Sensor and Simulation Notes

Note 106

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Capacitance and Equivalent Area of a Disk in a Circular Aperture

by

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Abstract

The capacitance between a circular disk and an infinite plane with a circular aperture is computed for the case where the disk lies in the plane of the aperture and is concentric with it. The equivalent area of this structure when used as an electric-field sensor is also computed.

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I. Introduction

In note 98 of this series[1] a "circular flush-plate dipole" sensor is described. This sensor consists of a circular disk at the center of a circular aperture in a large ground plane. For mathematical simplicity the disk and the ground plane are assumed to be infinitely thin and perfectly conducting and the ground plane is assumed to be infinite in extent.

An incident electromagnetic wave generates an electric field in the annular slot between the disk and the ground plane. Such a wave also induces some current flow between the disk and the ground plane. If the integral of the radial component of the electric field along a radial line between the disk and the ground plane is zero, the current flowing is called the short-circuit current. The ratio of this short-circuit current to the displacement current per unit area in the incident field is a parameter having the dimension of area. This parameter is called the equivalent area of the device[2].

It should be noted that, except in the low-frequency limit, the condition that the line integral of the electric field be zero is not unique. In note 98 the short-circuit current is computed by assuming the radial electric field to be zero everywhere in the aperture. At low frequencies, the definition of the short-circuit current may be made unique by considering the plane of the disk to be a plane of symmetry of the device. One of the two main purposes of this note is to give this alternative explicit definition of the short-circuit current and, based on this definition, to compute accurately the equivalent area of the device in the low-frequency limit. The second main purpose of this note is to compute accurately the electrostatic capacitance between the disk and the ground plane for the symmetric device. This parameter is useful in the low-frequency limit if one wishes to consider the interaction between the sensor and some output cable in an approximate manner. Both of these computations will be carried out for an arbitrary ratio of disk radius to aperture radius; in note 98 this ratio is assumed to be close to unity.

In essence then, the work in this note complements the computational part of note 98. In note 98, approximations are made that are appropriate for a narrow slot at any frequency. Here, we treat the low-frequency limit very accurately and for arbitrary slot width.

In the next section, we give a little more detailed discussion of the meaning of the equivalent area and input admittance of the device in the low-frequency limit. This will lead to a heuristic justification and a precise statement of the boundary-value problems we will solve and the parameters we will calculate from their solution.

In the third section, the boundary-value problems arrived at in the second section will be formulated in terms of a pair of coupled Fredholm integral equations of the second kind. The kernels of these equations will have properties such that the equations are very suitable for numerical solution. This coupled integral equation formulation will be based on standard methods from the theory of mixed boundary-value problems[3].

In the fourth section, we discuss the analytical solution of the equations developed in the third section in certain limiting cases. These analytical solutions will lead to asymptotic forms for the capacitance and equivalent area of the device for the case of small disk radius.

The fifth section is devoted to a reformulation of the capacitance problem in terms of a single integral equation of the first kind. This kind of integral equation is not as susceptible to precise numerical solution as an integral equation of the second kind. However, the equation of the first kind leads to a variational expression for the capacitance of the sensor. This, in turn, leads to a reasonably simple approximate representation for the capacitance which is very accurate for small disk radii. In fact, for the disk radius less than half the aperture radius, the variational expression for the capacitance is correct to five significant figures.

In the last section, we return to the integral equations of the third section and discuss the particular numerical method chosen for their solution. Also, reasons are given for our confidence in the accuracy of the six-figure tables of capacitance and equivalent area.

II. The Boundary-Value Problems

In this section we wish to arrive at a precise mathematical statement of the two physical quantities to be calculated, namely the capacitance and the equivalent area.

Concerning the capacitance calculation there can be no confusion. We merely determine the electrostatic capacitance between the disk and ground plane shown in figure 1. It should be pointed out that the low-frequency limit of the input admittance of the sensor is shown in note 98 to be purely capacitive, and that one of the quantities calculated there is

$$\Omega_o \equiv \frac{C}{2\varepsilon\sqrt{ab}} \tag{1}$$

where C is the total capacitance between the disk and the ground plane, "a" is the radius of the disk, and "b" is the radius of the aperture, as shown in figure 1. In this note we will compute Ω_o for comparison purposes, but first we will compute the capacitance normalized in a slightly different manner -- we will normalize to the capacitance of the disk isolated in free space, i.e. $8\varepsilon a$. This new normalization will lead to a quantity which varies very slowly with disk radius. We will compute, then,

$$c \equiv \frac{C}{8\varepsilon a} \quad . \tag{2}$$

This will be done by first solving, in the geometry shown in figure 1,

$$\nabla^2\psi = 0 \tag{3}$$

for the electrostatic potential ψ, which vanishes at infinity and is subject to the following conditions on the $z = 0$ plane:

$$\begin{aligned} \psi(\rho,0) &= 0 \qquad \rho > b \\ \psi(\rho,0) &= 1 \qquad \rho < a \\ \frac{\partial\psi(\rho,0)}{\partial z} &= 0 \qquad a < \rho < b \end{aligned}$$

The last condition listed above is necessary because of the assumed symmetry of the sensor about the z = 0 plane. Once this boundary-value problem is solved, it is clear that c is determined by the equation

$$c = \frac{-\pi}{2a} \int_{o}^{a} \frac{\partial\psi(\rho,0_{+})}{\partial z} \rho d\rho \quad . \tag{4}$$

We turn now to the slightly less familiar concept of equivalent area, A_{eq}. The parameter that would be really nice to calculate is the one which, no matter what kind of "short circuit" is applied to the sensor, would give the short-circuit current when multiplied by the incident displacement current per unit area. Unfortunately, such a parameter does not exist. The reason for this is the non-uniqueness, except in the low-frequency limit, of the definition of short-circuit current mentioned in the introduction. At the low-frequency limit, however, the definition of short-circuit current <u>is</u> unique once the geometry is completely specified, because at the low-frequency limit the electric field may be derived from a potential and the potential on the disk can be set equal to the potential on the ground plane. Again we will choose the geometry to be symmetric about the z plane. This approximation involves a neglect of the details of the cable feed but has become standard in these sensor problems.

There are two further details to mention in order to completely specify the equivalent area computations we will make. The first of these is that we will divide the short-circuit current not by the displacement current in the incident field alone but by the displacement current in the incident plus reflected field. This division may be the most appropriate one when large ground planes are involved; it has been used before (note 98); and in the low-frequency limit leads to a simple division by the field at infinity on one side of the z-plane. The second detail to be stated is that we will normalize the equivalent area to the area of a disk whose radius is the geometrical mean of the disk radius and the aperture radius. As with our normalization of the capacitance, this choice leads to a quantity that varies very slowly with the disk radius.

In order to relate current to an electrostatic problem we will say that we can determine the short-circuit current by integrating the surface current flowing out of the disk and then using the surface divergence theorem, i.e.

$$A_{eq} = \frac{I_{SC}}{i\omega\varepsilon E_z} = \frac{\int_{-\pi}^{\pi} K_\rho a d\phi}{i\omega\varepsilon E_z} = \frac{i\omega \int_0^a \int_{-\pi}^{\pi} \sigma(\rho)\rho d\rho d\phi}{i\omega\varepsilon E_z} = \frac{Q_d}{\varepsilon E_z} \tag{5}$$

where Q_d is the total charge on the disk and E_z is the electrostatic field at infinity.

It is clear that we may write the total electrostatic potential, Φ, as

$$\begin{aligned} \Phi &= - E_z z + E_z \psi(x,y,z) a \qquad z > 0 \\ &= E_z \psi(x,y,-z) a \qquad\qquad z < 0 \quad . \end{aligned} \tag{6}$$

From this and our previous discussion it follows that the normalized equivalent area is

$$A = \frac{a}{b} - \frac{4}{b} \int_0^a \rho \frac{\partial \psi(\rho,0)}{\partial z} d\rho \tag{7}$$

where

$$\nabla^2 \psi = 0$$

and

$$\psi(\rho,0) = 0 \qquad \rho > b; \ \rho < a$$

$$\frac{\partial \psi(\rho,0)}{\partial z} = \frac{1}{2a} \qquad a < \rho < b \quad .$$

The last condition is brought about by the continuity of the total field through the slot.

III. The Coupled Integral Equations

There are several ways one could think of to solve the two potential problems stated in the previous section. For instance, one could readily write down a pair of coupled integral equations for the charge density on the disk and ground plane using the free space electrostatic Green's function. Alternatively, one might use a half-space Green's function to write an integral equation for the electric field in the slot. The method that we will actually use, based on the well-developed theory of dual integral equations[3], may seem rather roundabout and unnecessary, but it will lead in the end to a pair of equations that are extremely well suited to numerical solution. Almost any other method one could think of would lead to much greater difficulty in the extraction of numerical results.

We begin by noticing that, for both our potential problems, the potential in the upper half space may be represented as

$$\psi(\rho,z) = \int_0^\infty A(\alpha) J_o(\alpha\rho) e^{-\alpha z} d\alpha \tag{8}$$

where $A(\alpha)$ is determined by the conditions on ψ on the $z = 0$ plane. In particular, for the capacitance problem $A(\alpha)$ is determined by

$$\begin{aligned} \int_0^\infty A(\alpha) J_o(\alpha\rho) d\alpha &= 1 \qquad \rho < a \\ \int_0^\infty \alpha A(\alpha) J_o(\alpha\rho) d\alpha &= 0 \qquad a < \rho < b \\ \int_0^\infty A(\alpha) J_o(\alpha\rho) d\alpha &= 0 \qquad \rho > b \end{aligned} \tag{9}$$

while for the equivalent area problem $A(\alpha)$ is determined through

$$\int_0^\infty A(\alpha)J_o(\alpha\rho)d\alpha = 0 \qquad \rho < a$$

$$\int_0^\infty \alpha A(\alpha)J_o(\alpha\rho)d\alpha = -1/(2a) \qquad a < \rho < b \tag{10}$$

$$\int_0^\infty A(\alpha)J_o(\alpha\rho)d\alpha = 0 \qquad \rho > b$$

Both of the above sets of equations may be written as special cases of the set

$$\int_0^\infty A(\alpha)J_o(\alpha\rho)d\alpha = g(\rho) \qquad \rho < a$$

$$\int_0^\infty \alpha A(\alpha)J_o(\alpha\rho)d\alpha = f(\rho) \qquad a < \rho < b \tag{11}$$

$$\int_0^\infty A(\alpha)J_o(\alpha\rho)d\alpha = g(\rho) \qquad \rho > b$$

and from equations (4) and (7) it can be seen that what we would like to have is $f(\rho)$ for $\rho < a$.

We may derive a relation between $f(\rho)$ for $\rho < a$ and $f(\rho)$ for $\rho > b$ by assuming for the moment that $f(\rho)$ is known for $\rho < a$ and writing the pair of dual integral equations

$$\int_0^\infty \alpha A(\alpha)J_o(\alpha\rho)d\alpha = f(\rho) \qquad \rho < b$$

$$\int_0^\infty A(\alpha)J_o(\alpha\rho)d\alpha = g(\rho) \qquad \rho > b$$

If we normalize the independant variables in these equations so that $\rho = xb$, it is seen that they are the pair solved in appendix A, and since, for both our special cases, $g(\rho)$ is zero for ρ greater than b we may write, from equation (A-20),

$$f(\rho) = \frac{-2}{\pi(\rho^2-b^2)^{\frac{1}{2}}} \int_0^a \frac{tf(t)(b^2-t^2)^{\frac{1}{2}}dt}{\rho^2-t^2} - \frac{2}{\pi(\rho^2-b^2)^{\frac{1}{2}}} \int_a^b \frac{tf(t)(b^2-t^2)^{\frac{1}{2}}dt}{\rho^2-t^2}, \quad \rho > b \quad (12)$$

where we have separated the two intervals of integration to remind ourselves that f(t) is known in the second integral but unknown in the interval of the first integral.

We may derive a second relation between $f(\rho)$ for $\rho < a$ and $f(\rho)$ for $\rho > b$ by assuming that $f(\rho)$ is known for $\rho > b$ and writing the pair of dual integral equations.

$$\int_0^\infty \alpha A(\alpha) J_o(\alpha\rho) d\alpha = f(\rho) \qquad \rho > a$$

$$\int_0^\infty A(\alpha) J_o(\alpha\rho) d\alpha = g(\rho) \qquad \rho < a$$

This pair corresponds to the pair solved in appendix B, and so we may use equation (B-15) to write

$$f(\rho) = \frac{-2}{\pi(a^2-\rho^2)^{\frac{1}{2}}} \int_b^\infty \frac{t(t^2-a^2)^{\frac{1}{2}}f(t)dt}{t^2-\rho^2} - \frac{2}{\pi(a^2-\rho^2)^{\frac{1}{2}}} \int_a^b \frac{t(t^2-a^2)^{\frac{1}{2}}f(t)dt}{t^2-\rho^2}$$

$$- \frac{2}{\pi\rho} \frac{d}{d\rho} \int_\rho^a \frac{tdt}{(t^2-\rho^2)^{\frac{1}{2}}} \frac{d}{dt} \int_0^t \frac{sg(s)ds}{(t^2-s^2)^{\frac{1}{2}}} \qquad 0 \leq \rho < a \qquad (13)$$

where we have again separated the known and unknown portions of f(t).

Now in equations (12) and (13) we can make the substitutions

$$f(\rho) = \frac{\sqrt{ab}}{\rho(a^2-\rho^2)^{\frac{1}{2}}} P_1(\rho/a) \qquad \rho < a \qquad (14)$$

$$f(\rho) = \frac{b}{\rho(\rho^2-b^2)^{\frac{1}{2}}} P_2(b/\rho) \qquad \rho > b \qquad (15)$$

and achieve the simpler pair

$$p_1(x) + \int_0^1 K(x,y)p_2(y)dy = h_1(x) \quad , \quad 0 \le x < 1 \tag{16}$$

$$p_2(x) + \int_0^1 K(x,y)p_1(y)dy = h_2(x) \quad , \quad 0 \le x < 1 \tag{17}$$

where

$$K(x,y) = \frac{2x\sqrt{k}\,(1-k^2y^2)^{\frac{1}{2}}}{\pi(1-y^2)^{\frac{1}{2}}(1-k^2x^2y^2)} \tag{18}$$

and we have defined

$$k \equiv a/b \quad .$$

In equations (16) and (17) h_1 and h_2 are defined by the known parts of equations (12 and (13). Inserting our special known values of g and f from equations (9) and (10) into (12) and (13) it is found that for the capacitance problem

$$h_1^c(x) = \frac{2\sqrt{k}x}{\pi} \tag{19}$$

$$h_2^c(x) = 0 \quad , \tag{20}$$

while for the equivalent area problem

$$h_1^q(x) = \frac{x}{\pi\sqrt{k}}\left\{k' - k\sqrt{1-x^2}\,\tan^{-1}\left(\frac{k'}{k(1-x^2)^{\frac{1}{2}}}\right)\right\} \tag{21}$$

$$h_2^a(x) = \frac{1}{\pi x k}\left\{k' - \frac{\sqrt{1-x^2}}{x}\,\tan^{-1}\left(\frac{xk'}{(1-x^2)^{\frac{1}{2}}}\right)\right\} \tag{22}$$

where

$$k' \equiv (1-k^2)^{\frac{1}{2}} \quad .$$

Also, recalling the definitions of c and A from equations (4) and (7) and the fact that, on the z = 0 plane,

$$f(\rho) = -\frac{\partial\psi(\rho,0_+)}{\partial z}$$

we can use the substitutions (14) and (15) to write

$$c = \frac{\pi}{2\sqrt{k}} \int_0^1 \frac{p_1^c(x)dx}{(1-x^2)^{\frac{1}{2}}} \tag{23}$$

and

$$A = k + 4\sqrt{k} \int_0^1 \frac{p_1^a(x)dx}{(1-x^2)^{\frac{1}{2}}} \tag{24}$$

We note here that, since the kernels of the pair, (16) - (17), are identical, we can write uncoupled equations for the pair

$$p_+ = p_1 + p_2$$

$$p_- = p_1 - p_2$$

in the form

$$p_\pm(x) \pm \int_0^1 K(x,y)p_\pm(y) = h_\pm(x)$$

where

$$h_+ = h_1 + h_2$$

$$h_- = h_1 - h_2$$

but we make no further use of this fact since the rest of the analytical discussion is simpler without this transformation and, although the time required for numerical solution could be reduced by such a transformation, the total computation time will be negligible even without it.

We will, however, make one further transformation of equations (16) and

(17). We will also normalize the capacitance and equivalent area problems slightly differently in order to simplify equations (23) and (24) a little further and at the same time keep the integral equations of the two problems as similar as possible. In particular, for the capacitance problem we set

$$p_1(x) = \frac{2\sqrt{k}}{\pi} P_1(\theta) \sin\theta$$

$$p_2(x) = \frac{2\sqrt{k}}{\pi} P_2(\theta) \sin\theta$$

$$x = \sin\theta$$
$$y = \sin\theta'$$

and for the equivalent area problem we set

$$p_1(x) = \frac{1}{\sqrt{k}} P_2(\theta) \sin\theta$$

$$p_2(x) = \frac{1}{\sqrt{k}} P_1(\theta) \sin\theta$$

$$x = \sin\theta$$
$$y = \sin\theta'$$

This final substitution results in the following equations, which are the ones used for numerical work

$$P_1(\theta) + k \int_0^{\pi/2} G(\theta,\theta') P_2(\theta') d\theta' = H_1(\theta) \tag{25}$$

$$P_2(\theta) + \int_0^{\pi/2} G(\theta,\theta') P_1(\theta') d\theta' = H_2(\theta) \tag{26}$$

where

$$G(\theta,\theta') = \frac{2}{\pi} \cdot \frac{\sin\theta'(1-k^2\sin^2\theta')^{\frac{1}{2}}}{1-k^2\sin^2\theta\ \sin^2\theta'} \quad . \tag{27}$$

For the capacitance problem we set

$$H_1^c(\theta) = 1 \tag{28}$$

$$H_2^c(\theta) = 0 \tag{29}$$

and compute

$$c = \int_o^{\pi/2} P_1^c(\theta)\sin\theta\ d\theta \tag{30}$$

while for the equivalent area problem we set

$$H_1^a(\theta) = \frac{1}{\pi\ \sin^2\theta}\left\{k' - \cot\theta\ \tan^{-1}(k'\ \tan\theta)\right\} \tag{31}$$

$$H_2^a(\theta) = \frac{1}{\pi}\left\{k' - k\cos\theta\ \tan^{-1}\left(\frac{k'}{k\cos\theta}\right)\right\} \tag{32}$$

and compute

$$A = k + 4\int_o^{\pi/2} P_2^a(\theta)\sin\theta\ d\theta \quad . \tag{33}$$

Ir is easy to show that a second expression for c, which is equivalent to (30) and which can be derived by integrating the charge over the ground plane rather than over the disk, is

$$c = \int_o^{\pi/2} P_2^c(\theta)d\theta \quad . \tag{34}$$

In the next section we will discuss the solution of equations (25) and (26) for small k. In the last section we will discuss a numerical technique of solving them with high accuracy.

IV. Limiting Cases

Equations (25) and (26) are readily solved if we keep only first order terms in k. The integrals then become independent of θ and we can write

$$P_1^c + \frac{2k}{\pi}\int_0^{\pi/2} \sin\theta' P_2^c d\theta' = 1$$

$$P_2^c + \frac{2}{\pi}\int_0^{\pi/2} \sin\theta' P_1^c d\theta' = 0$$

From these equations it is clear that, in the limit we are considering, both P_1^c and P_2^c are independent of θ and that

$$\begin{aligned} P_1^c + \frac{2k}{\pi} P_2^c &= 1 \\ P_2^c + \frac{2}{\pi} P_1^c &= 0 \quad . \end{aligned} \tag{35}$$

Solving these equations for P_1^c we obtain from equation (30)

$$c = P_1^c = \frac{\pi^2}{\pi^2 - 4k} \tag{36}$$

This equation is in error by less than one part in a thousand for k less than a fifth.

For a similar treatment of the equivalent area problem we must also expand the right hand sides of equations (25) and (26) to first order in k. We then have

$$\begin{aligned} P_1^a(\theta) + \frac{2k}{\pi}\int_0^{\pi/2} \sin\theta' P_2^a(\theta') d\theta' &= \frac{1-\theta\cot\theta}{\pi\sin^2\theta} \\ P_2^a(\theta) + \frac{2}{\pi}\int_0^{\pi/2} \sin\theta' P_1^a(\theta') d\theta' &= \frac{1}{\pi} - \frac{k\cos\theta}{2} \end{aligned} \tag{37}$$

From these two equations it is clear that, to first order in k, both

$$P_1^a(\theta) - \frac{1-\theta\cot\theta}{\pi\sin^2\theta}$$

and

$$P_2^a(\theta) - \frac{1}{\pi} + \frac{k\cos\theta}{2} \tag{38}$$

are independent of θ. If we denote these two constants by kX_1 and X_2 respectively, we can use equations (37) to write

$$X_1 + \frac{2}{\pi}\int_0^{\pi/2} \sin\theta'\left(X_2 + \frac{1}{\pi} - \frac{k\cos\theta'}{2}\right)d\theta' = 0$$

$$X_2 + \frac{2}{\pi}\int_0^{\pi/2} \sin\theta'\left(kX_1 + \frac{1-\theta'\cot\theta'}{\pi\sin^2\theta'}\right)d\theta' = 0$$

or, carrying out the integrations,

$$X_1 + \frac{2}{\pi}X_2 = -\frac{2}{\pi^2} + \frac{k}{2\pi}$$

$$X_2 + \frac{2k}{\pi}X_1 = -\frac{1}{\pi} + \frac{2}{\pi^2}$$

Solving these equations to first order in k we find

$$X_2 = -\frac{1}{\pi} + \frac{2}{\pi^2} + \frac{8k}{\pi^4}$$

Using this, we can easily get $P_2^a(\theta)$ from (38) and then substitute P_2^a in (33) to get

$$A = \frac{8}{\pi^2} + \frac{32k}{\pi^4} \tag{39}$$

This equation is plotted in figure 3, along with the exact value of A.

It follows from the work in note 98 that the asymptotic form of the capacitance as k approaches unity may be written as a function of the parameter

$$x = -(\ln k)/2$$

in the form

$$\Omega_o = 4c\sqrt{k} = 2[\ln(\frac{16}{x}) - 2]$$

This expression is accurate to better than one part in a thousand for x less than a tenth. On figure 4, where x only goes up to a tenth, the plot of the above expression is indistinguishable from the true curve.

V. A Variational Expression

We turn now to a reformulation of the capacitance problem in terms of an integral equation of the first kind. This will lead, through a variational representation, to a fairly simple explicit expression for the capacitance which is nevertheless quite accurate for most values of k.

We start by developing an expression for the potential of a ring charge in the plane of a circular aperture in an infinite plane at zero potential. The ring charge is concentric with the aperture. If the radius of the aperture is unity and the radius of the ring charge is ρ', it can be seen that the potential is given by

$$\phi(\rho,z) = \int_0^\infty A(\alpha)J_o(\alpha\rho)e^{-\alpha\rho}d\alpha$$

where $A(\alpha)$ is determined by

$$\int_0^\infty A(\alpha)J_o(\alpha\rho)d\alpha = 0 \qquad \rho > 1 \tag{40}$$

$$\int_0^\infty \alpha A(\alpha)J_o(\alpha\rho)d\alpha = \frac{Q\delta(\rho-\rho')}{4\pi\varepsilon_o\rho'} \qquad \rho < 1 \tag{41}$$

and Q is the charge on the ring. Equations (40) and (41) are special cases of the pair of dual integral equations solved in appendix A, and so we may write immediately, from equation (A-16),

$$\phi(\rho,0) = \frac{Q}{2\pi^2\varepsilon_o}\int_\rho^1 \frac{dt}{(t^2-\rho^2)^{\frac{1}{2}}}\int_0^t \frac{x'\delta(x'-\rho')dx'}{(t^2-x'^2)^{\frac{1}{2}}}$$

$$= \frac{Q}{2\pi^2\varepsilon_o}\int_{\max(\rho,\rho')}^1 \frac{dt}{(t^2-\rho^2)^{\frac{1}{2}}(t^2-\rho'^2)^{\frac{1}{2}}}$$

We may now use this expression to write the potential, within the aperture, of any axisymmetric distribution of charge over a disk in the aperture in the form

$$\phi(\rho) = \int_0^k \frac{\rho'\sigma(\rho')d\rho'}{\pi\varepsilon_o} \left\{ \int_{\max(\rho,\rho')}^1 \frac{dt}{(t^2-\rho^2)^{\frac{1}{2}}(t^2-\rho'^2)^{\frac{1}{2}}} \right\}$$

where σ is the surface charge density within the charged disk and k is the radius of the charged disk. If the charged disk is a conductor its potential must be independent of ρ; so an integral equation determining the charge density on the disk is

$$V = \int_0^k \frac{\rho'\sigma(\rho')d\rho'}{\pi\varepsilon_o} \left\{ \int_{\max(\rho,\rho')}^1 \frac{dt}{(t^2-\rho^2)^{\frac{1}{2}}(t^2-\rho'^2)^{\frac{1}{2}}} \right\} .$$

Now we define

$$f(\rho) \equiv \frac{2\rho\sigma(\rho)}{V\varepsilon}$$

which brings the integral equation to the form

$$1 = \frac{1}{2\pi} \int_0^k \left\{ \int_{\max(\rho,\rho')}^1 \frac{dt}{(t^2-\rho^2)^{\frac{1}{2}}(t^2-\rho'^2)^{\frac{1}{2}}} \right\} f(\rho')d\rho' \quad ,$$

and the total charge on the disk to the form

$$Q_d = 2\pi \int_0^k \rho\sigma(\rho)d\rho = \pi\varepsilon V \int_0^k f(\rho)d\rho \quad .$$

Thus the capacitance of the disk is

$$C = \frac{Q_d}{V} = \pi\varepsilon \int_0^k f(\rho)d\rho \quad .$$

Up to now in this section we have been assuming that the radius of the aperture is unity, but this is only a matter of normalization and it is easy to show that the normalized capacitance defined by equation (2) is given by

$$c = \frac{\pi}{8k} \int_0^k f(x)dx \tag{42}$$

where k has now assumed its former meaning of ratio of inner to outer annular radii, and f(x) is determined by

$$1 = \int_0^k K(x,x')f(x')dx' \tag{43}$$

where

$$K(x,x') = K(x',x) = \frac{1}{2\pi} \int_{\max(x,x')}^{1} \frac{dt}{(t^2-x^2)^{\frac{1}{2}}(t^2-x'^2)^{\frac{1}{2}}}$$

Because of the form of equations (42) and (43) we may use the standard theory of variational representations[4] to write a stationary representation for c in the form

$$c = \frac{\pi/8\ [\int_0^k f(x)dx]^2}{k \int_0^k\int_0^k f(x)K(x,x')f(x')dx'dx} \tag{44}$$

This expression possesses the usual stationary property that first order errors in the form of the function f(x) introduce only second order and smaller errors in the value of the functional c.

To make any further progress we must assume some explicit form for f(x). We choose to make f(x) proportional to the function it would be if the disk were isolated in free space. That is, we assume

$$f(x) = \frac{x}{(k^2-x^2)^{\frac{1}{2}}}$$

This assumed form increases in accuracy as the disk decreases in size. Inserting this in (44) we see that

$$c = \pi^2\left(\int_0^k \frac{xdx}{(k^2-x^2)^{\frac{1}{2}}}\right)^2 [4kD(k)]^{-1}$$

$$= \pi^2 k/[4D(k)] \tag{45}$$

where

$$D(k) = \int_0^k\!\!\int_0^k \frac{x}{(k^2-x^2)^{\frac{1}{2}}} \cdot \int_{\max(x,x')}^{1} \frac{dt}{(t^2-x^2)(t^2-x'^2)} \cdot \frac{x'}{(k^2-x'^2)^{\frac{1}{2}}}\, dx'dx$$

$$= \int_0^k \frac{x'dx'}{(k^2-x'^2)^{\frac{1}{2}}} \int_{x'}^{k} \frac{xdx}{(k^2-x^2)^{\frac{1}{2}}} \int_x^1 \frac{dt}{(t^2-x^2)^{\frac{1}{2}}(t^2-x'^2)^{\frac{1}{2}}}$$

$$+ \int_0^k \frac{x'dx'}{(k^2-x'^2)^{\frac{1}{2}}} \int_0^{x'} \frac{xdx}{(k^2-x^2)^{\frac{1}{2}}} \int_{x'}^1 \frac{dt}{(t^2-x^2)^{\frac{1}{2}}(t^2-x'^2)^{\frac{1}{2}}}$$

$$= 2\int_0^k \frac{xdx}{(k^2-x^2)^{\frac{1}{2}}} \int_0^{x} \frac{x'dx'}{(k^2-x'^2)^{\frac{1}{2}}} \int_x^1 \frac{dt}{(t^2-x^2)^{\frac{1}{2}}(t^2-x'^2)^{\frac{1}{2}}}$$

$$= 2\int_0^k \frac{xdx}{(k^2-x^2)^{\frac{1}{2}}} \int_x^1 \frac{dt}{(t^2-x^2)^{\frac{1}{2}}} \int_0^x \frac{x'dx'}{(k^2-x'^2)^{\frac{1}{2}}(t^2-x'^2)^{\frac{1}{2}}}$$

$$= 2\left\{\int_0^k dt \int_0^t dx + \int_k^1 dt \int_0^k dx\right\} \frac{x}{(k^2-x^2)^{\frac{1}{2}}(t^2-x^2)^{\frac{1}{2}}} \int_0^x \frac{x'dx'}{(k^2-x'^2)^{\frac{1}{2}}(t^2-x'^2)^{\frac{1}{2}}}$$

or defining

$$S(x,t) = \int_0^x \frac{x'dx'}{(k^2-x'^2)^{\frac{1}{2}}(t^2-x'^2)^{\frac{1}{2}}}$$

we have

$$D(k) = 2\int_0^k dt \int_0^t S(x,t)\,\frac{dS(x,t)}{dx}\,dx + 2\int_k^1 dt \int_0^k S(x,t)\,\frac{dS(x,t)}{dx}\,dx$$

$$= \int_0^k dt\,[S^2(t,t) - S^2(0,t)] + \int_k^1 dt\,[S^2(k,t) - S^2(0,t)] \quad .$$

But

$$S(x,t) = \ln \frac{(k^2-x^2)^{\frac{1}{2}}-(t^2-x^2)^{\frac{1}{2}}}{k-t}$$

and so

$$D(k) = \int_{o}^{k} dt \ \ln^2(\frac{k+t}{k-t})^{\frac{1}{2}} + \int_{k}^{1} dt \ \ln^2(\frac{t+k}{t-k})^{\frac{1}{2}}$$

$$= \qquad I_1 \qquad + \qquad I_2$$

Now

$$I_1 = \frac{k}{4} \int_{o}^{1} \ln^2(\frac{1+x}{1-x}) dx$$

$$= \frac{k}{4} \cdot \frac{\pi^2}{3}$$

while

$$I_2 = \frac{k}{4} \int_{1}^{1/k} \ln^2(\frac{x+1}{x-1}) dx$$

$$= \frac{k}{4} \int_{k}^{1} \ln^2(\frac{1+u}{1-u}) \frac{du}{u^2}$$

$$= \frac{k}{4} \cdot \frac{2\pi^2}{3} - \frac{k}{4} \int_{o}^{k} \ln^2(\frac{1+u}{1-u}) \frac{du}{u^2}$$

So, substituting I_1 and I_2 back in equation (45) we find

$$c^{-1} = 1 - \frac{1}{\pi^2} \int_{o}^{k} \ln^2(\frac{1+u}{1-u}) \frac{du}{u^2} \qquad (46)$$

This value of c is accurate to five figures for all values of k less than a half. It is plotted in figure 2 as a function of k. There it can be seen that equation (46) is quite accurate for almost all values of k.

A series representation for c can be readily obtained from equation (46) by making the substitution

$$\frac{1+u}{1-u} = e^s$$

and integrating the s integral. The result is

$$c^{-1} = 1 - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\{2-p^n[1+(1-n \ln p)^2]\}}{n^2}$$

where

$$p = \frac{1-k}{1+k} \quad .$$

VI. Numbers

We return now to equations (25) and (26). These equations have integrals whose integrands are smooth functions of θ'; so we may approximate these integrals to a high degree of accuracy by using a Gaussian integration procedure. That is to say, if θ_j and $w(\theta_j)$ are the Gaussian points and weights[5] appropriate to the interval $(0,\pi/2)$, to a high degree of accuracy we may approximate equations (25) and (26) by

$$P_1(\theta) + k \sum_{j=1}^{N} G(\theta,\theta_j)w(\theta_j)P_2(\theta_j) = H_1(\theta)$$

$$P_2(\theta) + \sum_{j=1}^{N} G(\theta,\theta_j)w(\theta_j)P_1(\theta_j) = H_2(\theta)$$

where N is the number of points chosen for the Gaussian integration.

In particular the above equations must be true at the N Gaussian points, θ_i, and so we may write the following two coupled sets of linear algebraic equations:

$$P_1(\theta_i) + k \sum_{j=1}^{N} G(\theta_i,\theta_j)w(\theta_j)P_2(\theta_j) = H_1(\theta_i) \qquad i = 1,\cdots N$$

$$P_2(\theta_i) + \sum_{j=1}^{N} G(\theta_i,\theta_j)w(\theta_j)P_2(\theta_j) = H_2(\theta_i) \qquad i = 1,\cdots N$$

These equations may be solved numerically on a digital computer and we may again use Gaussian integration to say, using the equations of the third section, that if

$$H_1(\theta_i) = 1 \qquad i = 1,\cdots N$$

$$H_2(\theta_i) = 0 \qquad i = 1,\cdots N$$

then

$$c = \sum_{i=1}^{N} P_1(\theta_i)\sin\theta_i w(\theta_i) \tag{47}$$

or

$$c = \sum_{i=1}^{N} P_2(\theta_i)w(\theta_i) \tag{48}$$

and that if

$$H_1(\theta_i) = \frac{1}{\pi \sin^2\theta_i}\{k' - \cot\theta_i \tan^{-1}(k' \tan\theta_i)\}$$

$$H_2(\theta_i) = \frac{1}{\pi}\{k' - k\cos\theta_i \tan^{-1}(\frac{k'}{k\cos\theta_i})\}$$

then

$$A = k + 4\sum_{i=1}^{N} P_2(\theta_i)\sin\theta_i w(\theta_i) \quad . \tag{49}$$

The two independent representations for the capacitance, (47) and (48), were used as a check on the correctness of the computer code while the accuracy of the numerical results was assured by doubling the number of Gaussian integration points until the values for c and A stayed the same to several significant figures when the number of points was doubled once more. It was never necessary to use more than twenty-four integration points to obtain the six-figure accuracy of the tables. This relatively small number can be attributed to the accuracy of the Gaussian integration procedure.

Figures 2 and 3 give c and A as a function of k while for easier comparison with the results of note 98, figure 4 gives Ω_o, as defined by equation (1) or through

$$\Omega_o = 4c\sqrt{k} \quad ,$$

as a function of the parameter used in note 98, since the b/a of note 98 is equal to one half the natural log of our b/a or, in other words, $(\frac{1}{2})\ln(1/k)$. Figure 5 gives the normalized equivalent area as a function of the parameter of note 98.

Tables 1 through 4 give the same information as figures 2 through 5, but to a much higher degree of accuracy. It could be argued with some justification that the high accuracy of the tables is unnecessary. Nevertheless, the computer time is insignificant in any case and there is always a certain amount of satisfaction, on the infrequent occasions when it is possible, in solving a problem to a high degree of accuracy. It took less than three minutes of CDC 6400 computer time to obtain all the numbers in the tables.

In closing, we state without proof that the self-inductance of an annular disk of inner radius a and outer radius b is related to the normalized capacitance across the annular slot of the same dimensions, which we have calculated in this note, through

$$L = 2\mu_o ac \quad .$$

Also, the open-circuit voltage at low frequency which is induced in such an annular disk by a magnetic field, B^{inc}, perpendicular to it is related to the normalized equivalent area we have calculated here through

$$V = i\omega\pi ab\ B^{inc}A \quad .$$

The general dual nature of sensors made up of complementary areas of a flat plane may be discussed in detail in a future note. The above two equations are presented here so that our tables may also be used to determine the low-frequency characteristics of an annular disk sensor for magnetic field.

TABLE 1

Normalized Capacitance For Various Disk Radii

This table gives values of $c \equiv C/8\varepsilon a$ as a function of the ratio of the disk radius to the aperture radius $k \equiv a/b$. The first decimal point of k should be read from the left column while the second decimal point of k should be read from the top row.

k	0	1	2	3	4	5	6	7	8	9
.0	1.00000	1.00407	1.00817	1.01231	1.01648	1.02070	1.02494	1.02923	1.03356	1.03793
.1	1.04234	1.04697	1.05129	1.05584	1.06043	1.06508	1.06977	1.07451	1.07931	1.08417
.2	1.08908	1.09405	1.09907	1.10417	1.10932	1.11454	1.11983	1.12519	1.13062	1.13613
.3	1.14172	1.14738	1.15313	1.15896	1.16488	1.17089	1.17700	1.18320	1.18951	1.19592
.4	1.20244	1.20907	1.21582	1.22269	1.22969	1.23682	1.24408	1.25149	1.25905	1.26676
.5	1.27463	1.28267	1.29089	1.29929	1.30789	1.31668	1.32569	1.33491	1.34437	1.35408
.6	1.36404	1.37427	1.38479	1.39562	1.40676	1.41824	1.43009	1.44231	1.45495	1.46802
.7	1.48155	1.49558	1.51015	1.52529	1.54106	1.55749	1.57465	1.59261	1.61144	1.63121
.8	1.65203	1.67401	1.69728	1.72199	1.74832	1.77649	1.80675	1.83942	1.87492	1.91372
.9	1.95649	2.00408	2.05764	2.11882	2.18999	2.27488	2.37973	2.51627	2.71092	3.04810

TABLE 2

Normalized Equivalent Area For Various Disk Radii

This table gives values of $A \equiv A_{eq}/\pi ab$ as a function of the ratio of the disk radius to the aperture radius $k \equiv a/b$. The first decimal point of k should be read from the left column while the second decimal point of k should be read from the top row.

k	0	1	2	3	4	5	6	7	8	9
.0	.810569	.813841	.817085	.820302	.823492	.826654	.829790	.832899	.835982	.839037
.1	.842066	.845068	.848044	.850993	.853916	.856813	.859683	.862527	.865345	.868137
.2	.870903	.873642	.876356	.879043	.881705	.884340	.886949	.889533	.892090	.894621
.3	.897127	.899606	.902059	.904486	.906888	.909263	.911612	.913934	.916231	.918501
.4	.920746	.922963	.925155	.927319	.929458	.931570	.933655	.935713	.937744	.939749
.5	.941726	.943676	.945599	.947495	.949363	.951203	.953016	.954800	.956557	.958285
.6	.959985	.961656	.963298	.964911	.966494	.968048	.969573	.971067	.972531	.973964
.7	.975367	.976738	.978077	.979385	.980660	.981903	.983112	.984288	.985429	.986536
.8	.987607	.988643	.989642	.990604	.991528	.992413	.993258	.994063	.994826	.995545
.9	.996221	.996850	.997432	.997464	.998444	.998869	.999236	.999541	.999777	.999937

TABLE 3

Renormalized Capacitance In Terms Of An Old Parameter

This table gives values of $\Omega_o \equiv C/2\varepsilon(ab)^{\frac{1}{2}}$ as a function of $x \equiv 1/2 \ln(b/a)$, for values of x between zero and one tenth. The first two decimal points of x should be read from the left column while the third decimal point of x should be read from the top row.

$\frac{1}{2} \ln \frac{1}{k}$	0	1	2	3	4	5	6	7	8	9
.00	∞	15.3606	13.9744	13.1634	12.5880	12.1416	11.7769	11.4685	11.2014	10.9657
.01	10.7549	10.5641	10.3900	10.2298	10.0815	9.9434	9.8142	9.6928	9.5783	9.4701
.02	9.3673	9.2696	9.1764	9.0874	9.0021	8.9203	8.8417	8.7660	8.6931	8.6228
.03	8.5548	8.4891	8.4254	8.3637	8.3038	8.2456	8.1891	8.1341	8.0806	8.0284
.04	7.9976	7.9280	7.8797	7.8324	7.7862	7.7411	7.6969	7.6537	7.6114	7.5700
.05	7.5294	7.4895	7.4505	7.4122	7.3746	7.3377	7.3015	7.2658	7.2308	7.1964
.06	7.1626	7.1293	7.0966	7.0644	7.0327	7.0015	6.9707	6.9404	6.9106	6.8812
.07	6.8522	6.8236	6.7954	6.7676	6.7401	6.7131	6.6864	6.6600	6.6340	6.6083
.08	6.5829	6.5578	6.5330	6.5086	6.4844	6.4605	6.4369	6.4136	6.3905	6.3676
.09	6.3451	6.3227	6.3007	6.2788	6.2572	6.2358	6.2146	6.1937	6.1730	6.1524

TABLE 4

Normalized Equivalent Area In Terms Of An Old Parameter

This table gives values of $A \equiv A_{eq}/\pi ab$ as a function of $x \equiv 1/2 \ln(b/a)$ for value of x between zero and one tenth. The first two decimal points of x should be read from the left column while the third decimal point of x should be read from the top row.

$\frac{1}{2} \ln \frac{1}{k}$	0	1	2	3	4	5	6	7	8	9
.00	1.000000	.999997	.999988	.999975	.999958	.999937	.999912	.999884	.999853	.999818
.01	.999781	.999741	.999698	.999652	.999604	.999553	.999500	.999444	.999386	.999325
.02	.999263	.999198	.999131	.999062	.998991	.998918	.998842	.998765	.998686	.998606
.03	.998523	.998439	.998352	.998265	.998175	.998084	.997991	.997896	.997800	.997702
.04	.997603	.997502	.997400	.997296	.997191	.997084	.996976	.996867	.996756	.996644
.05	.996531	.996416	.996300	.996183	.996065	.995945	.995824	.995702	.995578	.995454
.06	.995328	.995202	.995074	.994945	.994815	.994684	.994551	.994418	.994284	.994149
.07	.994012	.993875	.993737	.993598	.993458	.993317	.993175	.993082	.992888	.992743
.08	.992598	.992451	.992304	.992156	.992007	.991857	.991706	.991555	.991403	.991250
.09	.991056	.990941	.990786	.990630	.990473	.990316	.990157	.989998	.989839	.989678

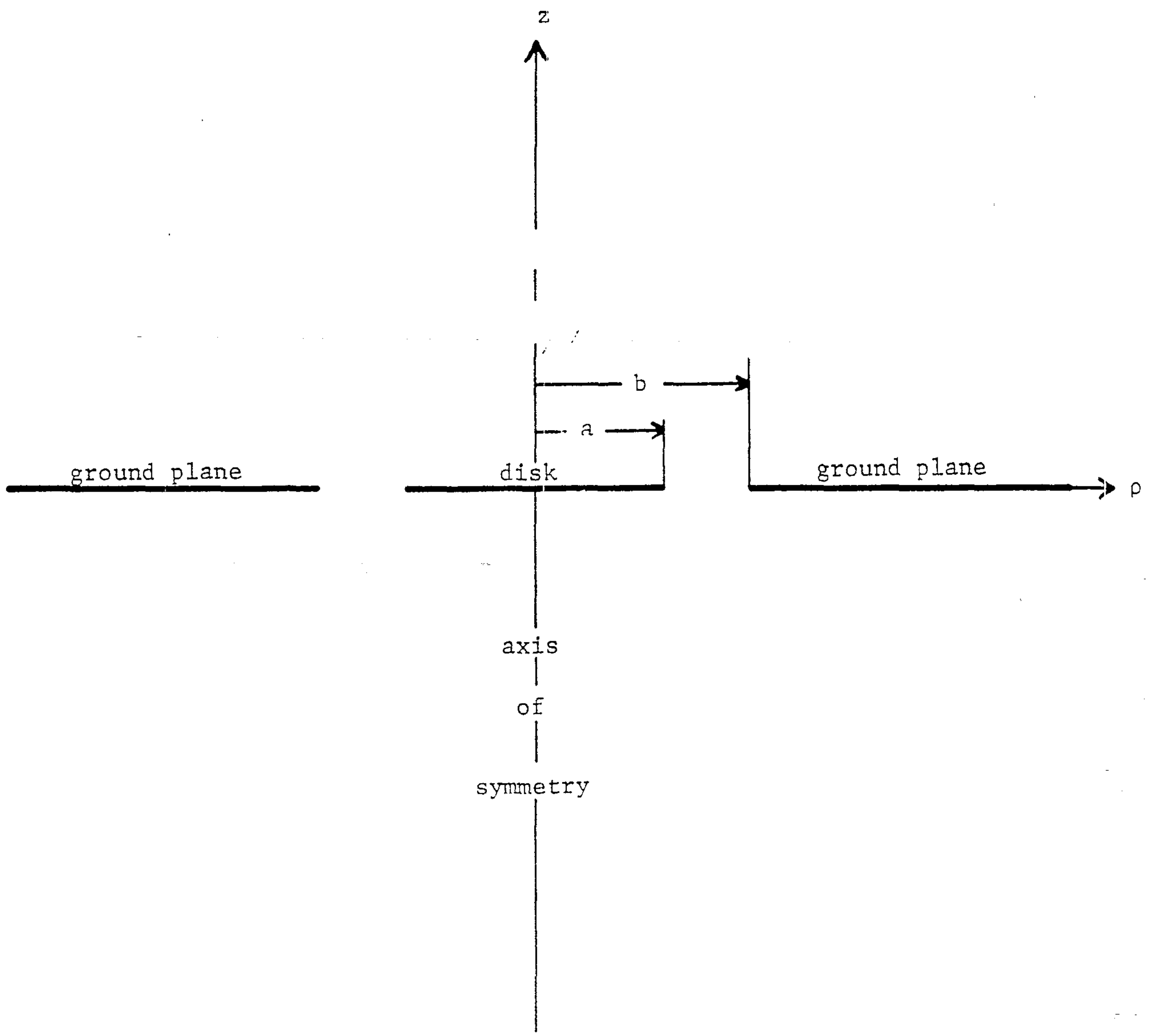

Figure 1: Idealized Sensor Geometry

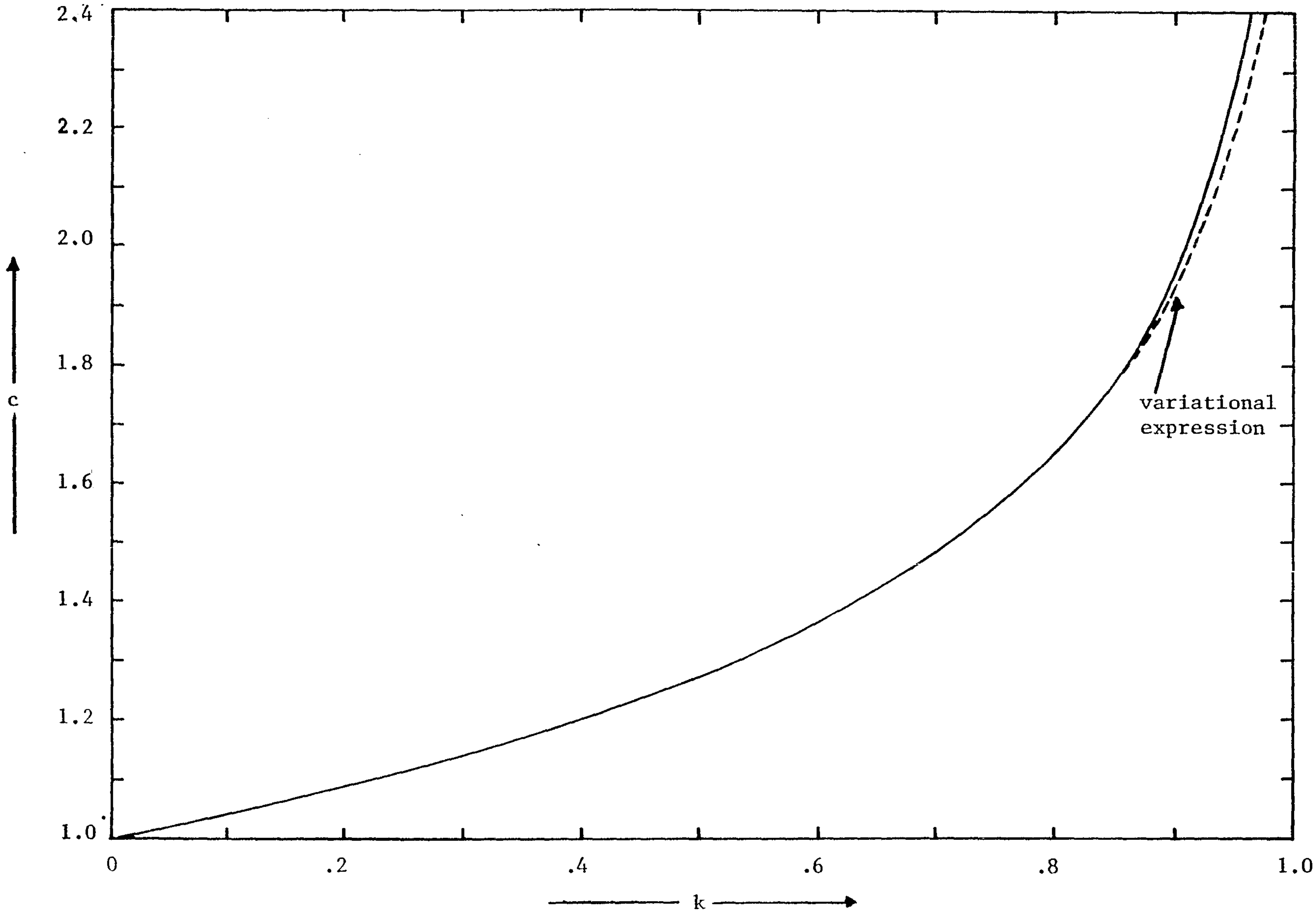

Figure 2: Normalized Capacitance for Various Disk Radii

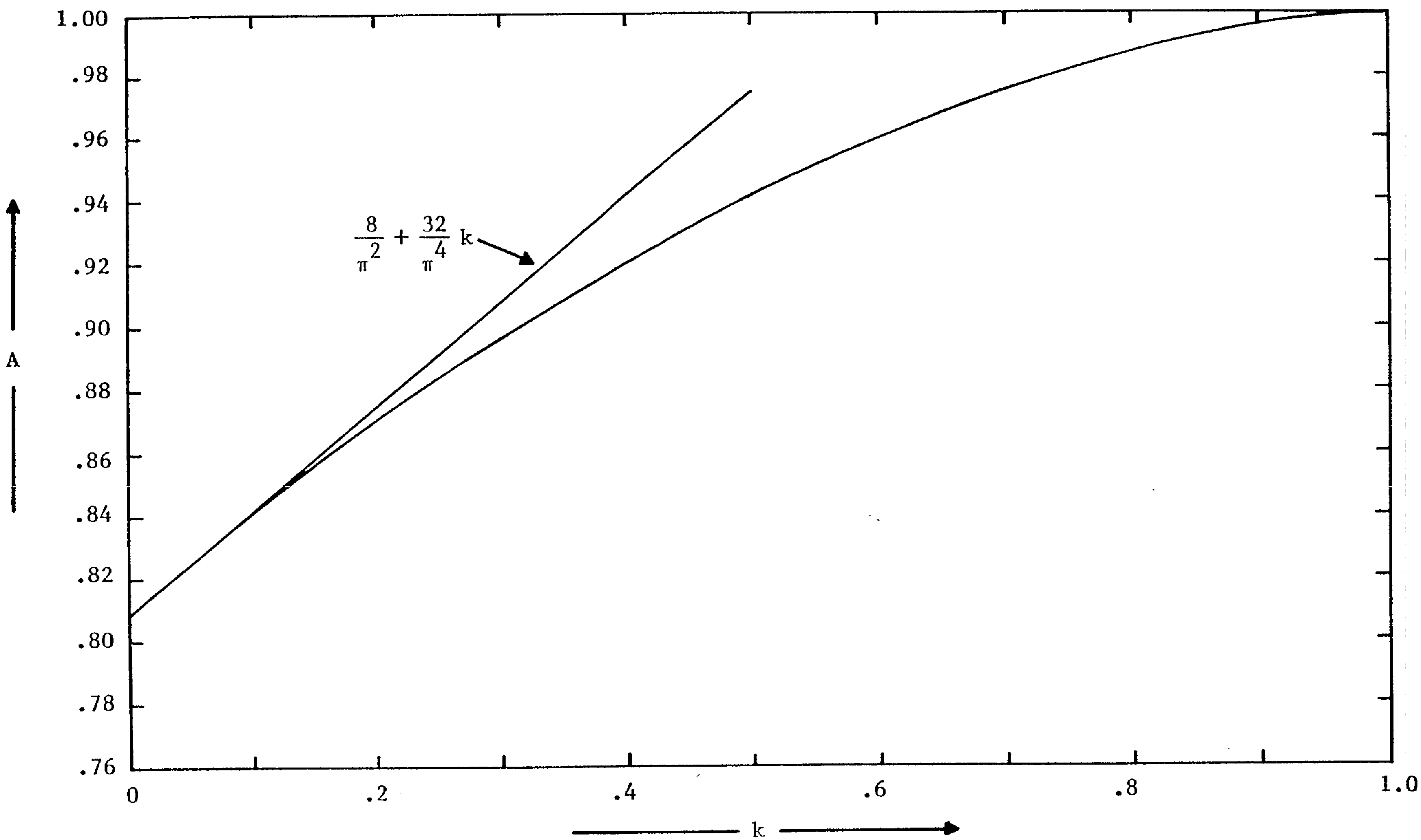

Figure 3: Normalized Equivalent Area for Various Disk Radii

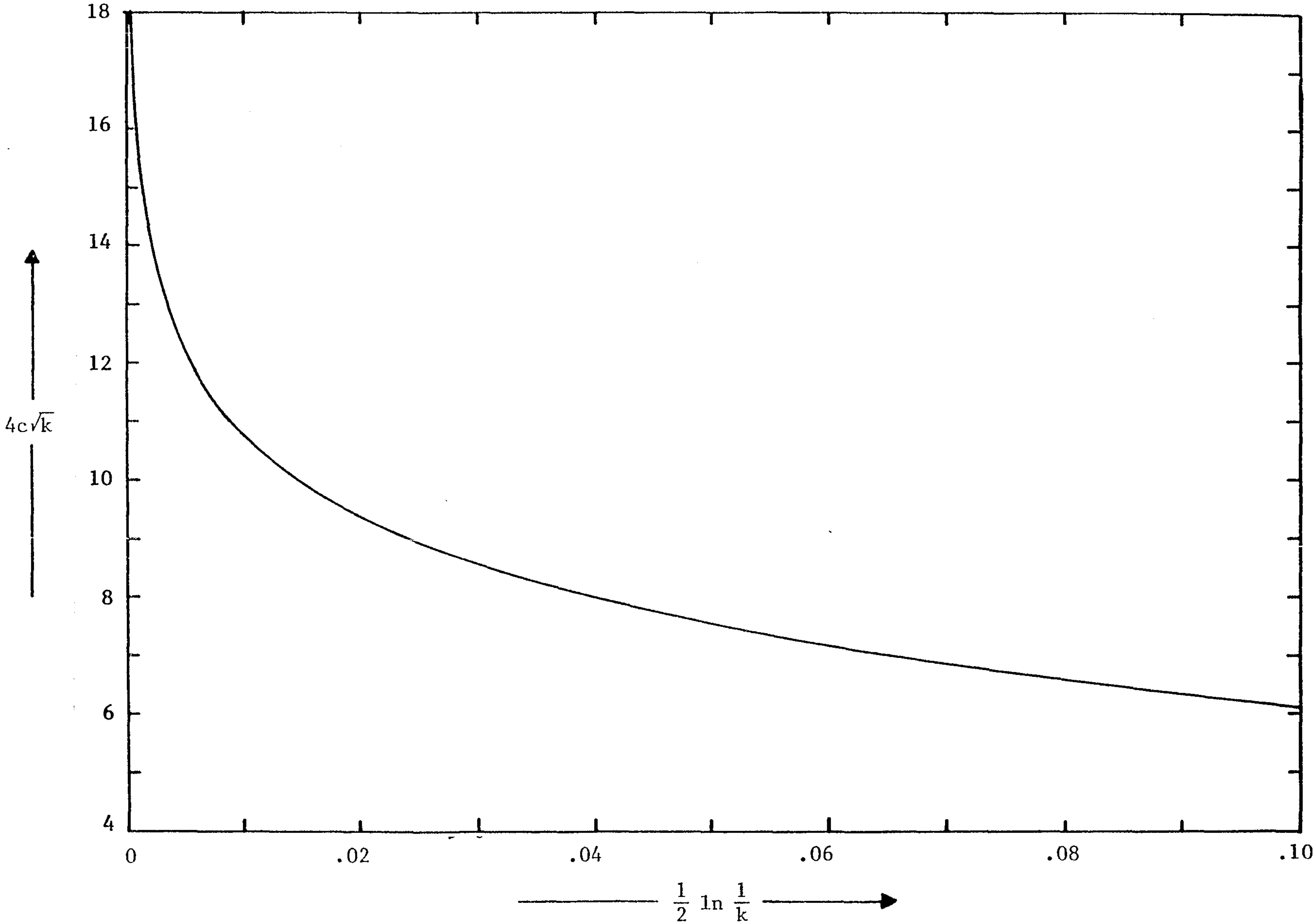

Figure 4: Renormalized Capacitance in Terms of An Old Parameter

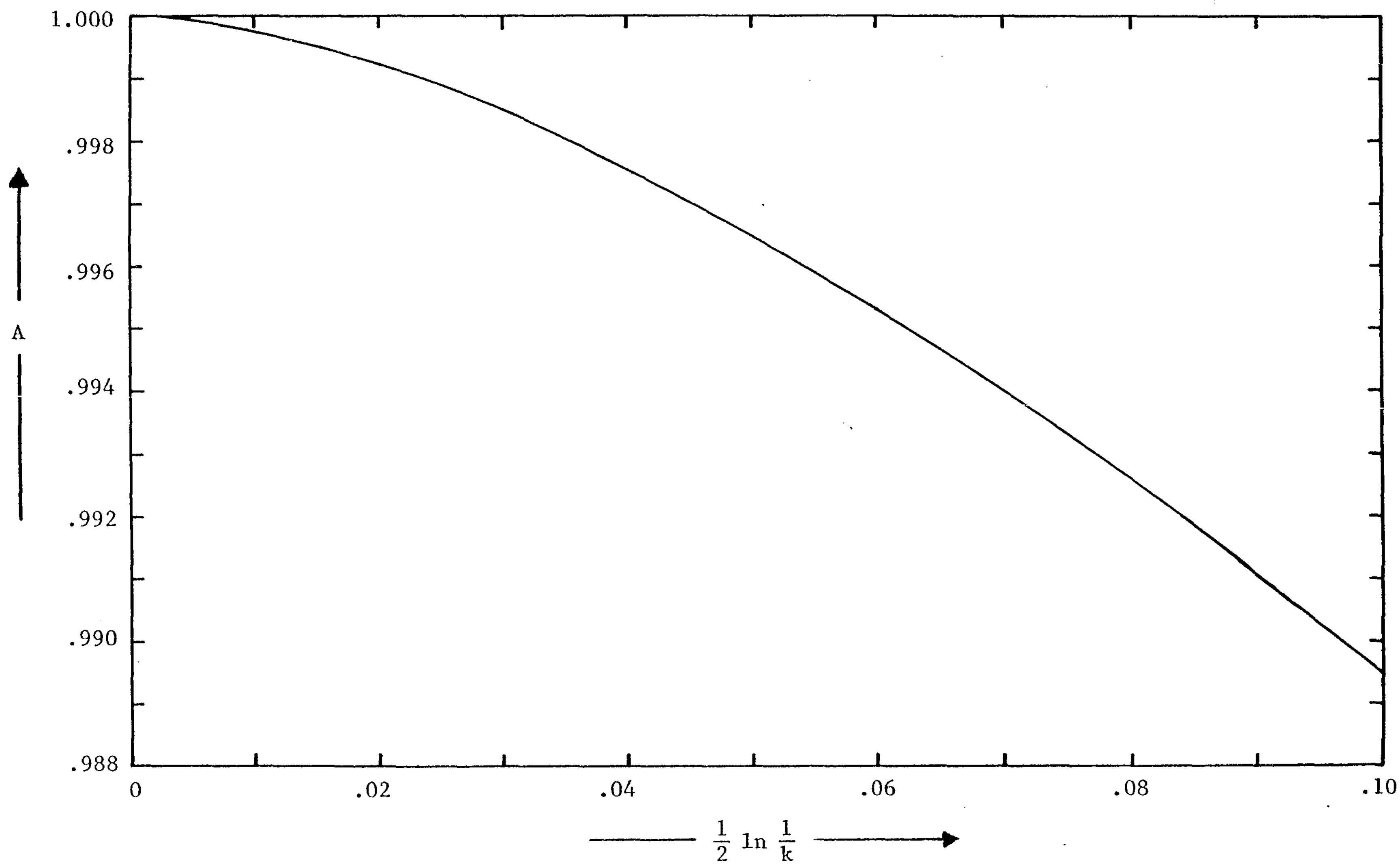

Figure 5: Normalized Equivalent Area in Terms of an Old Parameter

Appendix A

In this appendix we will show one way of solving the pair of dual integral equations

$$\int_0^\infty A(\alpha)J_o(\alpha x)d\alpha = g(x) \qquad x > 1 \tag{A-1}$$

$$\int_0^\infty \alpha A(\alpha)J_o(\alpha x)d\alpha = f(x) \qquad x < 1 \tag{A-2}$$

We also calculate the integral in (A-1) for $x < 1$ and the integral in (A-2) for $x > 1$.

We start by noting that, because of linearity, $A(\alpha)$ may be written as the sum of the solutions of two other pairs of dual equations, i.e.

$$A(\alpha) = B(\alpha) + C(\alpha) \tag{A-3}$$

where

$$\int_0^\infty B(\alpha)J_o(\alpha x)d\alpha = 0 \qquad x > 1 \tag{A-4}$$

$$\int_0^\infty \alpha B(\alpha)J_o(\alpha x)d\alpha = f(x) \qquad x < 1 \quad , \tag{A-5}$$

and

$$\int_0^\infty C(\alpha)J_o(\alpha x)d\alpha = g(x) \qquad x > 1 \tag{A-6}$$

$$\int_0^\infty \alpha C(\alpha)J_o(\alpha x)d\alpha = 0 \qquad x < 1 \quad . \tag{A-7}$$

In (A-4) and (A-5) we set

$$B(\alpha) = \int_0^1 \eta(t)\sin \alpha t dt \;; \quad \eta(0) = 0 \tag{A-8}$$

Interchanging the order of integration in (A-4) and making use of the integral[6]

$$\int_0^\infty J_o(\alpha x)\sin\alpha t\,d\alpha = (t^2 - x^2)^{-\frac{1}{2}} \qquad x < t$$

$$= 0 \qquad x > t \quad , \tag{A-8}$$

it can be seen that (A-4) is satisfied identically. Equation (A-5), after integrating by parts, becomes

$$\int_0^\infty J_o(\alpha x)d\alpha \left\{- \eta(1)\cos\alpha + \int_0^1 \eta'(t)\cos\alpha t dt\right\} = f(x) \tag{A-9}$$

Again interchanging the order of integration and making use of a second Bessel function integral, namely[6]

$$\int_0^\infty J_o(\alpha x)\cos\alpha t\,d\alpha = 0 \qquad x < t$$

$$= (x^2 - t^2)^{-\frac{1}{2}} \qquad x > t \quad , \tag{A-10}$$

equation (A-9) becomes

$$f_1(x) = \int_0^x \frac{\eta'(t)dt}{(x^2-t^2)^{\frac{1}{2}}} \quad .$$

From appendix C, the solution of this equation is

$$\eta'(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{xf(x)dx}{(t^2-x^2)^{\frac{1}{2}}}$$

or, integrating,

$$\eta(t) = \frac{2}{\pi}\int_0^t \frac{xf(x)dx}{(t^2-x^2)^{\frac{1}{2}}} \tag{A-11}$$

and we have

$$B(\alpha) = \frac{2}{\pi}\int_0^1 \sin \alpha t dt \int_0^t \frac{xf(x)dx}{(t^2-x^2)^{\frac{1}{2}}} \tag{A-12}$$

To solve the pair (A-6), (A-7) we substitute

$$C(\alpha) = \int_1^\infty \xi(t)\sin \alpha t dt \quad .$$

Interchanging orders of integration and making use of the derivative of equation (A-10) with respect to t, it is clear that (A-7) is satisfied identically while (A-6) becomes, with the use of (A-8)

$$\int_x^\infty \frac{\xi(t)dt}{(t^2-x^2)^{\frac{1}{2}}} = g(x) \quad .$$

The solution of this equation, from appendix C, is

$$\xi(t) = -\frac{2}{\pi}\frac{d}{dt}\int_t^\infty \frac{xg(x)dx}{(x^2-t^2)^{\frac{1}{2}}}$$

and so

$$C(\alpha) = -\frac{2}{\pi}\int_1^\infty \sin \alpha t dt \cdot \frac{d}{dt}\int_t^\infty \frac{xg(x)dx}{(x^2-t^2)^{\frac{1}{2}}} \quad . \tag{A-13}$$

Combining (A-12) and (A-13),

$$A(\alpha) = \frac{2}{\pi}\int_0^1 \sin \alpha t dt \int_0^t \frac{x'f(x')dx'}{(t^2-x'^2)^{\frac{1}{2}}} - \frac{2}{\pi}\int_1^\infty \sin \alpha t dt \frac{d}{dt}\int_t^\infty \frac{x'g(x')dx'}{(x'^2-t^2)^{\frac{1}{2}}} \tag{A-14}$$

Inserting this equation in (A-1), interchanging orders of integration, and again making use of (A-8), we find for $x < 1$

$$g(x) = \frac{2}{\pi}\int_x^1 \frac{dt}{(t^2-x^2)^{\frac{1}{2}}}\int_0^t \frac{x'f(x')dx'}{(t^2-x'^2)^{\frac{1}{2}}} - \frac{2}{\pi}\int_1^\infty \frac{dt}{(t^2-x^2)^{\frac{1}{2}}} \cdot \frac{d}{dt}\int_t^\infty \frac{x'g(x')dx'}{(x'^2-t^2)^{\frac{1}{2}}} \tag{A-15}$$

The inner integral of the second double integral in this equation may be integrated by parts. The remaining integral may then be interchanged with the t integration and the resulting t integration may then be carried out explicitly. The result of this manipulation is

$$g(x) = \frac{2}{\pi}\int_x^1 \frac{dt}{(t^2-x^2)^{\frac{1}{2}}}\int_0^t \frac{x'f(x')dx'}{(t^2-x'^2)^{\frac{1}{2}}} + \frac{2}{\pi}(1-x^2)^{\frac{1}{2}}\int_1^\infty \frac{x'g(x')dx'}{(x'^2-1)^{\frac{1}{2}}(x'^2-x^2)} \qquad x < 1 \qquad \text{(A-16)}$$

We may evaluate (A-2) for $x > 1$ where $A(\alpha)$ is given by (A-14) if we note that

$$J_o(\alpha x) = \frac{1}{\alpha x}\frac{d}{dx}(xJ_1(\alpha x)) \quad , \qquad \text{(A-17)}$$

which gives

$$f(x) = \frac{1}{x}\frac{d}{dx}\int_0^\infty xA(\alpha)J_1(\alpha x)d\alpha \qquad x < 1 \quad .$$

In this equation we substitute (A-14), interchange orders of integration, use the result[6]

$$\begin{aligned}\int_0^\infty J_1(\alpha x)\sin\alpha t d\alpha &= 0 \qquad x < t \\ &= \frac{t(x^2-t^2)^{-\frac{1}{2}}}{x} \qquad x > t \quad ,\end{aligned} \qquad \text{(A-18)}$$

and obtain

$$f(x) = \frac{2}{\pi x}\frac{d}{dx}\int_0^x \frac{tdt}{(x^2-t^2)^{\frac{1}{2}}}\int_0^t \frac{x'f(x')dx'}{(t^2-x'^2)^{\frac{1}{2}}} - \frac{2}{\pi x}\frac{d}{dx}\int_1^x \frac{tdt}{(x^2-t^2)^{\frac{1}{2}}}\frac{d}{dt}\int_t^\infty \frac{x'g(x')dx'}{(x'^2-t^2)^{\frac{1}{2}}} \qquad x > 1 \qquad \text{(A-19)}$$

Here we may interchange the order of integration in the first double integral and get the following simpler result:

$$f(x) = \frac{-2}{\pi(x^2-1)^{\frac{1}{2}}}\int_0^1 \frac{x'(1-x'^2)^{\frac{1}{2}}f(x')dx'}{x^2-x'^2} - \frac{2}{\pi x}\frac{d}{dx}\int_1^x \frac{tdt}{(x^2-t^2)^{\frac{1}{2}}}\frac{d}{dt}\int_t^\infty \frac{x'g(x')dx'}{(x'^2-t^2)^{\frac{1}{2}}} \qquad x > 1 \qquad \text{(A-20)}$$

Appendix B

In this appendix we will give a method of solving a pair of dual integral equations quite similar to those solved in appendix A, namely

$$\int_0^\infty A(\alpha)J_o(\alpha x)d\alpha = g(x) \qquad x < 1 \tag{B-1}$$

$$\int_0^\infty \alpha A(\alpha)J_o(\alpha x)d\alpha = f(x) \qquad x > 1 \tag{B-2}$$

We can write

$$A(\alpha) = B(\alpha) + C(\alpha) \tag{B-3}$$

where

$$\int_0^\infty \alpha B(\alpha)J_o(\alpha x)d\alpha = f(x) \qquad x > 1 \tag{B-4}$$

$$\int_0^\infty B(\alpha)J_o(\alpha x)d\alpha = 0 \qquad x < 1 \tag{B-5}$$

and

$$\int_0^\infty \alpha C(\alpha)J_o(\alpha x)d\alpha = 0 \qquad x > 1 \tag{B-6}$$

$$\int_0^\infty C(\alpha)J_o(\alpha x)d\alpha = g(x) \qquad x < 1 \ . \tag{B-7}$$

We now set

$$B(\alpha) = \int_1^\infty \eta(t)\cos \alpha t dt \ , \quad \eta(\infty) = 0 \tag{B-8}$$

in equations (B-4) and (B-5). Interchanging the order of integration in (B-5) and making use of (A-10) it is clear that (B-5) is satisfied identically;

while integrating (B-8) by parts and then substituting in (B-4) we obtain

$$\int_0^\infty \alpha J_o(\alpha x)d\alpha \left\{ \frac{\eta(t)\sin \alpha t}{\alpha}\bigg|_1^\infty - \int_1^\infty \frac{\eta'(t)\sin \alpha t dt}{\alpha} \right\} = f(x)$$

Now interchanging orders of integration and making use of (A-8) we get

$$-\int_x^\infty \frac{\eta'(t)dt}{(t^2-x^2)^{\frac{1}{2}}} = f(x)$$

and the solution of this equation, from appendix C, is

$$\eta'(t) = \frac{2}{\pi}\frac{d}{dt}\int_t^\infty \frac{xf(x)dx}{(x^2-t^2)^{\frac{1}{2}}}$$

or, integrating and substituting back in (B-8),

$$B(\alpha) = \frac{2}{\pi}\int_1^\infty \cos \alpha t dt \int_t^\infty \frac{xf(x)dx}{(x^2-t^2)^{\frac{1}{2}}} \quad . \tag{B-9}$$

We solve the pair (B-6), (B-7) by setting

$$C(\alpha) = \int_0^1 \xi(t)\cos \alpha t dt \quad .$$

Thus, from the derivative of (A-8) with respect to t, (B-6) is satisfied identically while (B-7) becomes

$$\int_0^x \frac{\xi(t)dt}{(x^2-t^2)^{\frac{1}{2}}} = g(x) \quad ,$$

whose solution, by appendix C, is

$$\xi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{xg(x)dx}{(t^2-x^2)^{\frac{1}{2}}}$$

and so

$$C(\alpha) = \frac{2}{\pi}\int_0^1 \cos \alpha t dt \cdot \frac{d}{dt}\int_0^t \frac{xg(x)dx}{(t^2-x^2)^{\frac{1}{2}}} \quad . \tag{B-10}$$

From (B-3), (B-9), and (B-10) we may now write

$$A(\alpha) = \frac{2}{\pi}\int_1^\infty \cos\alpha t dt \int_t^\infty \frac{xf(x)dx}{(x^2-t^2)^{\frac{1}{2}}} + \frac{2}{\pi}\int_0^1 \cos\alpha t dt \frac{d}{dt}\int_0^t \frac{xg(x)dx}{(t^2-x^2)^{\frac{1}{2}}} \quad . \tag{B-11}$$

Now we may insert this equation for $A(\alpha)$ into (B-1), interchange orders of integration, make use of equation (A-10), and arrive at the representation for $g(x)$ valid for $x > 1$:

$$g(x) = \frac{2}{\pi}\int_1^x \frac{dt}{(x^2-t^2)^{\frac{1}{2}}}\int_t^\infty \frac{x'f(x')dx'}{(x'^2-t^2)^{\frac{1}{2}}} + \frac{2}{\pi}\int_0^1 \frac{dt}{(x^2-t^2)^{\frac{1}{2}}}\frac{d}{dt}\int_0^t \frac{x'g(x')dx'}{(t^2-x'^2)^{\frac{1}{2}}} \qquad x > 1 \tag{B-12}$$

but we may simplify the second double integral in the manner used for equation (A-15) to obtain

$$g(x) = \frac{2}{\pi}\int_1^x \frac{dt}{(x^2-t^2)^{\frac{1}{2}}}\int_t^\infty \frac{x'f(x')dx'}{(x'^2-t^2)^{\frac{1}{2}}} + \frac{2}{\pi}(x^2-1)^{\frac{1}{2}}\int_0^1 \frac{x'g(x')dx'}{(1-x'^2)^{\frac{1}{2}}(x'^2-x^2)} \qquad x > 1 \tag{B-13}$$

Also, by substituting (A-17) and (B-11) in (B-2) and making use of the result[6]

$$\int_0^\infty J_1(\alpha x)\cos\alpha t d\alpha = \frac{1}{x} \qquad t < x$$

$$= \frac{1}{x} - \frac{t}{x(t^2-x^2)^{\frac{1}{2}}} \qquad t > x \tag{B-14}$$

it can be seen that

$$f(x) = \frac{2}{\pi x}\frac{d}{dx}\left\{\int_1^\infty dt \int_t^\infty \frac{x'f(x')dx'}{(x'^2-t^2)^{\frac{1}{2}}} - \int_x^\infty \frac{tdt}{(t^2-x^2)^{\frac{1}{2}}}\int_t^\infty \frac{x'f(x')dx'}{(x'^2-t^2)^{\frac{1}{2}}}\right\}$$

$$+ \frac{2}{\pi x}\frac{d}{dx}\left\{\int_0^1 dt \frac{d}{dt}\int_0^t \frac{x'g(x')dx'}{(t^2-x'^2)^{\frac{1}{2}}} - \int_x^1 \frac{tdt}{(t^2-x^2)^{\frac{1}{2}}}\frac{d}{dt}\int_0^t \frac{x'g(x')dx'}{(t^2-x'^2)^{\frac{1}{2}}}\right\} \qquad x < 1$$

The first double integral within each pair of braces is independent of x; so

$$f(x) = -\frac{2}{\pi x}\frac{d}{dx}\int_1^\infty \frac{tdt}{(t^2-x^2)^{\frac{1}{2}}}\int_t^\infty \frac{x'f(x')dx'}{(x'^2-t^2)^{\frac{1}{2}}} - \frac{2}{\pi x}\frac{d}{dx}\int_x^1 \frac{tdt}{(t^2-x^2)^{\frac{1}{2}}}\frac{d}{dt}\int_0^t \frac{x'g(x')dx'}{(t^2-x'^2)^{\frac{1}{2}}} \qquad x < 1$$

or, interchanging orders of integration in the first double integral,

$$f(x) = \frac{-2}{\pi(1-x^2)^{\frac{1}{2}}} \int_1^\infty \frac{x'(x'^2-1)^{\frac{1}{2}} f(x')dx'}{x'^2-x^2} - \frac{2}{\pi x}\frac{d}{dx}\int_x^1 \frac{tdt}{(t^2-x^2)^{\frac{1}{2}}}\frac{d}{dt}\int_o^t \frac{x'g(x')dx'}{(t^2-x'^2)^{\frac{1}{2}}} \quad x < 1 \tag{B-15}$$

Appendix C

In this appendix we will give brief derivations of the solutions of the two integral equations

$$\int_0^x \frac{f(t)dt}{(x^2-t^2)^{\frac{1}{2}}} = g(x) \tag{C-1}$$

and

$$\int_x^\infty \frac{F(t)dt}{(t^2-x^2)^{\frac{1}{2}}} = G(x) \quad . \tag{C-2}$$

First we treat equation (C-1). Operating on both sides of that equation with the integral operator,

$$\int_0^s \frac{xdx}{(s^2-x^2)^{\frac{1}{2}}} \quad ,$$

we obtain

$$\int_0^s \frac{xdx}{(s^2-x^2)^{\frac{1}{2}}} \cdot \int_0^x \frac{f(t)dt}{(x^2-t^2)^{\frac{1}{2}}} = \int_0^s \frac{xg(x)dx}{(s^2-x^2)^{\frac{1}{2}}} \quad .$$

Interchanging the order of integration in the double integral on the left results in

$$\int_0^s f(t)dt \int_t^s \frac{xdx}{(s^2-x^2)^{\frac{1}{2}}(x^2-t^2)^{\frac{1}{2}}} = \int_0^s \frac{xg(x)dx}{(s^2-x^2)^{\frac{1}{2}}} \quad .$$

The inner integral is equal to $\pi/2$, and so

$$\frac{\pi}{2}\int_0^s f(t)dt = \int_0^s \frac{xg(x)dx}{(s^2-x^2)^{\frac{1}{2}}} \quad ;$$

or, differentiating,

$$f(s) = \frac{2}{\pi} \cdot \frac{d}{ds}\int_0^s \frac{xg(x)dx}{(s^2-x^2)^{\frac{1}{2}}} \quad . \tag{C-3}$$

This equation is the solution of equation (C-1).

Equation (C-2) may be solved in a similar manner. We use the integral operator,

$$\int_s^\infty \frac{x\,dx}{(x^2-s^2)^{\frac{1}{2}}} \quad ,$$

and obtain

$$\int_s^\infty \frac{x\,dx}{(x^2-s^2)^{\frac{1}{2}}} \int_x^\infty \frac{F(t)\,dt}{(t^2-x^2)^{\frac{1}{2}}} = \int_s^\infty \frac{xG(x)\,dx}{(x^2-s^2)^{\frac{1}{2}}} \quad .$$

Again, interchanging the order of integration on the left, we get

$$\int_s^\infty F(t)\,dt \int_s^t \frac{x\,dx}{(t^2-x^2)^{\frac{1}{2}}(x^2-s^2)^{\frac{1}{2}}} = \int_s^\infty \frac{xG(x)\,dx}{(x^2-s^2)^{\frac{1}{2}}}$$

The rest of the procedure is the same as the derivation of (C-3), and we obtain finally

$$F(s) = -\frac{2}{\pi}\frac{d}{ds}\int_s^\infty \frac{xG(x)\,dx}{(x^2-s^2)^{\frac{1}{2}}} \quad . \qquad \text{(C-4)}$$

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