Sensor and Simulation Notes

Note 510

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Electrode Positions at a Dielectric Interface for Uniform Electric Field

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Abstract

This paper considers the placement of four electrodes at a dielectric interface (e.g., skin surface) to produce a nearly uniform electric field at a near-surface target (e.g., a tumor).

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1. Introduction

In some biomedical applications one desires to put a large electric field on a target near a skin surface. One can do this with two electrodes (needles) stradling the target, but the electric field near the target has significant nonuniformity. One may wish for a more uniform electric-field exposure of the target. So let us consider a 4-electrode system.

One approach is that of four wires as discussed in [1]. In that case consider the diagram in Fig. 1.1. Let the four wires be parallel to the z axis with equal radii r_e and with

$$(x, y) = (\pm a\, ,\ \pm b) \text{ (all combinations of + and -)} \tag{1.1}$$

These wires are considered infinitely long for the moment. With voltages $\pm V$ as indicated on the wires we find [1] that if we choose (for small r_e)

$$\frac{a}{b} = 3^{-1/2} \simeq 0.577\, ,\ \psi = 60^\circ \tag{1.2}$$

the electric field near $(x, y) = (0, 0)$ is oriented in the y direction. By symmetry all odd derivatives are zero, and also the second derivative is zero. This is a two-dimensional analog of a Helmholtz coil which is useful in EMP-simulator design.

The two-dimensional case, while giving a simple answer, is not the same as the three dimensional case.

As in Fig. 1.2, consider a three-dimensional geometry using spherical (or hemispherical) electrodes of small radius r_e . There are now two uniform, isotropic media separated by the $z = 0$ plane which forms a symmetry plane, the electric field on this plane being parallel to the plane. This simplifies the analysis, allowing us to calculate the fields as though the electrodes were in a uniform medium. The target is situated in medium 1 near $\vec{r} = 0$. This medium may be a conducting dielectric, but we can analyze this as an electrostatic problem. Medium two might be air, or some other medium one chooses to apply.

Referring back to Fig. 1.1, this gives the coordinates of our four spherical electrodes as seen normal to $z = 0$.

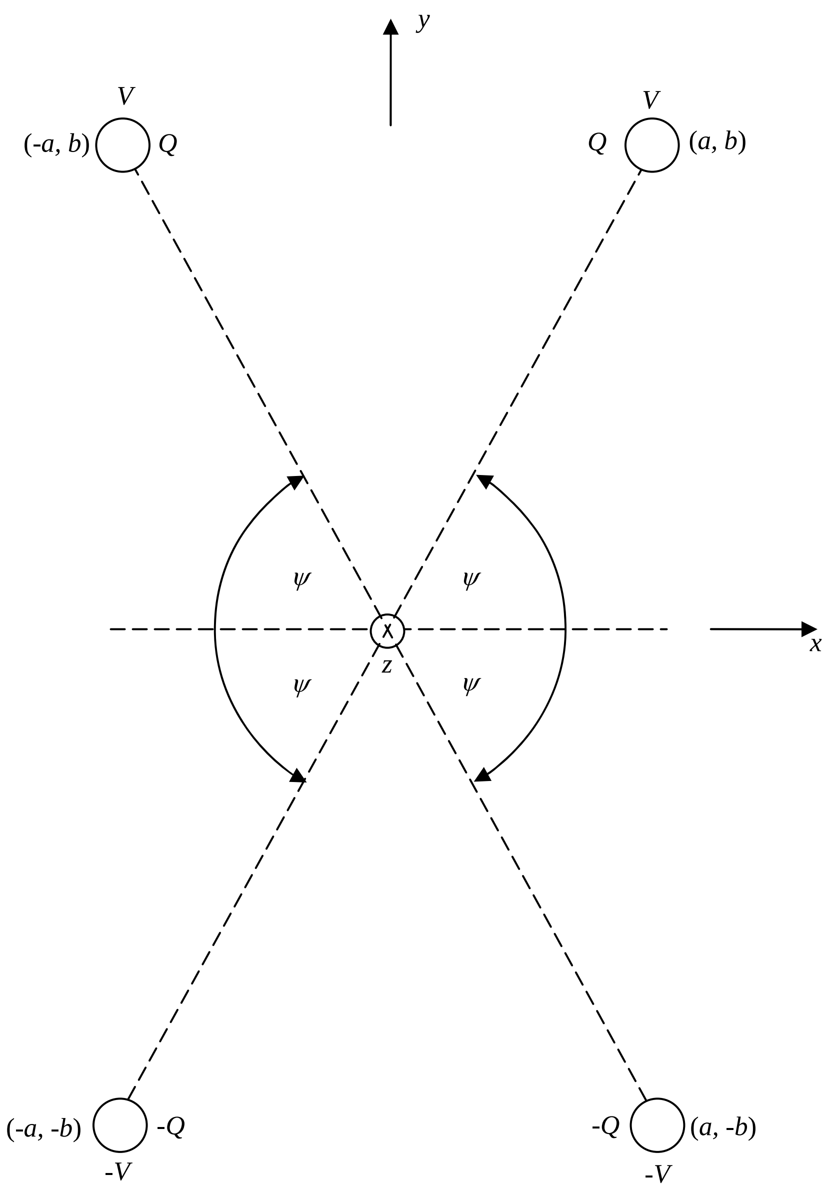

Fig. 1.1 Four-Electrode System

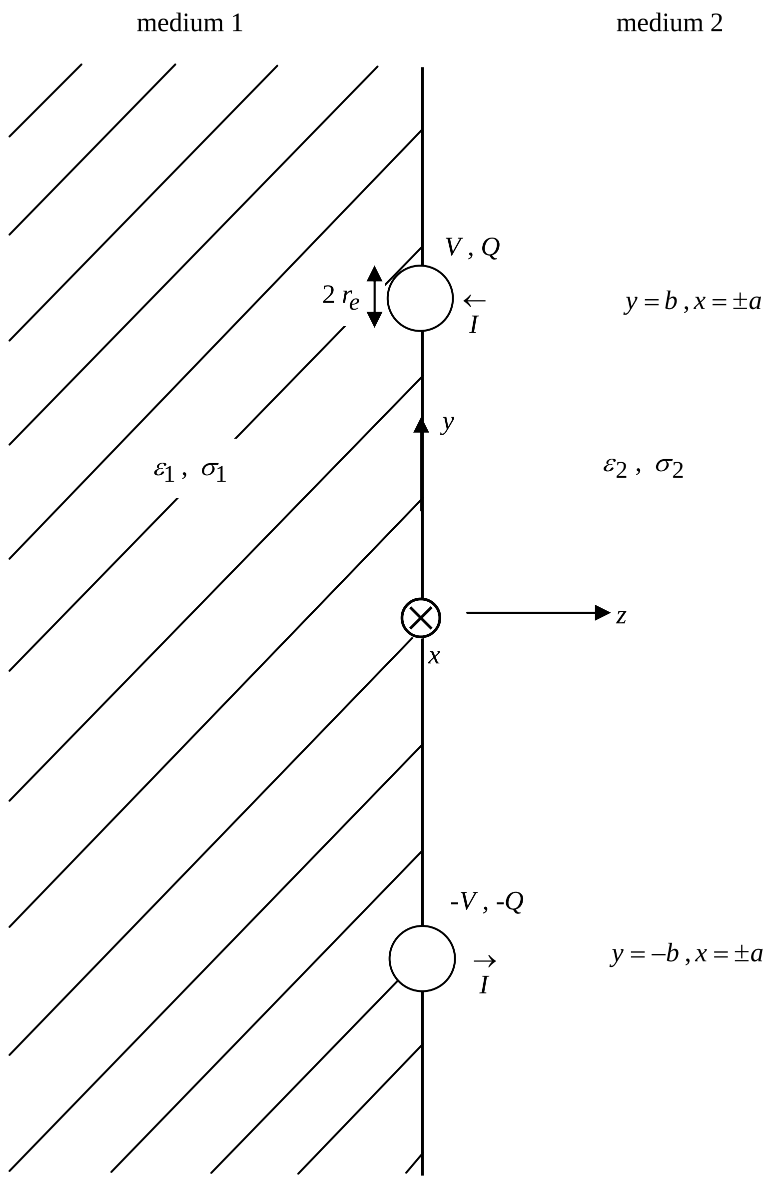

Fig 1.2 Four Electrode Three-Dimensional System: Side View

2. Field of Four Electrodes

Assuming small electrodes ($r_e << b$, a), we can write the electric field as a sum of the fields from the four electrodes. For large electrodes, the case of two electrodes is considered in [2].

Due to the symmetry of the geometry and excitation, the expressions simplify on the three axes: (x, 0, 0), (0, y, 0), and (0, 0, z). In these cases there is only a y component of the electric field. This effectively scalarizes the problem. By symmetry the first derivative (as well as all odd derivatives) is zero. So let us consider the second derivatives along the three axes as a measure of field nonuniformity.

3. z-Axis Field

Let us first consider the field along the z axis. This is

$$
\begin{aligned}
E_y &= -\frac{4Q}{4\pi\varepsilon}\frac{b}{\left[a^2 + b^2 + z^2\right]^{1/2}}\left[a^2 + b^2 + z^2\right]^{-1} = -\frac{2Q}{4\pi\varepsilon}\mathrm{X} \\
\mathrm{X} &= 2b\left[a^2 + b^2 + z^2\right]^{-3/2} \\
\mathrm{X} &= 2\sin(\psi)h^{-2} \qquad \text{at origin} \\
h &\equiv \left[a^2 + b^2\right]^{1/2}
\end{aligned}
\tag{3.1}
$$

Differentiating we have

$$
\begin{aligned}
\frac{d\mathrm{X}}{dz} &= -6bz\left[a^2 + b^2 + z^2\right]^{-5/2} \\
\frac{d^2\mathrm{X}}{dz^2} &= -6b\left[a^2 + b^2 + z^2\right]^{-5/2} + 30bz^2\left[a^2 + b^2 + z^2\right]^{-7/2} \\
\frac{d^2\mathrm{X}}{dz^2} &= -6\sin(\psi)h^{-4} \qquad \text{at origin}
\end{aligned}
\tag{3.2}
$$

From this we find that at $\vec{r} = 0$ (the origin) we have

$$
\frac{1}{E_y}\frac{\partial^2 E_y}{\partial z^2} = -3h^{-2} \tag{3.3}
$$

which is independent of ψ. We find that the electric field decreases for both positive and negative z, as expected. For finite h we cannot make the above expression zero. We must live with this, if the electrodes are all on the $z = 0$ plane.

4. *y*-Axis Field

Along the y axis we have

$$
\begin{aligned}
E_y &= -\frac{2Q}{4\pi\varepsilon}\frac{b-y}{\left[a^2+[b-y]^2\right]^{1/2}}\left[a^2+[b-y]^2\right]^{-1} \\
&\quad -\frac{2Q}{4\pi\varepsilon}\frac{b+y}{\left[a^2+[b+y]^2\right]^{1/2}}\left[a^2+[b+y]^2\right]^{-1} \\
&= -\frac{2Q}{4\pi\varepsilon}\,\mathrm{X} \\
\mathrm{X} &= [b-y]\left[a^2+[b-y]^2\right]^{-3/2} + [b+y]\left[a^2+[b+y]^2\right]^{-3/2}
\end{aligned}
\tag{4.1}
$$

Differentiating we have

$$
\begin{aligned}
\frac{d\mathrm{X}}{dy^2} &= -\left[a^2+[b-y]^2\right]^{-3/2} + 3[b-y]^2\left[a^2+[b-y]^2\right]^{-5/2} \\
&\quad + \left[a^2+[b+y]^2\right]^{-3/2} - 3[b+y]^2\left[a^2+[b+y]^2\right]^{-5/2} \\
\frac{d^2\mathrm{X}}{dy^2} &= -9[b-y]\left[a^2+[b-y]^2\right]^{-5/2} + 15[b-y]^3\left[a^2+[b-y]^2\right]^{-7/2} \\
&\quad -9\ [b+y]\left[a^2+[b+y]^2\right]^{-5/2} + 15[b+y]^3\left[a^2+[b+y]^2\right]^{-7/2} \\
\frac{d\mathrm{X}^2}{dy^2} &= -18bh^{-5} + 30b^3h^{-7} \qquad \text{at origin} \\
&= \left[-18\sin(\psi) + 30\sin^3(\psi)\right]h^{-4}
\end{aligned}
\tag{4.2}
$$

At the origin we have

$$
\frac{1}{E_y}\frac{\partial^2 E_y}{\partial y^2} = \left[-9 + 15\sin^2(\psi)\right]h^{-2} \tag{4.3}
$$

Since this is a function of ψ we can set this to zero at

$$
\sin(\psi) = \left[\frac{3}{5}\right]^{1/2}, \quad \psi \simeq 50.8^\circ \tag{4.4}
$$

For smaller ψ (4.3) is negative, while for larger ψ it is positive.

5. x = Axis Field

Along the x axis we have

$$
\begin{aligned}
E_y &= -\frac{2Q}{4\pi\varepsilon}\frac{b}{\left[b^2+[a-x]^2\right]^{1/2}}\left[b^2+[a-x]^2\right]^{-1} \\
&\quad -\frac{2Q}{4\pi\varepsilon}\frac{b}{\left[b^2+[b+x]^2\right]^{1/2}}\left[b^2+[a+x]^2\right]^{-1} = -\frac{2Q}{4\pi\varepsilon}\mathrm{X} \\
\mathrm{X} &= b\left[b^2+[a-x]^2\right]^{-3/2}+b\left[b^2+[a+x]^2\right]^{-3/2}
\end{aligned}
\tag{5.1}
$$

Differentiating we have

$$
\begin{aligned}
\frac{d\mathrm{X}}{dx} &= 3b[a-x]\left[b^2+[a-x]^2\right]^{-5/2}-3b[a+x]\left[b^2+[a+x]^2\right]^{-5/2} \\
\frac{d^2\mathrm{X}}{dx^2} &= -3b\left[b^2+[a-x]^2\right]^{-5/2}+15b[a-x]^2\left[b^2+[a-x]^2\right]^{-7/2} \\
&\quad -3b\left[b^2+[a+x]^2\right]^{-5/2}+15b[a+x]^2\left[b^2+[a+x]^2\right]^{-7/2} \\
\frac{d^2\mathrm{X}}{dx^2} &= -6bh^{-5} + 30ba^2h^{-7} \qquad \text{at origin} \\
&= \left[-6\sin(\psi) + 30\sin(\psi)\cos^2(\psi)\right]h^{-4}
\end{aligned}
\tag{5.2}
$$

At the origin we have

$$
\frac{1}{F_y}\frac{\partial^2 E_y}{\partial x^2} = \left[-3 + 15\cos^2(\psi)\right]h^{-2}
\tag{5.3}
$$

This can be set to zero at

$$
\cos(\psi) = \left[\frac{1}{5}\right]^{1/2} \quad , \qquad \psi \simeq 63.4^\circ
\tag{5.4}
$$

For smaller ψ (5.3) is positive, while for larger ψ it is negative.

6. Balancing *x* and *y* Variation

We can see from (4.3) and (5.3) that the *x* and *y* second derivatives cannot be simultaneously zero. Perhaps some compromise is called for ψ between the two values above. One approach is to minimize the magnitude of the sum of the two second derivatives (noting their opposite variation with ψ). We find

$$\frac{1}{E_y}\frac{\partial^2 E_y}{\partial x^2} + \frac{1}{E_y}\frac{\partial^2 E_y}{\partial y^2} = \left[-3+15\cos^2(\psi)\right]h^{-2} + \left[-9+15\sin^2(\psi)\right]h^{-2} = 3h^{-2} = \frac{1}{E_y}\frac{\partial^2 E_y}{\partial z^2} \tag{6.1}$$

So the sum of the three second derivatives is zero.

Another approach is to minimize the sums of the squares of the *x* and *y* second derivatives as

$$\begin{aligned}
0 &= \frac{d}{d\sin(\psi)}\left[12 - 15\sin^2(\psi)\right]^2 + \frac{d}{d\sin(\psi)}\left[-9 + 15\sin^2(\psi)\right] \\
&= 2\left[12 - 15\sin^2(\psi)\right][-30]\sin(\psi) + 2\left[-9 + 15\sin^2(\psi)\right]30\sin(\psi) \\
0 &= -\left[12 - 15\sin^2(\psi)\right] + \left[-9 + 15\sin^2(\psi)\right] \\
&= -21 + 30\sin(\psi) \\
\sin(\psi) &= \left[\frac{7}{10}\right]^{1/2} \quad , \quad \psi \simeq 56.8^\circ
\end{aligned} \tag{6.2}$$

which is between the two previous values. In this case we have

$$\begin{aligned}
\frac{1}{E_y}\frac{\partial^2 E_y}{\partial x^2} &= \left[12-\frac{21}{2}\right]h^{-2} = \frac{3}{2}h^{-2} \\
\frac{1}{E_y}\frac{\partial^2 E_y}{\partial y^2} &= \left[-9+\frac{21}{2}\right]h^{-2} = \frac{3}{2}h^{-2}
\end{aligned} \tag{6.3}$$

The two second derivatives are equal, each being half the magnitude of the second *z* derivative.

7. Concluding Remarks

Unlike the two-dimensional case, there is not a special value of ψ to make all the second derivatives zero with respect to the coordinate axes. The result of Section 6 can be considered a balanced design.

References

1. C. E. Baum, "Impedances and Field Distributions for Symmetrical Two Wire and Four Wire Transmission Line Simulators", Sensor and Simulation Note 27, October 1966.

2. W. R. Smythe, *Static and Dynamic Electricity, 3rd Ed*., Taylor and Francis, 1989.