Sensor and Simulation Notes

Note 511

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Mathematical Structure and Analysis of the Inductive and Electrostatic Fields of an Impulse Radiating Antenna

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Abstract

Impulse Radiating Antennas are noted for their large on-axis radiation field. Little attention has been paid to their non-radiation inductive and electrostatic components which become dominant at small ranges. These fields may become important sources of electromagnetic interference electronic equipment located at small ranges. This paper provides a detailed derivation of the inductive and electrostatic fields in the frequency domain for a simplified model of an Impulse Radiating Antenna (IRA), expressed in a spherical coordinate system. This derivation emphasizes meticulous execution in order to avoid apparent but, in practice, non-existing singularities. Such situations arise from the use of derived variables that appear in the denominator of transcendental functions.

1. Introduction

In a recent paper Kohlberg and Baum [1] derived expressions of the total electromagnetic field—radiation, induction, and electrostatic components for a simplified version of an IRA. Earlier, Baum developed an asymptotic expansion of the early-time field parallel to the aperture normal, and his results were focused on small angles about boresite [2]. Mikheev et. al. presented results for near and far zones, but their results in the near zone are limited to a cylindrical volume whose diameter is that of the reflector antenna [3]. Our results can be applied for all angles in the forward direction for ranges much greater than the appropriate linear dimension of the aperture (e.g., diameter for circular aperture, side for square aperture)

As shown in Figure 1, a source of uniform surface current pointing in the *z*-direction is assumed across a square aperture of side *D*. Contributions from the pre-pulse are not considered, and energy spillover from the reflector is not accounted for.

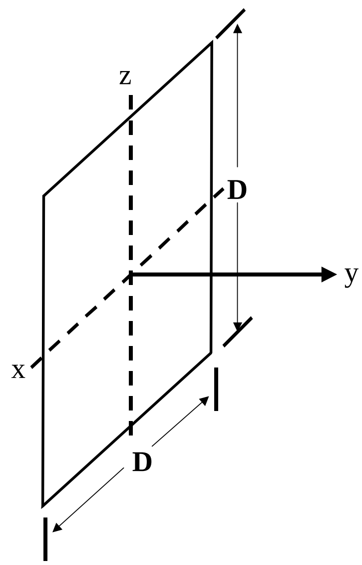

Figure 1. Geometry of Radiating Source

Baum has shown that this source produces uncertainties of about a factor of two because of the inhomogeneous Transverse Electromagnetic (TEM) wave launched on the aperture by the TEM transmission (conical with reflector). Other conditions for use of this model require many wavelengths across the aperture to minimize blockage of the feed arms, and fields outside the aperture, but on the aperture plane, are negligible. The geometry for the field calculations is shown in Figure 2.

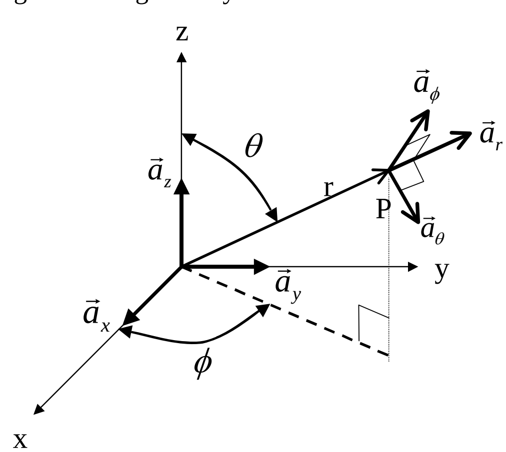

Figure 2. Geometry for Calculating Field at Point, P

The components of the electric field in the frequency domain are [1]

Electrostatic

$$\tilde{\vec{E}}_{el} = \left(-\frac{2c^2(P+\Psi)}{j\omega r^3}\vec{a}_r + \frac{c^2}{j\omega r^3}\frac{\partial(P+\Psi)}{\partial\theta}\vec{a}_\theta + \frac{c^2}{j\omega r^3}\frac{1}{\sin\theta}\frac{\partial(P+\Psi)}{\partial\phi}\vec{a}_\phi \right)\exp(-jkr) \qquad (1)$$

Induction

$$\tilde{\vec{E}}_{ind} = -\left(\frac{c\Psi}{r^2}\vec{a}_r + \frac{c}{r^2}\frac{\partial P}{\partial\theta}\vec{a}_\theta + \frac{c}{r^2}\frac{1}{\sin\theta}\frac{\partial P}{\partial\phi}\vec{a}_\phi \right)\exp(-jkr) \qquad (2)$$

Radiation

$$\tilde{\vec{E}}_{rad} = -j\omega(FQ)\vec{a}_\theta = \frac{-j\omega Q}{r}\exp(-jkr)\,\vec{a}_\theta \qquad (3)$$

Reference 1 dealt with determining the radiation term in the time domain. This paper provides a detailed computation of the electrostatic and induction terms, equations (1) and (2), in the frequency domain. As the reader will note, there is a substantial amount of analytical detail required and a demonstrations that resulting functions are analytic and well behaved.

2. Basic Definitions

Using the notation of Ref. 1 we have

$$T_x = \alpha\tau_D = \sin\theta\cos\phi\tau_D \qquad (4)$$

$$T_z = \beta\tau_D = \cos\theta\tau_D \qquad (5)$$

$$\tau_D = \frac{D}{2c} \qquad (6)$$

$$C = \frac{\mu\tilde{J}(\omega)D^2}{4\pi} \qquad (7)$$

$$\tilde{G}(\theta,\phi) = \left(\frac{\sin\omega T_x}{\omega T_x}\right)\left(\frac{\sin\omega T_z}{\omega T_z}\right) = \tilde{G}_x(\theta,\varphi)\tilde{G}_z(\theta,\varphi) \qquad (8)$$

$$\tilde{G}_x(\theta,\phi) \equiv \left(\frac{\sin\omega T_x}{\omega T_x}\right) \qquad (9)$$

$$\tilde{G}_z(\theta,\phi) \equiv \left(\frac{\sin\omega T_z}{\omega T_z}\right) \qquad (10)$$

$$P(\theta,\phi) = C\cos\theta\tilde{G}(\theta,\phi) \qquad (11)$$

$$Q(\theta,\phi) = -C\sin\theta\tilde{G}(\theta,\phi) \qquad (12)$$

$$\Psi(\theta,\phi) = \frac{1}{\sin\theta}\frac{\partial(\sin\theta\, Q(\theta,\varphi))}{\partial\theta} \equiv F_1 \qquad (13)$$

3. Calculation of Required Functions

Examination of equations (1) and (2) shows that in addition to $\Psi(\theta,\phi)$, the following functions need to be expressed in terms of basic functions: $\tilde{G}_x(\theta,\phi)$ and $\tilde{G}_z(\theta,\phi)$.

$$F_2 \equiv \frac{\partial P}{\partial \theta} \tag{14}$$

$$F_3 \equiv \frac{\partial P}{\partial \phi} \tag{15}$$

$$F_4 \equiv \frac{\partial \Psi}{\partial \phi} \tag{16}$$

$$F_5 \equiv \frac{\partial \Psi}{\partial \theta} \tag{17}$$

3.1 Calculation of F_2

From equation (11) we have

$$P(\theta,\phi) = C\cos\theta\tilde{G}(\theta,\phi) \tag{18}$$

$$\frac{\partial P}{\partial \theta} = -C\sin\theta\tilde{G}(\theta,\phi) + C\cos\theta\frac{\partial\tilde{G}(\theta,\phi)}{\partial\theta} \tag{19}$$

$$\tilde{G}(\theta,\phi) = \left(\frac{\sin\omega T_x}{\omega T_x}\right)\left(\frac{\sin\omega T_z}{\omega T_z}\right) = \tilde{G}_x(\theta,\varphi)\tilde{G}_z(\theta,\varphi) \tag{20}$$

We define the following time parameters:

$$T_x = \alpha\tau_D = \sin\theta\cos\phi\tau_D \tag{21}$$

$$T_z = \beta\tau_D = \cos\theta\tau_D \tag{22}$$

Using equation (20) there results

$$\frac{\partial\tilde{G}}{\partial\theta} = \tilde{G}_z\frac{\partial\tilde{G}_x}{\partial\theta} + \tilde{G}_x\frac{\partial\tilde{G}_z}{\partial\theta} \tag{23}$$

We first compute

$$\frac{\partial\tilde{G}_x}{\partial\theta} = \frac{\partial\tilde{G}_x}{\partial T_x}\frac{\partial T_x}{\partial\theta} = \cos\theta\cos\phi\tau_D\frac{\partial\tilde{G}_x}{\partial T_x}$$

$$\frac{\partial\tilde{G}_x}{\partial\theta} = \frac{\partial\tilde{G}_x}{\partial T_x}\frac{\partial T_x}{\partial\theta} = \cos\theta\cos\varphi\tau_D\frac{\partial\tilde{G}_x}{\partial T_x}$$

$$\frac{\partial \tilde{G}_x}{\partial T_x} = \frac{\omega^2 T_x \cos\omega T_x - \omega \sin\omega T}{(\omega T_x)^2} = \frac{\omega^2 T_x \cos\omega T_x}{(\omega T_x)^2} - \frac{\omega \sin\omega T}{(\omega T_x)^2} = \frac{\cos\omega T_x}{T_x} - \frac{\sin\omega T_x}{(\omega T_x) T_x}$$

$$\frac{\partial \tilde{G}_x}{\partial T_x} = \frac{1}{T_x}\left(\cos\omega T_x - \frac{\sin\omega T_x}{(\omega T_x)}\right) = \frac{1}{T_x}\left(\cos\omega T_x - \tilde{G}_x\right)$$

$$\frac{\partial \tilde{G}_x}{\partial \theta} = \frac{\cos\theta\cos\phi\tau_D}{T_x}\left(\cos\omega T_x - \tilde{G}_x\right) = \frac{\cos\theta}{\sin\theta}\left(\cos\omega T_x - \tilde{G}_x\right) \qquad (24)$$

Using the foregoing techniques we easily derive

$$\frac{\partial \tilde{G}_z}{\partial \theta} = \frac{\partial \tilde{G}_z}{\partial T_z}\frac{\partial T_z}{\partial \theta} = -\sin\theta\tau_D \frac{\partial \tilde{G}_z}{\partial T_z}$$

$$\frac{\partial \tilde{G}_z}{\partial T_z} = \frac{1}{T_z}\left(\cos\omega T_z - \frac{\sin\omega T_z}{(\omega T_z)}\right) = \frac{1}{T_z}\left(\cos\omega T_z - \tilde{G}_z\right)$$

$$\frac{\partial \tilde{G}_z}{\partial \theta} = \frac{-\sin\theta}{\cos\theta}\left(\cos\omega T_z - \tilde{G}_z\right) \qquad (25)$$

Putting all the terms together we deduce

$$\frac{\partial P}{\partial \theta} = -C\sin\theta\tilde{G} + C\cos\theta\frac{\partial \tilde{G}}{\partial \theta}$$

$$\frac{\partial P}{\partial \theta} = -C\sin\theta\tilde{G}_x\tilde{G}_z + C\cos\theta\tilde{G}_z\left(\frac{\cos\theta}{\sin\theta}\left(\cos\omega T_x - \tilde{G}_x\right)\right) - C\cos\theta\tilde{G}_x\left(\frac{\sin\theta}{\cos\theta}\left(\cos\omega T_z - \tilde{G}_z\right)\right)$$

$$F_2 = \frac{\partial P}{\partial \theta} = -C\sin\theta\tilde{G}_x\tilde{G}_z + C\tilde{G}_z\frac{\cos^2\theta}{\sin\theta}\left(\left(\cos\omega T_x - \tilde{G}_x\right)\right) - C\sin\theta\tilde{G}_x\left(\left(\cos\omega T_z - \tilde{G}_z\right)\right) \qquad (26a)$$

$$F_2 = \frac{\partial P}{\partial \theta} = C\tilde{G}_z\frac{\cos^2\theta}{\sin\theta}\cos\omega T_x - C\tilde{G}_z\tilde{G}_x\frac{\cos^2\theta}{\sin\theta} - C\sin\theta\tilde{G}_x\cos\omega T_z \qquad (26b)$$

3.2 Calculation of F_3

Starting with equation (20) in equation (18) we have

$$P = C\cos\theta\tilde{G}_x\tilde{G}_z$$

$$\frac{\partial P}{\partial \phi} = C\cos\theta\tilde{G}_z\frac{\partial \tilde{G}_x}{\partial \phi}$$

$$T_x = \alpha\tau_D = \sin\theta\cos\phi\tau_D$$

$$\frac{\partial \tilde{G}_x}{\partial \phi} = \frac{\partial \tilde{G}_x}{\partial T_x}\frac{\partial T_x}{\partial \phi} = -\sin\theta\sin\phi\tau_D\frac{\partial \tilde{G}_x}{\partial T_x}$$

$$\frac{\partial \tilde{G}_x}{\partial T_x} = \frac{1}{T_x}\left(\cos \omega T_x - \frac{\sin \omega T_x}{(\omega T_x)}\right) = \frac{1}{T_x}\left(\cos \omega T_x - \tilde{G}_x\right)$$

$$\frac{\partial \tilde{G}_x}{\partial \phi} = \frac{-\sin\theta \sin\phi}{\sin\theta \cos\phi}\left(\cos \omega T_x - \tilde{G}_x\right) = -\tan\phi\left(\cos \omega T_x - \tilde{G}_x\right)$$

$$F_3 = \frac{\partial P}{\partial \phi} = -\cos\theta \tan\phi C\tilde{G}_z\left(\cos \omega T_x - \tilde{G}_x\right) \tag{27}$$

3.3 Calculation of F_1

We have

$$\Psi = \frac{1}{\sin\theta}\frac{\partial(\sin\theta Q)}{\partial\theta}$$

$$Q = -C\sin\theta\tilde{G}$$

$$\Psi = \frac{-C}{\sin\theta}\frac{\partial(\sin^2\theta\tilde{G})}{\partial\theta} = \frac{-C}{\sin\theta}\left(2\sin\theta\cos\theta\tilde{G} + \sin^2\theta\frac{\partial\tilde{G}}{\partial\theta}\right)$$

$$\Psi = \frac{-C}{\sin\theta}\left(2\sin\theta\cos\theta\tilde{G} + \sin^2\theta\frac{\partial\tilde{G}}{\partial\theta}\right) = -2C\cos\theta\tilde{G} - C\sin\theta\frac{\partial\tilde{G}}{\partial\theta}$$

Using the expressions

$$\frac{\partial \tilde{G}_x}{\partial \theta} = \frac{\cos\theta}{\sin\theta}\left(\cos \omega T_x - \tilde{G}_x\right)$$

$$\frac{\partial \tilde{G}_z}{\partial \theta} = \frac{-\sin\theta}{\cos\theta}\left(\cos \omega T_z - \tilde{G}_z\right)$$

we initially deduce

$$\Psi = -2C\cos\theta\tilde{G} - C\sin\theta\left(\tilde{G}_z\frac{\cos\theta}{\sin\theta}\left(\cos \omega T_x - \tilde{G}_x\right) - \tilde{G}_x\frac{\sin\theta}{\cos\theta}\left(\cos \omega T_z - \tilde{G}_z\right)\right) \tag{28a}$$

$$\Psi = -C(\cos\theta + \frac{\sin^2\theta}{\cos\theta})\tilde{G} - C\tilde{G}_z\cos\theta\cos\omega T_x + C\tilde{G}_x\frac{\sin^2\theta}{\cos\theta}\cos\omega T_z \tag{28b}$$

Combining terms yields the final result

$$F_1 = \Psi = -C\frac{1}{\cos\theta}\tilde{G} - C\tilde{G}_z\cos\theta\cos\omega T_x + C\tilde{G}_x\frac{\sin^2\theta}{\cos\theta}\cos\omega T_z \tag{28c}$$

3.4 Calculation of F_4

We first write

$$\Psi = -2C\cos\theta\tilde{G} - C\sin\theta\frac{\partial\tilde{G}}{\partial\theta}$$

There results

$$\frac{\partial\Psi}{\partial\phi} = -2C\cos\theta\frac{\partial\tilde{G}}{\partial\phi} - C\sin\theta\frac{\partial^2\tilde{G}}{\partial\theta\partial\phi}$$

$$\frac{\partial\tilde{G}}{\partial\phi} = \tilde{G}_z\frac{\partial\tilde{G}_x}{\partial\phi}$$

$$\frac{\partial\Psi}{\partial\phi} = -2C\cos\theta\tilde{G}_z\frac{\partial\tilde{G}_x}{\partial\phi} - C\sin\theta\frac{\partial}{\partial\theta}\left(\tilde{G}_z\frac{\partial\tilde{G}_x}{\partial\phi}\right)$$

$$\frac{\partial}{\partial\theta}\left(\tilde{G}_z\frac{\partial\tilde{G}_x}{\partial\phi}\right) = \frac{\partial\tilde{G}_z}{\partial\theta}\frac{\partial\tilde{G}_x}{\partial\phi} + \tilde{G}_z\frac{\partial^2\tilde{G}_x}{\partial\theta\partial\phi}$$

$$\frac{\partial\tilde{G}_x}{\partial\phi} = -\tan\phi\left(\cos\omega T_x - \tilde{G}_x\right)$$

$$\frac{\partial\tilde{G}_z}{\partial\theta} = \frac{-\sin\theta}{\cos\theta}\left(\cos\omega T_z - \tilde{G}_z\right)$$

$$\frac{\partial\tilde{G}_x}{\partial\theta} = \frac{\cos\theta}{\sin\theta}\left(\cos\omega T_x - \tilde{G}_x\right)$$

$$\frac{\partial^2\tilde{G}_x}{\partial\theta\partial\phi} = \frac{\partial}{\partial\theta}\left(\frac{\partial\tilde{G}_x}{\partial\phi}\right) = \frac{\partial}{\partial\theta}\left(-\tan\phi\left(\cos\omega T_x - \tilde{G}_x\right)\right)$$

$$\frac{\partial^2\tilde{G}_x}{\partial\theta\partial\phi} = -\tan\phi\frac{\partial}{\partial\theta}\left(\cos\omega T_x - \tilde{G}_x\right) = -\tan\phi\left(-\omega\sin\omega T_x - \frac{\partial\tilde{G}_x}{\partial T_x}\right)\frac{\partial T_x}{\partial\theta}$$

$$\frac{\partial^2\tilde{G}_x}{\partial\theta\partial\phi} = -\tan\phi\left(-\omega\sin\omega T_x - \frac{1}{T_x}\left(\cos\omega T_x - \tilde{G}_x\right)\right)\cos\theta\cos\phi\tau_D$$

$$\frac{\partial^2\tilde{G}_x}{\partial\theta\partial\phi} = \cos\theta\sin\phi\tau_D\left(\frac{\omega^2T_x^2\sin\omega T_x}{\omega T_x^2} + \frac{1}{T_x}\cos\omega T_x - \tilde{G}_x\right)$$

$$\frac{\partial^2\tilde{G}_x}{\partial\theta\partial\phi} = \cot\theta\tan\phi\omega^2T_x^2\tilde{G}_x + \cot\theta\tan\phi\cos\omega T_x - \cot\theta\tan\phi\tilde{G}_x$$

Putting everything together we obtain

$$\frac{\partial^2\tilde{G}}{\partial\phi\partial\theta} = \frac{\partial}{\partial\theta}\left(\tilde{G}_z\frac{\partial\tilde{G}_x}{\partial\phi}\right) = \frac{\partial\tilde{G}_z}{\partial\phi}\frac{\partial\tilde{G}_x}{\partial\theta} + \tilde{G}_z\frac{\partial^2\tilde{G}_x}{\partial\theta\partial\phi}$$

$$= \tan\phi\left(\tan\theta\left[\left(\cos\omega T_z - \tilde{G}_z\right)\left(\cos\omega T_x - \tilde{G}_x\right)\right]\right) + \tan\phi\left(\cot\theta\left[\left(\omega^2T_x^2 - 1\right)\tilde{G} + \cos\omega T_x\tilde{G}_z\right]\right)$$

There results

$$\frac{\partial \Psi}{\partial \phi} = -2C\cos\theta \tilde{G}_z \frac{\partial \tilde{G}_x}{\partial \phi} - C\sin\theta \frac{\partial}{\partial \theta}\left(\tilde{G}_z \frac{\partial \tilde{G}_x}{\partial \phi}\right)$$

$$\frac{\partial \Psi}{\partial \phi} = -2C\cos\theta \tilde{G}_z \left(-\tan\phi\left[\cos\omega T_x - \tilde{G}_x\right]\right)$$
$$-C\sin\theta\begin{pmatrix}\tan\phi\left(\tan\theta\left[\left(\cos\omega T_z - \tilde{G}_z\right)\left(\cos\omega T_x - \tilde{G}_x\right)\right]\right) \\ +\tan\phi\left(\cot\theta\left[\left(\omega^2 T_x^2 - 1\right)\tilde{G} + \cos\omega T_x \tilde{G}_z\right]\right)\end{pmatrix}$$

$$\begin{aligned} F_4 = \frac{\partial \Psi}{\partial \phi} &= -C\sec\theta\tan\phi\left(\cos\omega T_x - \tilde{G}_x\right)\left(\cos\omega T_z - \tilde{G}_z\right) \\ &+ C\cos\theta\tan\phi\cos\omega T_z\left(\cos\omega T_x - \tilde{G}_x\right) \\ &- C\cos\theta\tan\phi\,\omega^2 T_x^2 \tilde{G} \end{aligned} \quad (29)$$

3.5 Calculation of F_5

We have

$$\Psi = -2C\cos\theta \tilde{G} - C\sin\theta \frac{\partial \tilde{G}}{\partial \theta}$$

$$\frac{\partial \Psi}{\partial \theta} = 2C\sin\theta \tilde{G} - 2C\cos\theta \frac{\partial \tilde{G}}{\partial \theta} - C\cos\theta \frac{\partial \tilde{G}}{\partial \theta} - C\sin\theta \frac{\partial^2 \tilde{G}}{\partial \theta^2}$$

$$\frac{\partial \Psi}{\partial \theta} = 2C\sin\theta \tilde{G} - 3C\cos\theta \frac{\partial \tilde{G}}{\partial \theta} - C\sin\theta \frac{\partial^2 \tilde{G}}{\partial \theta^2}$$

$$\frac{\partial \tilde{G}}{\partial \theta} = \tilde{G}_x \frac{\partial \tilde{G}_z}{\partial \theta} + \tilde{G}_z \frac{\partial \tilde{G}_x}{\partial \theta}$$

$$\frac{\partial^2 \tilde{G}}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial \tilde{G}}{\partial \theta}\right) = \tilde{G}_z \frac{\partial^2 \tilde{G}_x}{\partial \theta^2} + 2\frac{\partial \tilde{G}_x}{\partial \theta}\frac{\partial \tilde{G}_z}{\partial \theta} + \tilde{G}_x \frac{\partial^2 \tilde{G}_z}{\partial \theta^2}$$

$$\frac{\partial \tilde{G}_z}{\partial \theta} = \frac{-\sin\theta}{\cos\theta}\left(\cos\omega T_z - \tilde{G}_z\right) = -\tan\theta\left(\cos\omega T_z - \tilde{G}_z\right)$$

$$\frac{\partial \tilde{G}_x}{\partial \theta} = \frac{\cos\theta}{\sin\theta}\left(\cos\omega T_x - \tilde{G}_x\right) = \frac{1}{\tan\theta}\left(\cos\omega T_x - \tilde{G}_x\right)$$

$$\frac{\partial^2 \tilde{G}_x}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial \tilde{G}_x}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(\cot\theta\left[\cos\omega T_x - \tilde{G}_x\right]\right)$$

$$\frac{\partial^2 \tilde{G}_x}{\partial \theta^2} = -\left(\csc^2\theta + \cot^2\theta\right)\left(\cos\omega T_x - \tilde{G}_x\right) - \cot^2\theta\left(\omega T_x\right)^2 \tilde{G}_x$$

Similarly,

$$\frac{\partial^2 \tilde{G}_z}{\partial \theta^2} = -\left(\sec^2\theta + \tan^2\theta\right)\left(\cos\omega T_z - \tilde{G}_z\right) - \tan^2\theta\left(\omega T_z\right)^2 \tilde{G}_z$$

$$\frac{\partial \tilde{G}}{\partial \theta} = \tilde{G}_x\left(-\tan\theta\left[\cos\omega T_z - \tilde{G}_z\right]\right) + \tilde{G}_z\left(\cot\theta\left[\cos\omega T_x - \tilde{G}_x\right]\right)$$

$$\begin{aligned}\frac{\partial^2 \tilde{G}}{\partial \theta^2} = & -\tilde{G}_x\left(\sec^2\theta + \tan^2\theta\right)\left(\cos\omega T_z - \tilde{G}_z\right) - \tan^2\theta\left(\omega T_z\right)\tilde{G} \\ & + 2\left(\cos\omega T_x - \tilde{G}_x\right)\left(\cos\omega T_z - \tilde{G}_z\right) \\ & - \tilde{G}_z\left(\csc^2\theta + \cot^2\theta\right)\left(\cos\omega T_x - \tilde{G}_x\right) - \cot^2\theta\left(\omega T_x\right)\tilde{G}\end{aligned}$$

We finally obtain

$$\begin{aligned}F_5 = \frac{\partial \Psi}{\partial \theta} = & C\left(\frac{2\cos^2\theta - 1}{\sin\theta} - \frac{\sin\theta\left(1+\sin^2\theta\right)}{\cos^2\theta} + \omega^2\tau_D^2\sin^3\theta + \omega^2\tau_D^2\sin\theta\cos^2\theta\cos^2\phi\right)\tilde{G} \\ & - C\left(\sin\theta - \frac{\sin\theta\left(1+\sin^2\theta\right)}{\cos^2\theta}\right)\cos\omega T_z\tilde{G}_x \\ & - C\left(\frac{1}{\sin\theta}\right)\cos\omega T_x\tilde{G}_z + 2C\sin\theta\cos\omega T_x\cos\omega T_z\end{aligned} \tag{30}$$

4. Calculation of Induction and Electrostatic Fields

Using the general formulas of Section 1 and the details from Section 3 we obtain the following results for the induction field.

$$\tilde{\tilde{E}}_{ind} = -\left(\frac{c\Psi}{r^2}\vec{a}_r + \frac{c}{r^2}\frac{\partial P}{\partial \theta}\vec{a}_\theta + \frac{c}{r^2}\frac{1}{\sin\theta}\frac{\partial P}{\partial \phi}\vec{a}_\phi\right)\exp(-jkr) \tag{31}$$

$$\tilde{\tilde{E}}_{ind} = -\frac{c}{r^2}\left(\tilde{E}^r_{ind}\vec{a}_r + \tilde{E}^\theta_{ind}\vec{a}_\theta + \tilde{E}^\phi_{ind}\vec{a}_\phi\right)\exp(-jkr) \tag{32}$$

The scaled fields are

$$\tilde{E}^r_{ind} = \Psi = -C\frac{1}{\cos\theta}\tilde{G} - C\tilde{G}_z\cos\theta\cos\omega T_x + C\tilde{G}_x\frac{\sin^2\theta}{\cos\theta}\cos\omega T_z \tag{33}$$

$$\tilde{E}^\theta_{ind} = \frac{\partial P}{\partial \theta} = C\tilde{G}_z\frac{\cos^2\theta}{\sin\theta}\cos\omega T_x - C\tilde{G}_z\tilde{G}_x\frac{\cos^2\theta}{\sin\theta} - C\sin\theta\tilde{G}_x\cos\omega T_z \tag{34}$$

$$\tilde{E}^\phi_{ind} = \frac{1}{\sin\theta}\frac{\partial P}{\partial \phi} = -\cot\theta\tan\phi C\tilde{G}_z(\cos\omega T_x - \tilde{G}_x) \tag{35}$$

In a similar way we obtain

$$\tilde{\tilde{\vec{E}}}_{el} = \left(-\frac{2c^2(P+\Psi)}{j\omega r^3}\vec{a}_r + \frac{c^2}{j\omega r^3}\frac{\partial(P+\Psi)}{\partial\theta}\vec{a}_\theta + \frac{c^2}{j\omega r^3}\frac{1}{\sin\theta}\frac{\partial(P+\Psi)}{\partial\phi}\vec{a}_\phi \right)\exp(-jkr) \tag{36}$$

$$\tilde{\tilde{\vec{E}}}_{el} = \frac{c^2}{j\omega r^3}\left(\tilde{E}^r_{el}\vec{a}_r + \tilde{E}^\theta_{el}\vec{a}_\theta + \tilde{E}^\phi_{el}\vec{a}_\phi\right)\exp(-jkr) \tag{37}$$

$$\begin{aligned}\tilde{E}^r_{el} &= -2(P+\Psi)\\ &= -2C\cos\theta\tilde{G} + 2C\sec\theta\tilde{G} + 2C\tilde{G}_z\cos\theta\cos\omega T_x - 2C\tilde{G}_x\frac{\sin^2\theta}{\cos\theta}\cos\omega T_z\end{aligned} \tag{38}$$

$$\begin{aligned}\tilde{E}^\theta_{el} &= \frac{\partial(P+\Psi)}{\partial\theta} = \frac{\partial P}{\partial\theta} + \frac{\partial\Psi}{\partial\theta}\\ &= C\tilde{G}_z\frac{\cos^2\theta}{\sin\theta}\cos\omega T_x - C\tilde{G}_z\tilde{G}_x\frac{\cos^2\theta}{\sin\theta} - C\sin\theta\tilde{G}_x\cos\omega T_z +\\ &+ C\left(\frac{2\cos^2\theta-1}{\sin\theta} - \frac{\sin\theta\left(1+\sin^2\theta\right)}{\cos^2\theta} + \omega^2\tau_D^2\sin^3\theta + \omega^2\tau_D^2\sin\theta\cos^2\theta\cos^2\phi\right)\tilde{G}\\ &- C\left(\sin\theta - \frac{\sin\theta\left(1+\sin^2\theta\right)}{\cos^2\theta}\right)\cos\omega T_z\tilde{G}_x - \left(\frac{1}{\sin\theta}\right)C\cos\omega T_x\tilde{G}_z\\ &+ 2C\sin\theta\cos\omega T_x\cos\omega T_z\end{aligned} \tag{39}$$

$$\tilde{E}^\phi_{el} = \frac{1}{\sin\theta}\frac{\partial(P+\Psi)}{\partial\phi} = \frac{1}{\sin\theta}\frac{\partial P}{\partial\phi} + \frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\phi}$$

$$\begin{aligned}\tilde{E}^\phi_{el} &= -\cot\theta\tan\phi C\tilde{G}_z(\cos\omega T_x - \tilde{G}_x)\\ &- C\sec\theta\csc\theta\tan\phi\left(\cos\omega T_x - \tilde{G}_x\right)\left(\cos\omega T_z - \tilde{G}_z\right)\\ &+ C\cot\theta\tan\phi\cos\omega T_z\left(\cos\omega T_x - \tilde{G}_x\right)\\ &- C\cot\theta\tan\phi\omega^2 T_x^2\tilde{G}\end{aligned} \tag{40}$$

5. Analytical Properties of Induction and Electrostatic Fields

A cursory examination of the fields shows terms that individually blow up at the values $\theta = 0, \pi/2$ or $\phi = 0, \pi/2$. We now show that collectively there are cancellations that render the combined field well behaved functions of angles θ and ϕ.

5.1 Considerations Regarding F_2

The starting point is equation (26a)

$$F_2 = \frac{\partial P}{\partial\theta} = -C\sin\theta\tilde{G}_x\tilde{G}_z + C\tilde{G}_z\frac{\cos^2\theta}{\sin\theta}\left(\cos\omega T_x - \tilde{G}_x\right) - C\sin\theta\tilde{G}_x\left(\cos\omega T_z - \tilde{G}_z\right)$$

The potential troublesome looking spot is the second term which goes as

$$h_1 \to \frac{\cos^2\theta}{\sin\theta}\left(\cos\omega T_x - \tilde{G}_x\right)$$

We need to verify $h_1 < \infty$ as $\sin\theta \to 0$. Using the formulas

$$T_x = \sin\theta\cos\phi\tau_D \tag{41a}$$

$$\tilde{G}_x = \frac{\sin\omega T_x}{\omega T_x} \tag{41b}$$

$$\tilde{G}_x = \frac{\sin\omega T_x}{\omega T_x} \to 1 - \frac{(\omega T_x)^2}{6} \quad \text{as } \sin\theta \to 0 \tag{41c}$$

$$\cos\omega T_x \to 1 - \frac{(\omega T_x)^2}{2} \quad \text{as } \sin\theta \to 0 \tag{41d}$$

we deduce in the limit

$\sin\theta \to 0$:

$$h_1 \to \frac{1}{\sin\theta}\left(1 - \frac{(\omega T_x)^2}{2} - 1 + \frac{(\omega T_x)^2}{6}\right) \to \frac{\sin^2\theta}{\sin\theta} \to 0 \tag{42}$$

We conclude that there is no problem with the function F_2 .

5.2 Considerations Regarding F_3

The starting point is equation (27)

$$F_3 = \frac{\partial P}{\partial\phi} = -\cos\theta\tan\phi C\tilde{G}_z\left(\cos\omega T_x - \tilde{G}_x\right)$$

Since $\tan\phi = (\sin\phi / \cos\phi)$ potential troublesome looking term is the expression

$$h_2 \to \frac{\sin\phi}{\cos\phi}\left(\cos\omega T_x - \tilde{G}_x\right)$$

This must be examined in the limit as $\cos\phi \to 0$. We need to verify $h_2 < \infty$ as $\cos\phi \to 0$. Again, using equation (41) we arrive at the result

In limit

$\cos\phi \to 0$:

$$h_2 \to \frac{1}{\cos\phi}\left(1 - \frac{(\omega T_x)^2}{2} - 1 + \frac{(\omega T_x)^2}{6}\right) \to \frac{\cos^2\phi}{\cos\phi} \to 0 \tag{43}$$

We conclude that there is no problem with the function F_3 .

5.3 Considerations Regarding F_1

The starting point is equation (28)

$$F_1 = \Psi = -C\frac{1}{\cos\theta}\tilde{G} - C\tilde{G}_z\cos\theta\cos\omega T_x + C\tilde{G}_x\frac{\sin^2\theta}{\cos\theta}\cos\omega T_z$$

Combining terms shows that the potential troublesome looking spot is the expression

$$h_3 \to \frac{1}{\cos\theta}\tilde{G} - \tilde{G}_x\frac{\sin^2\theta}{\cos\theta}\cos\omega T_z \to \frac{1}{\cos\theta}\left(\tilde{G}_z - \sin^2\theta\cos\omega T_z\right)$$

This must be examined in the limit as $\cos\phi \to 0$. We need to verify $h_3 < \infty$ as $\cos\theta \to 0$. In this limit $\sin\theta \to 1$ and

$$h_3 \to \frac{1}{\cos\theta}\left(\tilde{G}_z - \cos\omega T_z\right)$$

Recalling,

$$T_z = \cos\theta\tau_D$$

$$\tilde{G}_z = \frac{\sin\omega T_z}{\omega T_z}$$

and using equations

$$\tilde{G}_z = \frac{\sin\omega T_z}{\omega T_z} \to 1 - \frac{(\omega T_z)^2}{6}, \quad \text{as } \cos\theta \to 0$$

$$\cos\omega T_z \to 1 - \frac{(\omega T_z)^2}{2}, \text{ as the function } \cos\theta \to 0$$

we get

$$h_3 \to \frac{1}{\cos\theta}\left(1 - \frac{(\omega T_z)^2}{2} - 1 + \frac{(\omega T_z)^2}{6}\right) \to \frac{\cos^2\theta}{\cos\theta} \to 0 \qquad (44)$$

We conclude that there is no problem with F_1.

5.4 Considerations Regarding F_4

The starting point is equation (29)

$$\begin{aligned} F_4 = \frac{\partial\Psi}{\partial\phi} &= -C\sec\theta\tan\phi\left(\cos\omega T_x - \tilde{G}_x\right)\left(\cos\omega T_z - \tilde{G}_z\right) \\ &+ C\cos\theta\tan\phi\cos\omega T_z\left(\cos\omega T_x - \tilde{G}_x\right) \\ &- C\cos\theta\tan\phi\,\omega^2 T_x^2\tilde{G} \end{aligned}$$

All potential troublesome looking spots deal with the $\tan\phi = \sin\phi/\cos\phi$ terms. The first two terms in the expression for F_4 vary as the previously examined function

$$h_2 \to \frac{\sin\phi}{\cos\phi}\left(\cos\omega T_x - \tilde{G}_x\right) \to \frac{\cos^2\phi}{\cos\phi} \to 0 \text{, as the function} \cos\phi \to 0$$

The last term in the expression for F_4 varies as

$$h_4 \to \tan\phi T_x^2 \to \frac{\cos^2\phi}{\cos\phi} \to 0 \text{ as } \cos\phi \to 0 \tag{45}$$

We conclude that there is no problem with the function F_1.

5.5 Considerations Regarding *F₅*

The starting point is equation (30)

$$F_5 = \frac{\partial\Psi}{\partial\theta} = C\left(\frac{2\cos^2\theta - 1}{\sin\theta} - \frac{\sin\theta\left(1+\sin^2\theta\right)}{\cos^2\theta} + \omega^2\tau_D^2\sin^3\theta + \omega^2\tau_D^2\sin\theta\cos^2\theta\cos^2\phi\right)\tilde{G}$$
$$-C\left(\sin\theta - \frac{\sin\theta\left(1+\sin^2\theta\right)}{\cos^2\theta}\right)\cos\omega T_z\tilde{G}_x$$
$$-C\left(\frac{1}{\sin\theta}\right)\cos\omega T_x\tilde{G}_z + 2C\sin\theta\cos\omega T_x\cos\omega T_z$$

Here there are two potential troublesome looking spots dealing the behavior as $\sin\theta \to 0$ and as $\cos\theta \to 0$ respectively. We'll handle one at a time. In the limit: $\sin\theta \to 0$, F_5 behaves as

$$h_5^1 \to \left(\frac{2\cos^2\theta - 1}{\sin\theta}\right)\tilde{G} - \left(\frac{1}{\sin\theta}\right)\cos\omega T_x\tilde{G}_z \to \frac{1}{\sin\theta}\left(\tilde{G} - \cos\omega T_x\tilde{G}_z\right)$$

Using $\tilde{G} = \tilde{G}_x\tilde{G}_z$ we get

$$h_5^{(1)} \to \frac{\tilde{G}_z}{\sin\theta}\left(\tilde{G}_x - \cos\omega T_x\right) \to \frac{\sin^2\theta}{\sin\theta} \to 0 \text{ In the limit } \sin\theta \to 0 \tag{46}$$

For the case $\cos\theta \to 0$, F_5 behaves as

$$h_5^{(2)} \to \left(-1 - \frac{2}{\cos^2\theta} + \omega^2\tau_D^2\right)\tilde{G} - \left(1 - \frac{2}{\cos^2\theta}\right)\cos\omega T_z\tilde{G}_x - \cos\omega T_x\tilde{G}_z + 2\cos\omega T_x\cos\omega T_z$$

The potential troublesome looking spot arises from the combination of terms

$$h_5^{(2)} \to \left(-\frac{2}{\cos^2\theta}\right)\tilde{G} - \left(-\frac{2}{\cos^2\theta}\right)\cos\omega T_z\tilde{G}_x \to \frac{2}{\cos^2\theta}\left(\cos\omega T_z\tilde{G}_x - \tilde{G}\right) \tag{47}$$

Using $\tilde{G} = \tilde{G}_x\tilde{G}_z$ we get

$$h_5^{(2)} \to \frac{2\tilde{G}_x}{\cos^2\theta}\left(\cos\omega T_z - \tilde{G}_z\right)$$

In the limit $\cos\theta \to 0$ we have

$$\tilde{G}_z = \frac{\sin\omega T_z}{\omega T_z} \to 1 - \frac{(\omega T_z)^2}{6} \quad \text{as } \cos\theta \to 0$$

$$\cos\omega T_z \to 1 - \frac{(\omega T_z)^2}{2} \quad \text{as } \cos\theta \to 0$$

$$\cos\omega T_z - \tilde{G}_z \to \frac{\omega^2 T_z^2}{3} \to \cos^2\theta$$

$$h_5^{(2)} \to 2\tilde{G}_x < \infty \tag{48}$$

We conclude that there is no problem with the function F_5 .

6. Conclusion

In this paper we derive detailed formulas in the frequency domain for the inductive and electrostatic components of a simplified model of an Impulse Radiating Antenna (IRA), expressed in a spherical coordinate system. These fields are the dominant ones at small ranges. These fields may become important sources of electromagnetic interference on electronic equipment located at small ranges. The mathematical expressions are analytic in the range of interest.

7. References

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