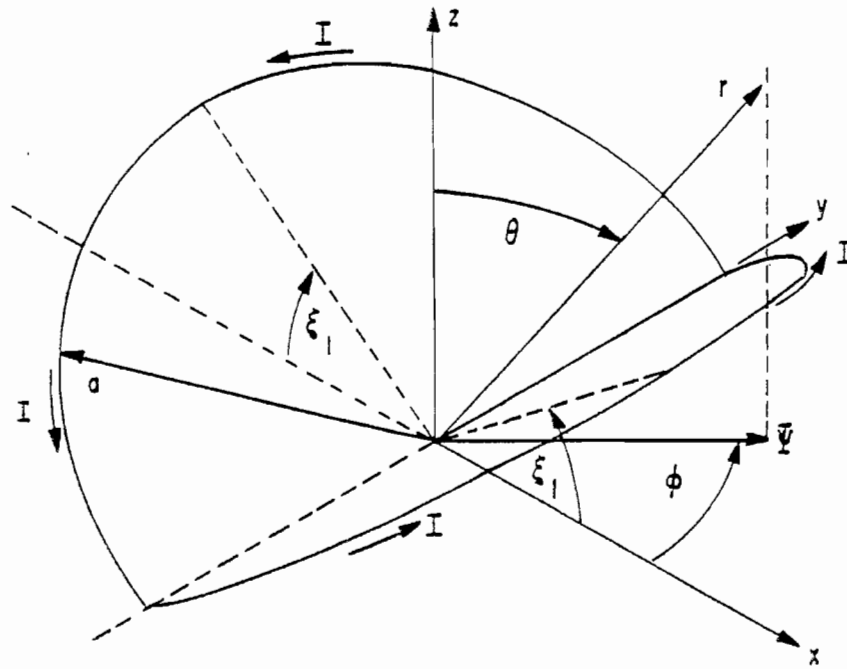


A. SIDE VIEW

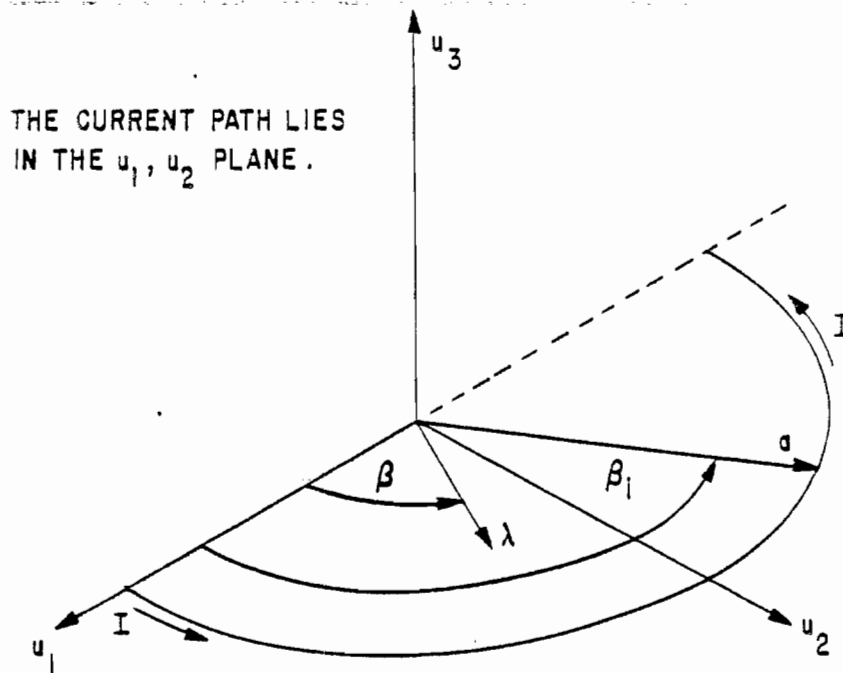


B. TOP VIEW

FIGURE 2. HALF TOROID WITH IMAGE

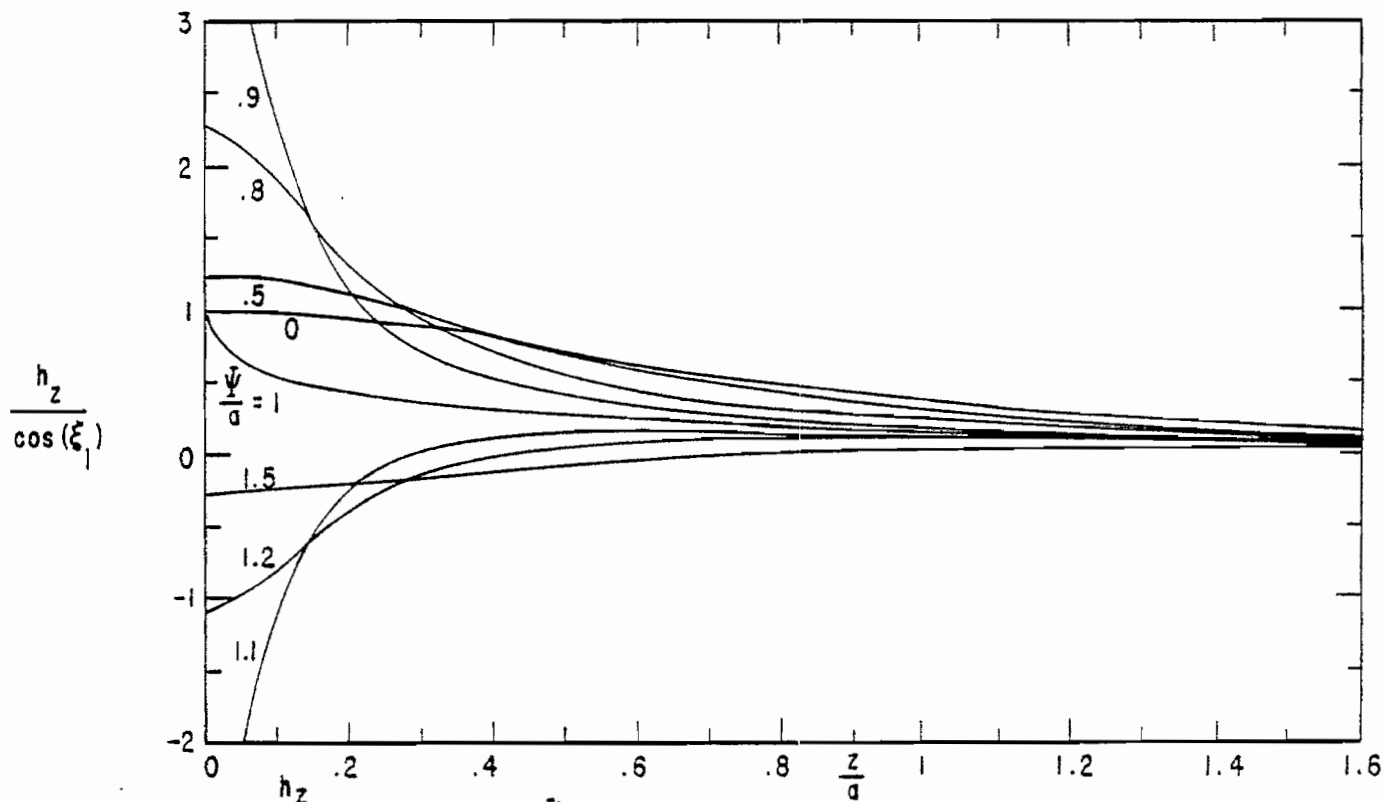


A. BENT CIRCULAR CURRENT PATH WITH COORDINATES

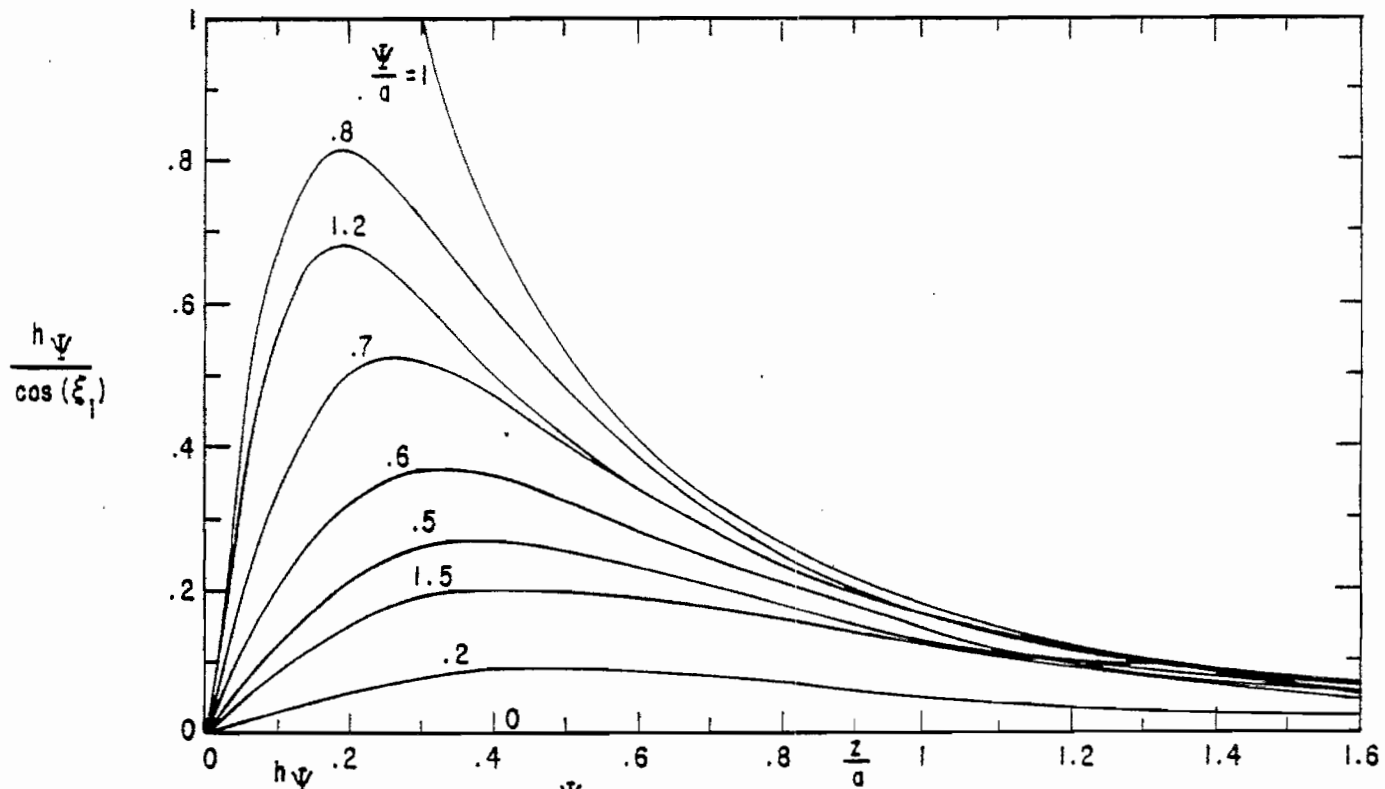


B. HALF OF BENT CIRCULAR CURRENT PATH WITH SPECIAL COORDINATES

FIGURE 3. COORDINATES FOR SIMULATOR WITH IMAGE



A. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{\Psi}{a}$ AS A PARAMETER



B. $\frac{h_\Psi}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{\Psi}{a}$ AS A PARAMETER

FIGURE 4. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF $z: \frac{2\xi_1}{\pi} = 0$

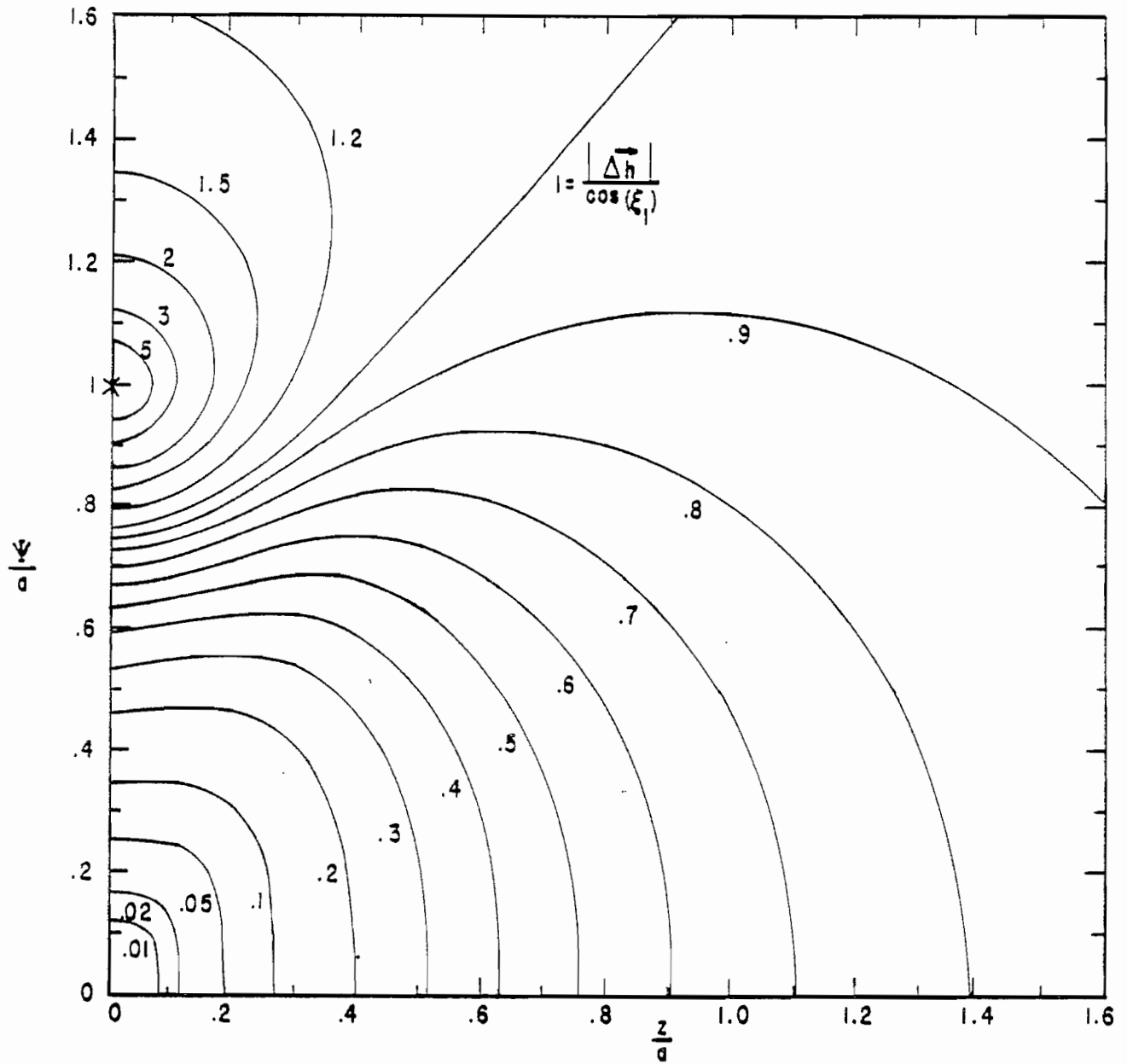


FIGURE 5. CONTOUR PLOT OF $\frac{|\Delta \bar{h}|}{\cos(\xi_1)} : \frac{2\xi_1}{\pi} = 0$

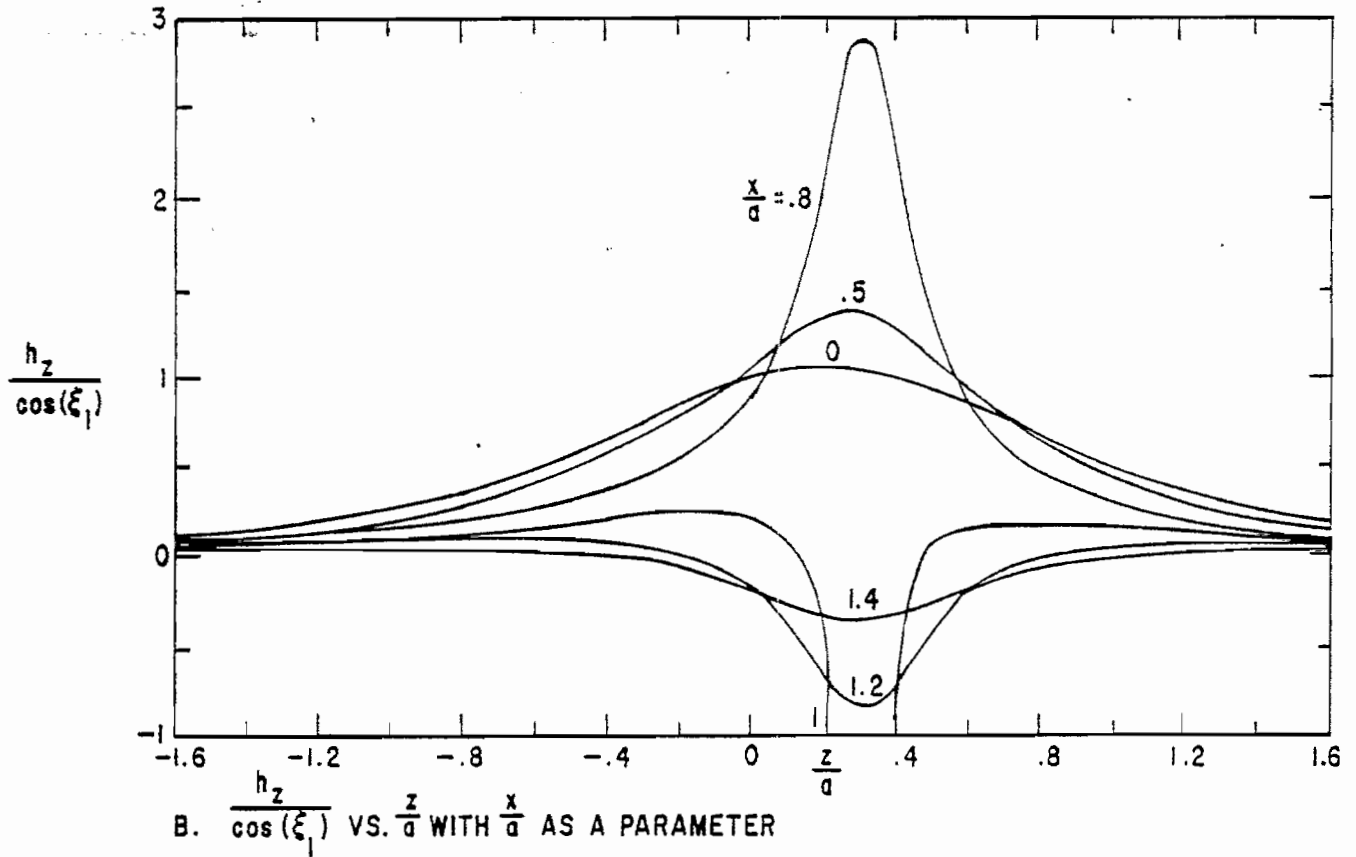
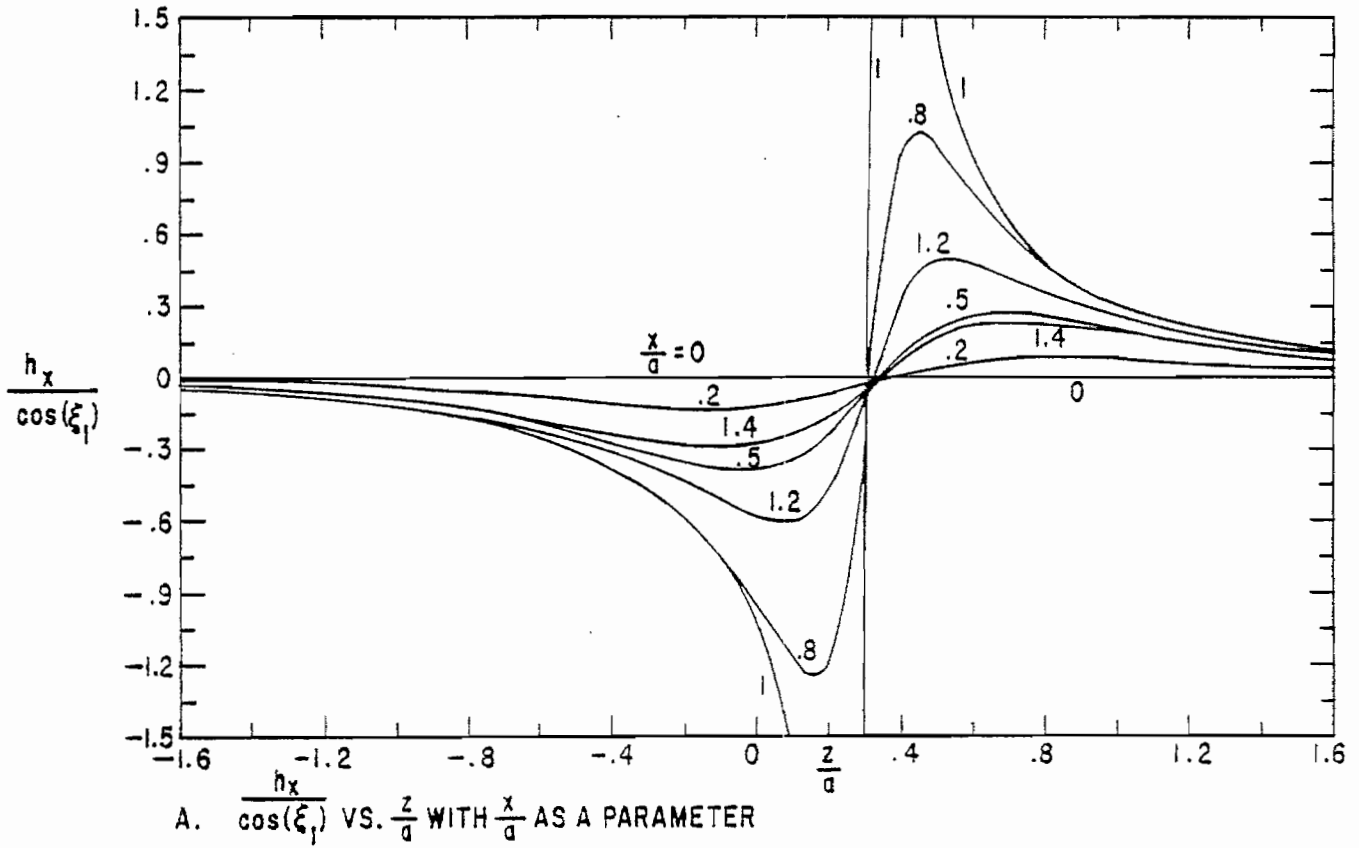


FIGURE 6. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF $z: \frac{2\xi_1}{\pi} = .2; \frac{y}{a} = 0$
 ($h_y \equiv 0$)

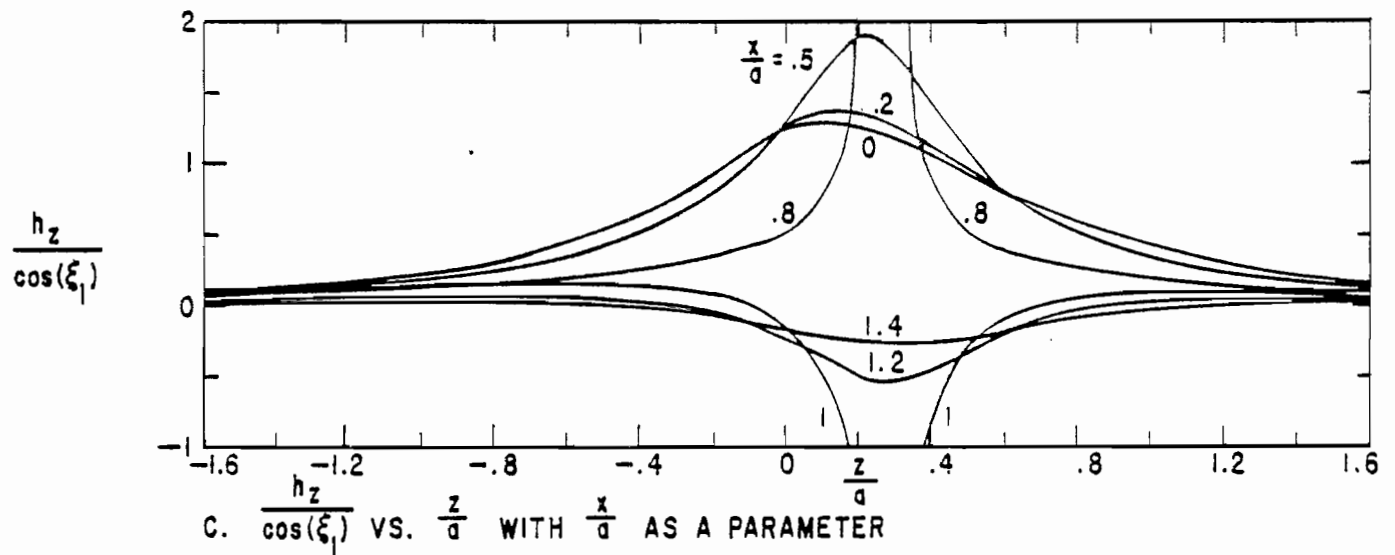
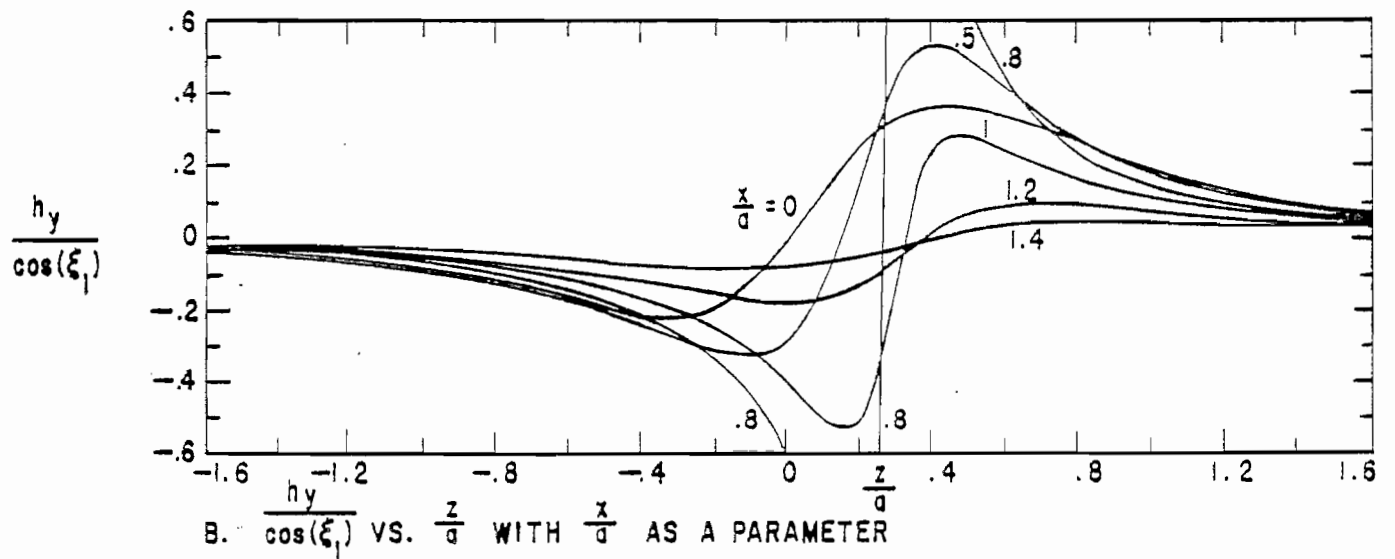
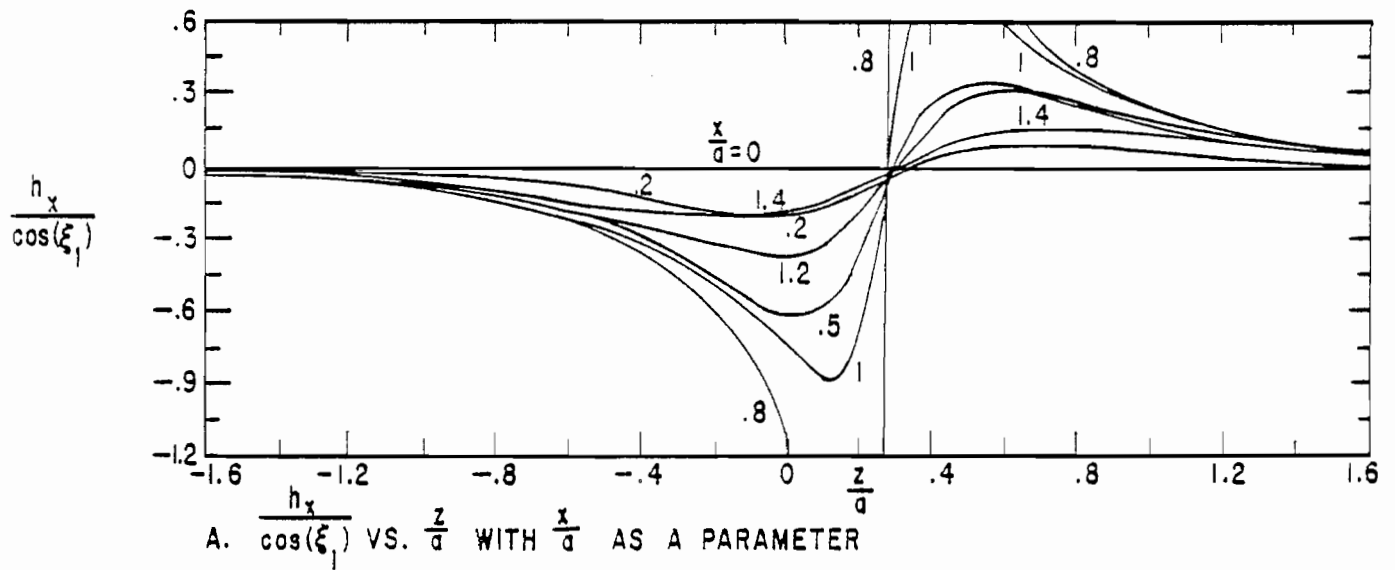
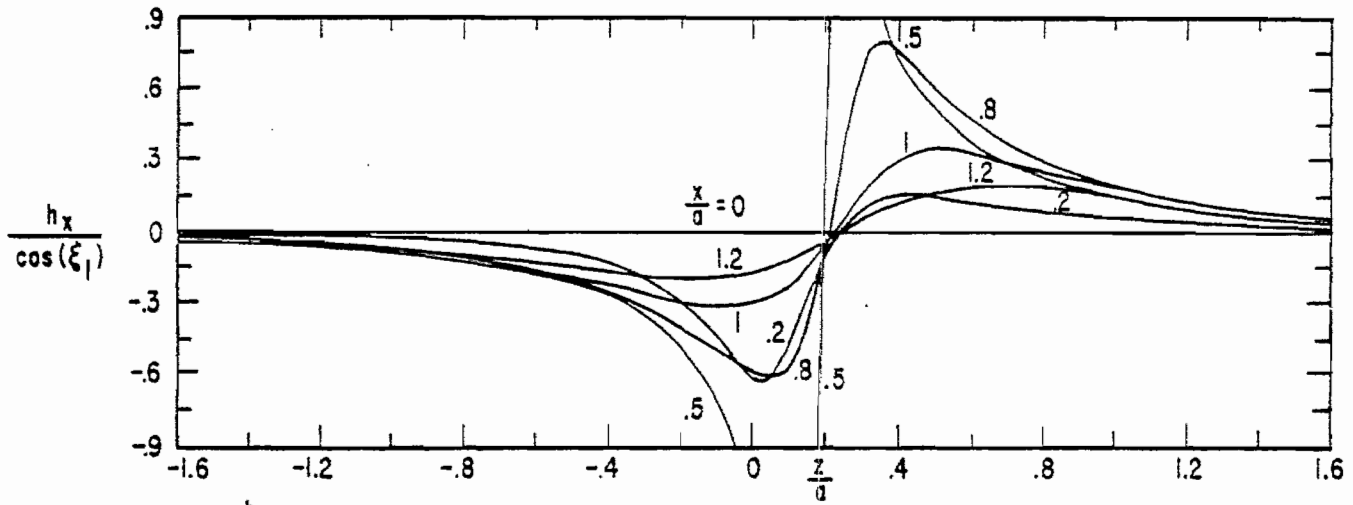
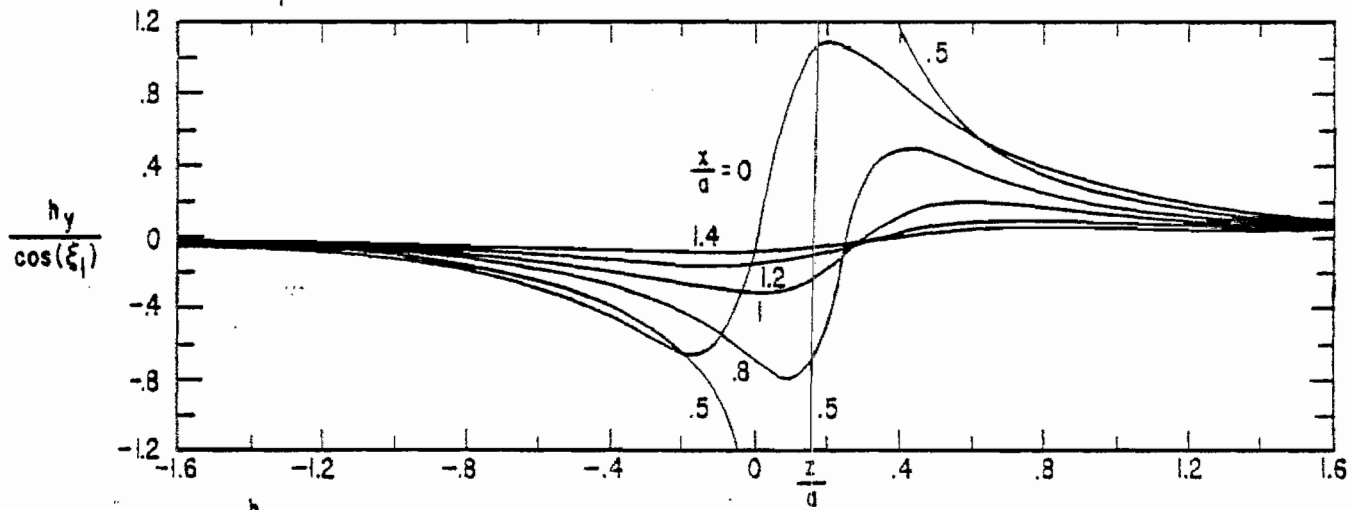


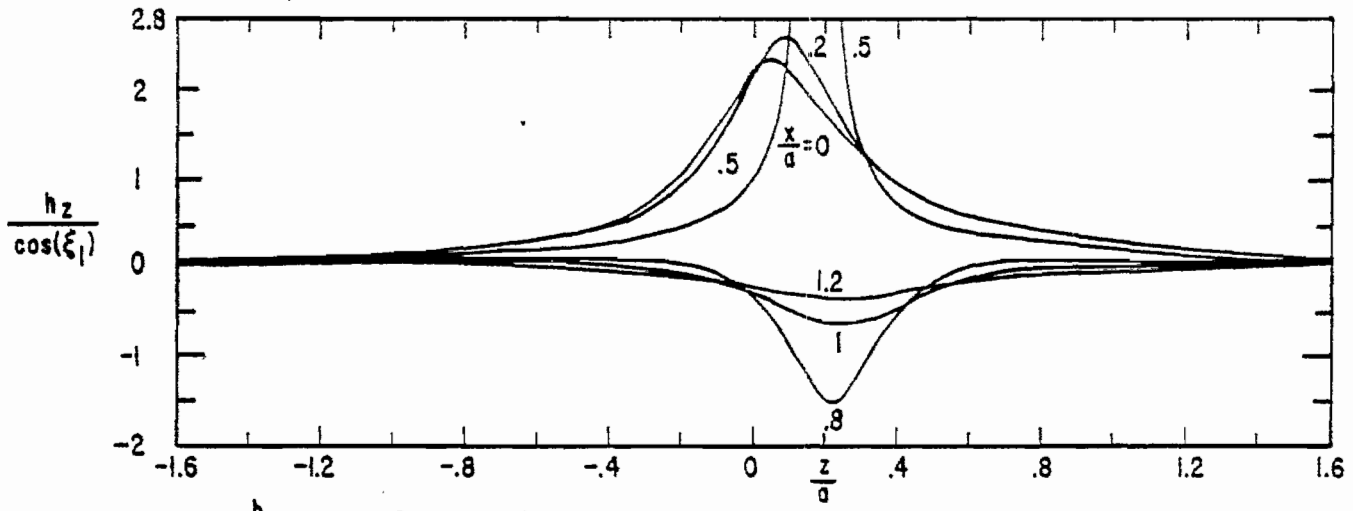
FIGURE 7. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF $z: \frac{2\xi_1}{\pi} = .2; \frac{y}{a} = .5$



A. $\frac{h_x}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

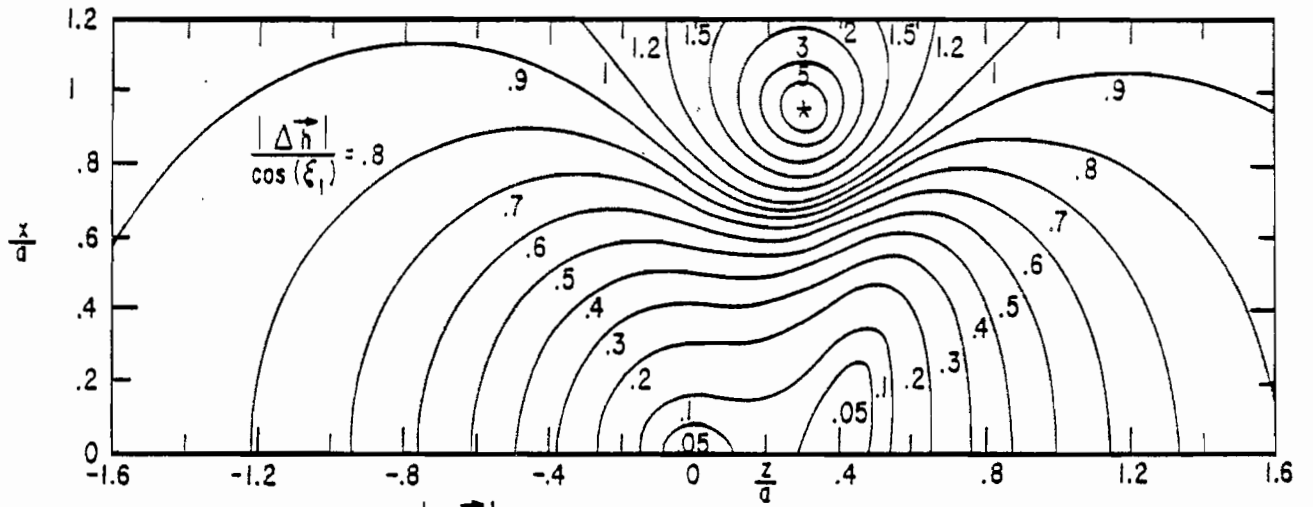


B. $\frac{h_y}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

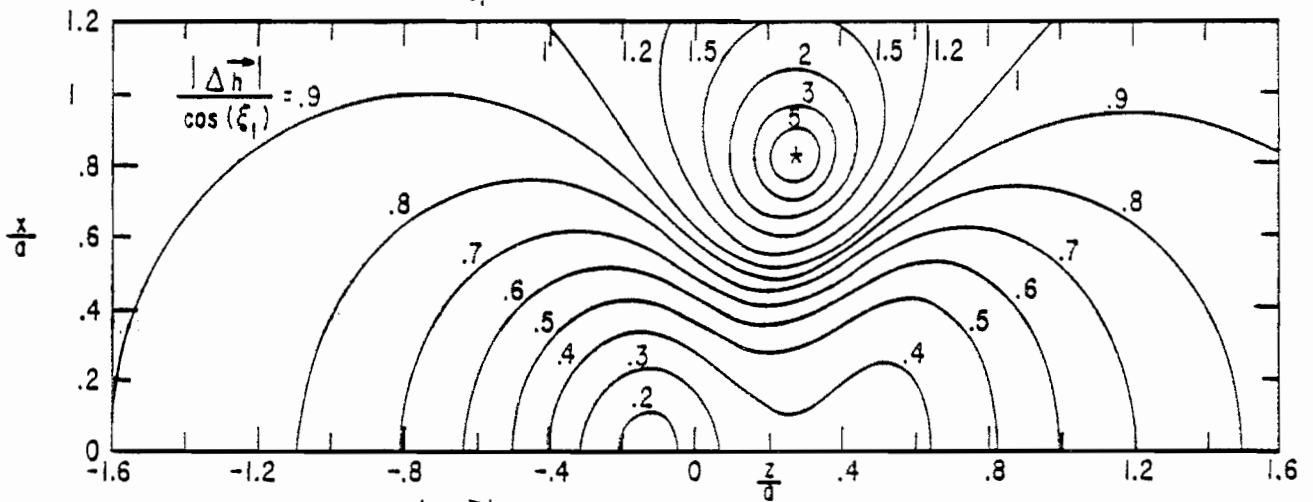


C. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

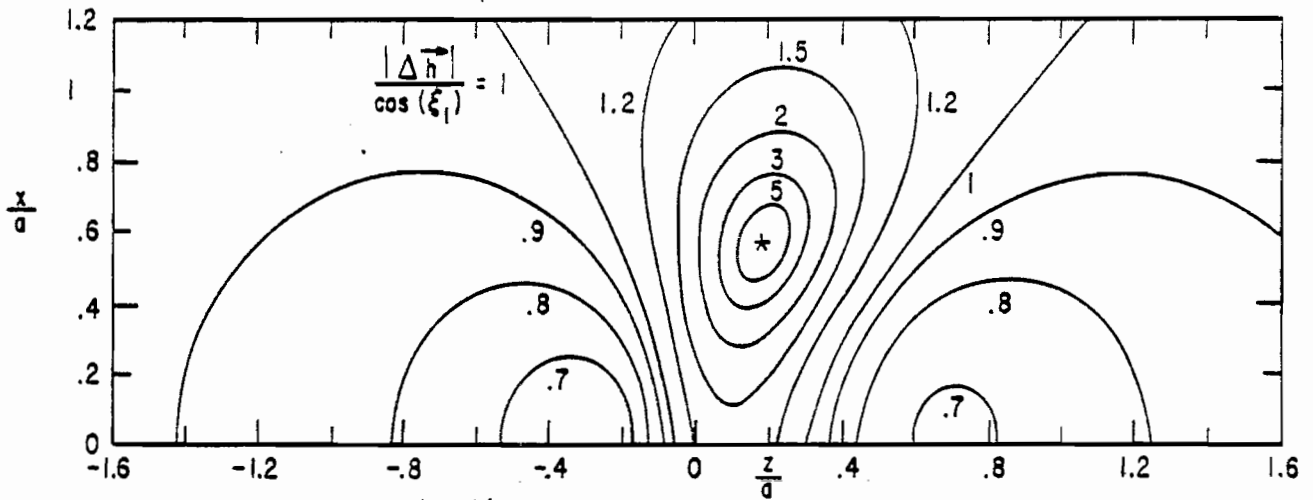
FIGURE 8. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF z : $\frac{2\xi_1}{\pi} = .2$; $\frac{y}{a} = .8$



A. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{a} = 0$

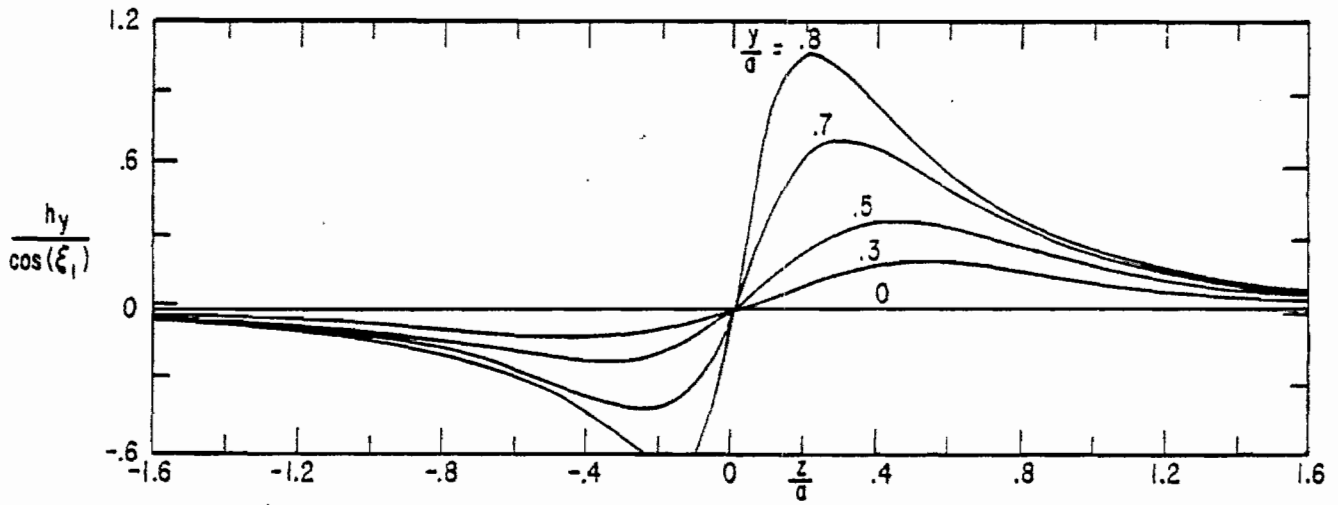


B. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{a} = .5$

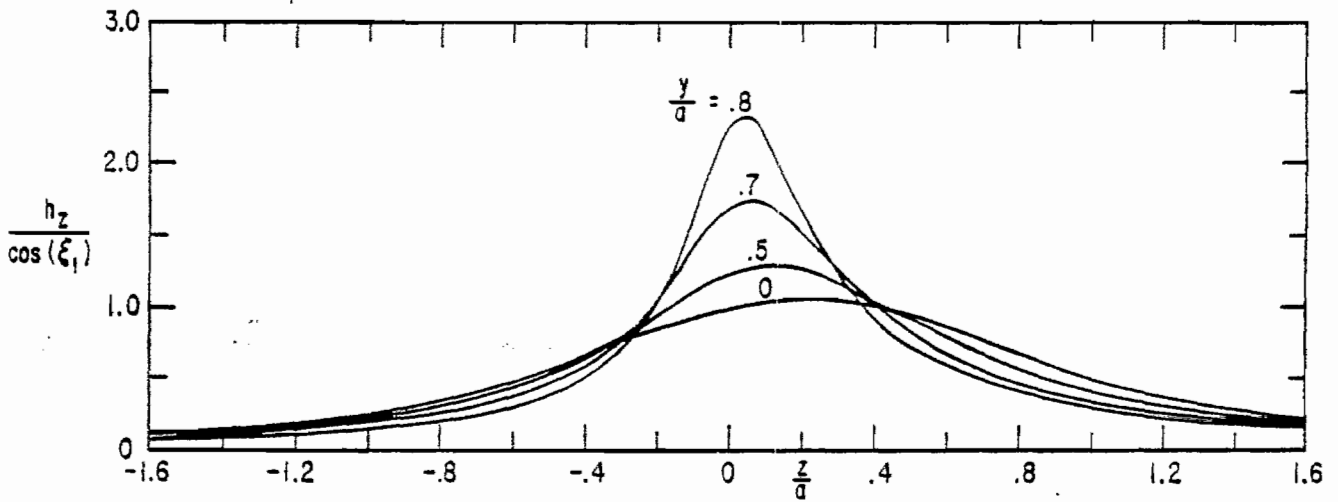


C. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{a} = .8$

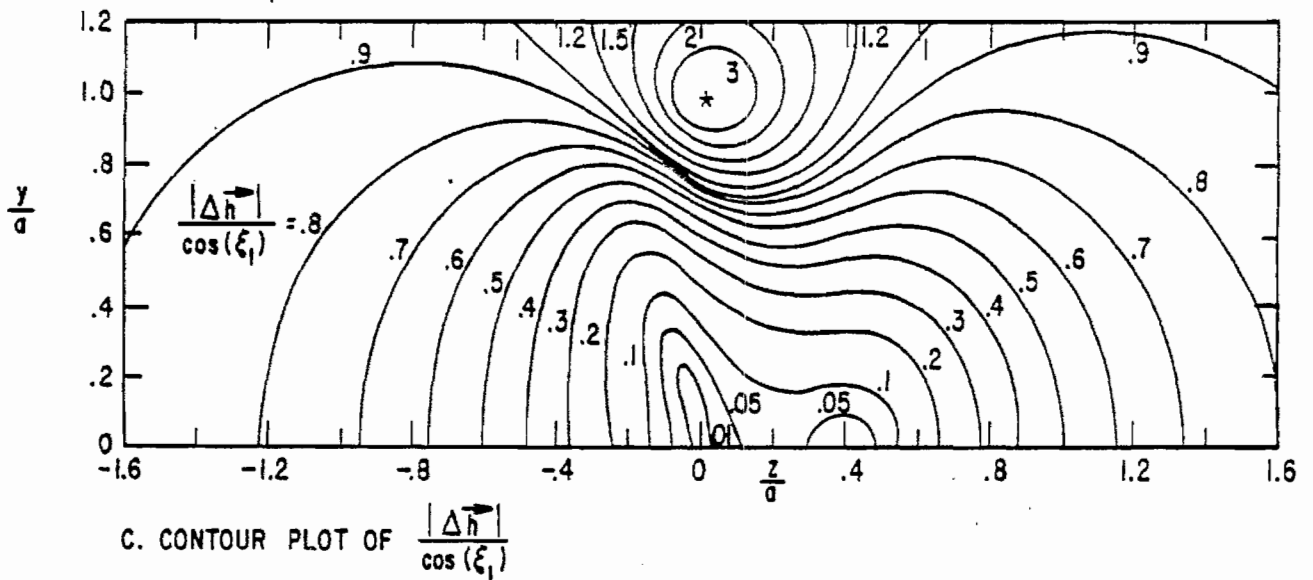
FIGURE 9. CONTOUR PLOTS OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{2\xi_1}{\pi} = .2$



A. $\frac{h_y}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{y}{a}$ AS A PARAMETER

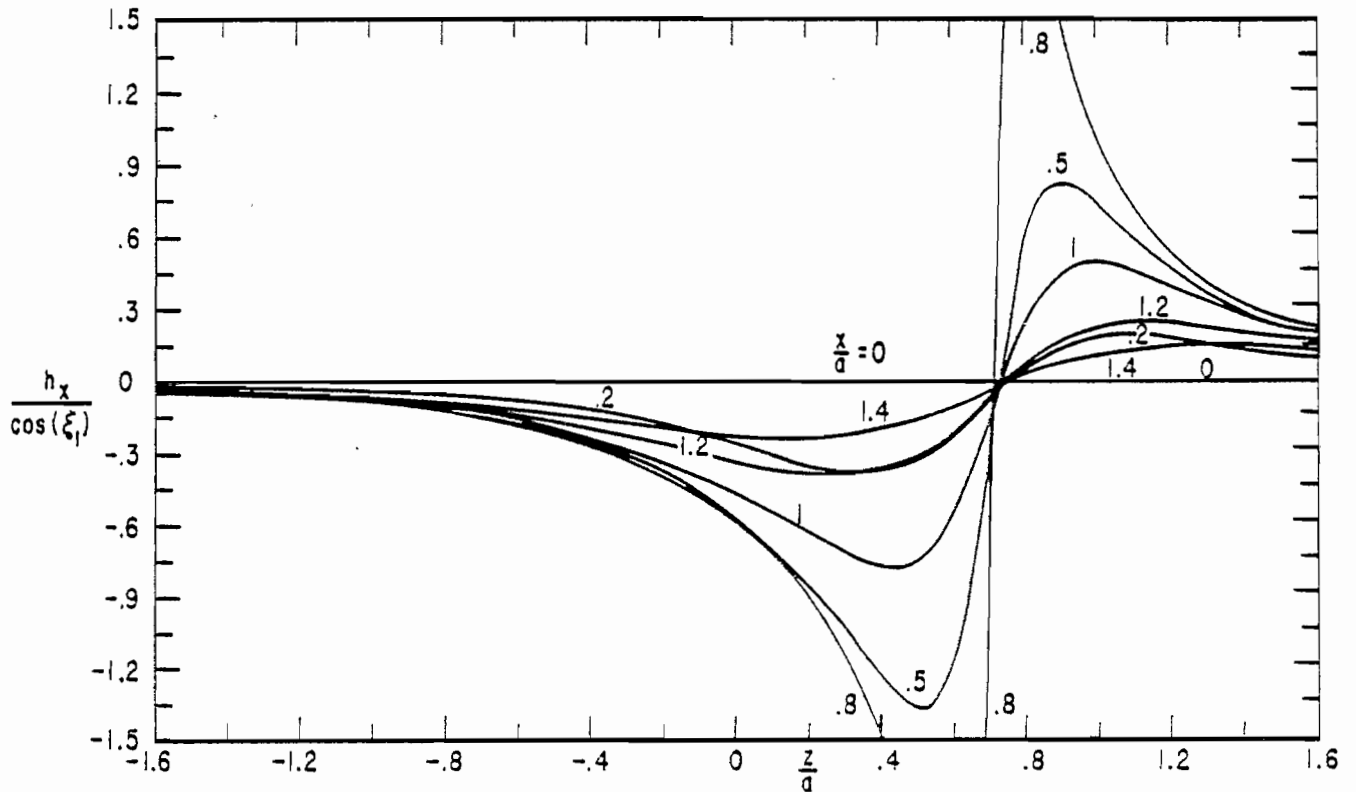


B. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{y}{a}$ AS A PARAMETER

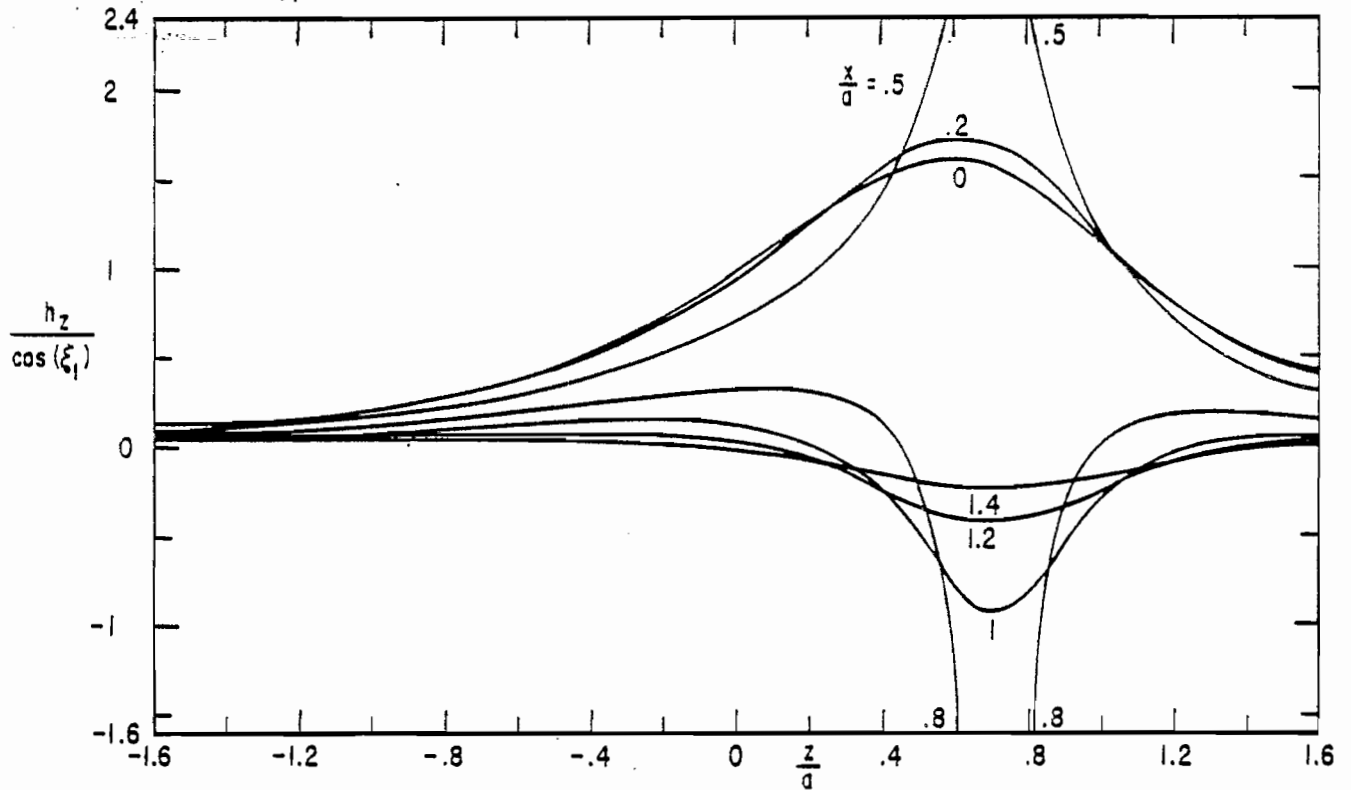


C. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$

FIGURE 10. MAGNETIC FIELD COMPONENTS AND CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{x}{a} = 0: \frac{2\xi_1}{\pi} = .2$

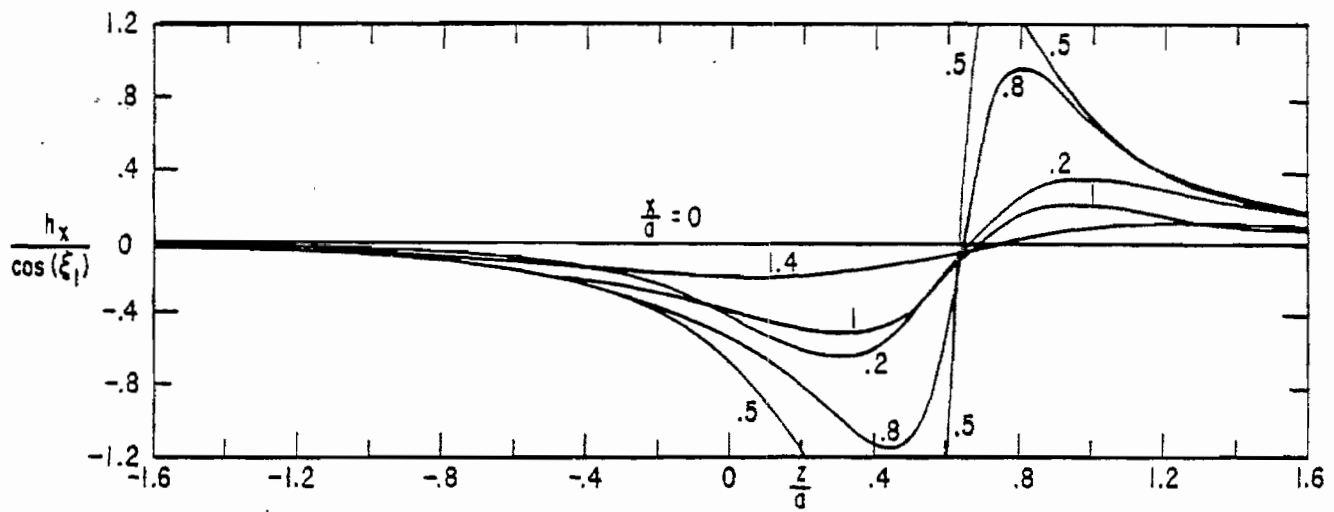


A. $\frac{h_x}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

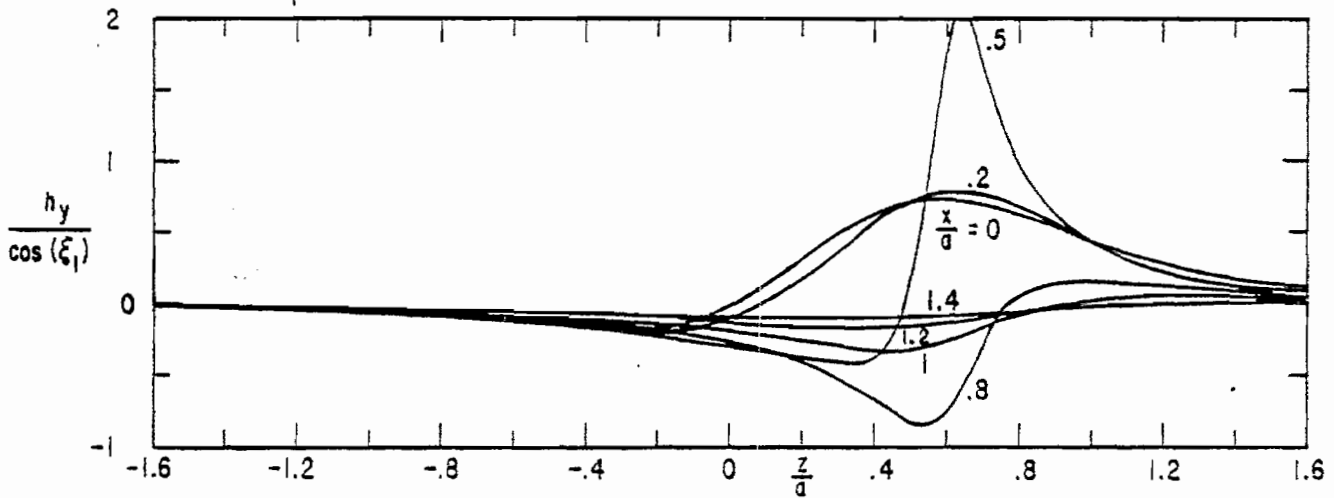


B. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

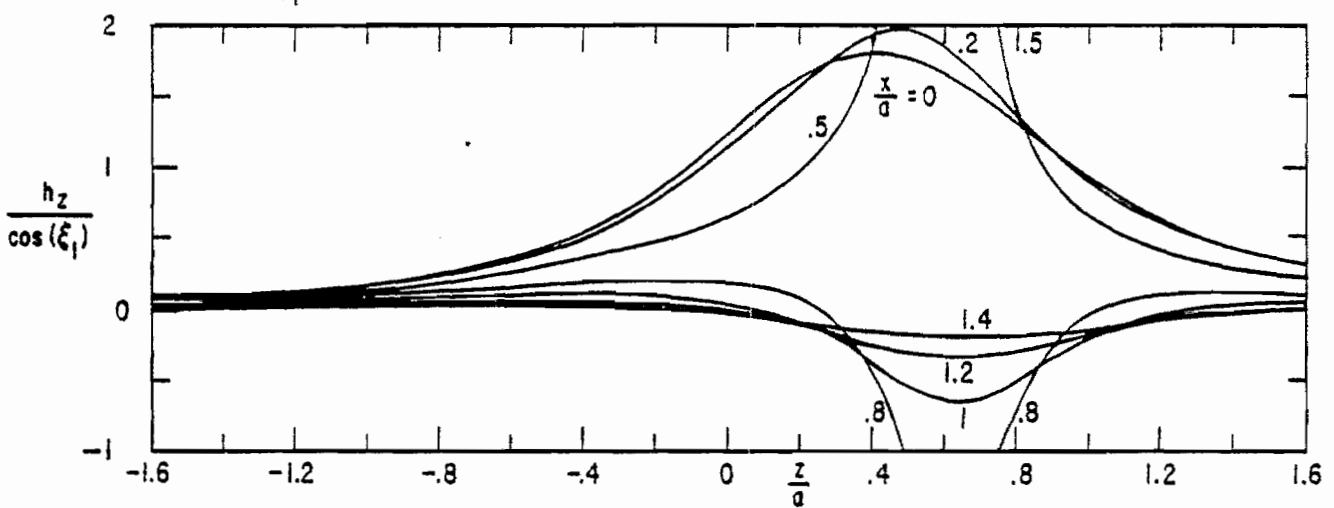
FIGURE 11. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF $z: \frac{2\xi_1}{\pi} = .5; \frac{y}{a} = 0$
($h_y = 0$)



A. $\frac{h_x}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

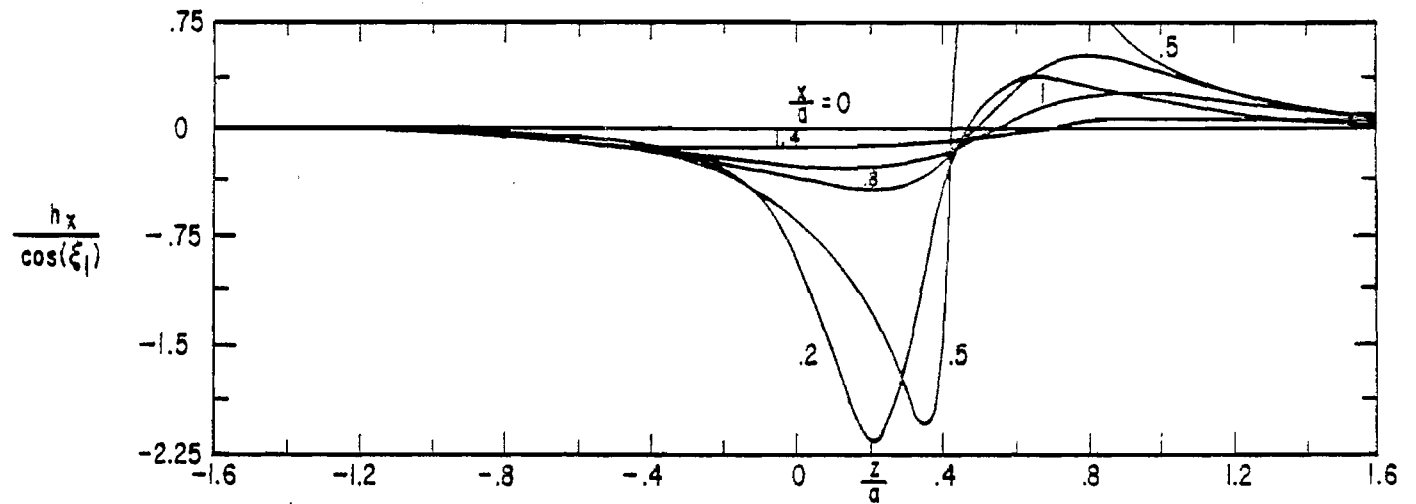


B. $\frac{h_y}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

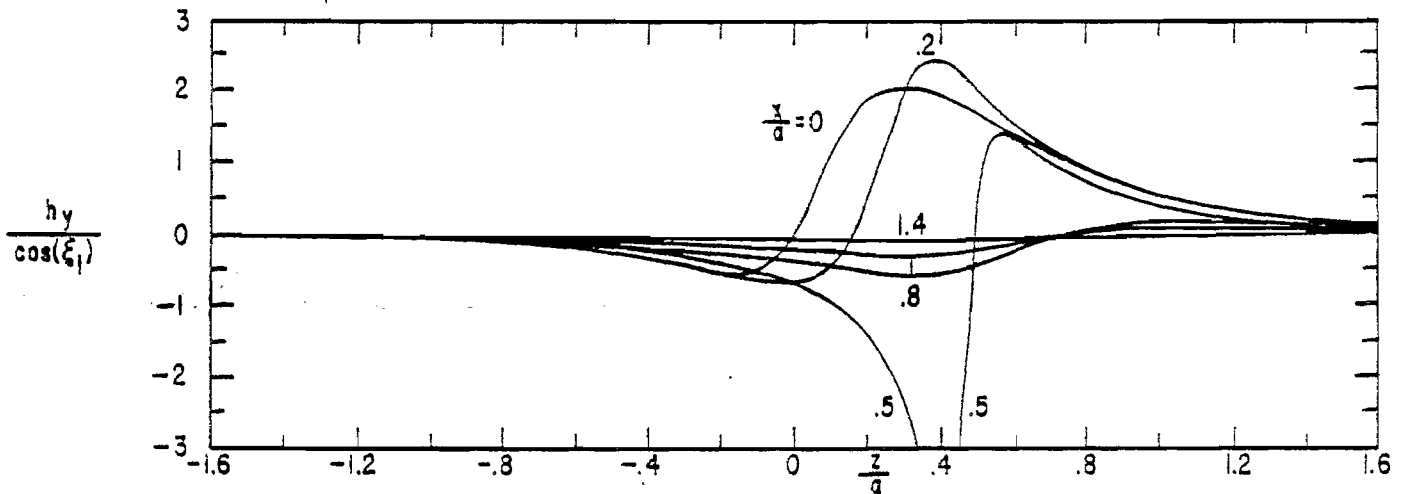


C. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

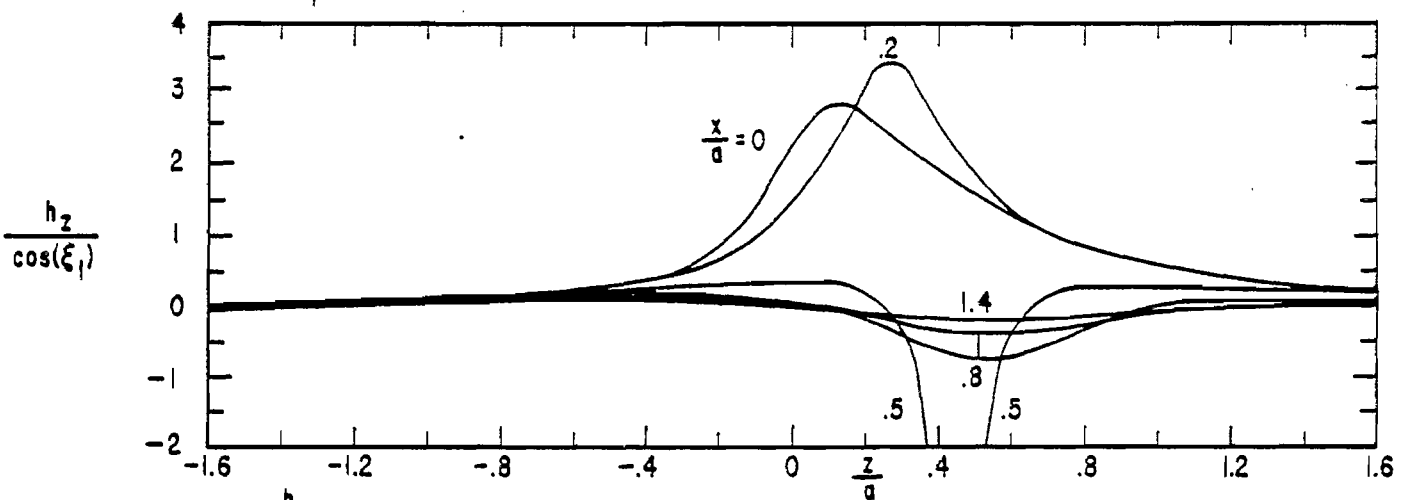
FIGURE 12. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF $z: \frac{2\xi_1}{\pi} = .5; \frac{y}{a} = .5$



A. $\frac{h_x}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

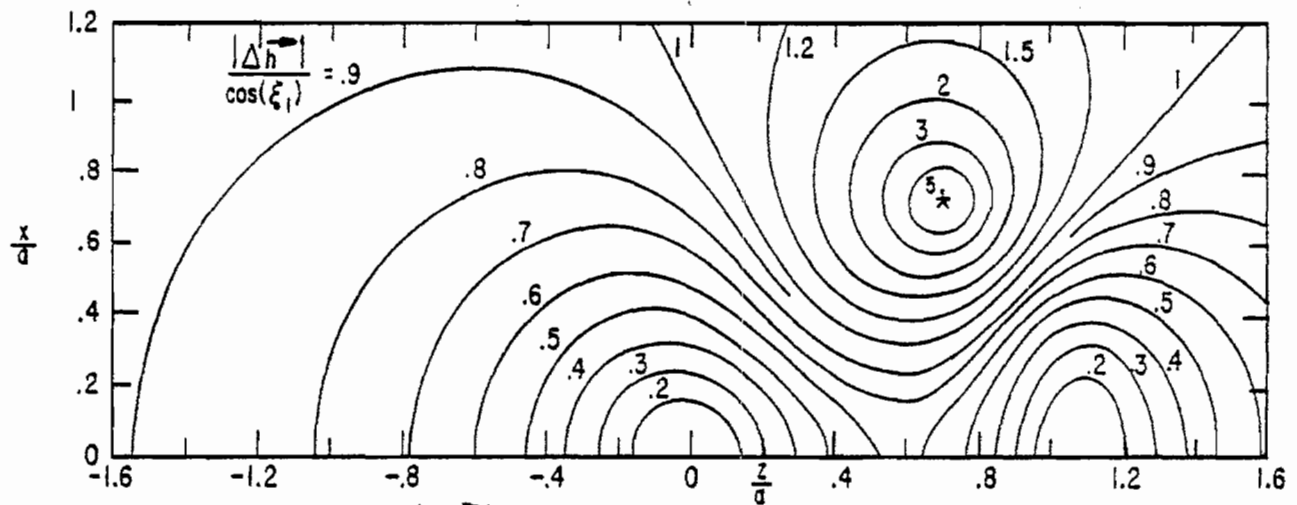


B. $\frac{h_y}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

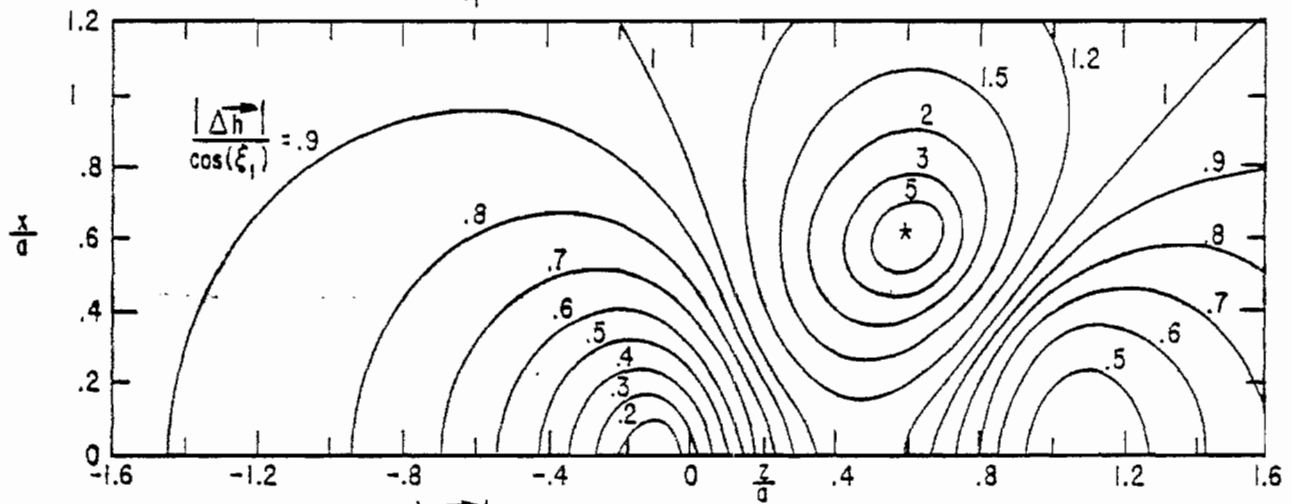


C. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

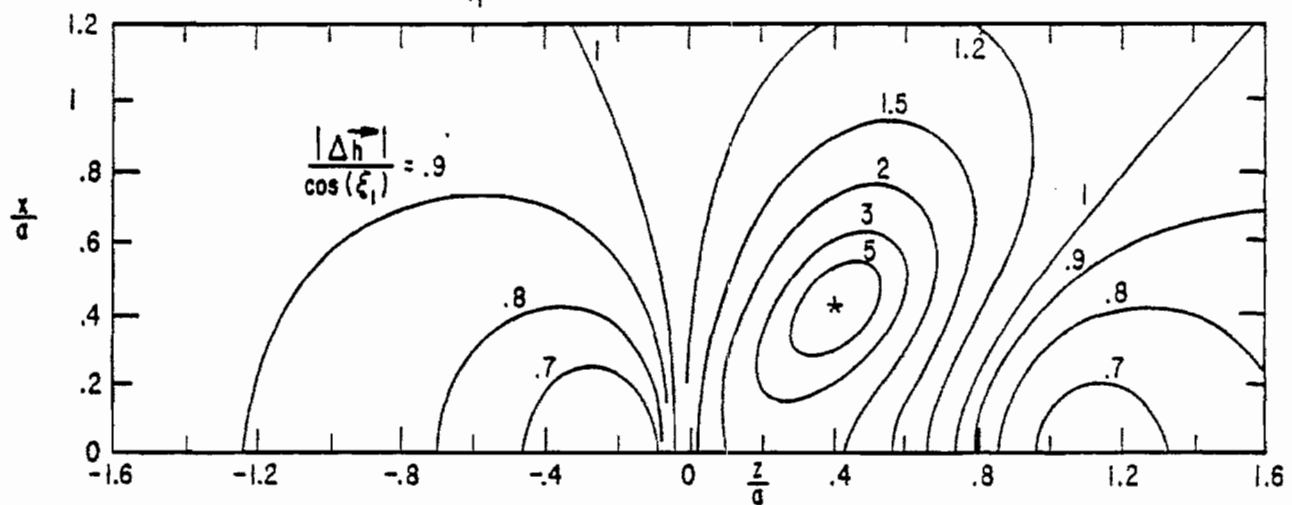
FIGURE 13. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF $z: \frac{2\xi_1}{r} = .5; \frac{y}{a} = .8$



A. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{d} = 0$



B. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{d} = 0.5$



C. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{d} = 0.8$

FIGURE 14. CONTOUR PLOTS OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{2\xi_1}{\pi} = 0.5$

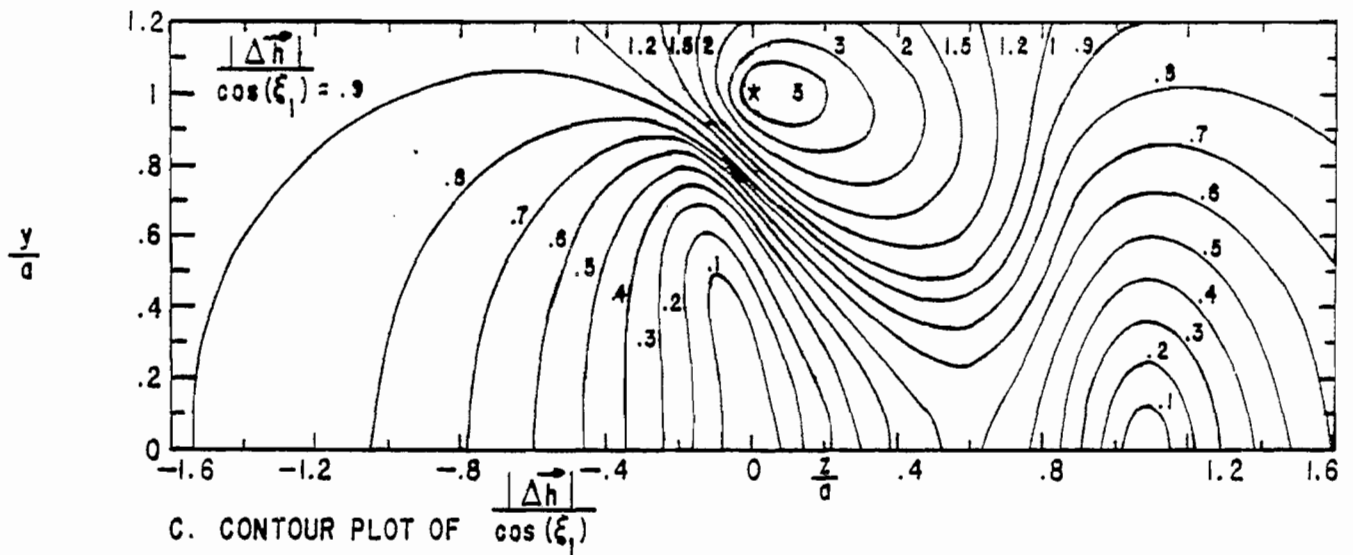
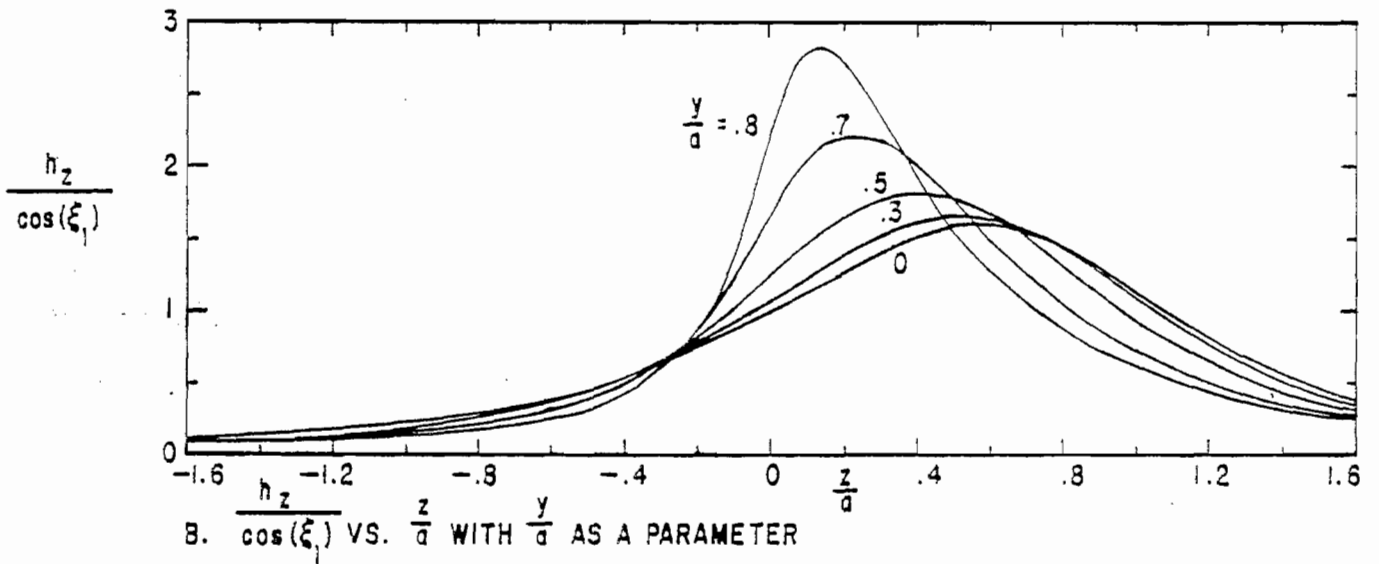
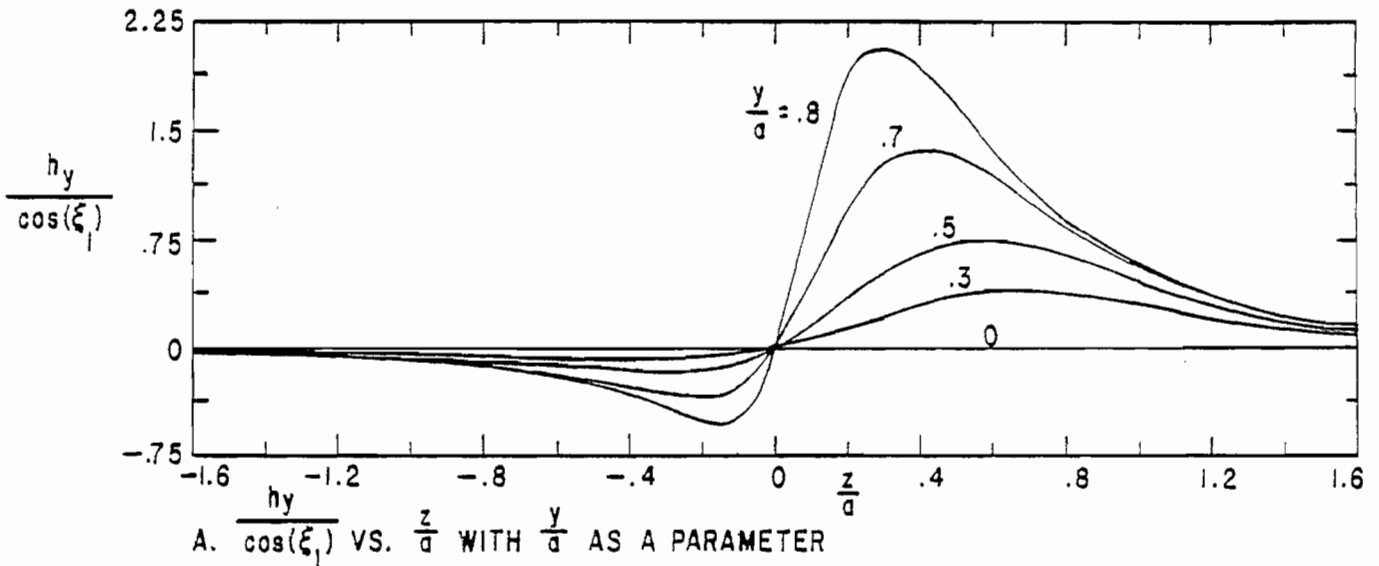


FIGURE 15. MAGNETIC FIELD COMPONENTS AND CONTOUR PLOT OF $\frac{|\Delta h|}{\cos(\xi_1)}$ FOR $\frac{x}{a} = 0 : \frac{2\xi_1}{\pi} = .5$

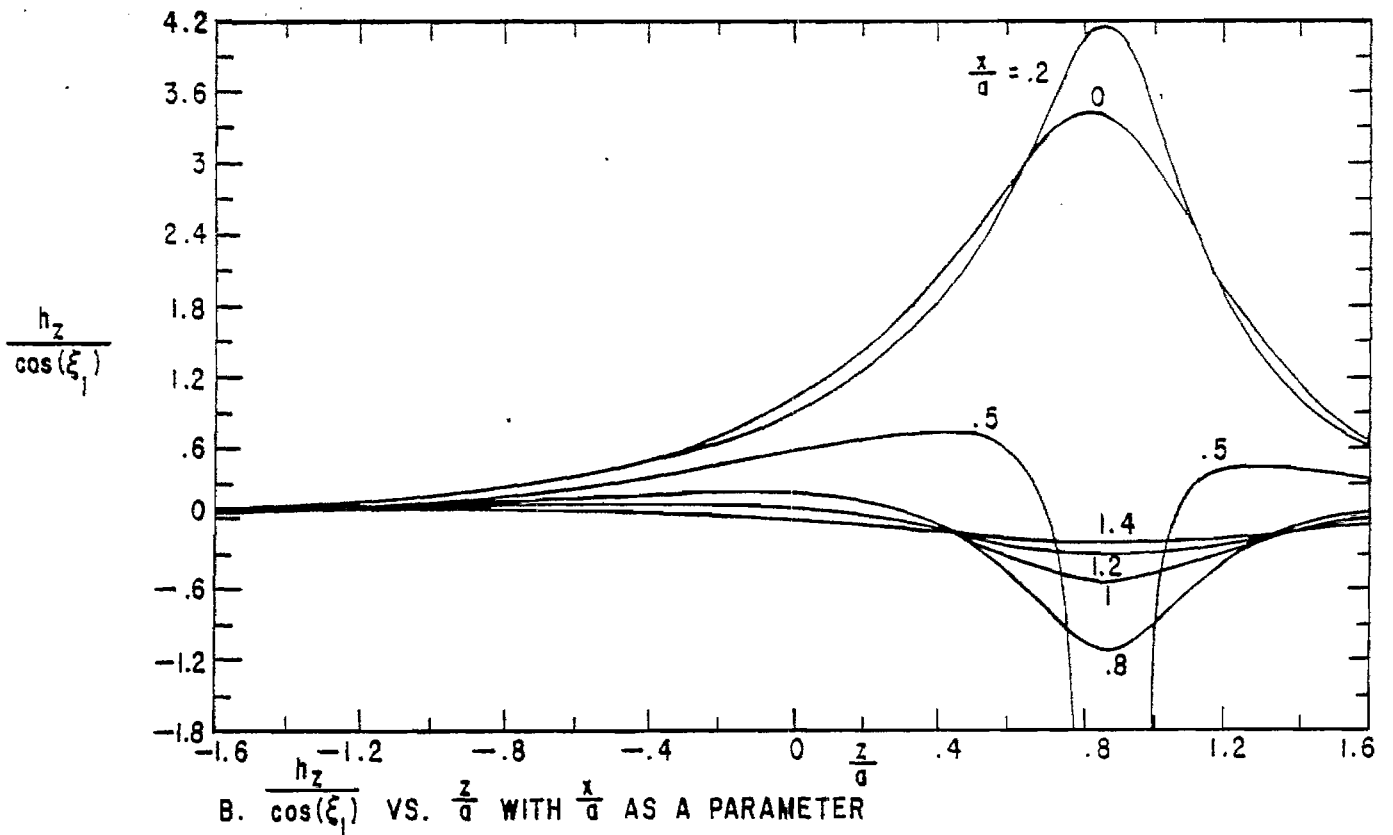
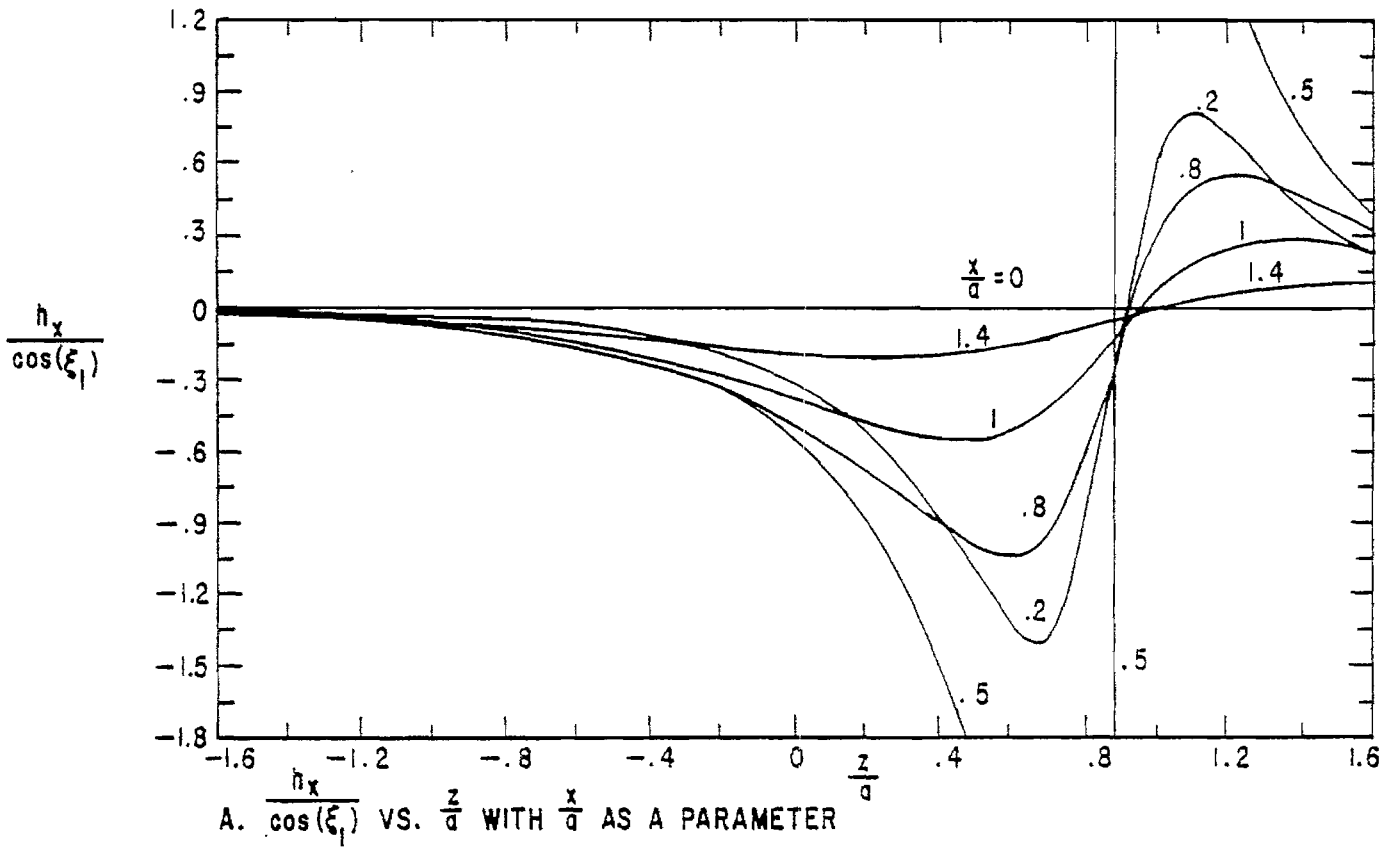
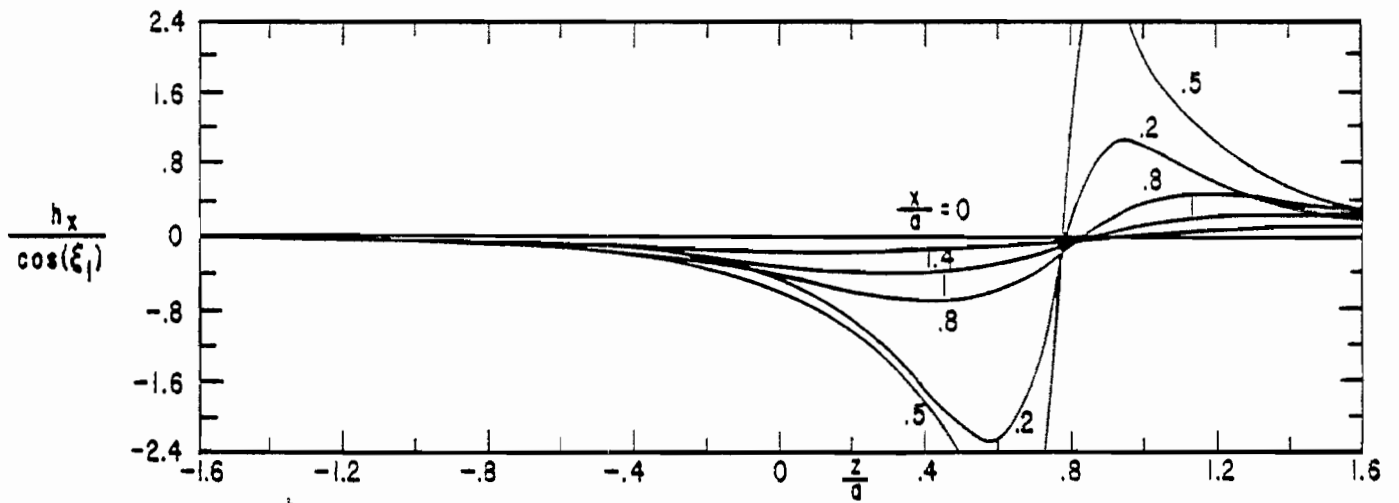
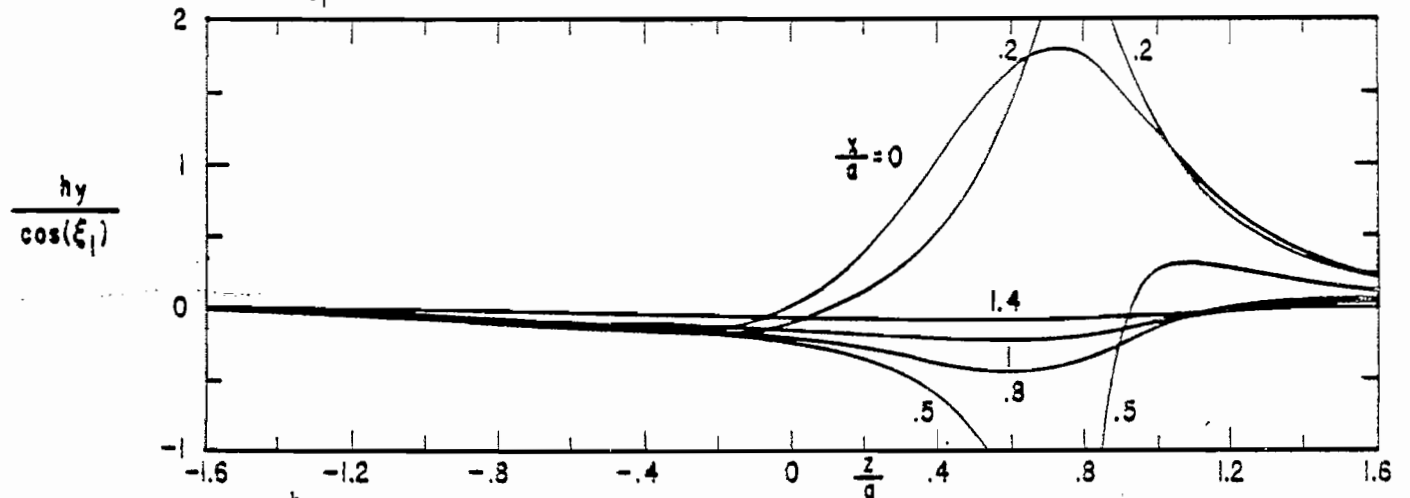


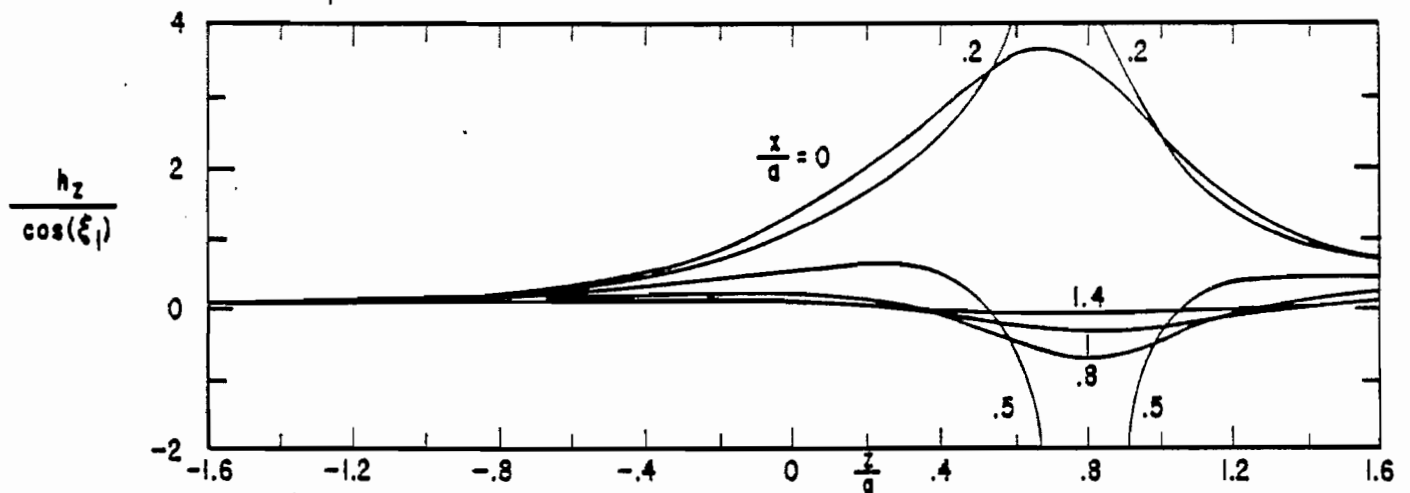
FIGURE 16. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF $z: \frac{2\xi_1}{\pi} = .7; \frac{y}{a} = 0$
 ($h_y \equiv 0$)



A. $\frac{h_x}{\cos(\xi_1)}$ VS. $\frac{z}{d}$ WITH $\frac{x}{d}$ AS A PARAMETER

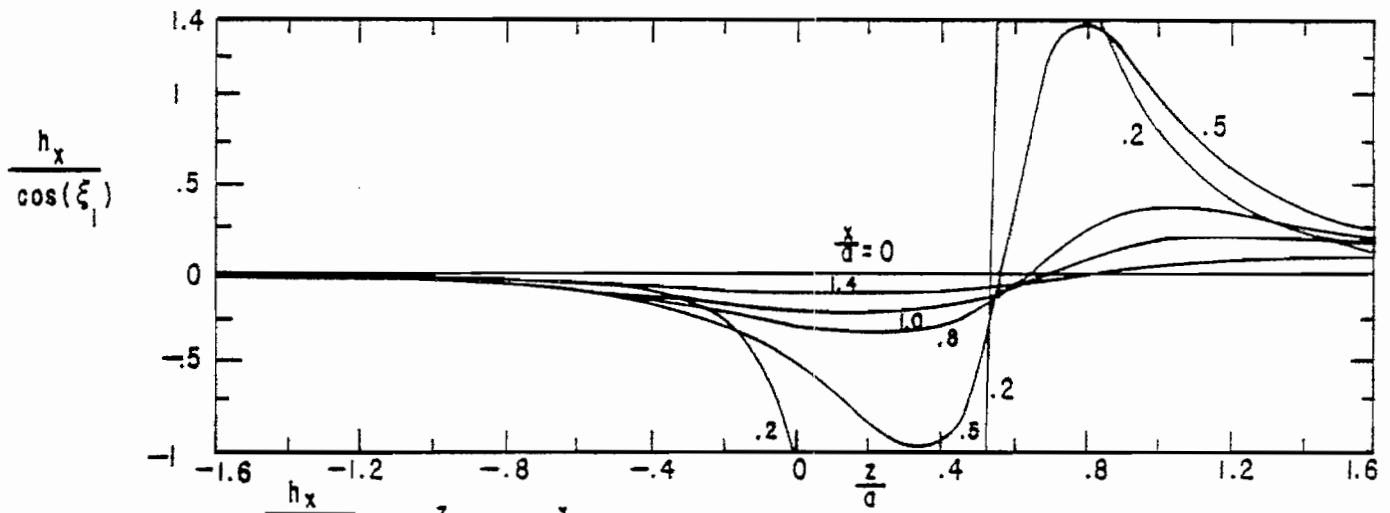


B. $\frac{h_y}{\cos(\xi_1)}$ VS. $\frac{z}{d}$ WITH $\frac{x}{d}$ AS A PARAMETER

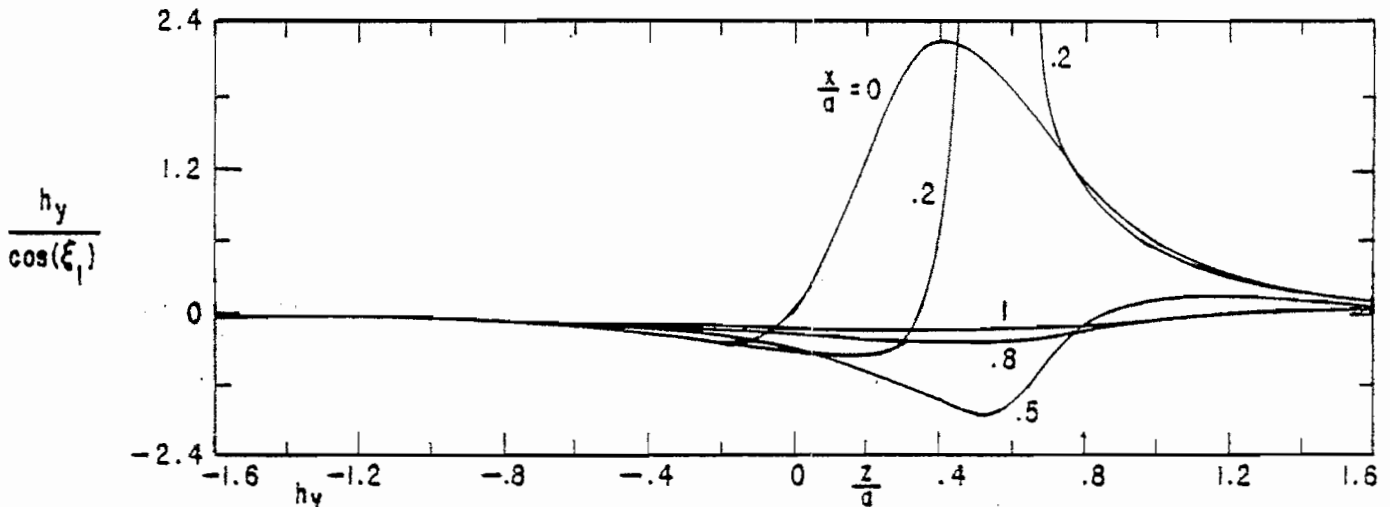


C. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{d}$ WITH $\frac{x}{d}$ AS A PARAMETER

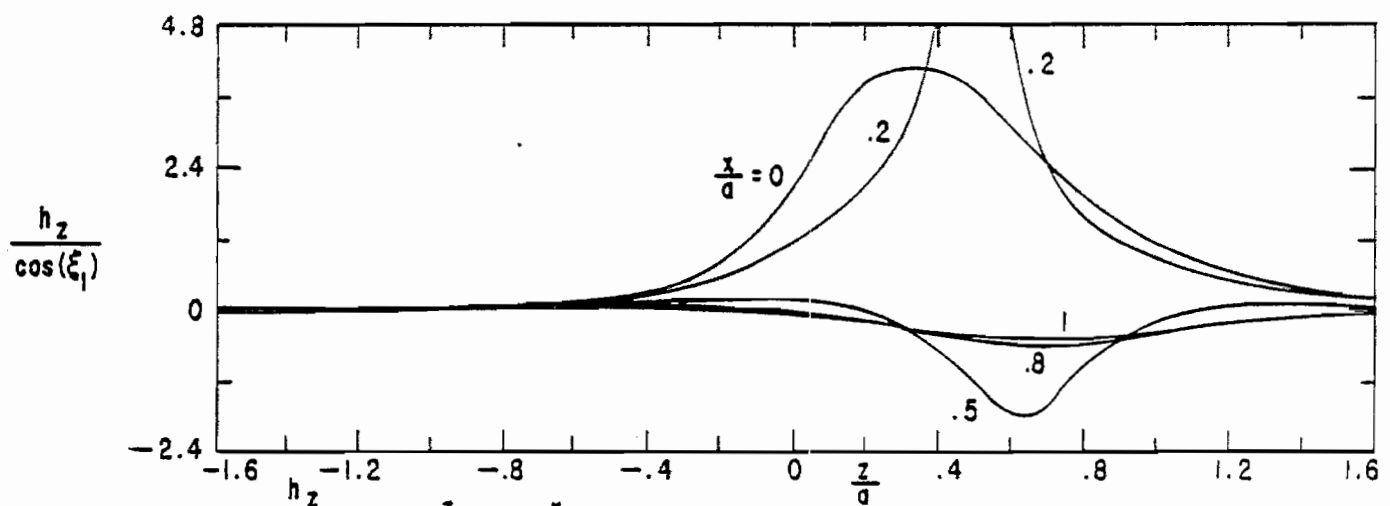
FIGURE 17. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF z : $\frac{2\xi_1}{\tau} = .7$; $\frac{y}{b} = .5$



A. $\frac{h_x}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

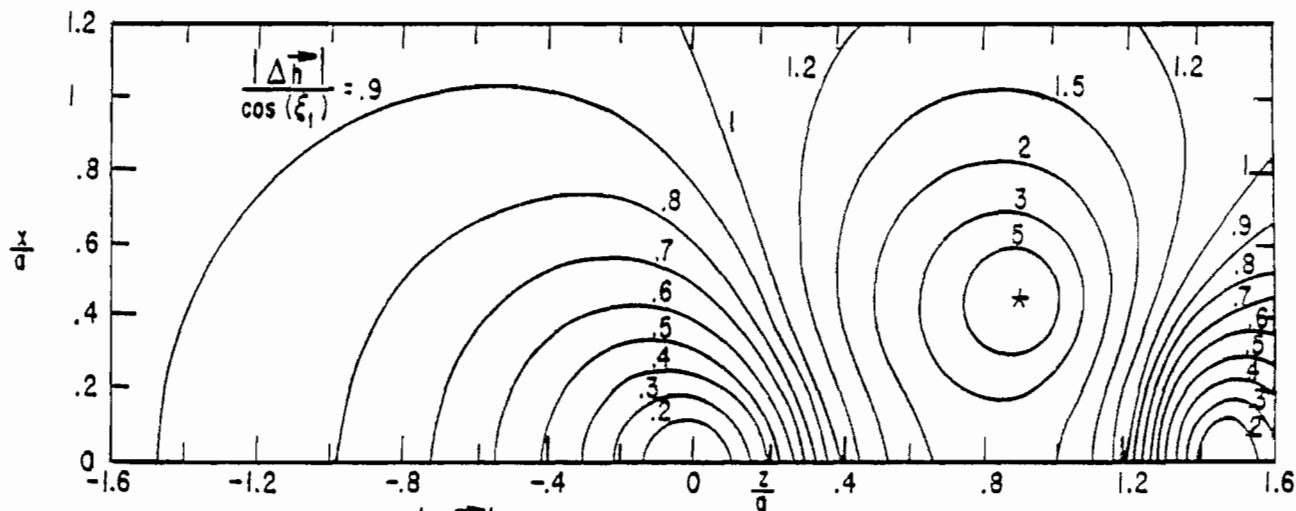


B. $\frac{h_y}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

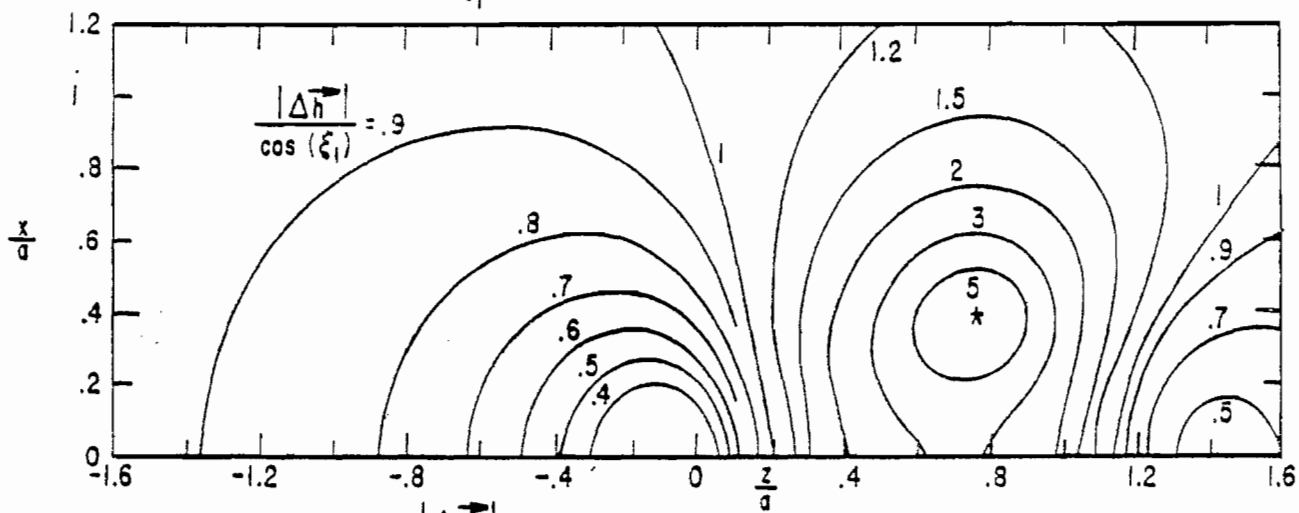


C. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

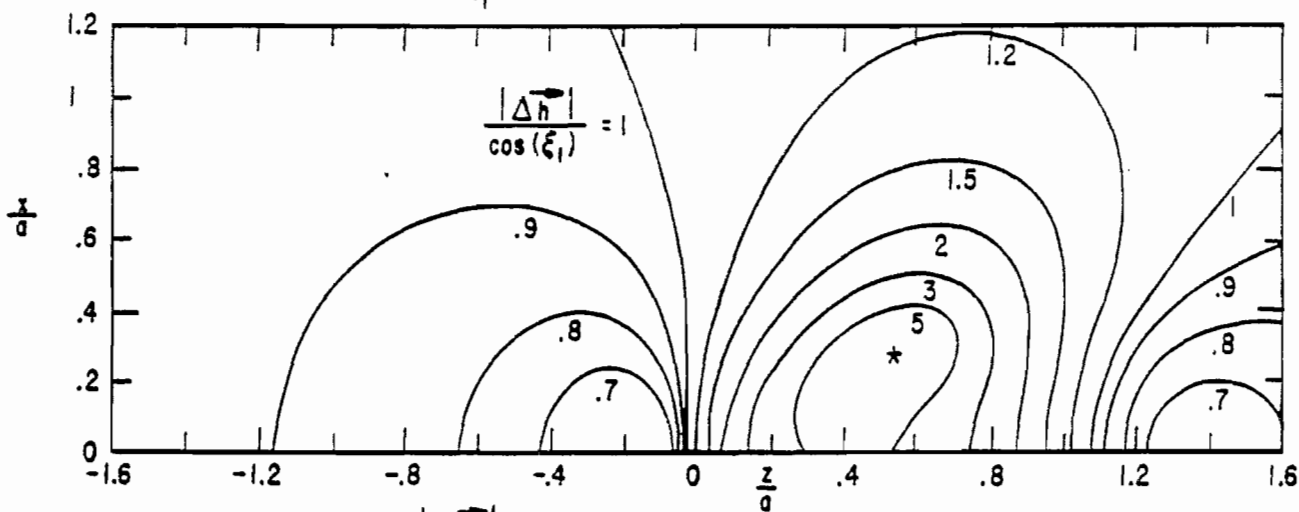
FIGURE 18. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF $z: \frac{2\xi_1}{\pi} = .7; \frac{y}{a} = .8$



A. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{a} = 0$

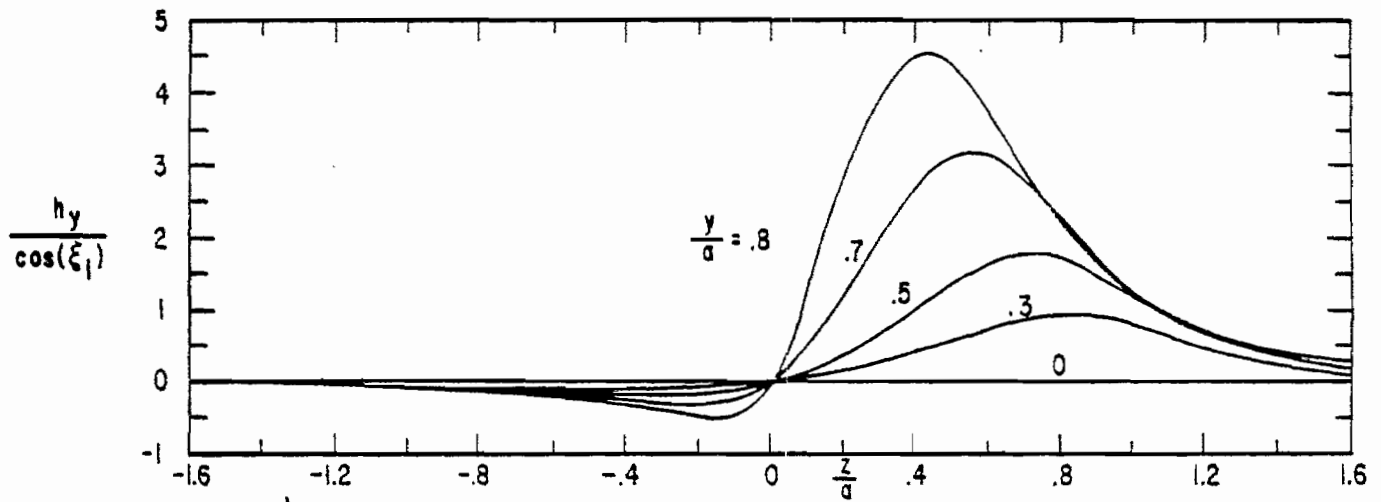


B. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{a} = .5$

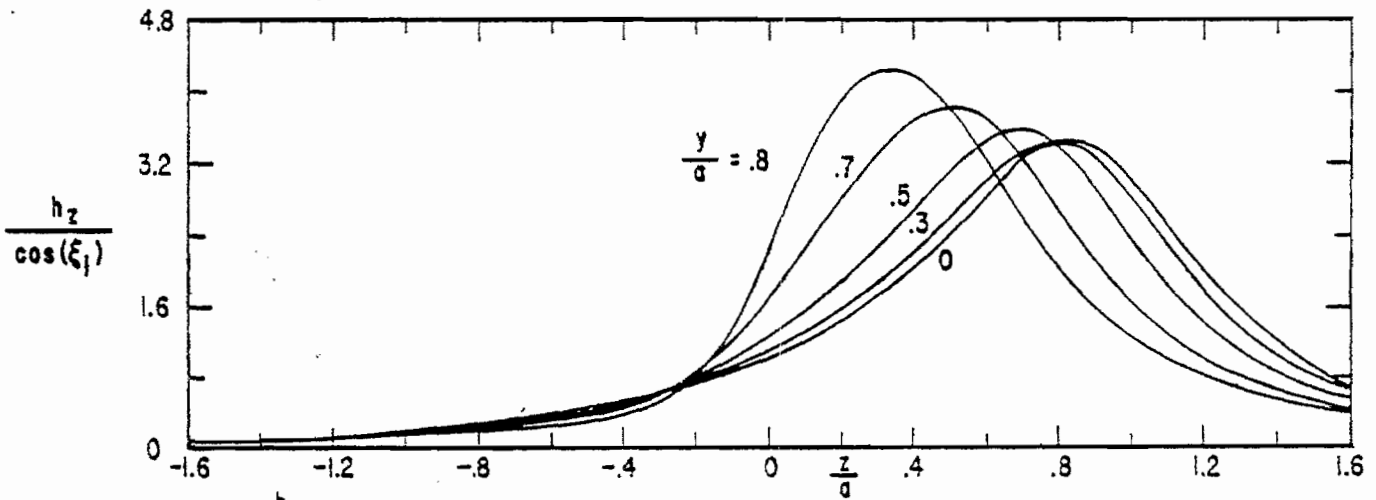


C. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{a} = .8$

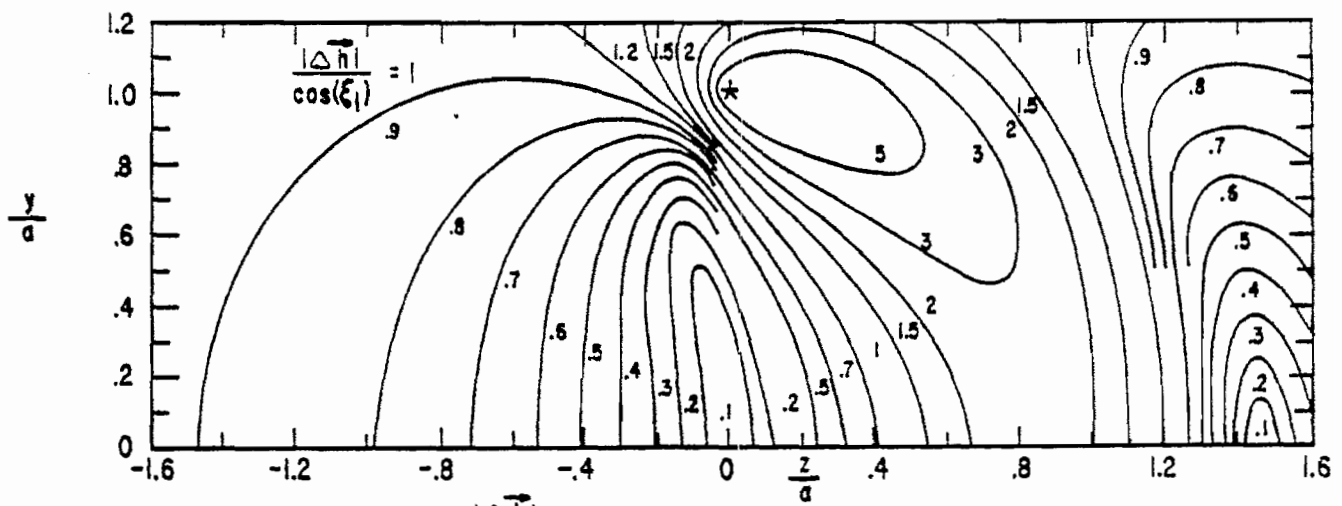
FIGURE 19. CONTOUR PLOTS OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{2\xi_1}{\pi} = .7$



A. $\frac{h_y}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{y}{a}$ AS A PARAMETER

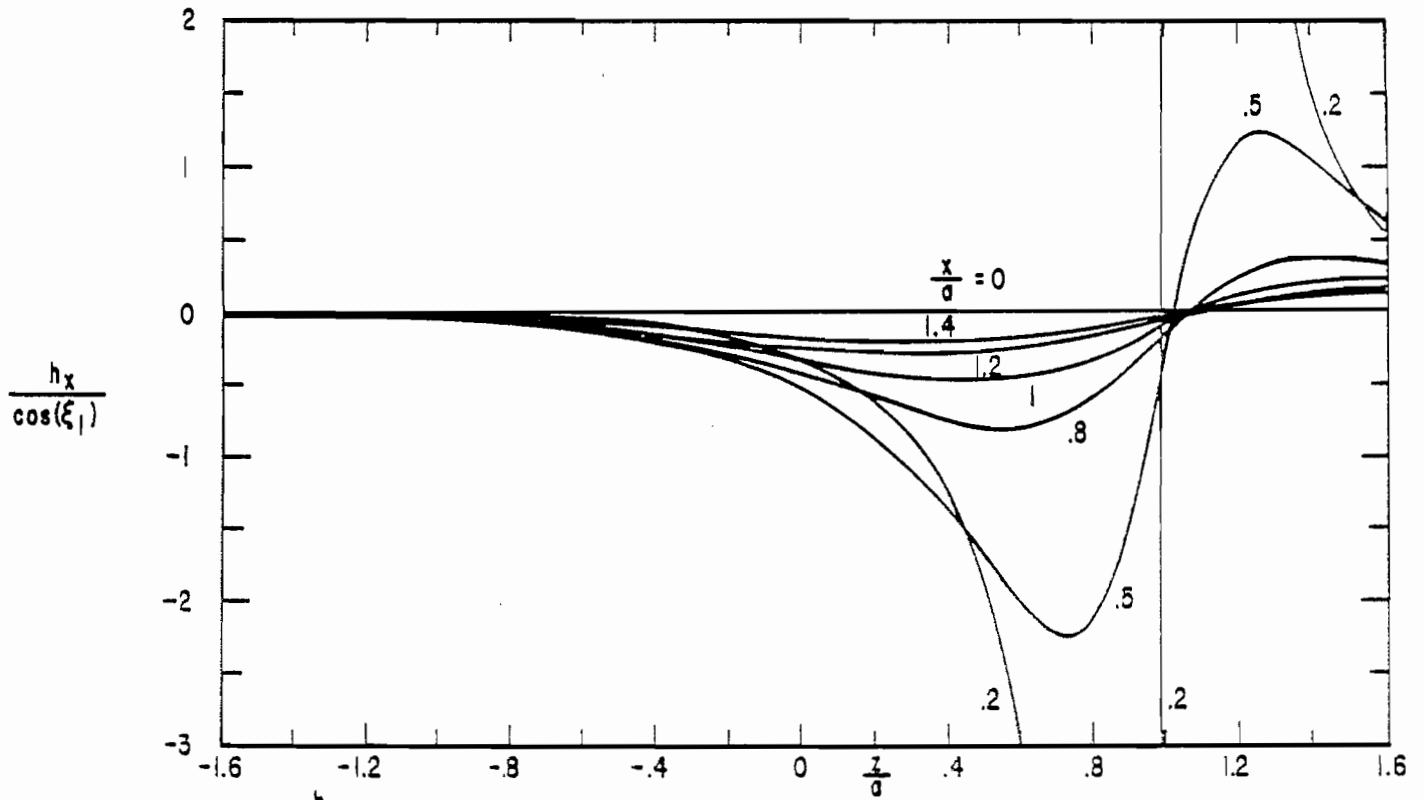


B. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{y}{a}$ AS A PARAMETER

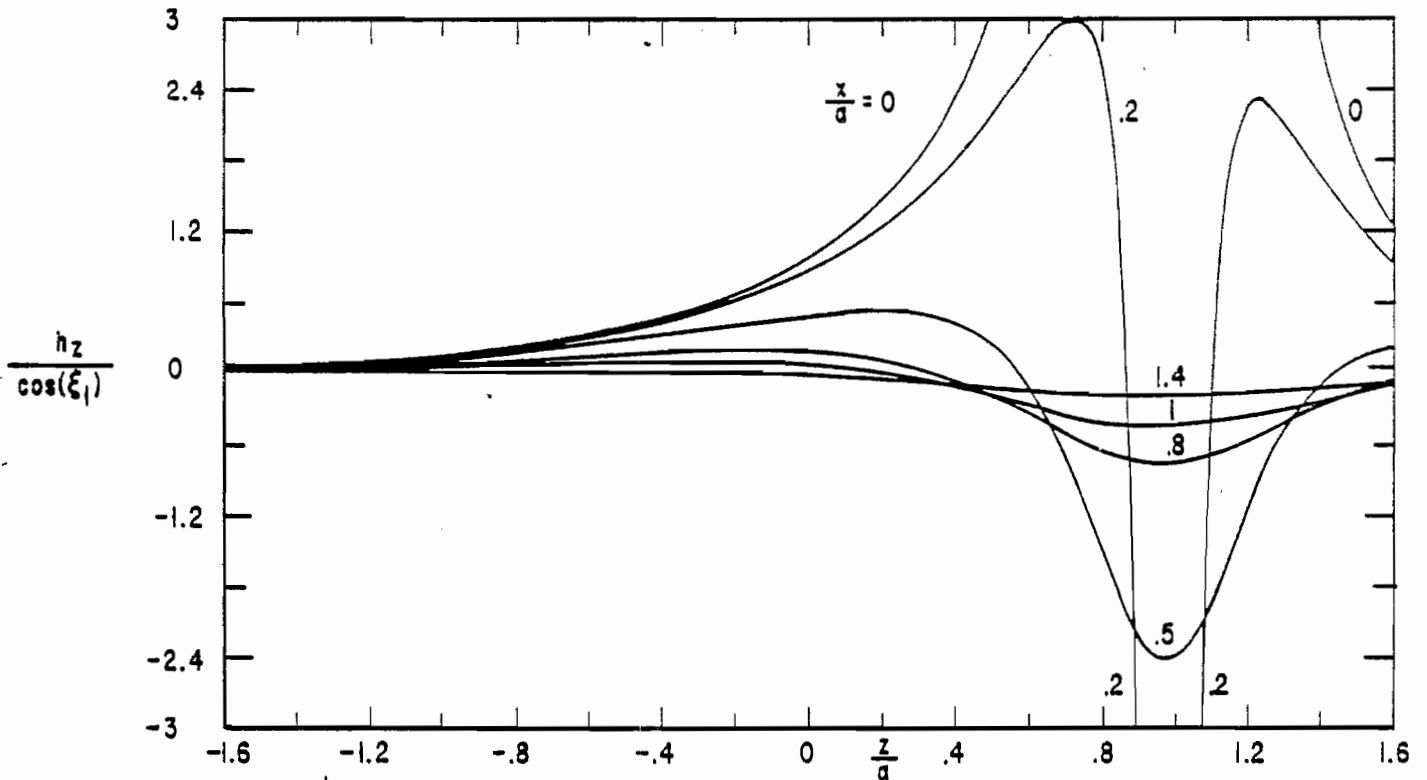


C. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$

FIGURE 20. MAGNETIC FIELD COMPONENTS AND CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR $\frac{x}{a} = 0: \frac{2\xi_1}{\pi} = .7$

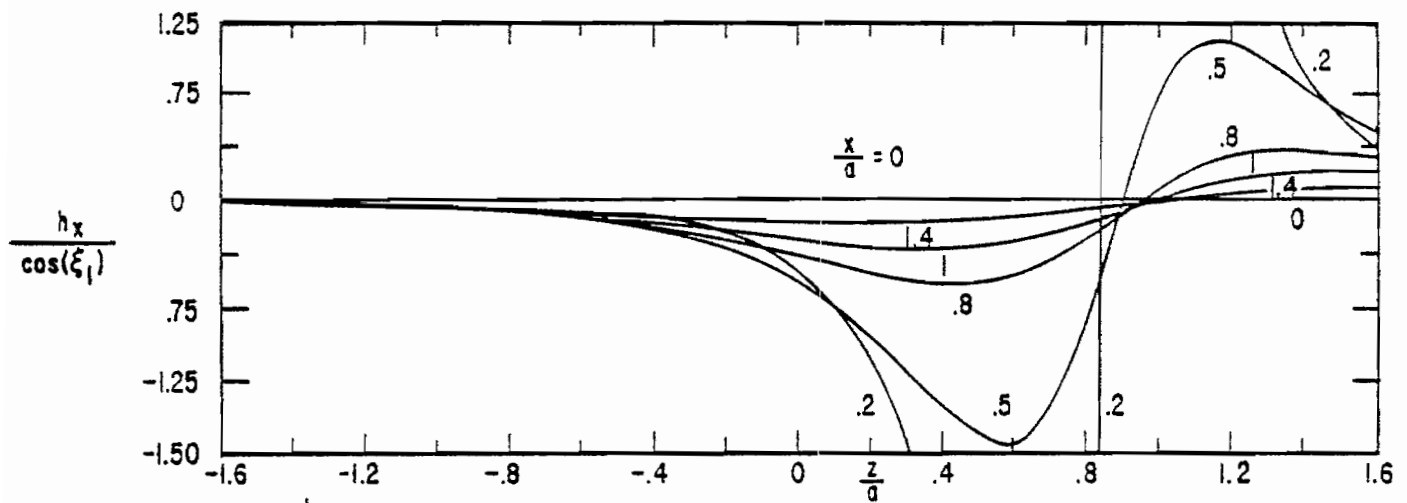


A. $\frac{h_x}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

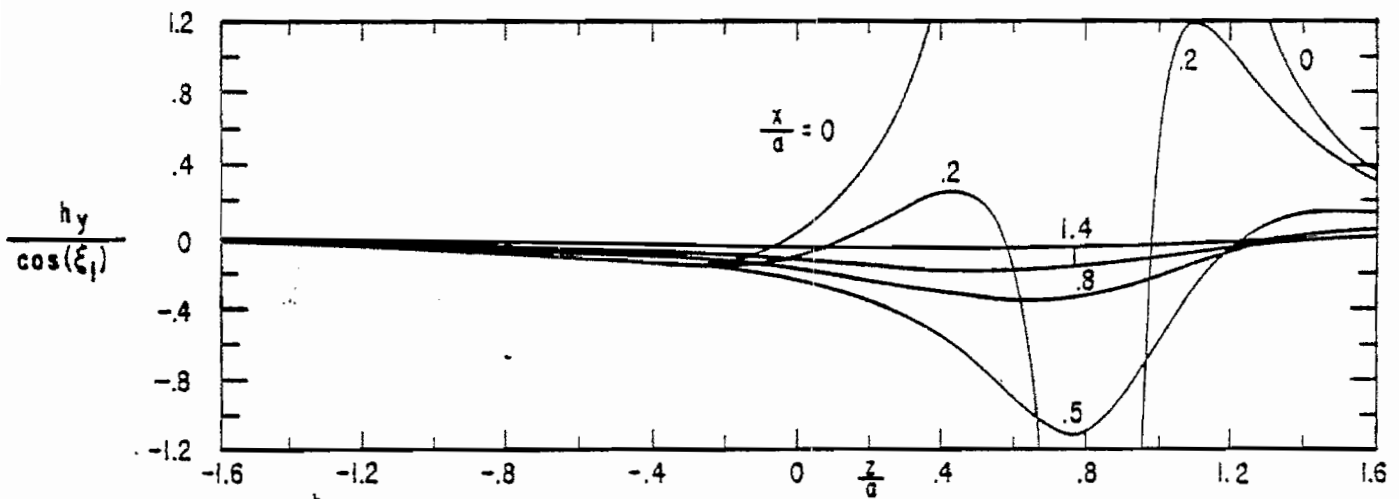


B. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

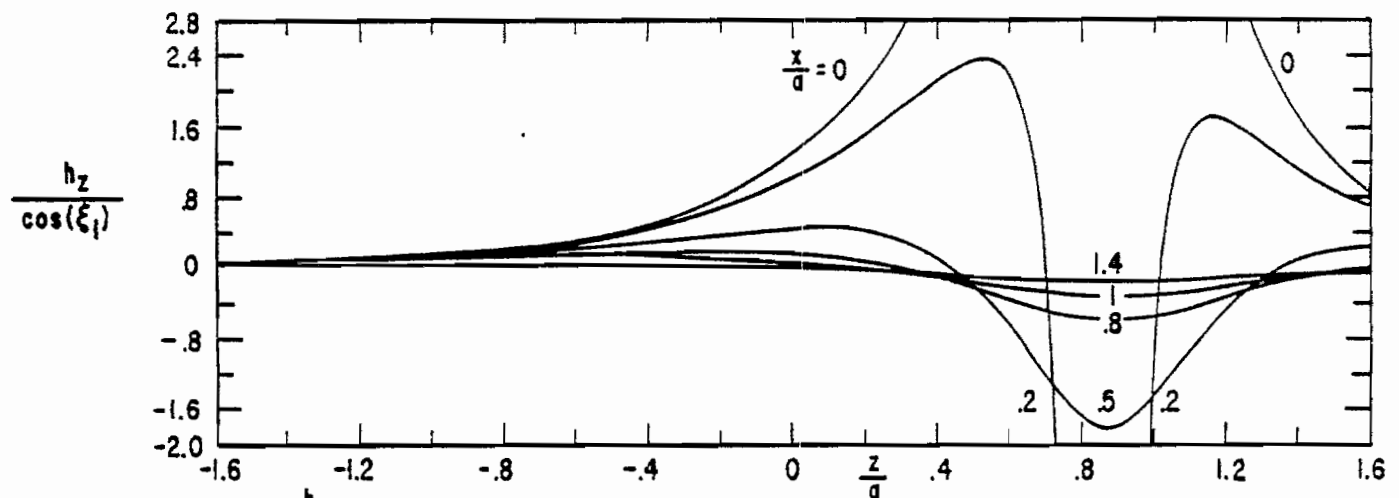
FIGURE 21. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF z : $\frac{2\xi_1}{r} = .9$; $\frac{y}{a} = 0$
 ($h_y \equiv 0$)



A. $\frac{h_x}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER



B. $\frac{h_y}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER



C. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{x}{a}$ AS A PARAMETER

FIGURE 22. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF z : $\frac{2\xi_1}{r} = .9$; $\frac{y}{a} = .5$

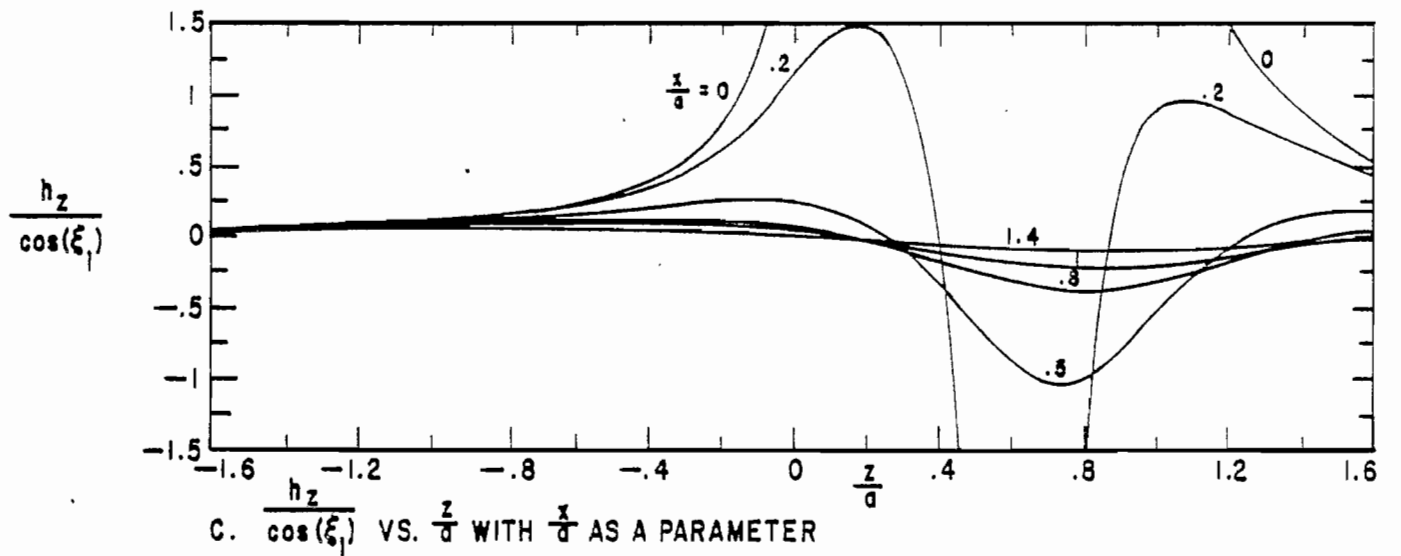
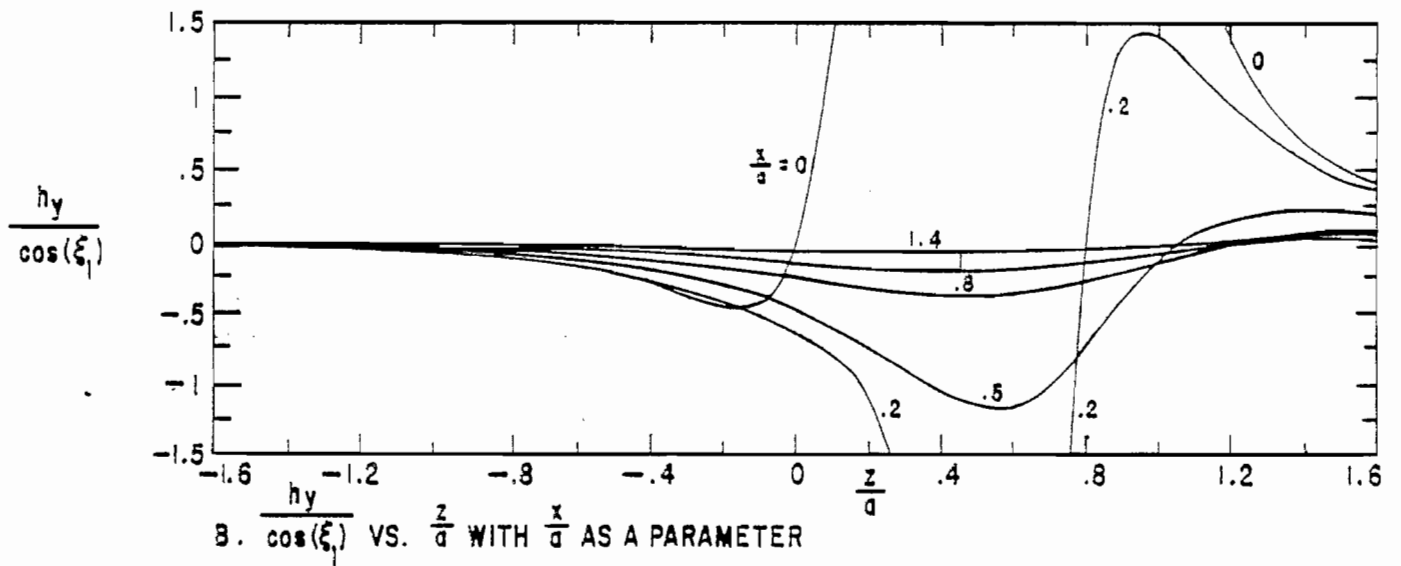
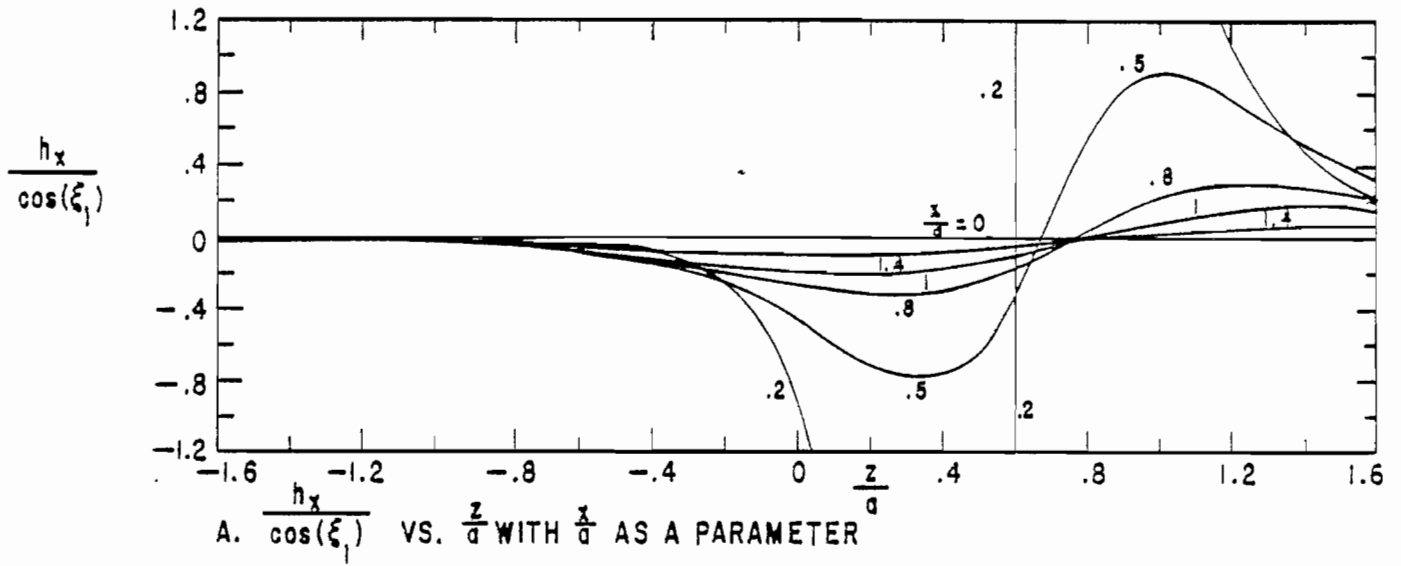
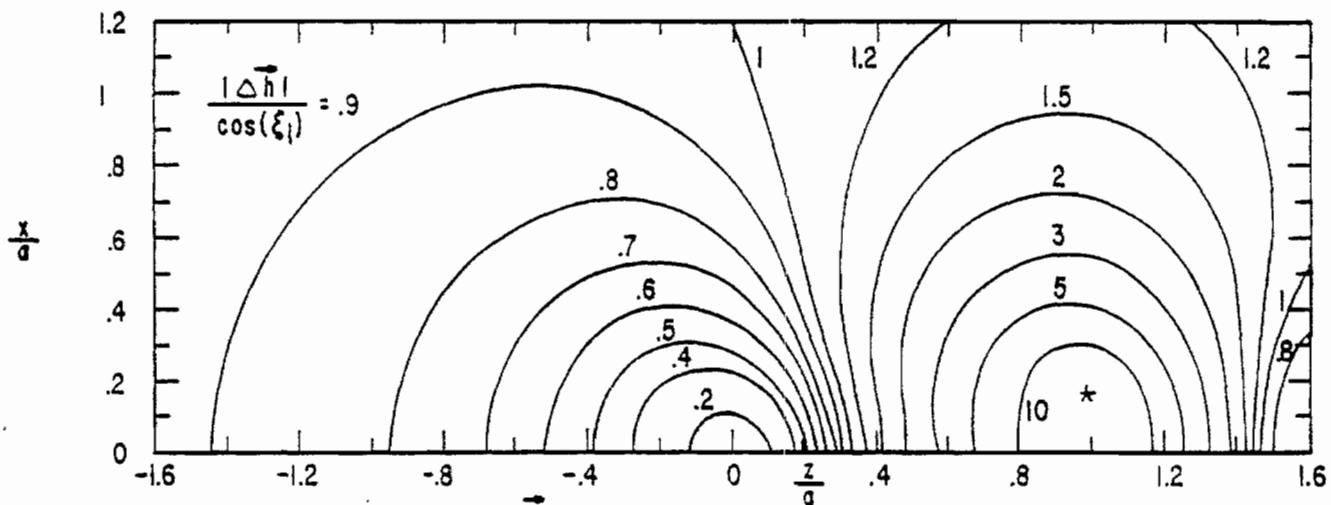
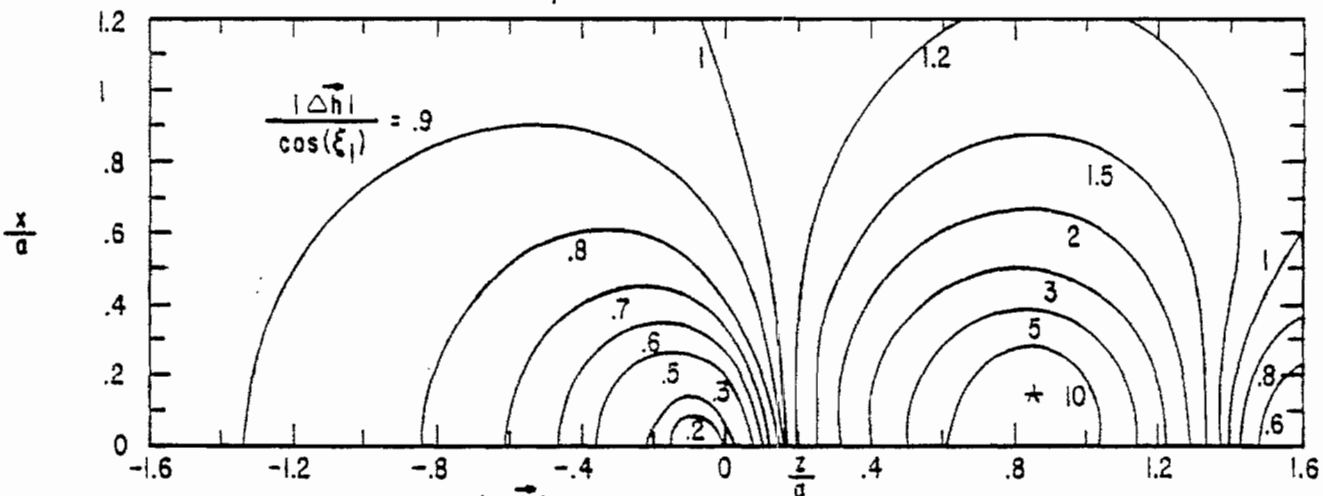


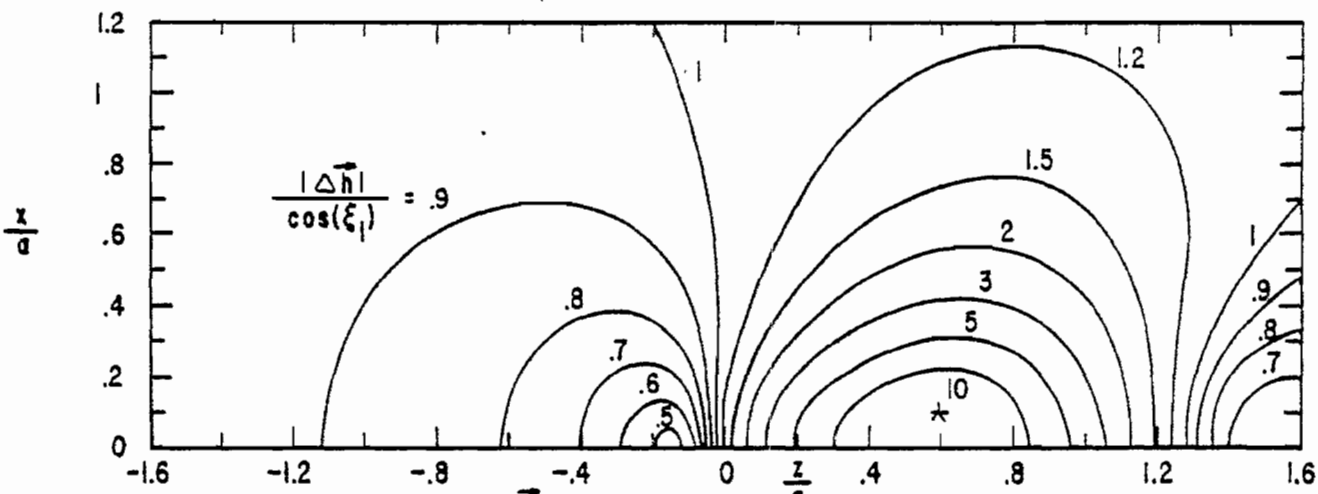
FIGURE 23. MAGNETIC FIELD COMPONENTS AS A FUNCTION OF $z: \frac{2\xi_1}{\pi} = .9; \frac{y}{a} = .8$



A. CONTOUR PLOT OF $\frac{|\Delta \bar{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{a} = 0$

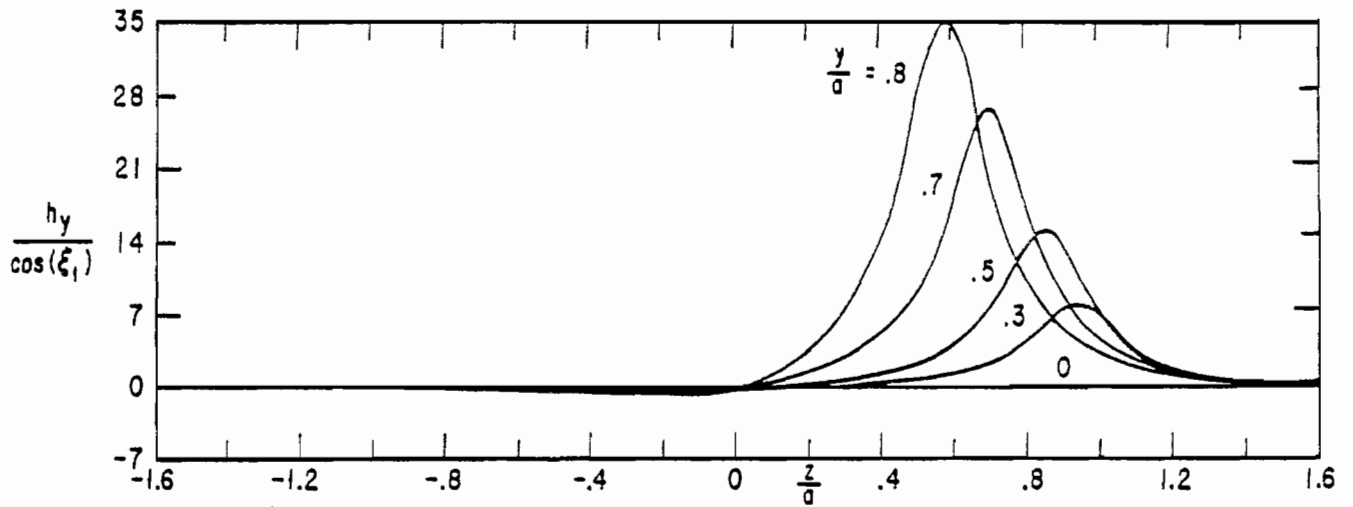


B. CONTOUR PLOT OF $\frac{|\Delta \bar{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{a} = .5$

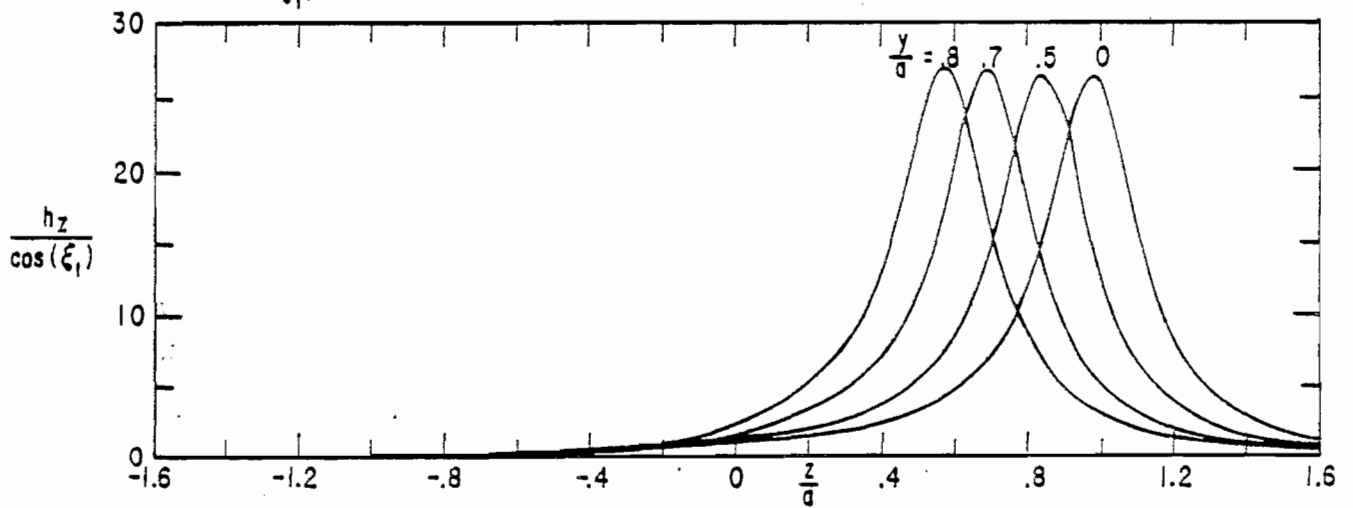


C. CONTOUR PLOT OF $\frac{|\Delta \bar{h}|}{\cos(\xi_1)}$ FOR $\frac{y}{a} = .8$

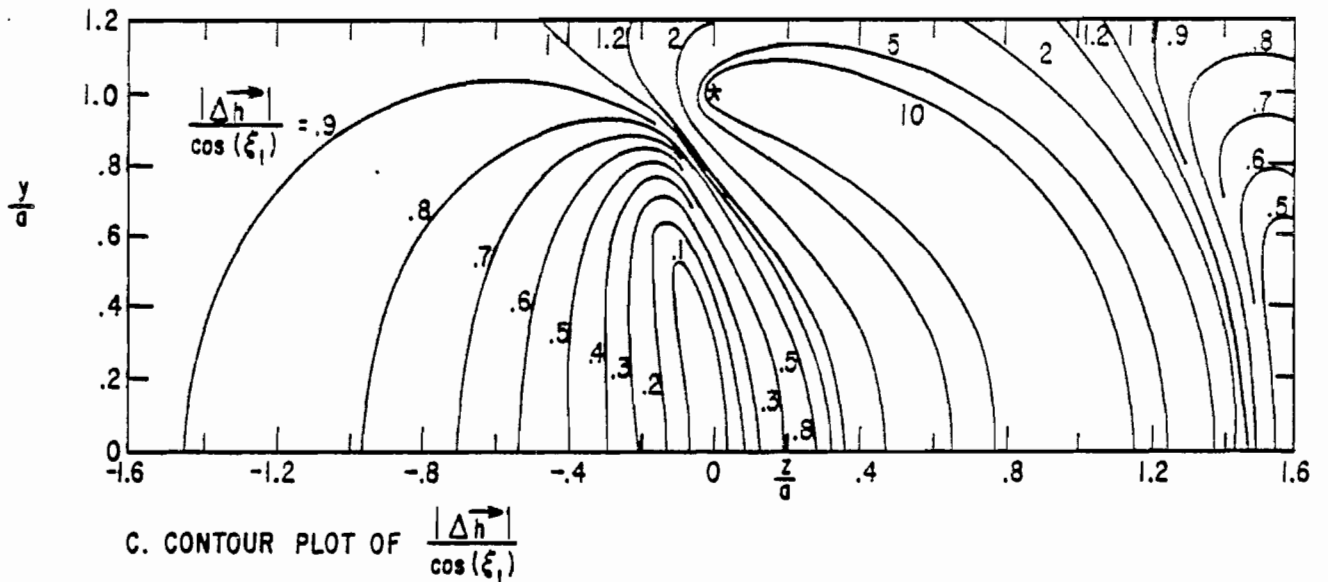
FIGURE 24. CONTOUR PLOTS OF $\frac{|\Delta \bar{h}|}{\cos(\xi_1)}$ FOR $\frac{2\xi_1}{\pi} = .9$



A. $\frac{h_y}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{y}{a}$ AS A PARAMETER



B. $\frac{h_z}{\cos(\xi_1)}$ VS. $\frac{z}{a}$ WITH $\frac{y}{a}$ AS A PARAMETER



C. CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$

FIGURE 25. MAGNETIC FIELD COMPONENTS AND CONTOUR PLOT OF $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ FOR

$$\frac{x}{a} = 0 : \frac{2\xi_1}{\pi} = .9$$

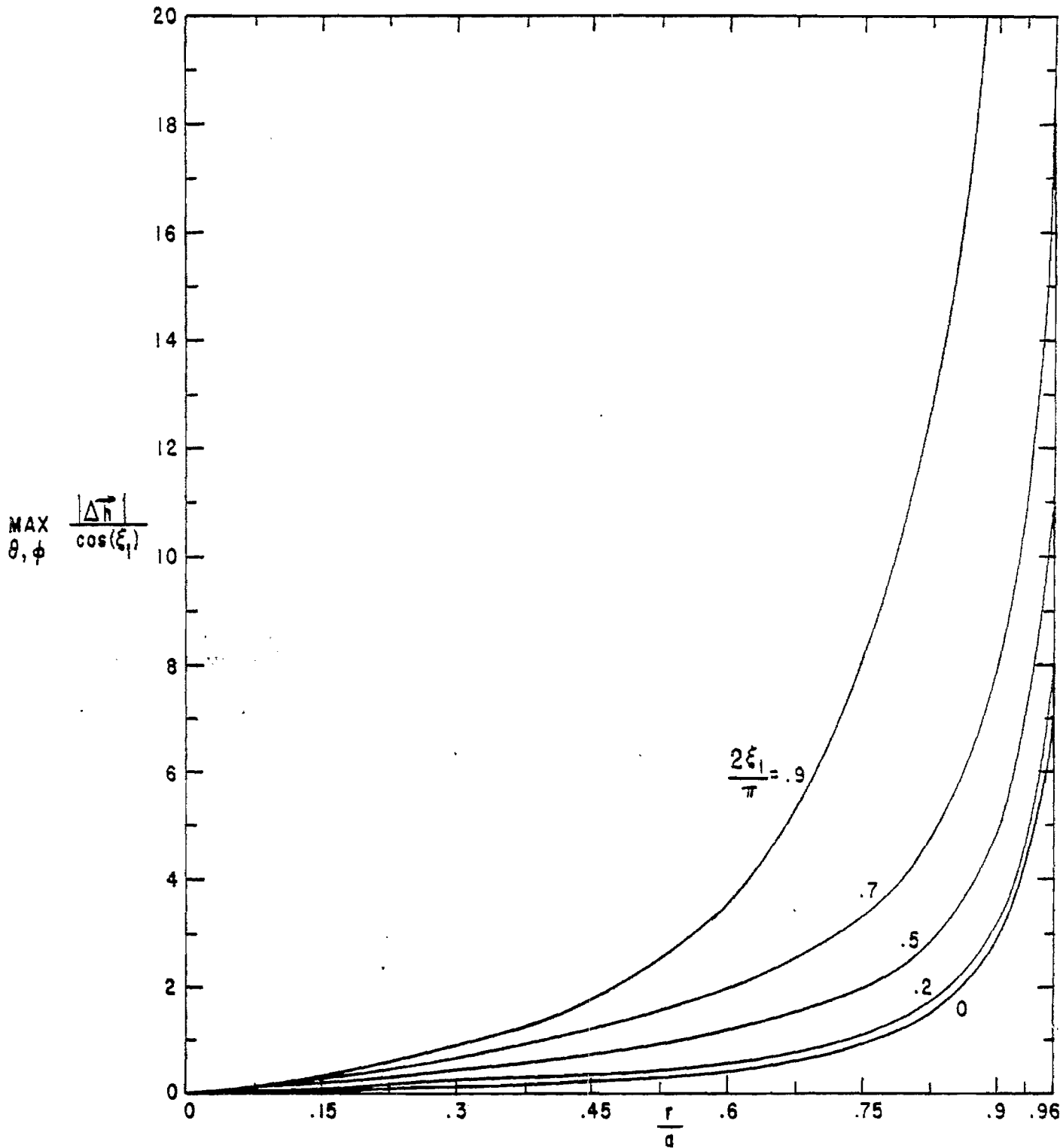


FIGURE 26. MAXIMUM $\frac{|\Delta h|}{\cos(\xi_1)}$ ON THE SURFACE OF A SPHERE OF RADIUS r/a CENTERED AT THE ORIGIN.

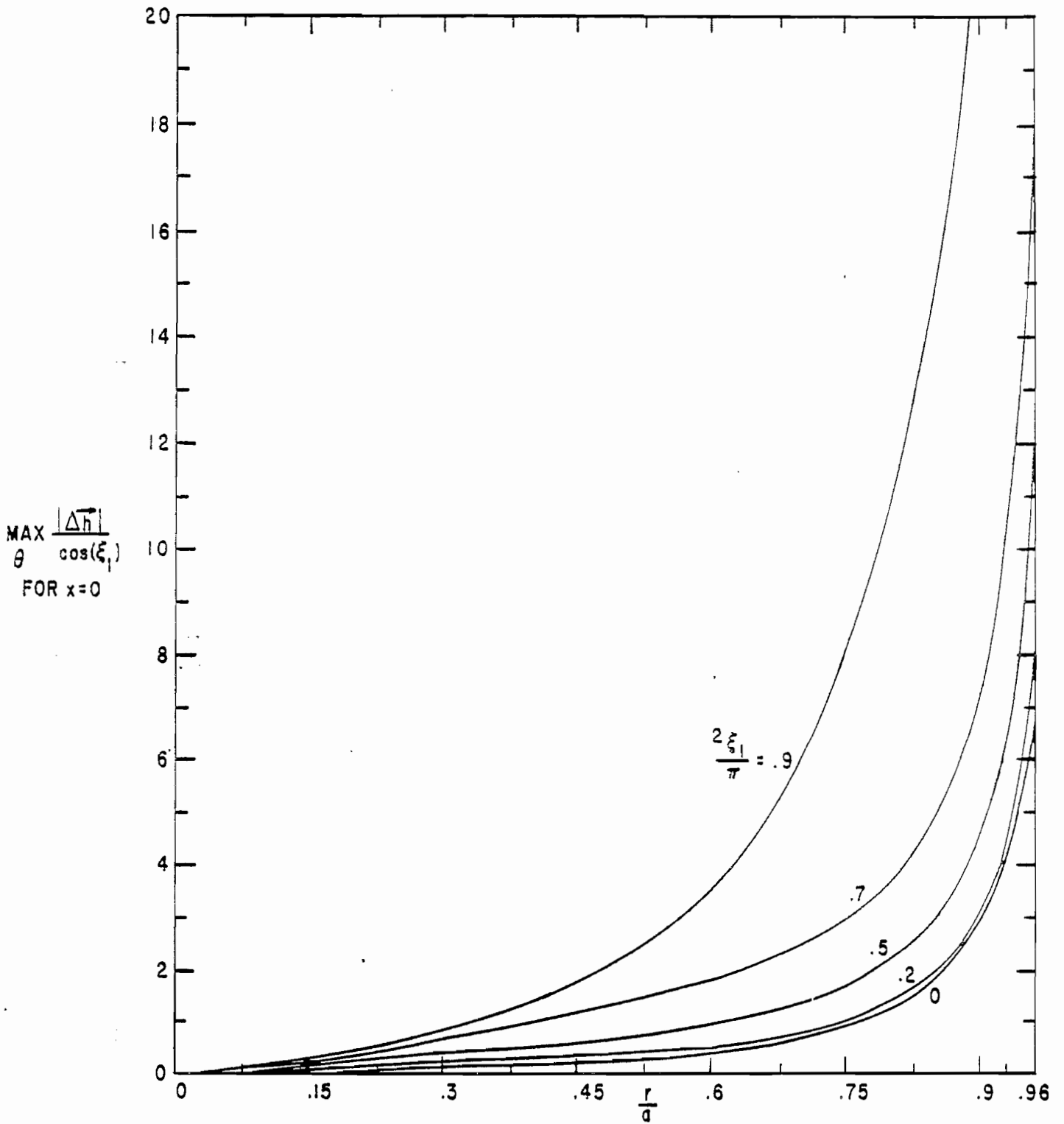


FIGURE 27. MAXIMUM $\frac{|\Delta h|}{\cos(\xi_1)}$ ON THE PERIMETER OF A CIRCLE ON THE GROUND SURFACE OF RADIUS r CENTERED AT THE ORIGIN: $\frac{x}{a} = 0$

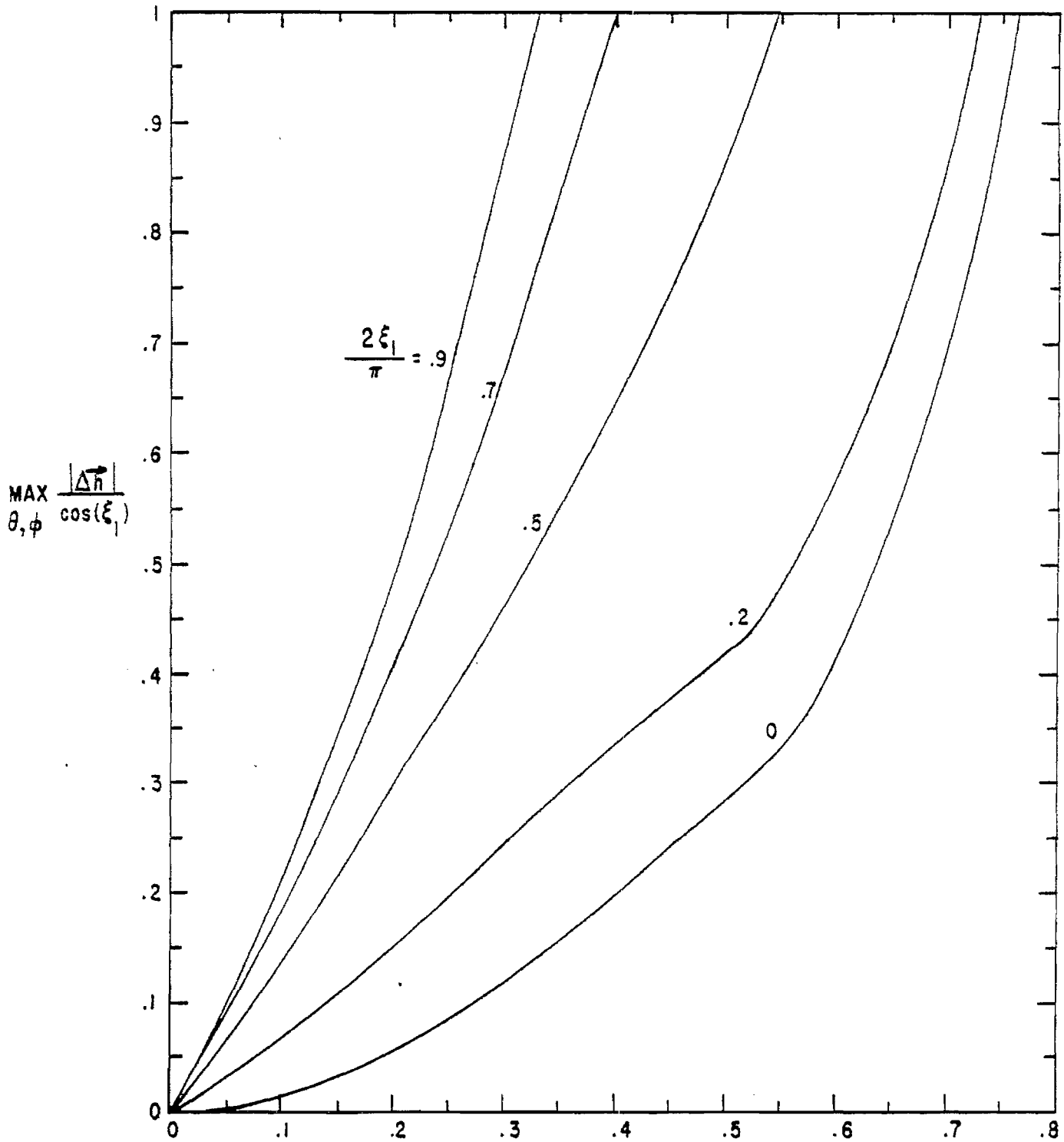


FIGURE 28. MAXIMUM $\frac{|\Delta \vec{h}|}{\cos(\xi_1)}$ ON THE SURFACE OF A SPHERE OF RADIUS r CENTERED AT THE ORIGIN (SCALE OF FIGURE 26 EXPANDED)

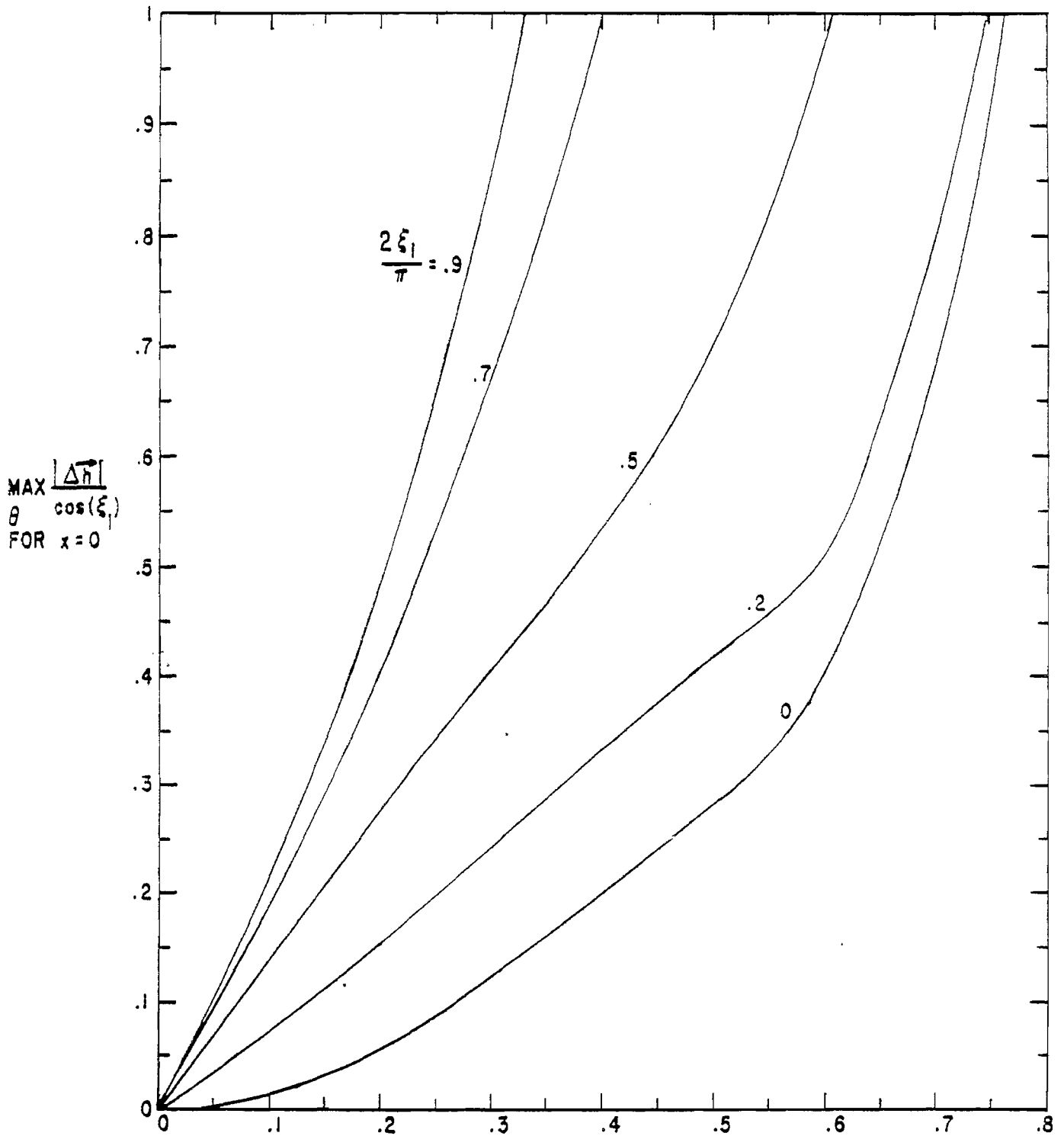


FIGURE 29. MAXIMUM $\frac{|\Delta h|}{\cos(\xi_1)}$ ON THE PERIMETER OF A CIRCLE ON THE GROUND SURFACE OF RADIUS r CENTERED AT THE ORIGIN: $\frac{x}{a} = 0$ (SCALE OF FIGURE 27 EXPANDED)

Appendix A: Elliptic Integrals and Their Derivatives

For the elliptic integral notation we follow a standard reference.^{1A} There are three standard forms of incomplete elliptic integrals; these are the first kind

$$F(\zeta|m) \equiv \int_0^{\zeta} [1 - m \sin^2(v)]^{-1/2} dv \quad (A1)$$

second kind

$$E(\zeta|m) \equiv \int_0^{\zeta} [1 - m \sin^2(v)]^{1/2} dv \quad (A2)$$

and third kind

$$\Pi(n;\zeta|m) \equiv \int_0^{\zeta} [1 - n \sin^2(v)]^{-1} [1 - m \sin^2(v)]^{-1/2} dv \quad (A3)$$

where ζ is called the amplitude, m is called the parameter, and n is called the characteristic. There are also the delta amplitude defined by

$$\Delta(\zeta) \equiv [1 - m \sin^2(\zeta)]^{1/2} \quad (A4)$$

and the complementary parameter defined by

$$m_1 \equiv 1 - m \quad (A5)$$

When ζ is taken as $\pi/2$ the elliptic integrals are called complete and can be written in a shorter form; these are the first kind

^{1A} M. Abramowitz and I. A. Stegun, ed., Handbook of Mathematical Functions, AMS55, National Bureau of Standards, 1964, chapter 17.

$$K(m) \equiv F\left(\frac{\pi}{2} \middle| m\right) \quad (\text{A6})$$

second kind

$$E(m) \equiv E\left(\frac{\pi}{2} \middle| m\right) \quad (\text{A7})$$

and third kind

$$\Pi(n|m) \equiv \Pi\left(n; \frac{\pi}{2} \middle| m\right) \quad (\text{A8})$$

Now consider the derivatives of the first two kinds of elliptic functions. First consider ζ . The derivatives are obtained from equations A1 and A2 to give

$$\frac{\partial}{\partial \zeta} F(\zeta|m) = [\Delta(\zeta)]^{-1} = [1 - m \sin^2(\zeta)]^{-1/2} \quad (\text{A9})$$

and

$$\frac{\partial}{\partial \zeta} E(\zeta|m) = \Delta(\zeta) = [1 - m \sin^2(\zeta)]^{1/2} \quad (\text{A10})$$

For completeness from equation A3 we add

$$\begin{aligned} \frac{\partial}{\partial \zeta} \Pi(n; \zeta|m) &= [1 - n \sin^2(\zeta)]^{-1} [\Delta(\zeta)]^{-1} \\ &= [1 - n \sin^2(\zeta)]^{-1} [1 - m \sin^2(\zeta)]^{-1/2} \end{aligned} \quad (\text{A11})$$

The complete elliptic integrals are not functions of ζ and so such derivatives are not considered.

Next consider the derivatives of the first two kinds of elliptic integrals with respect to m . Using equation A1 we have for the first kind

$$\begin{aligned}
\frac{\partial}{\partial m} F(\zeta|m) &= \frac{\partial}{\partial m} \int_0^\zeta [1 - m \sin^2(v)]^{-1/2} dv \\
&= \frac{1}{2} \int_0^\zeta \sin^2(v) [1 - m \sin^2(v)]^{-3/2} dv \\
&= \frac{1}{2m} [-F(\zeta|m) + \Pi(m; \zeta|m)] \tag{A12}
\end{aligned}$$

The special elliptic integral of the third kind in this result can be written as^{2A}

$$\begin{aligned}
\Pi(m; \zeta|m) &= \frac{1}{m_1} E(\zeta|m) - \frac{m}{2m_1} \frac{\sin(2\zeta)}{\Delta(\zeta)} \\
&= \frac{1}{m_1} E(\zeta|m) - \frac{m}{m_1} \frac{\sin(\zeta) \cos(\zeta)}{[1 - m \sin^2(\zeta)]^{1/2}} \tag{A13}
\end{aligned}$$

Thus we can write this derivative in terms of incomplete elliptic integrals of the first and second kind as

$$\begin{aligned}
\frac{\partial}{\partial m} F(\zeta|m) &= \frac{1}{2m} \left\{ \frac{1}{m_1} E(\zeta|m) - F(\zeta|m) - \frac{m}{2m_1} \frac{\sin(2\zeta)}{[1 - m \sin^2(\zeta)]^{1/2}} \right\} \\
&= \frac{1}{2m} \left\{ \frac{1}{m_1} E(\zeta|m) - F(\zeta|m) - \frac{m}{m_1} \frac{\sin(\zeta) \cos(\zeta)}{[1 - m \sin^2(\zeta)]^{1/2}} \right\} \tag{A14}
\end{aligned}$$

In the case of the complete elliptic integral of the first kind this result reduces to

$$\frac{\partial}{\partial m} K(m) = \frac{1}{2m} \left\{ \frac{1}{m_1} E(m) - K(m) \right\} \tag{A15}$$

^{2A}. Ref. 1A, eqn. 17.7.24.

For the incomplete elliptic integral of the second kind we have

$$\begin{aligned}\frac{\partial}{\partial m} E(\zeta|m) &= \frac{\partial}{\partial m} \int_0^\zeta [1 - m \sin^2(v)]^{1/2} dv \\ &= -\frac{1}{2} \int_0^\zeta \sin^2(v) [1 - m \sin^2(v)]^{-1/2} dv \\ &= \frac{1}{2m} [E(\zeta|m) - F(\zeta|m)]\end{aligned}\tag{A16}$$

For the complete elliptic integral of the second kind this result becomes

$$\frac{\partial}{\partial m} E(m) = \frac{1}{2m} [E(m) - K(m)]\tag{A17}$$

Appendix B: Two Integrals T_λ and T_β and Their Derivatives

From equations 34 we have the integral

$$T_\lambda = 2 \int_{\frac{\beta}{2}}^{\frac{\pi+\beta}{2}} \frac{\sin(\psi) \cos(\psi)}{[1 - m \sin^2(\psi)]^{1/2}} d\psi \quad (B1)$$

Let

$$\eta \equiv \Delta^2(\psi) = 1 - m \sin^2(\psi) \quad (B2)$$

$$d\eta = -2m \sin(\psi) \cos(\psi) d\psi$$

giving

$$\begin{aligned} T_\lambda &= -\frac{1}{m} \int_{\Delta^2(\frac{\beta}{2})}^{\Delta^2(\frac{\pi+\beta}{2})} \eta^{-1/2} d\eta \\ &= -\frac{2}{m} \eta^{1/2} \Big|_{\Delta^2(\frac{\beta}{2})}^{\Delta^2(\frac{\pi+\beta}{2})} \\ &= \frac{2}{m} \left[\Delta\left(\frac{\beta}{2}\right) - \Delta\left(\frac{\pi+\beta}{2}\right) \right] \quad (B3) \end{aligned}$$

Thus for T_λ we have

$$\begin{aligned}
T_\lambda &= \frac{2}{m} \left\{ \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{1/2} - \left[1 - m \sin^2\left(\frac{\pi+\beta}{2}\right) \right]^{1/2} \right\} \\
&= \frac{2}{m} \left\{ \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{1/2} - \left[1 - m \cos^2\left(\frac{\beta}{2}\right) \right]^{1/2} \right\} \quad (B4)
\end{aligned}$$

From equations 34 we have another integral

$$\begin{aligned}
T_\beta &= \int_{\frac{\beta}{2}}^{\frac{\pi+\beta}{2}} \frac{2 \sin^2(\psi) - 1}{[1 - m \sin^2(\psi)]^{1/2}} d\psi \\
&= \int_{\frac{\beta}{2}}^{\frac{\pi+\beta}{2}} \left\{ -\frac{2}{m} [1 - m \sin^2(\psi)]^{1/2} + \left(\frac{2}{m} - 1\right) [1 - m \sin^2(\psi)]^{-1/2} \right\} d\psi \quad (B5)
\end{aligned}$$

Thus T_β can be written in terms of incomplete elliptic integrals as

$$\begin{aligned}
T_\beta &= -\frac{2}{m} \left[E\left(\frac{\pi+\beta}{2} \middle| m\right) - E\left(\frac{\beta}{2} \middle| m\right) \right] \\
&\quad + \left(\frac{2}{m} - 1\right) \left[F\left(\frac{\pi+\beta}{2} \middle| m\right) - F\left(\frac{\beta}{2} \middle| m\right) \right] \quad (B6)
\end{aligned}$$

These two integrals T_λ and T_β are both functions of β and m . Consider first the derivatives with respect to β . From equation B1 we have .

$$\begin{aligned}
\frac{\partial}{\partial \beta} T_\lambda &= \sin\left(\frac{\pi+\beta}{2}\right) \cos\left(\frac{\pi+\beta}{2}\right) \left[1 - m \sin^2\left(\frac{\pi+\beta}{2}\right) \right]^{-1/2} \\
&\quad - \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \\
&= -\sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \left\{ \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} + \left[1 - m \cos^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right\} \quad (B7)
\end{aligned}$$

and from equation B5 we have

$$\begin{aligned} \frac{\partial}{\partial \beta} T_{\beta} &= \left[\sin^2\left(\frac{\pi+\beta}{2}\right) - \frac{1}{2} \right] \left[1 - m \sin^2\left(\frac{\pi+\beta}{2}\right) \right]^{-1/2} \\ &\quad - \left[\sin^2\left(\frac{\beta}{2}\right) - \frac{1}{2} \right] \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \end{aligned} \quad (B8)$$

Now consider the derivative of T_{λ} with respect to m . From equation B4 we have

$$\begin{aligned} \frac{\partial}{\partial m} T_{\lambda} &= -\frac{2}{m^2} \left\{ \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{1/2} - \left[1 - m \cos^2\left(\frac{\beta}{2}\right) \right]^{1/2} \right\} \\ &\quad + \frac{1}{m} \left\{ -\sin^2\left(\frac{\beta}{2}\right) \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} + \cos^2\left(\frac{\beta}{2}\right) \left[1 - m \cos^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right\} \\ &= \frac{1}{m^2} \left\{ -\left[2 - m \sin^2\left(\frac{\beta}{2}\right) \right] \left[1 - m \sin^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right. \\ &\quad \left. + \left[2 - m \cos^2\left(\frac{\beta}{2}\right) \right] \left[1 - m \cos^2\left(\frac{\beta}{2}\right) \right]^{-1/2} \right\} \end{aligned} \quad (B9)$$

Finally consider the derivative of T_{β} with respect to m . From equation B6 and using the elliptic function derivatives from appendix A we have

$$\begin{aligned} \frac{\partial}{\partial m} T_{\beta} &= \frac{2}{m^2} \left[E\left(\frac{\pi+\beta}{2} \middle| m\right) - E\left(\frac{\beta}{2} \middle| m\right) \right] \\ &\quad - \frac{1}{m^2} \left[E\left(\frac{\pi+\beta}{2} \middle| m\right) - F\left(\frac{\pi+\beta}{2} \middle| m\right) - E\left(\frac{\beta}{2} \middle| m\right) + F\left(\frac{\beta}{2} \middle| m\right) \right] \\ &\quad - \frac{2}{m^2} \left[F\left(\frac{\pi+\beta}{2} \middle| m\right) - F\left(\frac{\beta}{2} \middle| m\right) \right] \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{m^2} - \frac{1}{2m} \right) \left\{ \frac{1}{m_1} E \left(\frac{\pi+\beta}{2} \middle| m \right) - F \left(\frac{\pi+\beta}{2} \middle| m \right) - \frac{m}{m_1} \frac{\sin \left(\frac{\pi+\beta}{2} \right) \cos \left(\frac{\pi+\beta}{2} \right)}{\left[1 - m \sin^2 \left(\frac{\pi+\beta}{2} \right) \right]^{1/2}} \right. \\
& \quad \left. - \frac{1}{m_1} E \left(\frac{\beta}{2} \middle| m \right) + F \left(\frac{\beta}{2} \middle| m \right) + \frac{m}{m_1} \frac{\sin \left(\frac{\beta}{2} \right) \cos \left(\frac{\beta}{2} \right)}{\left[1 - m \sin^2 \left(\frac{\beta}{2} \right) \right]^{1/2}} \right\} \quad (B10)
\end{aligned}$$

Collecting terms, this becomes

$$\begin{aligned}
\frac{\partial}{\partial m} T_\beta &= \frac{1}{m^2} \frac{4-3m}{2m_1} \left\{ E \left(\frac{\pi+\beta}{2} \middle| m \right) - E \left(\frac{\beta}{2} \middle| m \right) \right\} \\
& - \frac{1}{m^2} \frac{4-m}{2} \left\{ F \left(\frac{\pi+\beta}{2} \middle| m \right) - F \left(\frac{\beta}{2} \middle| m \right) \right\} \\
& + \frac{1}{m} \frac{2-m}{2m_1} \sin \left(\frac{\beta}{2} \right) \cos \left(\frac{\beta}{2} \right) \left\{ \left[1 - m \sin^2 \left(\frac{\beta}{2} \right) \right]^{-1/2} + \left[1 - m \cos^2 \left(\frac{\beta}{2} \right) \right]^{-1/2} \right\} \quad (B11)
\end{aligned}$$