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Magnetic-Field Distortion by a Specific Axisymmetric, Semi-Infinite, Perfectly Conducting Body

by

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Abstract

A consideration of the magnetic lines of a semi-infinite line of static magnetic point dipoles parallel to a uniform external magnetic field leads to a profile of a semi-infinite, perfectly conducting body which lends itself to exact analysis of the scattering problem. The solution of this scattering problem may yield some information concerning the distortion of the magnetic field by a rocket platform or a boom which approximates this specific shape. The flux through a coaxial loop is calculated and plotted against the distance from the end of the scatterer for several values of the loop's radius. The plot of the magnetic field lines outside the body is also given. The corresponding electrostatic problem, where the incident electric field vector is perpendicular to the axis of the body, is discussed.

I. Introduction

The field distortion by a perfectly conducting, solid cylinder (which could be a rocket platform or a boom) and a method of reducing it have been discussed previously.[1,2] The present note is concerend with a detailed calculation of the magnetic-field distortion near the end of such a cylinder where an EMP sensor is likely to be placed.[1] Since only the field near the cylinder's end concerns us here, we may assume the cylinder to be semi-infinite and then treat the problem as a magnetostatic one provided that we are only interested in frequencies of the incident energy smaller than the lowest resonance frequency of the cylinder. One might first think of a semi-infinite solid circular cylinder exposed to a uniform magnetic field and try to solve the magnetostatic boundary-value problem with the normal component of the magnetic field vanishing on the cylindrical surface as well as on the end. Although this problem can be formulated by the Wiener-Hopf technique,[3] it becomes too complicated to obtain any quick numerical solution for the field near the cylinder's end. Other shapes like a semi-infinite cylinder with a spherical or conical end cap are at least equally difficult. Thus, instead of solving a boundary-value problem for a given shape of a body we shall employ an indirect approach in which we seek the shape of a body that would correspond to the known field of a given distribution of magnetic sources (e.g., magnetic multipoles) and yet would resemble a rocket platform or a boom.

In section II, we shall consider the magnetic field of a semi-infinite line of magnetic point dipoles and calculate the total magnetic field lines when the field of such a dipole distribution is superimposed on a uniform magnetic field parallel to the dipoles. This consideration leads to an axisymmetric body whose profile follows the field lines. The fractional flux, namely the ratio of the scattered to the incident flux, through a loop coaxial with the body is calculated in section III. A similar approach has been tried in the transverse case, where the incident magnetic field is perpendicular to the axis of a rocket platform, but these attempts have not yet met with success. In section IV, the corresponding electrostatic problem, where the external electric field is perpendicular to the axis of the body, is briefly discussed. It is shown how the solution of this electrostatic problem can be directly deduced from that of the corresponding magnetostatic problem.

II. Lines of Force

Consider the situation as depicted in Fig. 1 where a line of magnetic point dipoles, each having dipole moment mdz and pointing in the negative z direction, is immersed in a uniform magnetic field $\underset{\sim}{H}_o$. We wish to find the magnetic lines of force in this situation. The magnetic field $d\underset{\sim}{H}^s$ due to a magnetic point dipole situated at z = z' and pointing in the negative z direction (Fig. 1) is given by[4]

$$d\underset{\sim}{H}^s = \frac{mdz'}{4\pi} \nabla \frac{\cos\theta}{r^2} = \frac{mdz'}{4\pi} \nabla \frac{z - z'}{[(z - z')^2 + \rho^2]^{3/2}} \tag{1}$$

The total magnetic field due to a line of magnetic point dipoles extending from z = 0 to z = ∞ is obtained by integrating this expression. Thus,

$$\underset{\sim}{H}^s(\rho,z) = \frac{m}{4\pi} \nabla \int_0^\infty \frac{z - z'}{[(z - z')^2 + \rho^2]^{3/2}}\, dz' = -\frac{m}{4\pi} \nabla \frac{1}{r} \tag{2}$$

Writing (2) out in component form we have

$$H^s_\rho = \frac{m}{4\pi} \frac{\rho}{r^3}, \qquad H^s_z = \frac{m}{4\pi} \frac{z}{r^3} \quad . \tag{3}$$

The equation of the total magnetic-field (i.e. $\underset{\sim}{H}^s + \underset{\sim}{H}_o$) lines is obtained by solving

$$\frac{d\rho}{H^s_\rho} = \frac{dz}{H_o + H^s_z} \tag{4}$$

Setting $m = \pi a^2 H_o$, where a has the dimension of length, and substituting (3) into (4) we get

$$\frac{d\rho}{dz} = \frac{a^2\rho}{4(\rho^2 + z^2)^{3/2} + a^2 z} \tag{5}$$

To solve (5) we set $z = \rho \cot\theta$ and eliminate z from (5). Thus, (5) becomes

$$\frac{a^2}{\rho} \frac{d\theta}{d\rho} = -4 \csc\theta \tag{6}$$

which gives on integration

$$\cos\theta = \frac{2\rho^2}{a^2} + C \tag{7}$$

where C is the integration constant. Eliminating θ in favor of z, we have from (7)

$$\frac{z}{\sqrt{\rho^2 + z^2}} = \frac{2\rho^2}{a^2} + C \tag{8}$$

This equation could also have been obtained by setting the total flux (incident plus scattered) through a loop of radius ρ equal to a constant.

The constant C will be so determined that (8) will describe the profile of a semi-infinite body. To do this we observe from (5) that $d\rho/dz = \infty$ at $\rho = 0$, $z = -a/2$. Thus, setting $\rho = 0$, $z = -a/2$ in (8) we find that $C = -1$. Solving (8) for ρ/a we get

$$\rho/a = 2^{-\frac{1}{2}}\left[1 - (z/a)^2 + (z/a)\sqrt{2 + (z/a)^2}\right]^{\frac{1}{2}} \tag{9}$$

which is graphed in Fig. 2. Note that (9) gives $\rho = a$ when $z/a = \infty$.

Equation (9) describes, in the ρ-z plane, a family of curves with "a" as a parameter and is the equation of the lines of force we set out to find at the beginning of this section. If a body takes the form described by (9) and is immersed in a uniform magnetic field $\underset{\sim}{H}_o$ as shown in Fig. 1, then the scattered field is given by (3) when the boundary condition at the surface of the body is that the normal component of the total magnetic field vanishes. This boundary condition is in complete agreement with the one for a time-varying electromagnetic field on the surface of a perfectly conducting body. In the next section we shall calculate the magnetic flux through a loop coaxial with the scattering body (see Fig. 1).

III. Fractional Flux Through a Coaxial Loop

Referring to Fig. 1 we define Φ_o and $\Delta\Phi$ to be, respectively, the flux of the incident field $\underset{\sim}{H}_o$ and the scattered field $\underset{\sim}{H}^s$ through the loop. We now calculate the fractional flux, $-\Delta\Phi/\Phi_o$, as a function of z/a with b/a as a parameter. Using the second expression of (3) and $m = \pi a^2 H_o$ we have

$$-\frac{\Delta\Phi}{\Phi_o} = \frac{-\{\pi a^2 H_o/(4\pi)\}\int_0^b zr^{-3} 2\pi\rho\, d\rho}{\pi b^2 H_o}$$

$$= \frac{a^2}{2b^2} z \int_b^o \frac{\rho\, d\rho}{(\rho^2 + z^2)^{3/2}} = \frac{a^2}{2b^2}\{1 - [1 + (b/z)^2]^{-\frac{1}{2}}\} \tag{10}$$

which, of course, can be expressed in terms of the solid angle, Ω, subtended by the loop at the origin, i.e.,

$$-\frac{\Delta\Phi}{\Phi_o} = \frac{1}{\pi b^2 H_o}(\pi a^2 H_o)\left(\frac{\Omega}{4\pi}\right) = \frac{a^2}{b^2}\frac{\Omega}{4\pi} \quad . \tag{11}$$

For a loop of any shape situated anywhere outside the scattering body equation (11) can be easily generalized to be

$$-\frac{\Delta\Phi}{\Phi_o} = \frac{\pi a^2}{A}\frac{\Omega}{4\pi}$$

where A is the area of the loop projected onto a plane perpendicular to $\underset{\sim}{H}_o$.

Equation (10) is plotted in Fig. 3 for several representative values of b/a. The curve for b/a = 0 is actually the plot of $-H_z^s/H_o$ along the z axis. The total magnetic-field lines outside the scattering body are plotted in Fig. 4; this plot is obtained from (8) by appropriately choosing the values for the constant C.

IV. The Corresponding Electrostatic Problem

From the symmetry and the boundary conditions of two static problems, the longitudinal magnetic and the transverse electric, for a perfectly conducting body of revolution, one can deduce certain relationship between the solutions of the two problems.[6] Let V be the scalar potential of the transverse electric case and A_ϕ the ϕ-component of the vector potential of the longitudinal magnetic case. Then, from Ref. 6 we can immediately write down

$$V = -2E_o f(\rho,z)\cos\phi$$
$$A_\phi = H_o f(\rho,z) \tag{12}$$

where $\underset{\sim}{E}_o$ ($\underset{\sim}{H}_o$) is the external electric (magnetic) field perpendicular (parallel) to the axis of the body. At infinity f should vary as $\rho/2$ so that V and A_ϕ give, respectively, $\underset{\sim}{E}_o$ and $\underset{\sim}{H}_o$ there.

Let us now deduce f from the results of section II. From (3) with $m = \pi a^2 H_o$ and $\underset{\sim}{H} = \nabla \times (\underset{\sim}{e}_\phi A_\phi)$, we have

$$\frac{\partial f}{\partial z} = -\frac{a^2}{4}\frac{\rho}{r^3}$$
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f) = 1 + \frac{a^2}{4}\frac{z}{r^3}\ . \tag{13}$$

Integrating the second equation we get

$$f(\rho,z) = \frac{\rho}{2} - \frac{a^2}{4}\frac{z}{r\rho} + \frac{C'}{\rho} \tag{14}$$

which automatically satisfies the first equation of (13). To determine the integration constant C' we use the boundary condition that A_ϕ should vanish on the surface of the body defined by (9). We find that $C' = -a^2/4$. Thus,

$$f(\rho,z) = \frac{\rho}{2} - \frac{a^2}{4}\frac{z}{r\rho} - \frac{a^2}{4\rho}\ . \tag{15}$$

With (12) and (15) we can calculate the electric field everywhere. In particular, we shall calculate the normal component, E_n, of the field on the surface of the body. Referring to figure 1, we have

$$E_n = (E_\rho^2 + E_z^2)^{1/2} = 2E_o \cos\phi[(\frac{\partial f}{\partial \rho})^2 + (\frac{\partial f}{\partial z})^2]^{1/2} .$$

Evaluation of E_n on the surface of the body defined by (9) gives

$$E_n = 2E_o(\rho/a)\cos\phi(4 - 3\rho^2/a^2)^{1/2} \quad (16)$$

and the induced surface charge density σ is obtained from $\sigma = \varepsilon E_n$. From σ one can deduce the induced surface current density K_ϕ in the longitudinal magnetic case, since

$$K_\phi = \underset{\sim}{e}_\phi \cdot (\underset{\sim}{n} \times \underset{\sim}{H}) = (\underset{\sim}{e}_\phi \times \underset{\sim}{n}) \cdot (\nabla \times \underset{\sim}{A})$$

$$= H_o(\underset{\sim}{e}_\phi \times \underset{\sim}{n}) \cdot \nabla \times (\underset{\sim}{e}_\phi f) = - H_o \underset{\sim}{n} \cdot \nabla f ,$$

and

$$\sigma = \varepsilon E_n = - \varepsilon \underset{\sim}{n} \cdot \nabla V = 2\varepsilon E_o \cos\phi \, \underset{\sim}{n} \cdot \nabla f$$

from which

$$\frac{K_\phi}{H_o} = - \frac{\sigma}{2\varepsilon E_o \cos\phi} . \quad (17)$$

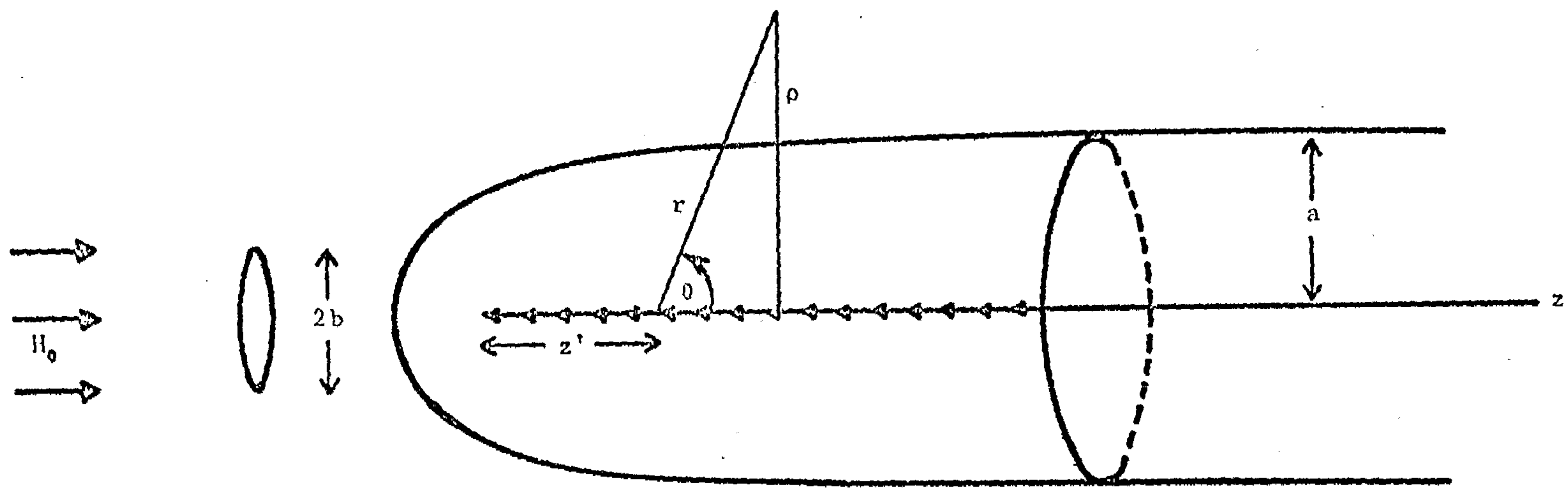

Figure 1. Geometry of the problem.

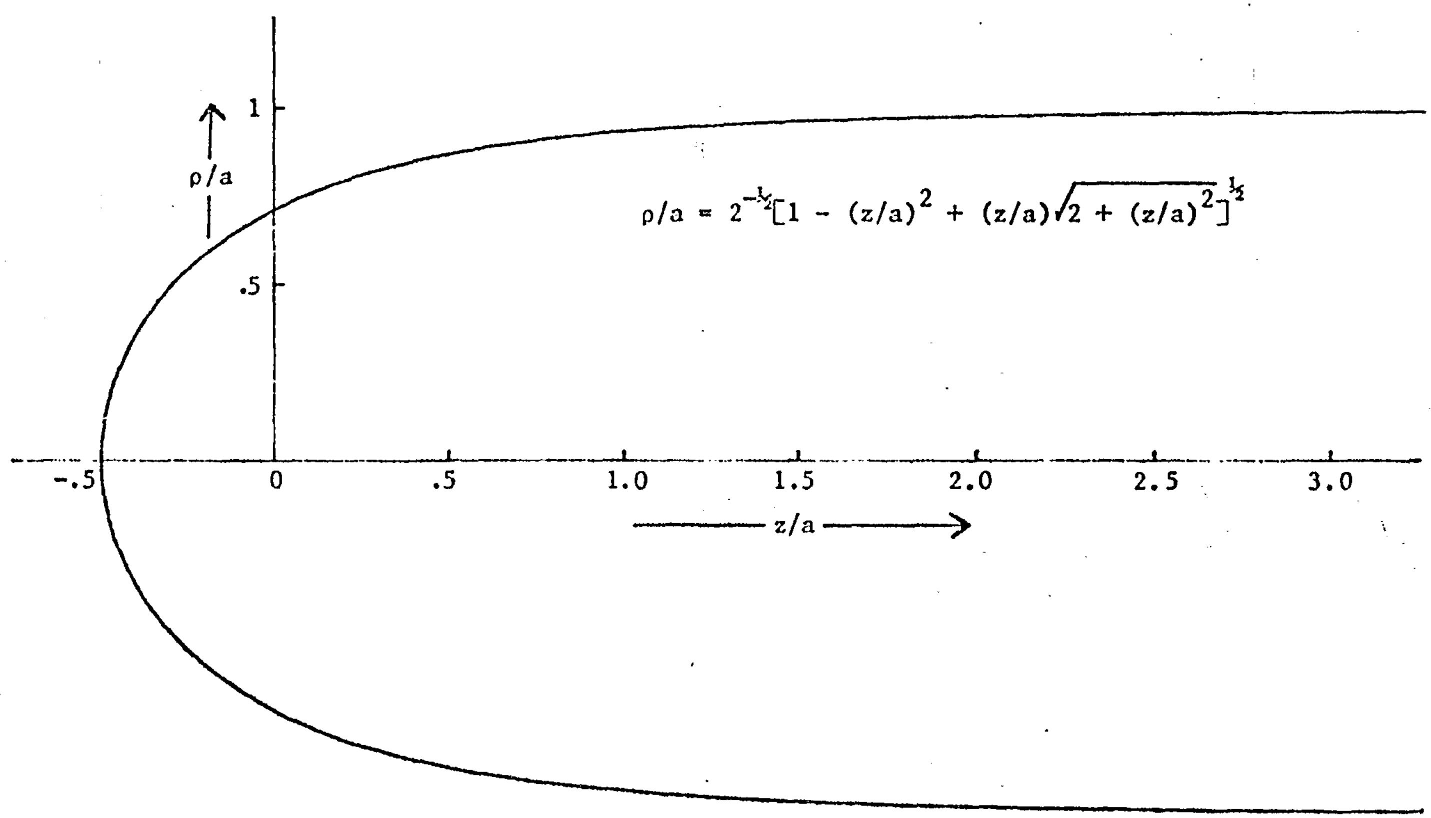

Figure 2. Profile of the scatterer.

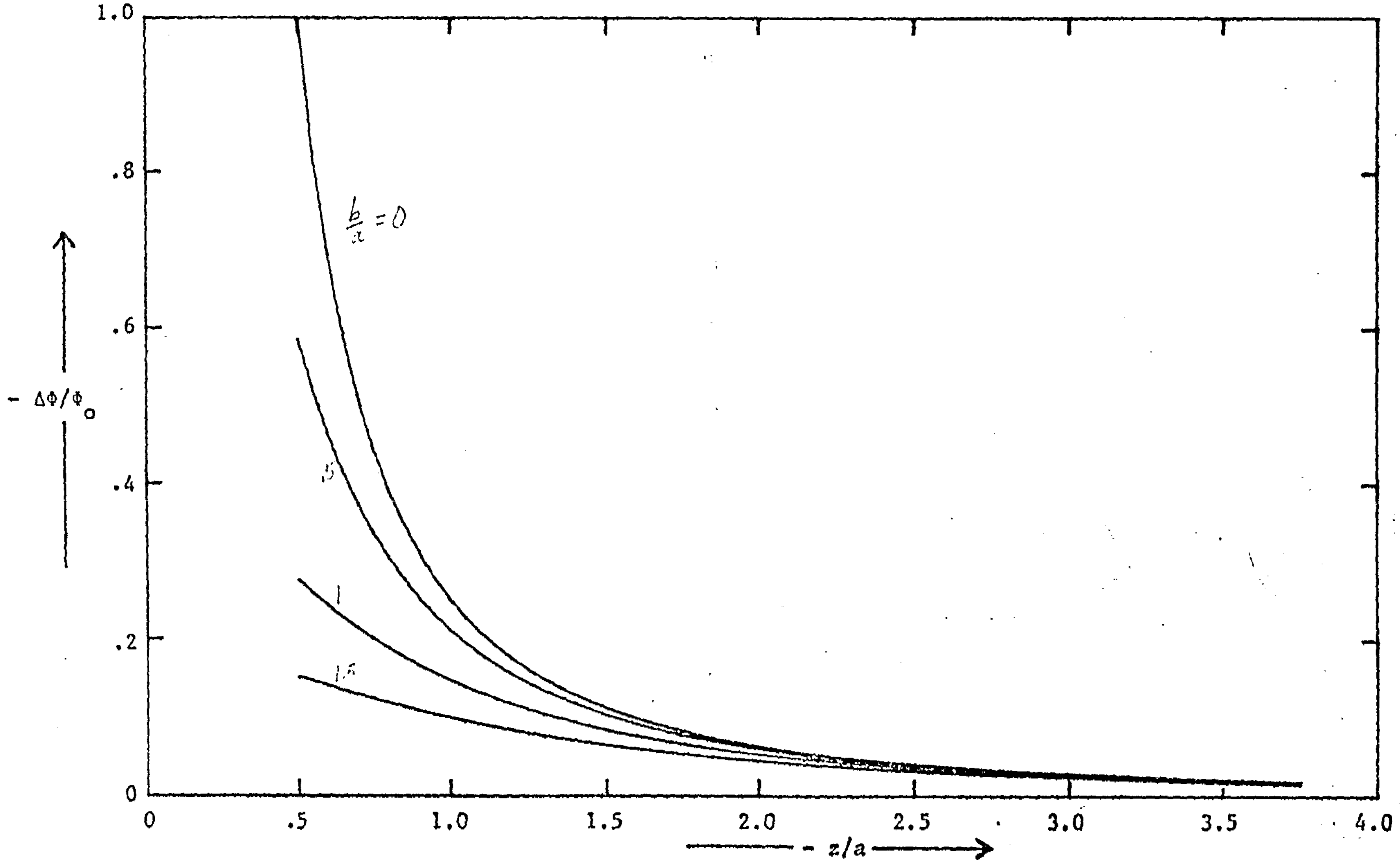

Figure 3. Fractional flux through a coaxial loop versus distance.

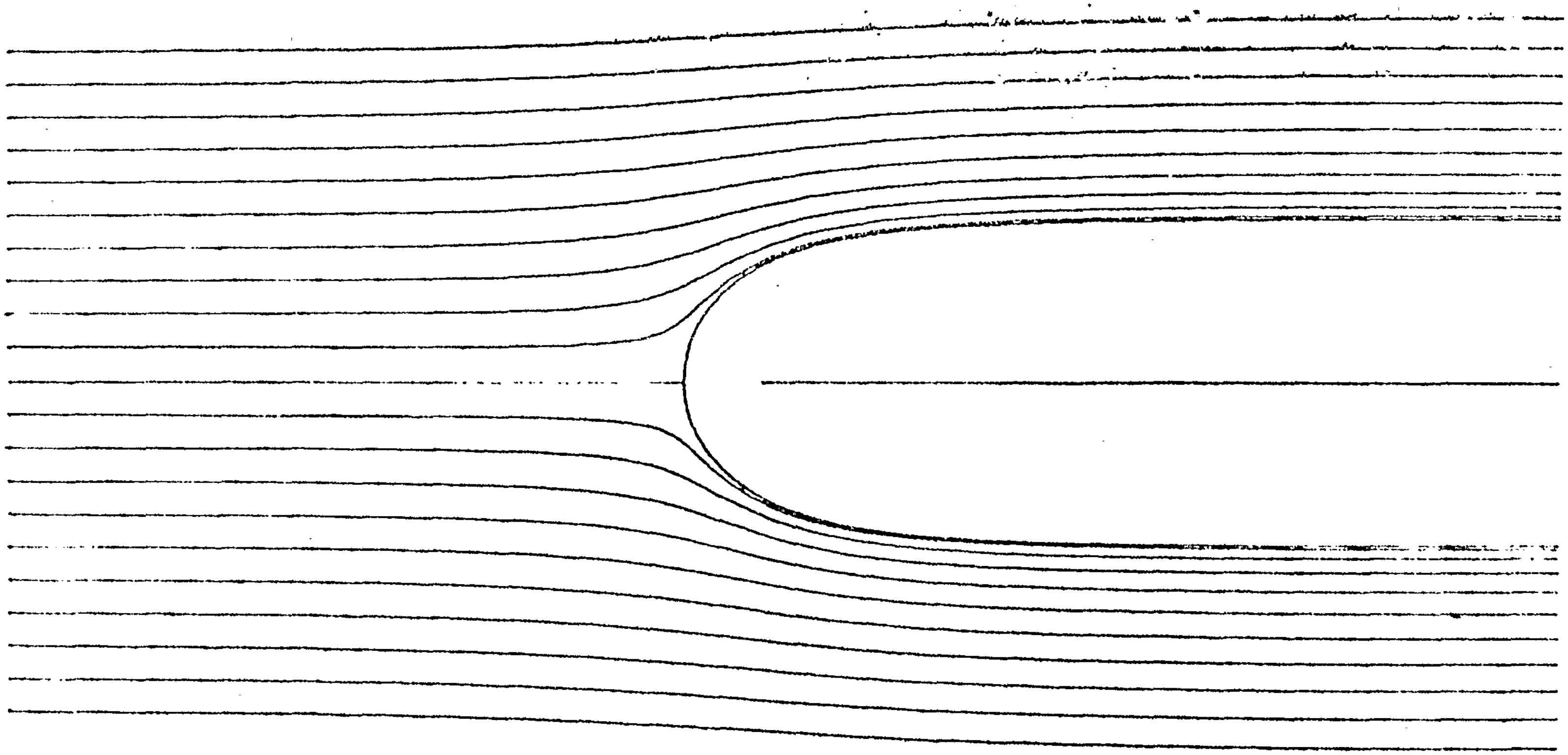

Figure 4. Magnetic field lines outside the scatterer.

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