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GENERALIZED TEM, E, AND H MODES*

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Abstract

Previous papers have considered transient lenses for propagating TEM modes without dispersion. This paper considers the properties of E and H modes in such lenses. The presence of longitudinal field components brings in additional constraints on the allowable coordinate systems, limiting the cases of transient lenses supporting E and H modes to a subset of those supporting TEM modes.

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Contents

1 Introduction

A technique developed by C. E. Baum[1] for the design of EM lenses utilizes the expression of the constitutive parameters $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}$ and Maxwell's equations in a general orthogonal curvilinear coordinate system, yielding what we will call the formal quantities. These are customarily denoted by affixing primes as superscripts. The line element is

$$(ds)^2 = h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2 \tag{1.1}$$

and the coordinates are $(u_1,\ u_2,\ u_3)$. The scale factors, h_i, relate the formal parameters ϵ_i' and μ_i' to the real world parameters ϵ_i and μ_i in the diagonal case via the equations

$$(\epsilon_{ij}') = \begin{pmatrix} \frac{h_2h_3}{h_1}\epsilon_1 & 0 & 0 \\ 0 & \frac{h_1h_3}{h_2}\epsilon_2 & 0 \\ 0 & 0 & \frac{h_1h_2}{h_3}\epsilon_3 \end{pmatrix} \tag{1.2}$$

$$(\mu_{ij}') = \begin{pmatrix} \frac{h_2h_3}{h_1}\mu_1 & 0 & 0 \\ 0 & \frac{h_1h_3}{h_2}\mu_2 & 0 \\ 0 & 0 & \frac{h_1h_2}{h_3}\mu_3 \end{pmatrix} \tag{1.3}$$

The scale factors also relate the formal fields E_i' and H_i' to the real fields E_i and H_i via the equations

$$E_i' = h_iE_i, \quad H_i' = h_iH_i \tag{1.4}$$

for $i = 1, 2, 3$. Maxwell's equations for the formal fields are

$$\begin{aligned} \nabla' \times \vec{E}' &= -s\vec{B}' = -s\overleftrightarrow{\mu}' \cdot \vec{H}' \\ \nabla' \times \vec{H}' &= s\vec{D}' = s\overleftrightarrow{\epsilon}' \cdot \vec{E}' \end{aligned} \tag{1.5}$$

Here we use $s = \Omega + j\omega$, the two-sided Laplace-transform variable or complex frequency. This suppresses the time-derivatives for our convenience in notation, and furthermore allows the constitutive parameters to be frequency dependent if desired. This last point is significant only in the case of dispersive media, which need not concern us here. Since we are not going back and forth between the time and frequency domains, we do not need to indicate the fact that all fields are Laplace transforms (functions of complex frequency).

Thus if we assume diagonal forms for the tensors $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}$, Maxwell's equations in expanded form become

$$\begin{aligned} \frac{\partial E_3'}{\partial u_2} - \frac{\partial E_2'}{\partial u_3} &= -s\mu_1'H_1' \\ \frac{\partial E_1'}{\partial u_3} - \frac{\partial E_3'}{\partial u_1} &= -s\mu_2'H_2' \\ \frac{\partial E_2'}{\partial u_1} - \frac{\partial E_1'}{\partial u_2} &= -s\mu_3'H_3' \end{aligned} \tag{1.6}$$

and

$$\begin{aligned}
\frac{\partial H_3'}{\partial u_2} - \frac{\partial H_2'}{\partial u_3} &= s\epsilon_1' E_1' \\
\frac{\partial H_1'}{\partial u_3} - \frac{\partial H_3'}{\partial u_1} &= s\epsilon_2' E_2' \\
\frac{\partial H_2'}{\partial u_1} - \frac{\partial H_1'}{\partial u_2} &= s\epsilon_3' E_3'
\end{aligned} \tag{1.7}$$

These equations will be the starting point in our search for conditions on the parameters $\overleftrightarrow{\mu}$ and $\overleftrightarrow{\epsilon}$ in the case of E (or TM) and H (or TE) modes.

2 TEM Case (Formal Fields)

In this section we recapitulate the results obtained, in the formal case, for a TEM wave propagating in the u_3 coordinate direction. These results, which are consistent with examples described in earlier work [3, 5], will suggest the approach to be taken in the case of an E-wave or an H-wave. We begin with the assumption that the parameters $\overleftrightarrow{\mu}'$ and $\overleftrightarrow{\epsilon}'$ are in the form

$$\begin{aligned} \overleftrightarrow{\mu}' &= \begin{pmatrix} \mu_1' & 0 & 0 \\ 0 & \mu_2' & 0 \\ 0 & 0 & \mu_3' \end{pmatrix} \\ \overleftrightarrow{\epsilon}' &= \begin{pmatrix} \epsilon_1' & 0 & 0 \\ 0 & \epsilon_2' & 0 \\ 0 & 0 & \epsilon_3' \end{pmatrix} \end{aligned} \tag{2.1}$$

We will think of our (u_1, u_2, u_3) coordinates as though they are Cartesian and allow $\overleftrightarrow{\mu}'$ and $\overleftrightarrow{\epsilon}'$ to be inhomogeneous and anisotropic. Our TEM plane wave is to propagate in the u_3 direction and all fields will be assumed to have propagation factors which account for this. Thus if u_3 is the propagation direction, then μ_3' and ϵ_3' are irrelevant.

2.1 Factored Form of Parameters

At this point we assume a form for the fields and constitutive parameters that factors the dependence into products of the form a function of u_1 and u_2 (transverse coordinates) times a function u_3 (longitudinal or propagation coordinate). For the constitutive parameters we have (all terms real and positive), for $n = 1, 2$,

$$\begin{aligned} \mu_n' &= \mu_n^{(0)'}(u_1, u_2) g_\mu(u_3) \\ \epsilon_n' &= \epsilon_n^{(0)'}(u_2, u_2) g_\epsilon(u_3) \\ \overleftrightarrow{\mu}' &= \overleftrightarrow{\mu}_n^{(0)'}(u_1, u_2) g_\mu(u_3) \\ \overleftrightarrow{\epsilon}' &= \overleftrightarrow{\epsilon}_n^{(0)'}(u_1, u_2) g_\epsilon(u_3) \end{aligned} \tag{2.2}$$

Note that there are not separate functions of u_3 for each of the 1 and 2 components, this being an assumption of invariance to transformation (e.g., rotation) of the u_1, u_2 coordinates.

We then seek TEM solutions of the form

$$\vec{E}' = \vec{E}^{(0)'}(u_1, u_2) g_e(u_3), \quad \vec{H}' = \vec{H}^{(0)'}(u_1, u_2) g_h(u_3) \tag{2.3}$$

with

$$
\begin{aligned}
\vec{E}^{(0)'} \cdot \vec{1}_3 = 0 = & E_3^{(0)'}, \quad \vec{H}^{(0)'} \cdot \vec{1}_3 = 0 = H_3^{(0)'} \\
\vec{D}' = \overleftrightarrow{\epsilon}' \cdot \vec{E}' = & \overleftrightarrow{\epsilon}^{(0)'}(u_1, u_2) \cdot \vec{E}^{(0)'}(u_1, u_2) g_\epsilon(u_3) g_e(u_3) \\
= & \vec{D}^{(0)'}(u_1, u_2) g_\epsilon(u_3) g_e(u_3) \\
\vec{B}' = \overleftrightarrow{\mu}' \cdot \vec{H}' = & \overleftrightarrow{\mu}^{(0)'}(u_1, u_2) \cdot \vec{H}^{(0)'}(u_1, u_2) g_\mu(u_3) g_h(u_3) \\
= & \vec{B}^{(0)'}(u_1, u_2) g_\mu(u_3) g_h(u_3)
\end{aligned}
\tag{2.4}
$$

2.2 Maxwell Equations and Separation of Variables

Maxwell's equations then have the form

$$
\begin{aligned}
\frac{\partial E_2'}{\partial u_3} &= s\mu_1' H_1' \\
\frac{\partial E_1'}{\partial u_3} &= - s\mu_2' H_2' \\
\frac{\partial E_2'}{\partial u_1} - \frac{\partial E_1'}{\partial u_2} &= 0
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
\frac{\partial H_2'}{\partial u_3} &= - s\epsilon_1' E_1' \\
\frac{\partial H_1'}{\partial u_3} &= s\epsilon_2' E_2' \\
\frac{\partial H_2'}{\partial u_1} - \frac{\partial H_1'}{\partial u_2} &= 0.
\end{aligned}
\tag{2.6}
$$

These equations may then be rewritten as, using (2.2) and (2.3),

$$
\begin{aligned}
E_2^{(0)'} \frac{dg_e}{du_3} &= s\mu_1^{(0)'} g_\mu g_h H_1^{(0)'} \\
E_1^{(0)'} \frac{dg_e}{du_3} &= - s\mu_2^{(0)'} g_\mu g_h H_2^{(0)'} \\
\frac{\partial E_2^{(0)'}}{\partial u_1} - \frac{\partial E_1^{(0)'}}{\partial u_2} &= 0
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
H_2^{(0)'} \frac{dg_h}{du_3} &= - s\epsilon_1^{(0)'} g_\epsilon g_e E_1^{(0)'} \\
H_1^{(0)'} \frac{dg_h}{du_3} &= s\epsilon_2^{(0)'} g_\epsilon g_e E_2^{(0)'} \\
\frac{\partial H_2^{(0)'}}{\partial u_1} - \frac{\partial H_1^{(0)'}}{\partial u_2} &= 0.
\end{aligned}
\tag{2.8}
$$

We note that in the first two equations in both (2.7) and (2.8) we have "separability" in the sense that these equations may be reexpressed in a form where we have a function of u_1 and u_2 equal to a function of u_3 only. The immediate result is then the fact that both functions are equal to a constant (i.e., independent of the *spatial coordinates*). This same reasoning, it will be recalled, is used in the "separation of variables" technique in partial differential equations. Thus we define constants K_e and K_h, from (2.7) and (2.8), by

$$K_e = -sg_\mu g_h \left(\frac{dg_e}{du_3}\right)^{-1} = -\frac{E_2^{(0)'}}{\mu_1^{(0)'} H_1^{(0)'}} = \frac{E_1^{(0)'}}{\mu_2^{(0)'} H_2^{(0)'}} \tag{2.9}$$

$$K_h = -sg_\epsilon g_e \left(\frac{dg_h}{du_3}\right)^{-1} = -\frac{H_1^{(0)'}}{\epsilon_2^{(0)'} E_2^{(0)'}} = \frac{H_2^{(0)'}}{\epsilon_1^{(0)'} E_1^{(0)'}}. \tag{2.10}$$

Thus

$$\begin{aligned} K_e K_h =& s^2 g_\mu g_h g_\epsilon g_e \left(\frac{dg_e}{du_3}\right)^{-1} \left(\frac{dg_h}{du_3}\right)^{-1} \\ =& \frac{1}{\mu_1^{(0)'} \epsilon_2^{(0)'}} = \frac{1}{\mu_2^{(0)'} \epsilon_1^{(0)'}} \equiv \left(v^{(0)'}\right)^2 \\ =& \text{ constant} \\ v^{(0)'} =& \left[\mu_1^{(0)'} \epsilon_2^{(0)'}\right]^{-\frac{1}{2}} = \left[\mu_2^{(0)'} \epsilon_1^{(0)'}\right]^{-\frac{1}{2}} \\ \equiv& \text{ propagation speed in } u_n \text{ coordinates.} \end{aligned} \tag{2.11}$$

The formal parameters, ϵ'_n and μ'_n, then satisfy

$$\mu'_1 \epsilon'_2 = \mu_1^{(0)'} \epsilon_2^{(0)'} g_\mu g_\epsilon = \mu_2^{(0)'} \epsilon_1^{(0)'} g_\mu g_\epsilon = \mu'_2 \epsilon'_1 \tag{2.12}$$

and hence

$$(v'(u_3))^{-2} = \mu'_1 \epsilon'_2 = \mu'_2 \epsilon'_1 = (v^{(0)'})^{-2} g_\mu g_\epsilon \tag{2.13}$$

where $v'(u_3)$ is the formal propagation speed and is a function of u_3 only. We also can form

$$\begin{aligned} \frac{K_e}{K_h} =& \frac{g_\mu g_h \dfrac{dg_h}{du_3}}{g_\epsilon g_e \dfrac{dg_e}{du_3}} \\ =& \frac{\epsilon_1^{(0)'}}{\mu_2^{(0)'}} \left[\frac{E_1^{(0)'}}{H_2^{(0)'}}\right]^2 = \frac{\epsilon_2^{(0)'}}{\mu_1^{(0)'}} \left[\frac{E_2^{(0)'}}{H_1^{(0)'}}\right]^2 \\ =& \text{ constant.} \end{aligned} \tag{2.14}$$

Thus we can define wave impedances

$$Z_1^{(0)'} \equiv \left[\frac{\mu_2^{(0)'}}{\epsilon_1^{(0)'}}\right]^{\frac{1}{2}}, \quad Z_2^{(0)'} \equiv \left[\frac{\mu_1^{(0)'}}{\epsilon_2^{(0)'}}\right]^{\frac{1}{2}}. \tag{2.15}$$

Normalize the constants K_e and K_h for convenience such that

$$K_e = K_h \equiv K_f \quad (f \text{ for field}) \tag{2.16}$$

This is merely a scale on the factors in the above equations (constants being absorbed as parts of g_ϵ and/or g_μ as well as the field components) giving

$$K_f = v^{(0)'} \quad (\text{not a function of any } u_n) \tag{2.17}$$

with the plus sign chosen for propagation in the $+u_3$ direction, and our two constants K_e and K_h are now determined.

2.3 Two-Dimensional Vectors/Matrices for Transverse Field Components

The third equation in each of (2.7) and (2.8) leads to the result that there exist differentiable functions $\Phi'_e(u_1, u_2)$ and $\Phi'_h(u_1, u_2)$ such that

$$\begin{aligned} \vec{E}^{(0)'} &= -\nabla'_t \Phi'_e(u_1, u_2) \\ \vec{H}^{(0)'} &= -\nabla'_t \Phi'_h(u_1, u_2) \end{aligned} \tag{2.18}$$

because of the Poincaré lemma [8]. The operator ∇'_t is the (formal) transverse gradient

$$\nabla'_t = \vec{1}_1 \frac{\partial}{\partial u_1} + \vec{1}_2 \frac{\partial}{\partial u_2} \tag{2.19}$$

Thus

$$\begin{aligned} \vec{E}' &= -\nabla'_t \Phi'_e(u_1, u_2) g_e(u_3) \\ \vec{H}' &= -\nabla'_t \Phi'_h(u_1, u_2) g_h(u_3) \end{aligned} \tag{2.20}$$

The first pair of equations in each of (2.7) and (2.8) can be written in vector form as

$$\begin{aligned} \vec{1}_3 \times \vec{E}^{(0)'} &= v^{(0)'} \overleftrightarrow{\mu}^{(0)'} \cdot \vec{H}^{(0)'} = v^{(0)'} \vec{B}^{(0)'} \\ \vec{1}_3 \times \vec{H}^{(0)'} &= -v^{(0)'} \overleftrightarrow{\epsilon}^{(0)'} \cdot \vec{E}^{(0)'} = -v^{(0)'} \vec{D}^{(0)'} \end{aligned} \tag{2.21}$$

From these we can conclude

$$\begin{aligned} \vec{E}^{(0)'} \cdot \vec{B}^{(0)'} &= \vec{E}^{(0)'} \cdot \overleftrightarrow{\mu}^{(0)'} \cdot \vec{H}^{(0)'} = 0 \\ \vec{D}^{(0)'} \cdot \vec{H}^{(0)'} &= \vec{E}^{(0)'} \cdot \overleftrightarrow{\epsilon}^{(0)'} \cdot \vec{H}^{(0)'} = 0. \end{aligned} \tag{2.22}$$

Furthermore we also have

$$\begin{aligned}\frac{E_1^{(0)'}}{H_2^{(0)'}} &= v^{(0)'}\mu_2^{(0)'} = Z_1^{(0)'} = \frac{1}{\epsilon_1^{(0)'}v^{(0)'}}\\ \frac{E_2^{(0)'}}{H_1^{(0)'}} &= -\, v^{(0)'}\mu_1^{(0)'} = -Z_2^{(0)'} = -\frac{1}{\epsilon_2^{(0)'}v^{(0)'}}\end{aligned} \tag{2.23}$$

Again note the sign convention ($v^{(0)'}$ positive) for propagation in the $+u_3$ direction.

At this point we may introduce

$$\overleftrightarrow{\tau}_t \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;,\quad \overleftrightarrow{1}_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \text{transverse identity} \tag{2.24}$$

using only the transverse (u_1, u_2) coordinates so that

$$\overleftrightarrow{\tau}_t^{1} = \overleftrightarrow{\tau}_t^{T} = -\overleftrightarrow{\tau}_t \;,\quad \overleftrightarrow{\tau}_t^{2} = -\overleftrightarrow{1}_t \tag{2.25}$$

and we can replace the crossproduct with $\vec{1}_3$ by

$$\vec{1}_3 \times = \overleftrightarrow{\tau}_t \cdot \tag{2.26}$$

This allows us to restate (2.21) in terms of only transverse coordinates as

$$\begin{aligned}\overleftrightarrow{\tau}_t \cdot \vec{E}^{(0)'} &= v^{(0)'}\overleftrightarrow{\mu}^{(0)'} \cdot \vec{H}^{(0)'} = v^{(0)'}\vec{B}^{(0)'}\\ \overleftrightarrow{\tau}_t \cdot \vec{H}^{(0)'} &= -\, v^{(0)'}\overleftrightarrow{\epsilon}^{(0)'} \cdot \vec{E}^{(0)'} = -v^{(0)'}\vec{D}^{(0)'}\end{aligned} \tag{2.27}$$

This notation will also be useful in later sections.

2.4 Generalized TEM Wave

From (2.22) we see that $\vec{E}$ and $\vec{H}$ need not be mutually perpendicular for a TEM wave. Instead this property holds between $\vec{E}$ and $\vec{B}$ and between $\vec{D}$ and $\vec{H}$. This is associated with our allowing the medium to be *anisotropic*.

If we want $\vec{E}$ and $\vec{H}$ to be perpendicular, with general transverse orientation (polarization), then this leads to the requirement

$$\begin{aligned}&\overleftrightarrow{\mu}^{(0)'} = \mu^{(0)'}\,\overleftrightarrow{1} \;,\quad \overleftrightarrow{\epsilon}^{(0)'} = \epsilon^{(0)'}\,\overleftrightarrow{1}\\ &\overleftrightarrow{1} = \vec{1}_1\vec{1}_1 + \vec{1}_2\vec{1}_2 + \vec{1}_3\vec{1}_3 = \overleftrightarrow{1}_t + \vec{1}_3\vec{1}_3\\ &\vec{E}^{(0)'} \cdot \vec{H}^{(0)'} = 0\end{aligned} \tag{2.28}$$

where, of course, it is only the 1 and 2 components that are relevant. Such an isotropic medium is precisely the case considered in [3].

Whether one considers the uniform formal (prime) medium as anisotropic as in (2.22) or isotropic as in (2.24), there is still the question of scaling to the real coordinates and fields. One can then again ask the question of whether the formal permittivity and permeability can be isotropic or anisotropic.

2.5 Constraint of Wave Propagation in $+u_3$ Direction

Consistent with [3] let us now assume that the wave is propagating in the $+u_3$ direction with no reflections. Note that $\vec{E}'$ and $\vec{H}'$ are tangential to every surface of constant u_3, and that tangential components of these vectors must be continuous on passing through such boundaries. In particular, for low frequencies ($s \rightarrow 0$) the potential functions in (2.18) must apply for all u_3 (i.e. g_e and $g_h \rightarrow$ constants) or the change between u_3 surfaces will imply $E_3^{(0)'}$ and $H_3^{(0)'}$ nonzero – a contradiction.

So let us look for solutions of the form (consistent with [3], there expressed in time domain)

$$\begin{aligned} g_f(u_3) \equiv & g_e = g_h = \exp\left(-s\int_0^{u_3} \frac{du_3'}{v'(u_3')}\right) \quad \text{(subscript } f \text{ for fields)} \\ \frac{1}{g_f}\frac{\partial g_f}{\partial u_3} & = -\frac{s}{v'(u_3)} \equiv -\gamma'(u_3) \end{aligned} \tag{2.29}$$

From (2.9), (2.10), and (2.16) we directly have, as well,

$$\begin{aligned} g_c(u_3) \equiv & g_\epsilon = g_\mu \quad \text{(subscript } c \text{ for constitutive parameters)} \\ g_f = & \exp\left(-\gamma^{(0)'}\int_0^{u_3} \frac{du_3'}{g_c(u_3')}\right) \quad , \gamma^{(0)'} \equiv \frac{s}{v^{(0)'}} \end{aligned} \tag{2.30}$$

Note that only $g_c(u_3)$ (real and positive) is left to be specified here, $v^{(0)'}$ being a positive real constant. This result applies to the usual TEM wave in isotropic media (Section 2.3), as well as to the generalized form in anisotropic media (with the constraint in (2.11)) as well.

3 Scaling to Real Medium for TEM Modes

In [3, (Section 5)] an assumption of isotropic μ', ϵ', μ, and ϵ (u_3 associated components being irrelevant) led to the result that surfaces of constant u_3 could only be spheres and planes. Let us revisit this question now that we have the possibility of anisotropic $\overleftrightarrow{\mu}'$ and $\overleftrightarrow{\epsilon}'$.

Our general scaling relations [11] are

$$\begin{gathered}
\vec{E}' = (\alpha_{n,m})\cdot \vec{E}\ ,\ \vec{H}' = (\alpha_{n,m})\cdot \vec{H} \\
\overleftrightarrow{\epsilon}' = (\gamma_{n,m})\cdot \overleftrightarrow{\epsilon}\ ,\ \overleftrightarrow{\mu}' = (\gamma_{n,m})\cdot \overleftrightarrow{\mu} \\
(\alpha_{n,m}) = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}, (\gamma_{n,m}) = \begin{pmatrix} \frac{h_2 h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_3 h_1}{h_2} & 0 \\ 0 & 0 & \frac{h_1 h_2}{h_3} \end{pmatrix}
\end{gathered} \tag{3.1}$$

where the constitutive-parameter dyadics (matrices) are assumed diagonal in the (u_1, u_2, u_3) coordinate system. The scale factors and line element are

$$\begin{aligned}
h_n^2 &= \left[\frac{\partial x}{\partial u_n}\right]^2 + \left[\frac{\partial y}{\partial u_n}\right]^2 + \left[\frac{\partial z}{\partial u_n}\right]^2 \\
[d\ell]^2 &= \sum_{n=1}^{3} h_n^2 [du_n]^2
\end{aligned} \tag{3.2}$$

3.1 General Results

Considering only the first and second diagonal terms of the constitutive-parameter dyadics we have

$$\begin{aligned}
\epsilon_1^{(0)'}(u_1, u_2) g_c(u_3) &= \frac{h_2 h_3}{h_1} \epsilon_1 \\
\epsilon_2^{(0)'}(u_1, u_2) g_c(u_3) &= \frac{h_3 h_1}{h_2} \epsilon_2 \\
\mu_1^{(0)'}(u_1, u_2) g_c(u_3) &= \frac{h_2 h_3}{h_1} \mu_1 \\
\mu_2^{(0)'}(u_1, u_2) g_c(u_3) &= \frac{h_3 h_1}{h_2} \mu_2
\end{aligned} \tag{3.3}$$

Forming ratios we have functions of u_1, u_2 only as

$$\begin{aligned}
\frac{\mu_1^{(0)'}(u_1,u_2)}{\epsilon_1^{(0)'}(u_1,u_2)} &= \frac{\mu_1}{\epsilon_1} \\
\frac{\mu_2^{(0)'}(u_1,u_2)}{\epsilon_2^{(0)'}(u_1,u_2)} &= \frac{\mu_2}{\epsilon_2} \\
Z_1^{(0)'^2}(u_1,u_2) &= \frac{\mu_2^{(0)'}}{\epsilon_1^{(0)'}} = \left[\frac{h_1}{h_2}\right]^2 \frac{\mu_2}{\epsilon_1} \\
Z_2^{(0)'^2}(u_1,u_2) &= \frac{\mu_1^{(0)'}}{\epsilon_2^{(0)'}} = \left[\frac{h_2}{h_1}\right]^2 \frac{\mu_1}{\epsilon_2} \\
\frac{\epsilon_1^{(0)'}(u_1,u_2)}{\epsilon_2^{(0)'}(u_1,u_2)} &= \left[\frac{h_2}{h_1}\right]^2 \frac{\epsilon_1}{\epsilon_2} \\
\frac{\mu_1^{(0)'}(u_1,u_2)}{\mu_2^{(0)'}(u_1,u_2)} &= \left[\frac{h_2}{h_1}\right]^2 \frac{\mu_1}{\mu_2}
\end{aligned} \tag{3.4}$$

3.2 Isotropic Real Medium

At this point we can consider the special case that the real medium is *isotropic* so that

$$\begin{aligned}
\epsilon &= \epsilon_1 = \epsilon_2 \ (= \epsilon_3) \\
\mu &= \mu_1 = \mu_2 \ (= \mu_3)
\end{aligned} \tag{3.5}$$

which implies from the third pair in (3.6) (or from the first pair)

$$\frac{\epsilon_1^{(0)'}(u_1,u_2)}{\epsilon_2^{(0)'}(u_1,u_2)} = \frac{\mu_1^{(0)'}(u_1,u_2)}{\mu_2^{(0)'}(u_1,u_2)} \tag{3.6}$$

However, this is nothing more than what we have previously found in (2.11), after cross multiplying the denominators. On the other hand the first pair in (3.4) implies

$$\mu_2^{(0)'}\epsilon_2^{(0)'} = \mu_1^{(0)}\epsilon_1^{(0)'} \tag{3.7}$$

Multiplying by $\left(v^{(0)'}\right)^2$ from (2.12) implies

$$\frac{\epsilon_2^{(0)'}}{\epsilon_1^{(0)'}} = \frac{\epsilon_1^{(0)'}}{\epsilon_2^{(0)'}}, \quad \frac{\mu_2^{(0)'}}{\mu_1^{(0)'}} = \frac{\mu_1^{(0)'}}{\mu_2^{(0)'}} \tag{3.8}$$

and taking the positive square root gives

$$\epsilon_1^{(0)'} = \epsilon_2^{(0)'} \equiv \epsilon^{(0)'}, \quad \mu_1^{(0)'} = \mu_2^{(0)'} \equiv \mu^{(0)'} \tag{3.9}$$

and the formal medium is also *isotropic*. In this case the results of [3] apply and

$$h_1 = h_2 \equiv h_t \tag{3.10}$$

and surfaces of constant u_3 can only be spheres or planes [6 (pp. 114-117, 146-149), 1 (Appendix B), 11 (Section 2.4)]. Further we have

$$\begin{aligned}
\mu^{(0)'} g_c &= h_3 \mu \,,\ \epsilon^{(0)'} g_c = h_3 \epsilon \\
\frac{\mu^{(0)'}}{\epsilon^{(0)'}} &= \frac{\mu}{\epsilon} = \text{function of } u_1, u_2 \text{ only} \\
Z_1^{(0)'} = Z_2^{(0)'} \equiv Z^{(0)'} = Z' = Z &= \left[\frac{\mu}{\epsilon}\right]^{\frac{1}{2}} = \text{ function of } u_1, u_2 \text{ only} \\
v \equiv [\mu\epsilon]^{\frac{1}{2}} = h_3[\mu'\epsilon']^{\frac{1}{2}} &= h_3 g_c^{-1}\left[\mu^{(0)'}\epsilon^{(0)'}\right]^{-\frac{1}{2}} \\
&= h_3 g_c^{-1} v^{(0)'} = h_3 v' \\
&= \text{real TEM speed} \\
v' = g_c^{-1} v^{(0)'} &= \text{TEM speed in } u_n \text{ coordinates (as in (2.13))} \\
&= \text{function of } u_3 \text{ only}
\end{aligned} \tag{3.11}$$

If we do not assume that the real medium is isotropic, then the formal medium need not be isotropic, and surfaces of constant u_3 may possibly take more general shapes.

3.3 Isotropic μ Medium

Now let the real medium have an isotropic (scalar) μ (not necessarily uniform such as μ_0) but $\overleftrightarrow{\epsilon}$ be allowed to be anisotropic and a function of the spatial coordinates. Then the first pair of (3.4) give

$$\frac{\epsilon_1}{\epsilon_2} = \frac{\mu_2^{(0)'}\epsilon_1^{(0)'}}{\mu_1^{(0)'}\epsilon_2^{(0)'}} = 1 \tag{3.12}$$

This assumption then forces $\overleftrightarrow{\epsilon}$ to be isotropic as ϵ. This forces us back into the isotropic real medium with results as in Section 3.2.

A similar comment can be made regarding an isotropic (scalar) ϵ medium, not necessarily ϵ_0. The condition

$$\frac{\mu_2^{(0)'}\epsilon_1^{(0)'}}{\mu_1^{(0)}\epsilon_2^{(0)'}} = 1 \tag{3.13}$$

forces $\mu_2^{(0)'} = \mu_1^{(0)'}$ and hence $\overleftrightarrow{\mu}$ is isotropic as μ. Thus the medium is again real isotropic and the results of Section 3.2 are applicable.

3.4 Uniform Real Permeability

Adding the requirement that

$$\mu = \mu_0 \text{ (uniform and isotropic)} \tag{3.14}$$

so that the permeability is both uniform and isotropic we have a case of practical significance. From [3] we have

$$\begin{aligned} Z &= \left[\frac{\mu_0}{\epsilon}\right]^{\frac{1}{2}} \\ \epsilon &= \epsilon(u_1, u_2) = \text{ function of } u_1, u_2 \text{ only} \\ v &= [\mu_0 \epsilon]^{-\frac{1}{2}} = v(u_1, u_2) = \text{ function of } u_1, u_2 \text{ only} \\ h_3 &= \frac{v(u_1, u_2)}{v'(u_3)} = \frac{v(u_1, u_2)}{v^{(0)'}} g_c(u_3) \text{ factored form} \end{aligned} \tag{3.15}$$

From (3.3) and (3.12) we also have

$$\frac{h_1}{h_2} = \frac{\epsilon_2^{(0)'}(u_1, u_2)}{\epsilon_1^{(0)'}(u_1, u_2)} = \text{ function of } u_1, u_2 \text{ only} \tag{3.16}$$

4 E-Wave (Formal Fields)

We continue our investigation with the E-wave case (transverse magnetic field), and so we will take $H_3' = 0$ and seek conditions on the formal parameters ϵ_i' and μ_i' which lead to solutions of the formal Maxwell equations. As usual in the case of waveguides [2, 10] we seek solutions for the formal-field components in terms of some operator on E_3' which we will later take as some mode function of u_1 and u_2 (transverse coordinates) times some propagation function of u_3. Our starting point once again will be Maxwell's equations as they appear in (1.6) and (1.7), which come from $\nabla' \times \vec{E}' = -s\overset{\leftrightarrow}{\mu}' \cdot \vec{H}'$ and $\nabla \times \vec{H}' = s\overset{\leftrightarrow}{\epsilon}' \cdot \vec{E}'$.

4.1 Maxwell's Equations with $H_3' = 0$

If we put $H_3' = 0$ in (1.6) and (1.7) we obtain

$$\begin{aligned} \frac{\partial E_3'}{\partial u_2} - \frac{\partial E_2'}{\partial u_3} &= -\, s\mu_1' H_1' \\ \frac{\partial E_1'}{\partial u_3} - \frac{\partial E_3'}{\partial u_1} &= -\, s\mu_2' H_2' \\ \frac{\partial E_2'}{\partial u_1} - \frac{\partial E_1'}{\partial u_2} &= 0 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \frac{\partial H_2'}{\partial u_3} &= -s\epsilon_1' E_1' \\ \frac{\partial H_1'}{\partial u_3} &= s\epsilon_2' E_2' \\ \frac{\partial H_2'}{\partial u_1} - \frac{\partial H_1'}{\partial u_2} &= s\epsilon_3' E_3' \end{aligned} \tag{4.2}$$

The consequences of the condition that $H_3' = 0$ will lead to restrictions on the formal parameters ϵ_i' and μ_i' (except for μ_3' since $H_3' = 0$) as well as solutions for the formal fields E_i' and H_i'.

We assume that the formal constitutive parameters have the forms similar to those specified in Section 2.1, namely

$$\begin{aligned} \mu_n' &= \mu_n^{(0)'}(u_1, u_2) g_\mu(u_3) \\ \epsilon_n' &= \epsilon_n^{(0)'}(u_1, u_2) g_\epsilon(u_3) \end{aligned} \tag{4.3}$$

for $n = 1, 2$, with μ_3' irrelevant since u_3 is the assumed propagation direction. We take

$$\epsilon_3' = \epsilon_3^{(0)'}(u_1, u_2) g_{\epsilon 3}(u_3) \tag{4.4}$$

and try solutions of the form

$$\begin{aligned}
\vec{E}'_t &= \vec{E}_t^{(0)'}(u_1,u_2)g_e(u_3) \\
\vec{H}'_t &= \vec{H}_t^{(0)'}(u_1,u_2)g_h(u_3) \\
E'_3 &= E_3^{(0)'}(u_1,u_2)g_{e3}(u_3),\ H'_3 = 0 \\
\overleftrightarrow{D}' &= \overleftrightarrow{\epsilon}' \cdot \vec{E}',\ \overleftrightarrow{B}' = \overleftrightarrow{\mu}' \cdot \vec{H}'.
\end{aligned} \tag{4.5}$$

Now for physical realizability g_μ and g_ϵ are real, nonzero and frequency independent. The propagation functions g_e, g_h and g_{e3} are in general complex functions of the complex frequency s (exponential like) and all have u_3 derivatives nonzero except possibly at special frequencies like $s = 0$, or degenerate cases like propagation perpendicular to u_3 (waveguide cutoff). The functions of u_1, u_2 are taken as independent of the complex frequency s.

Thus we obtain

$$\begin{aligned}
g_{e3}\frac{\partial E_3^{(0)'}}{\partial u_2} - \frac{dg_e}{du_3}E_2^{(0)'} &= -s\mu_1^{(0)'}g_\mu g_h H_1^{(0)'} \\
E_1^{(0)'}\frac{dg_e}{du_3} - g_{e3}\frac{\partial E_3^{(0)'}}{\partial u_1} &= -s\mu_2^{(0)'}g_\mu g_h H_2^{(0)'} \\
\frac{\partial E_2^{(0)'}}{\partial u_1} - \frac{\partial E_1^{(0)'}}{\partial u_2} &= 0
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
H_2^{(0)'}\frac{dg_h}{du_3} &= -s\epsilon_1^{(0)'}g_\epsilon g_e E_1^{(0)'} \\
H_1^{(0)'}\frac{dg_h}{du_3} &= s\epsilon_2^{(0)'}g_\epsilon g_e E_2^{(0)'} \\
g_h\left[\frac{\partial H_2^{(0)'}}{\partial u_1} - \frac{\partial H_1^{(0)'}}{\partial u_2}\right] &= s\epsilon_3^{(0)'}g_{\epsilon 3}g_{e3}E_3^{(0)'}
\end{aligned} \tag{4.7}$$

Moreover, since $\nabla' \cdot \left[\overleftrightarrow{\epsilon}' \cdot \vec{E}'\right] = 0$, we have

$$\frac{\partial[\epsilon_1^{(0)'}E_1^{(0)'}]}{\partial u_1} + \frac{\partial[\epsilon_2^{(0)'}E_2^{(0)'}]}{\partial u_2} = -\frac{1}{g_\epsilon g_e}\frac{\partial[g_{\epsilon 3}g_{e3}]}{\partial u_3}\epsilon_3^{(0)'}E_3^{(0)'} \tag{4.8}$$

and also, from $\nabla' \cdot \left[\overleftrightarrow{\mu}' \cdot \vec{H}'\right] = 0$, we have

$$\nabla'_t \cdot \left[\overleftrightarrow{\mu}_t^{(0)'} \cdot \vec{H}_t^{(0)'}\right] = 0. \tag{4.9}$$

4.2 Maxwell's Equations and Separation of Variables

We may now define some constants since we can separate variables in (4.7) and (4.8). Thus we can set

$$K_h \equiv -sg_\epsilon g_e \left[\frac{dg_h}{du_3}\right]^{-1} = \frac{H_2^{(0)'}}{\epsilon_1^{(0)'} E_1^{(0)'}} = -\frac{H_1^{(0)'}}{\epsilon_2^{(0)'} E_2^{(0)'}} \tag{4.10}$$

$$v_0\gamma_{e3} \equiv -sg_{\epsilon 3}g_{e3}g_h^{-1} = \frac{1}{\epsilon_3^{(0)'} E_3^{(0)'}} \left[\frac{\partial H_1^{(0)'}}{\partial u_2} - \frac{\partial H_2^{(0)'}}{\partial u_1}\right] \tag{4.11}$$

$$\gamma_d \equiv \frac{1}{g_\epsilon g_e}\frac{d[g_{\epsilon 3}g_{e3}]}{du_3} = -\frac{1}{\epsilon_3^{(0)'} E_3^{(0)'}} \left[\frac{\partial[\epsilon_1^{(0)'} E_1^{(0)'}]}{\partial u_1} + \frac{\partial[\epsilon_2^{(0)'} E_2^{(0)'}]}{\partial u_2}\right]. \tag{4.12}$$

Here γ_{e3} and γ_d are included to give a dimension of inverse length (in the u_n coordinates) to balance the u_1, u_2 derivatives. Later we will find that these are related to the propagation constant or wave number. As such, γ_{e3} and γ_d may be functions of complex frequency s, while K_h is not a function of s. A constant v_0 (dimension velocity) is also included to make the units work out. This is later identified with $v^{(0)'}$. We note that γ_d can be expressed in terms of K_h and γ_{e3} since we have

$$\begin{aligned} v_0\gamma_{e3}\frac{dg_h}{du_3} &= -\,s\frac{d}{du_3}[g_{\epsilon 3}g_{e3}] \text{ (from (4.11))} \\ \gamma_d &= \frac{v_0}{K_h}\gamma_{e3} \text{ (from (4.10) and (4.12))} \end{aligned} \tag{4.13}$$

These constants are in general nonzero and bounded, except perhaps for special values of s (e.g., $s = 0$ or cutoff condition in a waveguide). Note now that γ_d and γ_{e3} have the *same* frequency dependence.

4.3 Two-Dimensional Vectors/Matrices for Transverse Field Components

Using the notation introduced in Section 2.3 the first two equations in (4.6) can be written in a more compact form as

$$g_{e3}\nabla_t' E_3^{(0)'} - \frac{\partial g_e}{\partial u_3}\vec{E}_t^{(0)'} = -sg_\mu g_h \overleftrightarrow{\tau}_t \cdot \overleftrightarrow{\mu}_t^{(0)'} \cdot \vec{H}^{(0)'} \tag{4.14}$$

where we can cast the transverse parts of the constitutive-parameter matrices in the 2×2 form

$$\overleftrightarrow{\mu}_t^{(0)'} = \begin{pmatrix} \mu_1^{(0)'}(u_1, u_2) & 0 \\ 0 & \mu_2^{(0)'}(u_1, u_2) \end{pmatrix}, \quad \overleftrightarrow{\epsilon}_t^{(0)'} = \begin{pmatrix} \epsilon_1^{(0)'}(u_1, u_2) & 0 \\ 0 & \epsilon_2^{(0)'}(u_1, u_2) \end{pmatrix} \tag{4.15}$$

simplifying some formulae. We also note that there is a function Φ'_e such that

$$\vec{E}_t^{(0)'} = -\nabla'_t \Phi'_e \tag{4.16}$$

The existence of Φ'_e follows from the third equation of (4.6) and the Poincaré lemma.

The first two equations of (4.7) may likewise be written in the condensed form

$$\begin{aligned} \overleftrightarrow{\tau}_t \cdot \vec{H}_t^{(0)'} &= - K_h \overleftrightarrow{\epsilon}^{(0)'} \cdot \vec{E}_t^{(0)'} \\ &= - K_h \vec{D}_t^{(0)'} \end{aligned} \tag{4.17}$$

and hence we have the orthogonality relation

$$\vec{D}_t^{(0)'} \cdot \vec{H}_t^{(0)'} = 0 \tag{4.18}$$

We can remove the magnetic field from (4.14) via (4.16) as

$$\begin{aligned} \overleftrightarrow{\tau}_t \cdot \overleftrightarrow{\mu}^{(0)'} \vec{H}_t^{(0)'} &= K_h \overleftrightarrow{\tau}_t \cdot \overleftrightarrow{\mu}^{(0)'} \cdot \overleftrightarrow{\tau}_t \cdot \overleftrightarrow{\epsilon}^{(0)'} \cdot \vec{E}_t^{(0)'} \\ &= - K_h \overleftrightarrow{p} \cdot \vec{E}_t^{(0)'} \\ \overleftrightarrow{p} \equiv - \overleftrightarrow{\tau}_t \cdot \overleftrightarrow{\mu}^{(0)'} \cdot \overleftrightarrow{\tau}_t \cdot \overleftrightarrow{\epsilon}^{(0)'} &= \begin{pmatrix} \mu_2^{(0)'} \epsilon_1^{(0)'} & 0 \\ 0 & \mu_1^{(0)'} \epsilon_2^{(0)'} \end{pmatrix} \end{aligned} \tag{4.19}$$

In (4.14) this gives

$$\begin{aligned} g_{e3} \nabla'_t E_3^{(0)'} - \frac{dg_e}{du_3} \vec{E}_t^{(0)'} &= s g_\mu g_h K_h \overleftrightarrow{p} \cdot \vec{E}_t^{(0)'} \\ \nabla'_t E_3^{(0)'} &= \left[\frac{1}{g_{e3}} \frac{dg_e}{du_3} \overleftrightarrow{1}_t + \frac{s g_\mu g_h}{g_{e3}} K_h \overleftrightarrow{p} \right] \cdot \vec{E}_t^{(0)'} \end{aligned} \tag{4.20}$$

which involves only the electric field.

From (4.9) and (4.19) we have another equation for the transverse electric field as

$$\begin{aligned} 0 &= \frac{1}{K_h} \nabla'_t \cdot \left[\overleftrightarrow{\mu}_t^{(0)'} \cdot \vec{H}_t^{(0)'} \right] \\ &= \nabla'_t \cdot \left[\overleftrightarrow{\tau}_t \cdot \overleftrightarrow{p} \cdot \vec{E}_t^{(0)'} \right] \end{aligned} \tag{4.21}$$

Furthermore, (4.8) can be rewritten to give

$$\nabla'_t \cdot \left[\overleftrightarrow{\epsilon}_t^{(0)'} \cdot \vec{E}_t^{(0)'} \right] = -\gamma_d \epsilon_3^{(0)'} E_3^{(0)'} \tag{4.22}$$

which also involves only the electric field. These are in addition to (4.9) which involves only the magnetic field.

4.4 Properties of $\overleftrightarrow{p}$

We would like the E modes to propagate in the same medium as the TEM modes in Section 2. There we found in (2.11) that

$$\overleftrightarrow{p} = \begin{pmatrix} \mu_2^{(0)'}\epsilon_1^{(0)'} & 0 \\ 0 & \mu_1^{(0)'}\epsilon_2^{(0)'} \end{pmatrix} = v^{(0)'^{-2}} \overleftrightarrow{1}_t$$
$$v^{(0)'^{-2}} = \mu_2^{(0)'}\epsilon_1^{(0)'} = \mu_1^{(0)'}\epsilon_2^{(0)'} = \text{constant} \tag{4.23}$$

We can impose this requirement now, but it is instructive to consider this from an E-mode viewpoint.

A first observation is that a limiting case of an E-mode is a TEM mode. As $s \to \infty$ in a typical waveguide with perfectly conducting walls the ratio of the longitudinal electric field to the transverse electric field tneds to zero for a given mode (with basically a fixed number of transverse wavelengths) [2, 10]. This leads to the result in the previous paragraph.

Appendix A shows that this result can be derived by requiring that two or more independent E modes exist.

4.5 Identification of Some Constants

At this point with the result of (4.23), like (2.11) and (2.15) for TEM modes we have for the transverse field components

$$\frac{E_1^{(0)'}}{H_2^{(0)'}} = \left[\frac{\mu_2^{(0)'}}{\epsilon_1^{(0)'}}\right]^{\frac{1}{2}} \equiv Z_1^{(0)'} = \frac{1}{\epsilon_1^{(0)'}v^{(0)'}} = v^{(0)'}\mu_2^{(0)'}$$
$$\frac{E_2^{(0)'}}{H_1^{(0)'}} = -\left[\frac{\mu_1^{(0)'}}{\epsilon_2^{(0)'}}\right]^{\frac{1}{2}} \equiv -Z_2^{(0)'} = -\frac{1}{\epsilon_2^{(0)'}v^{(0)'}} = -v^{(0)'}\mu_1^{(0)'} \tag{4.24}$$

This allows us to identify one of the separation constants from (4.10) as

$$K_h = v^{(0)'} = -s g_\epsilon g_e \left[\frac{dg_h}{du_3}\right]^{-1} \tag{4.25}$$

and also have

$$\gamma_{e3} = \gamma_d = \frac{s}{v^{(0)'}} \cdot \frac{g_{\epsilon 3}g_{e3}}{g_h} = \frac{1}{g_\epsilon g_e}\frac{d}{du_3}[g_{\epsilon 3}g_{e3}]$$
$$v_0 \equiv v^{(0)'} \text{ (allowed choice of } v_0) \tag{4.26}$$

The two "independent" constants have been reduced to one.

Next, noting that γ_d is a separation constant with units of $s/v^{(0)'}$, we can set

$$\gamma_{e3} = \gamma_d \equiv \frac{s}{v^{(0)'}} \tag{4.27}$$

implying

$$1 = \frac{g_{\epsilon 3} g_{e3}}{g_h}, \quad \frac{s}{v^{(0)'}} = \frac{1}{g_\epsilon g_e} \frac{d}{du_3}[g_{\epsilon 3} g_{e3}] \tag{4.28}$$

Referring to (4.7) and (4.8) whence these separation constants come, the above choice is merely establishing a scaling relationship between $E_3^{(0)'}$ and the other (transverse) field components, these having been related for their u_1, u_2 parts by (4.24). All three separation constants have now been determined.

4.6 Continuation of Separation of Variables

Substituting our result for $\overleftrightarrow{p}$ in (4.23) into (4.20) gives

$$\nabla_t' E_3^{(0)'} = \left[\frac{1}{g_{e3}} \frac{dg_e}{du_3} + \frac{s g_\mu g_h}{g_{e3} v^{(0)'}}\right] \vec{E}_t^{(0)'}. \tag{4.29}$$

Now separation of variables gives

$$\begin{aligned} \gamma_g &\equiv -\frac{1}{g_{e3}} \frac{dg_e}{du_3} - \frac{s}{v^{(0)'}} \frac{g_\mu g_h}{g_{e3}} \\ \nabla_t' E_3^{(0)'} &= -\gamma_g \vec{E}_t^{(0)'} = \gamma_g \nabla_t' \Phi_e' \end{aligned} \tag{4.30}$$

where γ_g has dimensions of length^{-1} (in u_n coordinates). This allows us to set

$$E_3^{(0)'} = \gamma_g \Phi_e' \tag{4.31}$$

with the integration constant taken as zero. Note that $\gamma_g \neq 0, \infty$ without one of the vectors in (4.30) vanishing.

Combining (4.31) with (4.22) we find

$$\begin{aligned} \Phi_e' &= \frac{E^{(0)'}}{\gamma_g} = -\frac{1}{\gamma_d \gamma_g} \frac{1}{\epsilon_3^{(0)'}} \nabla_t' \cdot \left[\overleftrightarrow{\epsilon}_t^{(0)'} \cdot \vec{E}^{(0)'}\right] \\ &= \frac{1}{\gamma_d \gamma_g} \frac{1}{\epsilon_3^{(0)'}} \nabla_t' \cdot \left[\overleftrightarrow{\epsilon}_t^{(0)'} \cdot \nabla_t' \Phi_e'\right]. \end{aligned} \tag{4.32}$$

This is a differental equation for Φ_e'. For appropriate bounday conditions (e.g. $\Phi_e' = 0$ on some closed contour in the u_1, u_2 plane) one can in principle solve a waveguide problem for Φ_e'.

From (4.19) with (4.23) we also find

$$\begin{aligned} \vec{B}_t^{(0)'} &= \overleftrightarrow{\mu}^{(0)'} \cdot \vec{H}_t^{(0)'} = K_h \overleftrightarrow{\tau}_t \cdot \overleftrightarrow{p} \cdot \vec{E}_t^{(0)'} \\ &= v^{(0)'^{-1}} \overleftrightarrow{\tau}_t \cdot \vec{E}_t^{(0)'} \end{aligned} \tag{4.33}$$

from which we have

$$\vec{E}_t^{(0)'} \cdot \vec{B}_t^{(0)'} = 0 \tag{4.34}$$

which complements the orthogonality relation (4.18), namely

$$\vec{D}_t^{(0)'} \cdot \vec{H}_t^{(0)'} = 0. \tag{4.35}$$

These results compare directly to (2.22) for TEM modes.

4.7 Constraint of wave propagation in $+u_3$ direction

In Section 2.5 we have found that for a TEM mode to propagate in a single direction ($+u_3$ only or $-u_3$ only) without reflection we need the constitutive parameters to be related as

$$g_c(u_3) \equiv g_\epsilon(u_3) = g_\mu(u_3). \tag{4.36}$$

Constraining this here as well the u_3 variation of the medium is then reduced to the two parameters g_c and g_{ϵ_3}. Note again that a limiting case of an E mode is a TEM mode with $E_3^{(0)'} \to 0$.

Now change variables from u_3 to U_3 such that

$$g_c du_3 = dU_3. \tag{4.37}$$

This makes our separation constants in (4.25) through (4.28) become

$$\begin{aligned} K_h &= v^{(0)'} = -s g_e \left[\frac{dg_h}{dU_3}\right]^{-1} \\ \gamma_{e3} &= \gamma_d = \frac{s}{v^{(0)'}} \\ 1 &= \frac{g_{\epsilon 3} g_{e3}}{g_h}, \quad \frac{s}{v^{(0)'}} = \frac{1}{g_e}\frac{d}{dU_3}\left[g_{\epsilon 3} g_{e3}\right]. \end{aligned} \tag{4.38}$$

Returning to (4.32) we have

$$\begin{aligned} \Phi'_e = \frac{E^{(0)'}}{\gamma_g} &= -\frac{1}{\gamma_d \gamma_g}\frac{1}{\epsilon_3^{(0)'}} \nabla'_t \cdot \left[\overleftrightarrow{\epsilon}_t^{(0)'} \cdot \vec{E}^{(0)'}\right] \\ &= \frac{1}{\gamma_d \gamma_g}\frac{1}{\epsilon_3^{(0)'}} \nabla'_t \cdot \left[\overleftrightarrow{\epsilon}_t^{(0)'} \cdot \nabla'_t \Phi'_e\right]. \end{aligned} \tag{4.39}$$

Defining

$$dU_n \equiv \frac{1}{\epsilon_n^{(0)'}} du_n \quad \text{for } n = 1, 2 \tag{4.40}$$

we have the alternate form

$$\Phi'_e = \frac{1}{\gamma_d\gamma_g}\left[\frac{1}{\epsilon_3^{(0)'}\epsilon_1^{(0)'}}\frac{\partial^2\Phi'_e}{\partial U_1^2} + \frac{1}{\epsilon_3^{(0)'}\epsilon_2^{(0)'}}\frac{\partial^2\Phi'_e}{\partial U_2^2}\right]. \tag{4.41}$$

Thus $\gamma_d\gamma_g$ is an eigenvalue giving a transverse wavenumber (propagation constant) which can be computed from (4.39) or (4.41). This applies to waveguide solutions with $\Phi'_e = 0$ on some closed contour C_0 in the u_1, u_2 plane. Note that if $\epsilon_3^{(0)'}\epsilon_1^{(0)'} = \epsilon_3^{(0)'}\epsilon_2^{(0)'} =$ constant then this reduces to an equation of the usual waveguide variety. In any event we have real-valued solutions for Φ'_e with negative eigenvalues as

$$\begin{aligned}\gamma_d\gamma_g &= -K_0^2 = \frac{s}{v^{(0)'}}\gamma_g \\ K_0^2 &= \text{ positive numbers of same dimension as } [s/v^{(0)'}]^2.\end{aligned} \tag{4.42}$$

This parameter is related to the usual waveguide cutoff frequencies for the various modes. If we do not impose a boundary condition on C_0 we can have transversely propagating solutions (complex) and cutoff corresponds to no propagation in the u_3 direction.

The eigenvalues take the form

$$\begin{aligned}K_0^2 = -\gamma_d\gamma_g &= \frac{s}{v^{(0)'}}\frac{g_{\epsilon 3}g_{e3}}{g_h}\left[-\frac{g_c}{g_{e3}}\frac{dg_e}{dU_3} - \frac{s}{v^{(0)'}}\frac{g_c g_h}{g_{e3}}\right] \\ &= -\frac{s}{v^{(0)'}}\frac{g_{\epsilon 3}g_c}{g_h}\frac{dg_e}{dU_3} - \left[\frac{s}{v^{(0)'}}\right]^2 g_{\epsilon 3}g_c \\ &= g_{\epsilon 3}g_c\left[\frac{1}{g_h}\frac{d^2g_h}{dU_3^2} - \left[\frac{s}{v^{(0)'}}\right]^2\right]\end{aligned} \tag{4.43}$$

with a substitution from the first of (4.36). Rearranging we have

$$\left[\left[\frac{s}{v^{(0)'}}\right]^2 + \frac{K_0^2}{g_{\epsilon 3}g_c}\right]g_h = \frac{d^2g_h}{dU_3^2}. \tag{4.44}$$

For a specified $g_{\epsilon 3}g_c$ as a function of u_3, and hence of U_3 we have a wave equation for g_h, and various forms of $g_{\epsilon 3}g_c$ can be chosen which lead to various special functions to describe g_h. Having g_h then (4.38) leads to g_e and g_{e3}.

Suppose we want the mode propagation of g_h to be the same at each U_3 (translation invariance). This constraint means that the cutoff frequency is the same everywhere along a waveguide defined by the contour C_0. Cutoff in (4.22) is defined by

$$\begin{aligned}\left[\frac{s_0}{v^{(0)'}}\right]^2 &= -\left[\frac{\omega_0}{v^{(0)'}}\right]^2 = -\frac{K_0^2}{g_{\epsilon 3}g_c} \\ \omega_0 = -js_0 &= \frac{v^{(0)'}K_0}{[g_{\epsilon 3}g_c]^{1/2}} \\ &\equiv \text{ cutoff (radian) frequency} \\ &\quad \text{(a set of same, real valued)}\end{aligned} \tag{4.45}$$

If these frequencies are to be independent of U_3 (and hence of u_3) then we require that $g_{\epsilon 3} g_c$ be a constant, which we take without loss of generality as

$$g_{\epsilon 3} g_c = 1 \tag{4.46}$$

With this choice then we have

$$\begin{aligned} g_h &= e^{-\gamma U_3} = \exp\left(-\gamma \int_0^{u_3} g_c du_3'\right) \\ \gamma &= \left[\left[\frac{s}{v^{(0)'}}\right]^2 + K_0^2\right]^{1/2} \\ &= \frac{s}{v^{(0)'}}\left[1 + \left[\frac{v^{(0)'} K_0}{s}\right]^2\right]^{1/2} \quad \text{(appropriate form for frequencies far above cutoff)} \end{aligned} \tag{4.47}$$

where the sign for γ has been chosen to give propagation in the $+u_3$ direction. From (4.38) we then have

$$\begin{aligned} g_e &= -\frac{v^{(0)'}}{s}\frac{dg_h}{dU_3} = \frac{v^{(0)'}}{s}\gamma g_h = \frac{v^{(0)'}}{s}\gamma e^{-\gamma U_3} \\ &= \frac{v^{(0)'}}{s}\gamma \exp\left(-\gamma \int_0^{u_3} g_c du_3'\right) \\ g_{e3} &= g_c g_h = g_c e^{-\gamma U_3} = g_c \exp\left(-\gamma \int_0^{u_3} g_c du_3'\right) . \end{aligned} \tag{4.48}$$

Returning to (4.5) we have all three propation functions for the fields. The transverse field components are related in (4.24), and related to $E_3^{(0)'}$ by $\gamma_d = \gamma_{e3}$ in (4.11), (4.12), and (4.27). The factors in the constitutive parameters are related in (4.23), (4.36), and (4.46). The Maxwell equations in (4.6) and (4.7) have been satisfied by construction. Together with (4.39) or (4.41) for computing the eigenvalues (giving cutoff frequencies for waveguides) and Φ_e' from which we can determine $\vec{E}_t^{(0)'}$. When specified, $\epsilon_3^{(0)'}(u_1, u_2)$ appears in the eigenvalue equation and $E_3^{(0)'}$ can be found from $\gamma_g \Phi_e'$. Our E modes are now determined.

Thus we can have E modes in the same media as the TEM modes in Section 2, provided we have an additional constraint on the u_3 part of the permittivity. Specifically $g_{\epsilon 3}$ varies reciprocally with respect to g_c (previously specified) for the special case of K_0 independent of u_3. However, the medium can now be both inhomogeneous and anisotropic.

5 H-Wave (Formal Fields)

In this section the H-wave case (transverse electric field) is studied. We will impose the condition that the formal field component, E_3', vanishes and then look for conditions on the formal parameters, ϵ_i' and μ_i', which lead to solutions of the formal Maxwell equations. The analysis is dual to that of section 4, in which the E-wave case was studied, and the results will be dual. (Duality is the symmetry on interchange of electric and magnetic parameters.) Thus solutions will be sought for the formal field components in terms of an operator on H_3' which will eventually be taken as some mode function of the transverse coordinates, u_1 and u_2, multiplied by a propagation function of u_3. Our analysis thus begins with Maxwell's curl equations, (1.6) and (1.7) which result from $\nabla' \times \vec{E}' = -s\overleftrightarrow{\mu}' \cdot \vec{H}'$ and $\nabla' \times \vec{H}' = s\overleftrightarrow{\epsilon}' \cdot \vec{E}'$.

5.1 Maxwell's Equations with $E_3' = 0$

We take $E_3' = 0$ in (1.6) and (1.7) and obtain the duals of (4.1) and (4.2)

$$\begin{aligned} \frac{\partial E_2'}{\partial u_3} &= s\mu_1' H_1' \\ \frac{\partial E_1'}{\partial u_3} &= -s\mu_2' H_2' \\ \frac{\partial E_2'}{\partial u_1} - \frac{\partial E_1'}{\partial u_2} &= -s\mu_3' H_3' \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \frac{\partial H_3'}{\partial u_2} - \frac{\partial H_2'}{\partial u_3} &= s\epsilon_1' E_1' \\ \frac{\partial H_1'}{\partial u_3} - \frac{\partial H_3'}{\partial u_1} &= s\epsilon_2' E_2' \\ \frac{\partial H_2'}{\partial u_1} - \frac{\partial H_1'}{\partial u_2} &= 0. \end{aligned} \tag{5.2}$$

From our assumption that $E_3' = 0$ we once again will obtain restrictions on the parameters, e_i' and μ_i' (except for ϵ_3' since $E_3' = 0$) as well as solutions for the formal fields E_i' and H_i'.

The assumptions made in Section 2.1 on the form of the formal parameters remain in effect and thus we take

$$\begin{aligned} \mu_n' &= \mu_n^{(0)'}(u_1, u_2) g_\mu(u_3) \\ \epsilon_n' &= \epsilon_n^{(0)'}(u_1, u_2) g_\epsilon(u_3) \end{aligned} \tag{5.3}$$

for $n = 1, 2$. The parameter ϵ_3' is irrelevant since we take u_3 as the propagation direction, while

$$\mu_3' = \mu_3^{(0)'}(u_1, u_2) g_{\mu 3}(u_3). \tag{5.4}$$

We try solutions of the form

$$\begin{aligned}
\vec{E}'_t &= \vec{E}^{(0)'}_t(u_1, u_2) g_e(u_3) \\
\vec{H}'_t &= \vec{H}^{(0)'}_t(u_1, u_2) g_h(u_3) \\
H'_3 &= H^{(0)'}_3(u_1, u_2) g_{h3}\ ,\ E'_3 = 0 \\
\vec{D}' &= \overleftrightarrow{\epsilon} \cdot \vec{E}'\ ,\ \vec{B}' = \overleftrightarrow{\mu} \cdot \vec{H}'
\end{aligned} \tag{5.5}$$

Just as in Section 4, g_μ and g_ϵ are real, nonzero, and frequency independent. Similarly the propagation functions g_e, g_h, and g_{h3} are in general complex functions of the complex frequency s (of exponential type).

Thus we obtain the H-wave duals of (4.6) through (4.9) as

$$\begin{aligned}
E^{(0)'}_2 \frac{dg_e}{du_3} &= s\mu^{(0)'}_1 g_\mu g_h H^{(0)'}_1 \\
E^{(0)'}_1 \frac{dg_e}{du_3} &= -s\mu^{(0)'}_2 g_\mu g_h H^{(0)'}_2 \\
g_e \left[\frac{\partial E^{(0)'}_2}{\partial u_1} - \frac{\partial E^{(0)'}_1}{\partial u_2} \right] &= -s\mu^{(0)'}_3 g_{h3} g_{\mu 3} H^{(0)'}_3
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
g_{h3} \frac{dH^{(0)'}_3}{du_3} - \frac{dg_h}{du_3} H^{(0)'}_2 &= s\epsilon^{(0)'}_1 g_\epsilon g_e E^{(0)'}_1 \\
\frac{dg_h}{du_3} H^{(0)'}_1 - g_{h3} \frac{\partial H^{(0)'}_3}{\partial u_1} &= s\epsilon^{(0)'}_2 g_\epsilon g_e E^{(0)'}_2 \\
\frac{\partial H^{(0)'}_2}{\partial u_1} - \frac{\partial H^{(0)'}_1}{\partial u_2} &= 0.
\end{aligned} \tag{5.7}$$

In addition, from $\nabla' \cdot \left[\overleftrightarrow{\mu}' \cdot \vec{H}' \right] = 0$, we have

$$\frac{\partial}{\partial u_1}(\mu^{(0)'}_1 H^{(0)'}_1) + \frac{\partial}{\partial u_2}(\mu^{(0)'}_2 H^{(0)'}_2) = -\frac{1}{g_\mu g_h} \frac{\partial}{\partial u_3}(g_{\mu 3} g_{h3}) \mu^{(0)'}_3 H^{(0)'}_3 . \tag{5.8}$$

Moreover from $\nabla' \cdot \left[\overleftrightarrow{\epsilon}' \cdot \vec{E}' \right] = 0$, we have

$$\nabla'_t \cdot \left[\overleftrightarrow{\epsilon}^{(0)'}_t \cdot \vec{E}^{(0)'}_t \right] = 0. \tag{5.9}$$

5.2 Summary of Results for H-Wave Case

In Section 4.2 constants K_h, v_0, γ_{e3}, and γ_d were obtained by separating variables in Maxwell's equation. Similarly, a dual collection of constants arise in the present case with the interchange of media parameters and electric and magnetic field components.

Summarizing, we have a first set of relations

$$
\begin{aligned}
v^{(0)'-2} &= \mu_2^{(0)'} \epsilon_1^{(0)'} = \mu_1^{(0)'} \epsilon_2^{(0)'} = \text{constant} \\
\vec{D}_t^{(0)'} \cdot \vec{H}_t^{(0)'} &= 0 \ , \quad \vec{E}_t^{(0)'} \cdot \vec{B}_t^{(0)'} = 0 \\
\frac{E_1^{(0)'}}{H_2^{(0)'}} &= \left[\frac{\mu_2^{(0)'}}{\epsilon_1^{(0)'}}\right]^{\frac{1}{2}} \equiv Z_1^{(0)'} = \frac{1}{\epsilon_1^{(0)'} v^{(0)'}} = v^{(0)'} \mu_2^{(0)'} \\
\frac{E_2^{(0)'}}{H_1^{(0)'}} &= -\left[\frac{\mu_1^{(0)'}}{\epsilon_2^{(0)'}}\right]^{\frac{1}{2}} \equiv Z_2^{(0)'} = -\frac{1}{\epsilon_2^{(0)'} v^{(0)'}} = -v^{(0)'} \mu_1^{(0)'}
\end{aligned}
\tag{5.10}
$$

Analogous to (4.32) there is a differential equation for $H_3^{(0)'}$ as

$$
\Phi_h' = \frac{H_3^{(0)'}}{\gamma_g} = \frac{1}{\gamma_d \gamma_g} \frac{1}{\mu_3^{(0)'}} \nabla_t' \cdot \left[\overleftrightarrow{\mu}_t^{(0)'} \cdot \nabla_t' \Phi_h'\right]
\tag{5.11}
$$

For appropriate boundary conditions for the magnetic field (normal derivative of Φ_h' zero on some closed contour (perfectly conducting boundary) in the u_1, u_2 plane) we have a waveguide problem for Φ_h' with $\gamma_d \gamma_g$ assuming the role of an eigenvalue (a transverse wave number or propagation constant). Again we set

$$
\begin{aligned}
\gamma_d \gamma_g &= -K_0^2 = \frac{s}{v_0'} \gamma_g \\
K_0^2 &= \text{ positive numbers of same} \\
&\qquad \text{dimension as } [s/v^{(0)'}]^2
\end{aligned}
\tag{5.12}
$$

As before various separation constants are determined, including

$$
K_e = v^{(0)'} \ , \quad \gamma_{h3} = \gamma_d = \frac{s}{v^{(0)'}} .
\tag{5.13}
$$

Making the wave propagate in a single direction without reflection (say $+u_3$) gives

$$
g_c(u_3) \equiv g_\mu(u_3) = g_\epsilon(u_3)
\tag{5.14}
$$

Like (4.44) we have the basic propagation equation

$$
\begin{aligned}
\left[\left[\frac{s}{v^{(0)'}}\right]^2 + \frac{K_0^2}{g_{\mu 3} g_c}\right] g_h &= \frac{d^2 g_h}{dU_3^2} \\
g_c du_3 &= dU_3
\end{aligned}
\tag{5.15}
$$

The cutoff frequency being independent of U_3 leads to

$$
\begin{aligned}
\omega_0 = -js_0 &= \frac{v^{(0)'} K_0}{[g_{\mu 3} g_c]^{\frac{1}{2}}} \\
&\equiv \text{ cutoff (radian) frequency} \\
&\qquad \text{(a set of same, real valued)} \\
g_{\mu 3} g_c &= 1
\end{aligned}
\tag{5.16}
$$

Then we have

$$
\begin{aligned}
g_e &= e^{-\gamma U_3} = \exp\left(-\gamma \int_0^{u_3} g_c du_3'\right) \\
\gamma &= \frac{s}{v^{(0)'}}\left[1 + \left[\frac{v^{(0)'} K_0}{s}\right]\right]^{\frac{1}{2}} \\
g_h &= \frac{v^{(0)'}}{s}\gamma \exp\left(-\gamma \int_0^{u_3} g_c du_3'\right) \\
g_{h3} &= g_c \exp\left(-\gamma \int_0^{u_3} g_c du_3'\right)
\end{aligned}
\tag{5.17}
$$

Thus H modes can propagate in the same media as the TEM modes provided $g_{\mu 3}(u_3)$ is constrained to vary inversely with respect to $g_c(u_3)$ (previously specified).

6 Scaling to Real Medium for E-Modes

As in Section 3 for the TEM case let us now consider the scaling of the u_n coordinates to something other than Cartesian coordinates (for which the results in Section 4 are directly applicable). The scaling relations are given in (3.1) and (3.2).

6.1 General Results

The transverse components (1 and 2 subscripts) are related as in (3.3) and (3.4) apply here as well. These are supplemented by

$$\begin{aligned} \epsilon_3^{(0)'}(u_1,u_2)\; g_c^{-1}(u_3) &= \frac{h_1 h_2}{h_3}\, \epsilon_3 \\ \epsilon_1^{(0)'}(u_1,u_2)\; \epsilon_3^{(0)'}(u_1,u_2) &= h_2^2 \epsilon_1 \epsilon_3 \\ \epsilon_2^{(0)'}(u_1,u_2)\; \epsilon_3^{(0)'}(u_1,u_2) &= h_1^2 \epsilon_2 \epsilon_3 \end{aligned} \tag{6.1}$$

6.2 Isotropic Real Medium

If the real medium is constrained to be isotropic we have

$$\begin{aligned} \epsilon &= \epsilon_1 = \epsilon_2 = \epsilon_3 \\ \mu &= \mu_1 = \mu_2 (= \mu_3) \end{aligned} \tag{6.2}$$

so that we have five relevant constitutive-parameter components to consider, one more than in the TEM case. The results of Section 3.2 all follow. In particular we have

$$\begin{aligned} \epsilon_1^{(0)'} &= \epsilon_2^{(0)'} \equiv \epsilon_\tau^{(0)'} \quad , \quad \mu_1^{(0)'} = \mu_2^{(0)'} \equiv \mu^{(0)'} \\ h_1 &= h_2 \equiv h_t \end{aligned} \tag{6.3}$$

with surfaces of constant u_3 limited to spheres and planes (Appendix B).

Now (6.1) reduces to

$$\begin{aligned} \epsilon_3^{(0)'}(u_1,u_2) g_c^{-1}(u_3) &= \frac{h_t^2}{h_3}\, \epsilon \\ \epsilon_t^{(0)'}(u_1,u_2) \epsilon_3^{(0)'}(u_1,u_2) &= h_t^2\, \epsilon^2 \end{aligned} \tag{6.4}$$

giving

$$\epsilon_t^{(0)'}(u_1,u_2) g_c(u_3) = h_3\, \epsilon. \tag{6.5}$$

Furthermore we have

$$\begin{aligned}\mu_t^{(0)'}(u_1,u_2)g_c(u_3) =& h_3\ \mu \\ \frac{\mu_t^{(0)'(u_1,u_2)}}{\epsilon_t^{(0)'}(u_1,u_2)} =& \frac{\mu}{\epsilon} = \text{function of } u_1, u_2 \text{ only} \\ v' = g_c(u_3)v^{(0)'} =& \text{ TEM speed in } u_n \text{ coordinates} \\ =& \text{ function of } u_3 \text{ only} \\ v = [\mu\epsilon]^{-\frac{1}{2}} = h_3 v' =& \text{ real TEM speed}\end{aligned} \tag{6.6}$$

6.3 Uniform Real Permeability

Add the requirement that

$$\mu = \mu_0 \quad \text{(uniform and isotropic)} \tag{6.7}$$

while ϵ is only isotropic and use (6.3) for isotropic transverse components $\epsilon_t^{(0)'}$ and $\mu_t^{(0)'}$. This brings in the results of Section 3.4, specifically

$$\begin{aligned}Z =& \left[\frac{\mu_0}{\epsilon}\right]^{\frac{1}{2}} \\ \epsilon =& \epsilon(u_1,u_2) \\ v =& [\mu_0\epsilon]^{-\frac{1}{2}} = v(u_1,u_2) \\ h_3 =& \frac{v(u_1,u_2)}{v^{(0)'}}\ g_c(u_3)\end{aligned} \tag{6.8}$$

From (6.4) we then have

$$h_t = \frac{\left[\epsilon_t^{(0)'}(u_1,u_2)\ \epsilon_3^{(0)'}(u_1,u_2)\right]^{\frac{1}{2}}}{\epsilon(u_1,u_2)} = \text{ function of } u_1, u_2 \text{ only} \tag{6.9}$$

From Appendix B we have that $h_t(u_1,u_2)$ implies that surfaces of constant u_3 can only be planes. It is interesting to note that such a case of a bending lens with constant-ϕ surfaces being planes is considered in [3].

At this point we can note that it is possible to have a simpler form for h_3 as

$$h_3 = h_3(u_1,u_2) = \frac{v(u_1,u_2)}{v^{(0)'}} = \text{ function of } u_1, u_2 \text{ only} \tag{6.10}$$

by setting

$$g_c \equiv 1 = \frac{1}{g_{\epsilon 3}}. \tag{6.11}$$

This corresponds to choosing the formal medium to be uniform with respect to the u_3 coordinate, a somewhat simpler form for the formal medium.

6.4 Isotropic Real and Formal Media

If one in addition were to force the formal permittivity to be isotropic, then

$$\begin{aligned} \epsilon_3' &= \epsilon_3^{(0)'} g_c^{-1} = \epsilon_t^{(0)'} g_c = \epsilon_t' \equiv \epsilon' \\ g_c^2 &= \frac{\epsilon_3^{(0)'}}{\epsilon_t^{(0)'}} \neq \text{function of } u_3. \end{aligned} \tag{6.12}$$

Then as a matter of convention we set

$$\begin{aligned} g_c &= 1 \\ h_t &= h_3 \equiv h \\ \epsilon' &= h\epsilon \ , \ \ \mu' = h\mu. \end{aligned} \tag{6.13}$$

This gives the case in [11, (Appendix C)] which admits of only two types of solutions: Cartesian coordinates and the inversion of Cartesian coordinates.

This is a very restrictive case, so the constraint in (6.6) is not very significant, and we can allow the formal permittivity to be anisotropic.

7 Scaling to Real Medium for H-Modes

In Sections 3 and 6 the introduction of a scaling of the u_n coordinates to general orthogonal curvilinear coordinates was considered, for the cases of real TEM fields and real E-modes. The basic scaling relations appear in (3.1) and (3.2). We now consider the results of Section 5, which relate to the formal fields in the H-wave case.

7.1 General Results

Analogous to the E-wave case, the transverse components (corresponding to the subscripts 1 and 2) are related as in (3.3) and (3.4). In addition we have

$$\begin{aligned} \mu_3^{(0)'}(u_1,u_2)g_c^{-1}(u_3) =& \frac{h_1h_2}{h_3}\mu_3 \\ \mu_1^{(0)'}(u_1,u_2)\mu_3^{(0)'}(u_1,u_2) =& h_2^2\mu_1\mu_3 \\ \mu_2^{(0)'}(u_1,u_2)\mu_3^{(0)'}(u_1,u_2) =& h_1^2\mu_2\mu_3 \end{aligned} \tag{7.1}$$

7.2 Isotropic Real Medium

If the real medium is constrained to be isotropic we have

$$\begin{aligned} \epsilon =& \epsilon_1 = \epsilon_2 \\ \mu =& \mu_1 = \mu_2 = \mu_3. \end{aligned} \tag{7.2}$$

Hence we have five constitutive parameters to consider as in the case of an E-wave. The results of Section 3.2 all follow. In particular we will have

$$\begin{aligned} \epsilon_1^{(0)'} =& \epsilon_2^{(0)'} = \epsilon^{(0)'} \quad , \quad \mu_1^{(0)'} = \mu_2^{(0)'} \equiv \mu_t^{(0)'} \\ h_1 =& h_2 \equiv h_t \end{aligned} \tag{7.3}$$

and surfaces of constant u_3 are spheres or planes. The results of (6.4) also directly apply.

7.3 Uniform Real Permeability

Now instead of constraining uniform real permittivity (leading to the same results as in Section 6.3), let us constrain the practical case of uniform real permeability as

$$\mu = \mu_0 \tag{7.4}$$

while ϵ (ϵ_3 being irrelevant) is isotropic and from (6.3) the transverse components $\epsilon_t^{(0)'}$ and $\mu_t^{(0)'}$ are isotropic. Section 3.4 still applies giving

$$\begin{aligned} Z &= \left[\frac{\mu_0}{\epsilon}\right]^{\frac{1}{2}} \\ \epsilon &= \epsilon(u_1, u_2) \\ v &= [\mu_0 \epsilon]^{-\frac{1}{2}} = v(u_1, u_2) \\ h_3 &= \frac{v(u_1, u_2)}{v^{(0)'}}\, g_c(u_3) \end{aligned} \tag{7.5}$$

From (7.1) we have

$$h_t = \frac{\left[\mu_t^{(0)'}(u_1, u_2)\mu_3^{(0)'}(u_1, u_2)\right]^{\frac{1}{2}}}{\mu_0} = \text{function of } u_1, u_2 \text{ only} \tag{7.6}$$

Again $h_t(u_1, u_2)$ implies that surfaces of constant u_3 can only be planes.

8 Concluding Remarks

We now have a significant set of results for TEM, E, and H modes. The basic form for those is found by separating out the u_3 (propagation) coordinate from the u_1, u_2 (transverse) coordinates, and requiring propagation in one direction without reflection. Various assumptions (constraints) on the constitutive parameters lead to constraints on the allowable coordinate systems. We can note that these results apply only to such modes, and not to all possible solutions of the Maxwell equations which may include additional contributions (e.g., hybrid HE modes).

A related problem is treated by Friedman[7]. Those results have some similarities to and differences from the present results. There only $\mu = \mu_0$ was considered. His results are based on those of Bromwich[12]. The results had the decomposition of the fields into unique E- and H-mode parts. There h_3 was found to be a function of u_3 only, but our present results allow for more general h_3, specifically as a function of u_1 and u_2 as well.

In order to understand the differences in these results, it is instructive to quote from Bromwich:

> "The proof given below refers specially to spherical polar coordinates, and to problems in which the whole of angular space is used. The earlier part of the proof is arranged so that it can be applied to other types of orthogonal coordinates; but the details of the final reasoning need modification, and must be adapted in other cases so as to suit the problem in hand."

The reader will note that spherical polar coordinates are assumed by Bromwich (and, hence, Friedman). Furthermore, they use the whole of angular (θ, ϕ) space. This would logically lead to the usual spherical modal decomposition. In our case, however, we allow for the introduction of perfectly conducting boundaries such as encountered in waveguide (including open TEM) problems. As Bromwich recognized (above) other types of solutions might be possible (e.g., ours).

Nevertheless, there are some remarkable similarities in the results. In particular h_1/h_2 is independent of u_3. Constraining the practical case of $\mu = \mu_0$ we all have $\epsilon_1 = \epsilon_2$ with some freedom for ϵ_3. In our case, however ϵ_1 can be a function of u_1 and u_2. This leads to a nontrivial example of a bending lens in which surfaces of constant u_3 are *nonparallel* planes.

The present results also allow for more general anisotropic real and/or formal media to be considered, including for the case of TEM modes. This may lead to other interesting cases for transient lens design.

Note the fundamental assumption of $\vec{E}$ and $\vec{H}$ each having both u_1 and u_2 components. This could be relaxed by allowing the fields to have only one transverse component (e.g., E_1 and H_2). So there are various possible other cases to consider.

Appendix A. Multiplicity of E-Modes

Another approach to the constraints on $\overleftrightarrow{p}$ (Section 4.4) is found by setting

$$\overleftrightarrow{p} \equiv \begin{pmatrix} \alpha_1(u_1,u_2) & 0 \\ 0 & \alpha_2(u_1,u_2) \end{pmatrix} = \begin{pmatrix} \mu_2^{(0)'}\epsilon_1^{(0)'} & 0 \\ 0 & \mu_1^{(0)'}\epsilon_2^{(0)'} \end{pmatrix} \tag{A.1}$$

Then consider a set of potentials $\Phi^{(n)}$ corresponding to Φ'_e in (4.16) and requiring the various equations to hold for all of those.

Expansion of (A.1) yields

$$\begin{aligned} 0 =& \nabla'_t \cdot \begin{pmatrix} -\alpha_2 & \dfrac{\partial \Phi'_e}{\partial u_2} \\ \alpha_1 & \dfrac{\partial \Phi'_e}{\partial u_1} \end{pmatrix} = -\frac{\partial}{\partial u_1}\left(\alpha_2 \frac{\partial \Phi'_e}{\partial u_2}\right) + \frac{\partial}{\partial u_2}\left(\alpha_1 \frac{\partial \Phi'_e}{\partial u_1}\right) \\ =& -\frac{\partial \alpha_2}{\partial u_1}\frac{\partial \Phi'_e}{\partial u_2} - \alpha_2 \frac{\partial^2 \Phi'_e}{\partial u_1 \partial U_2} + \frac{\partial \alpha_1}{\partial u_2}\frac{\partial \Phi'_e}{\partial u_1} + \alpha_1 \frac{\partial^2 \Phi'_e}{\partial u_2 \partial u_1} \\ =& \begin{pmatrix} \dfrac{\partial \alpha_1}{\partial u_2} \\ -\dfrac{\partial \alpha_2}{\partial u_1} \end{pmatrix} \cdot \nabla'_t \Phi'_e + (\alpha_1 - \alpha_2)\frac{\partial^2 \Phi'_e}{\partial u_1 \partial u_2}. \end{aligned} \tag{A.2}$$

Next, let $\nabla'_t\Phi^{(1)}$ and $\nabla'_t\Phi^{(2)}$ be nonzero, noncollinear vectors at an arbitrary point (u_1, u_2). These vectors span a two dimensional space. Hence we may take

$$\nabla'_t \Phi'_e = \delta_1 \nabla'_t \Phi^{(1)} + \delta_2 \nabla'_t \Phi^{(2)} \neq 0 \tag{A.3}$$

with δ_1 and δ_2 chosen at our convenience. Hence, we choose δ_1 and δ_2 so that

$$\begin{pmatrix} \dfrac{\partial \alpha_1}{\partial u_2} \\ -\dfrac{\partial \alpha_2}{\partial u_1} \end{pmatrix} cdot \nabla'_\tau \Phi'_e = \begin{pmatrix} \dfrac{\partial \alpha_1}{\partial u_2} \\ -\dfrac{\partial \alpha_2}{\partial u_1} \end{pmatrix} \cdot \left[\delta_1 \nabla \Phi^{(1)} + \delta_2 \nabla \Phi^{(2)}\right] = 0. \tag{A.4}$$

At least one of these dot products must be nonzero since, by hypothesis, $\nabla'_t\Phi^{(1)}$ and $\nabla'_t\Phi^{(2)}$ are not collinear. We note that one of δ_1, δ_2 may be zero if one of the dot products

$$\begin{pmatrix} \dfrac{\partial \alpha_1}{\partial u_2} \\ -\dfrac{\partial \alpha_2}{\partial u_1} \end{pmatrix} \cdot \nabla'_t \Phi^{(1)}, \quad \text{or} \quad \begin{pmatrix} \dfrac{\partial \alpha_1}{\partial u_2} \\ -\dfrac{\partial \alpha_2}{\partial u_1} \end{pmatrix} \cdot \nabla'_t \Phi^{(2)} \tag{A.5}$$

vanishes. Thus, from (A.3) and (A.4) we have

$$(\alpha_1 - \alpha_2)\left[\delta_1 \frac{\partial^2 \Phi^{(1)}}{\partial u_1 \partial u_2} + \delta_2 \frac{\partial^2 \Phi^{(2)}}{\partial u_1 \partial u_2}\right] = 0. \tag{A.6}$$

Since we have the freedom to choose $\Phi^{(n)}$ we can assume one of the two terms in (A.6) is nonzero. Hence

$$\alpha_1 - \alpha_2 = 0 \tag{A.7}$$

and thus we take

$$\alpha \equiv \alpha_1 = \alpha_2 \tag{A.8}$$

which yields [9]

$$\begin{aligned} 0 =& \nabla'_t \cdot \left[\alpha \begin{pmatrix} -\frac{\partial \Phi'_e}{\partial u_2} \\ \frac{\partial \Phi'_e}{\partial u_1} \end{pmatrix} \right] \\ 0 =& \alpha \nabla'_t \cdot \begin{pmatrix} -\frac{\partial \Phi'_e}{\partial u_2} \\ \frac{\partial \Phi'_e}{\partial u_1} \end{pmatrix} + (\nabla'_t \alpha) \cdot \begin{pmatrix} -\frac{\partial \Phi'_e}{\partial u_2} \\ \frac{\partial \Phi'_e}{\partial u_1} \end{pmatrix} \end{aligned} \tag{A.9}$$

and hence

$$(\nabla'_t \alpha) \cdot \begin{pmatrix} -\frac{\partial \Phi'_e}{\partial u_2} \\ \frac{\partial \Phi'_e}{\partial u_1} \end{pmatrix} = 0. \tag{A.10}$$

Again, because of the freedom to choose Φ, let us choose $\Phi^{(3)}$ and $\Phi^{(4)}$ as before so that $\nabla'_\tau \Phi^{(3)}$ and $\nabla'_\tau \Phi^{(4)}$ span another 2-dimensional space at the point u_1, u_2, which is arbitrary. We then conclude that

$$\nabla_t \alpha = \vec{0} \; . \tag{A.11}$$

Thus we have

$$\alpha = \text{ constant (neither a function of } u_1 \text{ nor } u_2). \tag{A.12}$$

While there may be isolated points where this does not apply, due to singularities in $\nabla'_\tau \Phi'_e$ this is of no consequence here. Hence the form in (4.23) results. This constraint need not be applicable in some special cases. For example, in the case of unipolarized transverse fields only one of $\mu_2^{(0)'} \epsilon_1^{(0)'}$ or $\mu_1^{(0)'} \epsilon_2^{(0)'}$ need be considered [4].

Appendix B. Characteristics of Coordinates for the Case of Equal Transverse Scale Factors

In [6, (pp. 114-117, 146-149), 1(Appendix B), 11(Section 2.4)] there is the result that surfaces of constant u_3 for the case of

$$h_1 = h_2 \equiv h_t \tag{B.1}$$

can only be planes or spheres. Here we carry the considerations a little further.

For the second fundamental form for a constant u_3 surface we have the coefficients

$$D_3 = -\frac{h_1}{h_3}\frac{\partial h_1}{\partial u_3} \;\; ; \;\; D_3'' = -\frac{h_2}{h_3}\frac{\partial h_2}{\partial u_3} \tag{B.2}$$

for an orthogonal curvilinear system ($D_3' = 0$). Following Eisenhart we have

$$\frac{D_3}{h_1^2} = -\frac{1}{h_3 h_t}\frac{\partial h_t}{\partial u_3} = \frac{D_3''}{h_2^2} = -\lambda \tag{B.3}$$

so that the first (h_t^2) and second fundamental coefficients are in proportion. Such a surface satisfies

$$[\lambda x + a_x]^2 + [\lambda y + a_y]^2 + [\lambda z + a_z]^2 = 1 \tag{B.4}$$

which is the general equation of a sphere with the special case of $\lambda = 0$ giving a plane.

Applying these results to the case of E modes in Section 6.3, we have the result for isotropic real media that

$$h_t = h_t(u_1, u_2) = \text{ function of } u_1, u_2 \text{ only.} \tag{B.5}$$

Placing this form in (B.3) we have

$$\frac{\partial h_t}{\partial u_3} = 0 \;\; , \;\; \lambda = 0 \tag{B.6}$$

and surfaces of constant u_3 can only be planes.

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