Sensor and Simulation Notes

Note 436

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Admittance of Bent TEM Waveguides in a CID Medium

Carl E. Baum
Air Force Research Laboratory
Directed Energy Directorate

Abstract

This paper considers the effect on the characteristic admittance of bending a TEM waveguide with the uniform dielectric medium replaced by a special cylindrically-inhomogeneous dielectric (CID) which preserves the TEM character of the propagation. First some general properties of guides with certain symmetries in their cross sections are considered, showing a second order correction in terms of bend curvature. Canonical H-plane and E-plane bends are solved in closed form. The case of a bent circular coax is solved up through second order in curvature.

1. Introduction

Various solutions have been developed for the propagation of TEM waves in an inhomogeneous dielectric medium with permittivity ε proportional to Ψ^{-2} in a cylindrical (Ψ, ϕ, z) coordinate system with propagation in the ϕ direction [1-5, 7]. Experimental work is underway to approximately synthesize such a medium with guiding conductors to form a TEM-transmission-line bend [6]. There are also cases of TEM waves propagating in other directions in such a medium [8]. The general procedures are differential-geometric lens synthesis as discussed in [11].

This medium then has very special properties. So let us give it a name: cylindrically-inhomogeneous dielectric or CID for short. (El Cid is a spanish title, roughly translating as lord or sir.) The particular form of inhomogeneity as

$$\frac{\varepsilon}{\varepsilon_m} = \left[\frac{\Psi_m}{\Psi}\right]^2 \quad , \quad \mu = \mu_0 \tag{1.1}$$

is the form to be implied by this name. Here ε_m are convenient reference permittivity and radius respectively. In the present context these refer to the middle of the lens cross section.

With Ψ_m as the center of the waveguide with respect to the Ψ coordinate, let the guide extend a maximum distance b on either side of Ψ_m. In the present cases the fields are all confined to $|\Psi - \Psi_m| \leq b$. Then we have

$$\begin{aligned} \Psi_m &\equiv \text{radius of curvature of guide (referenced to center)} \\ \kappa &\equiv \frac{b}{\Psi_m} \equiv \text{normalized curvature of guide} \end{aligned} \tag{1.2}$$

This κ is an important parameter and as we shall see the guide admittance has a correction proportional to κ^2 for appropriately symmetrical guide cross sections.

As discussed in [4, 5] the electric potential Φ for the TEM mode satisfies

$$\Psi \frac{\partial}{\partial \Psi}\left[\Psi^{-1} \frac{\partial \Phi}{\partial \Psi}\right] + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{1.3}$$

which is similar to a Laplace equation on a cross section (constant ϕ). Here Φ is a function of Ψ and z, but other coordinates such as cylindrical coordinates centered on the cross section are more appropriate for the circular coax case. Note that there is a propagation function for the ϕ coordinate not included in the above, as well as a waveform function. One can also write an analogous equation for the magnetic potential, but we will not need it here. Further details are included in the references.

In the limit that $\Psi_m \to \infty$, the waveguide becomes straight and the dielectric is uniform in the guide cross section, changing (1.3) into a true Laplace equation as

$$\nabla_t^2 \Phi_0 = 0 \tag{1.4}$$

with whatever transverse (subscript t) coordinates one wishes. This reference case is denoted by the subscript 0 and we later have occasion to write

$$\Phi = \Phi_0 + \Delta\Phi \tag{1.5}$$

as a perturbation. For our cross-section boundary value problem we take our electric conductors as V_0 or 0 potential (volts) giving boundary conditions for Φ and Φ_0, $\Delta\Phi$ having zero for boundary conditions. The current I, found from a line integral of the magnetic field is used to define the characteristics admittance as

$$Y_c = Z_c^{-1} = \frac{V_0}{I} \quad , \quad Y_{c_0} = Z_{c_0}^{-1} = \frac{V_0}{I_0}$$

$$I_0 = \text{current for } \Psi_m = \infty \text{ (straight guide)} \tag{1.6}$$

The electric field for the cross section is given by

$$\vec{E} = -\nabla_t \Phi \tag{1.7}$$

with coordinates as appropriate. The magnetic field for the cross section is given by

$$\vec{H} = Y \, \vec{1}_\phi \times \vec{E}$$

$$Y = Z^{-1} = \left[\frac{\varepsilon}{\mu_0}\right]^{\frac{1}{2}} = Y_m \frac{\Psi_m}{\Psi} = \text{wave admittance} \tag{1.8}$$

where propagation has been taken in the $+\phi$ direction. Note that for $\Psi_m = \infty$ the wave admittance become uniformly Y_m in the guide.

The basic problem considered in this paper is the variation of the characteristic admittance $Y_c(\kappa)$ as a function of κ. After two simple canonical cases we concentrate on the important case of a circular coax.

2. Symmetry Considerations

To aid in the analysis let us consider certain kinds of symmetrical guide cross sections. Specifically we consider symmetries under which reflection and or rotation of the bent guide reverses the direction of bend ($\kappa \to -\kappa$) while conserving the characteristic admittance of the guide.

Consider first reflection symmetry in the cross section ($\phi = 0$) about a symmetry line (z' axis) at $\Psi = \Psi_m$ as in Fig. 2.1A. Consider first that as looking into the page the guide is bent to the left (positive Ψ_m, positive κ). Now consider reflection of the entire bent guide through a plane containing the z' axis and perpendicular to the page. The guide has a rotation axis z'', a distance $2\Psi_m$ to the right of the original rotation axis. The guide now bends right as one moves into the page which we can consider as a negative normalized curvature κ, i.e., $\kappa \to -\kappa$ in the transformation. Using Ψ' on the $\phi = 0$ plane measured from the z' axis the potential transforms from case 1 (left band) to case 2 (right bend) as

$$\Phi^{(1)}(\Psi', z') = \Phi^{(2)}(-\Psi', z') \tag{2.1}$$

This is a mirroring of the potential as discussed in [9]. The associated fields transform as

$$\begin{aligned} E_{\Psi'}^{(1)}(\Psi',z') &= -E_{\Psi'}^{(2)}(-\Psi',z') \\ E_{z'}^{(1)}(\Psi',z') &= E_{z'}^{(2)}(-\Psi',z') \\ H_{\Psi'}^{(1)}(\Psi',z') &= H_{\Psi'}^{(2)}(-\Psi',z') \\ H_{z'}^{(1)}(\Psi',z') &= -H_{z'}^{(2)}(-\Psi',z') \end{aligned} \tag{2.2}$$

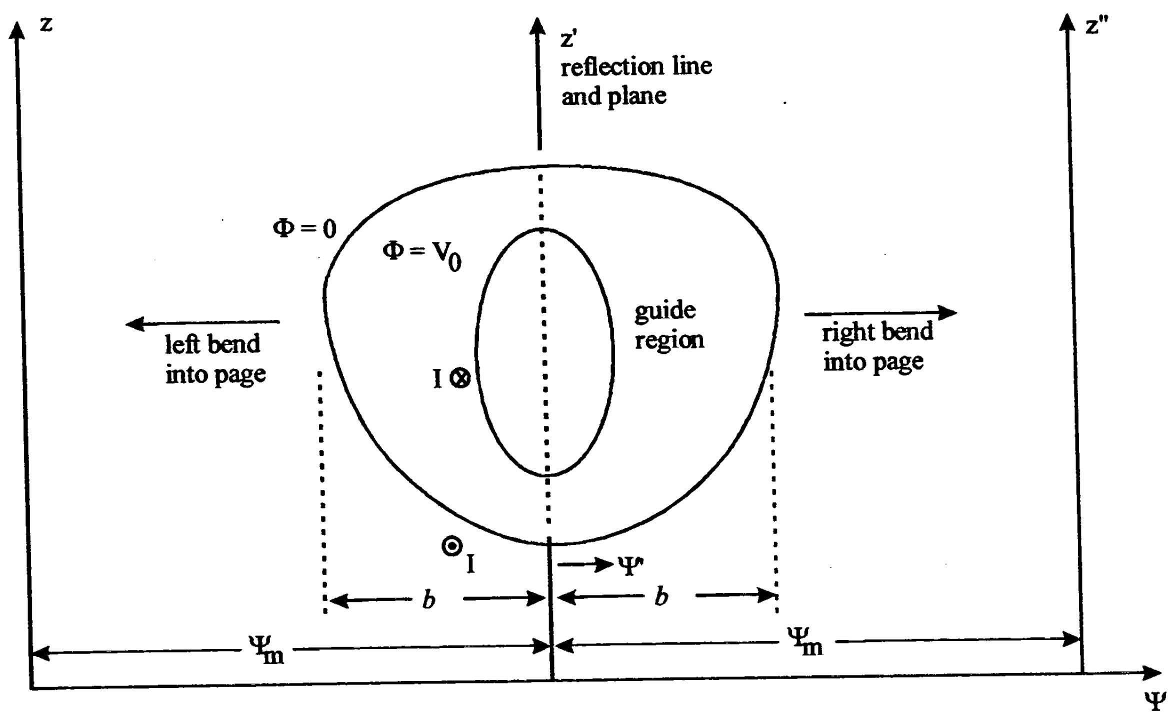
The permittivity transforms like the potential, i.e.,

$$\varepsilon^{(1)}(\Psi') = \varepsilon^{(2)}(-\Psi') \tag{2.3}$$

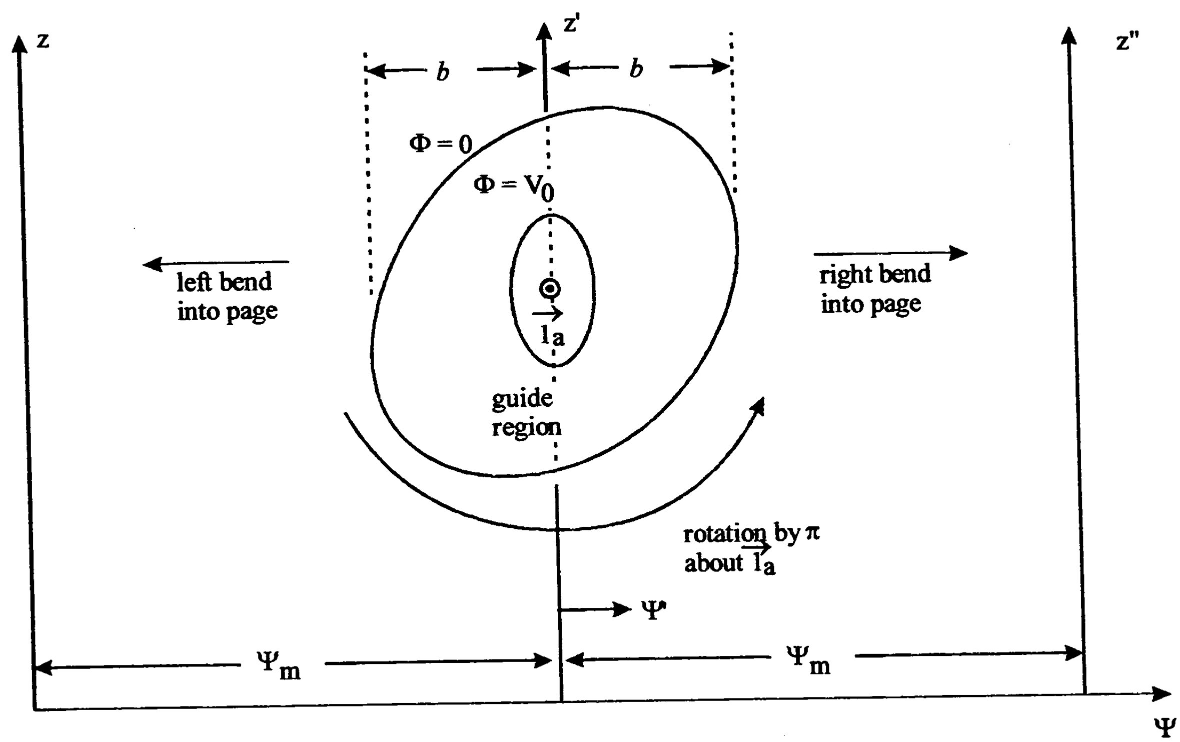
The potential V_0 and the current I (from a line integral around the center conductor) being unchanged in the $1 \to 2$ transformation, the characteristic admittance is unchanged, i.e.,

$$Y_c^{(1)}(\kappa) = Y_c^{(2)}(-\kappa) \tag{2.4}$$

which is an even function of the bend curvature.



A. Reflection symmetry on $\phi = 0$ plane about z' axis



B. C_2 rotation symmetry on $\phi = 0$ plane about $\vec{l}_a$ axis.

Fig. 2.1. Symmetrical Guide Cross Sections.

A similar thing happens if the guide cross section on the $\phi = 0$ plane has a two-fold rotation axis $\vec{1}_a$ ($\perp$ the plane), as illustrated in Fig. 2.1B. Again case 1 has the guide bending to the left. Case 2 is found by rotating the entire bent guide by π (180°) about $\vec{1}_a$. Now the guide has rotation symmetry about the z'' axis which we can interpret as a negative normalized curvature κ. The potential and fields on the cross section transform as

$$\begin{aligned}
\Phi^{(1)}(\Psi',z') &= \Phi^{(2)}(-\Psi',-z') \\
\vec{E}^{(1)}(\Psi',z') &= -\vec{E}^{(2)}(-\Psi',-z') \\
\vec{H}^{(1)}(\Psi',z') &= \vec{H}^{(2)}(-\Psi',-z')
\end{aligned} \tag{2.5}$$

The permittivity transforms as in (2.3). The characteristic impedance is unchanged and the conclusion in (2.4) applies again.

From (1.6) and (2.4) we have

$$Y_c(0) = Z_c^{-1}(0) = Y_{c_0} = Z_{c_0}^{-1} \tag{2.6}$$

Let us define a normalized form of the admittance as

$$y_c(\kappa) \equiv Y_c(\kappa)\, Z_{c_0}^{-1} \quad , \quad y_c(0) = 1 \tag{2.7}$$

When $\kappa = \pm 1$ the lens reaches the rotation axis where $\varepsilon = \infty$, a singularity occurs in the lens medium. For $|\kappa| < 1$ we can assume that y_c is an analytic function of κ, i.e., that it can be expanded in a power (Taylor) series in κ. The fact that y_c is even in κ means that only even powers are allowed in the expansion, giving

$$y_c(\kappa) = y_c(-\kappa) = \sum_{\ell=0}^{\infty,2} y_\ell\, \kappa^\ell \quad , \quad y_0 = 1 \tag{2.8}$$

where the second index above the summation indicates the increment (2 in this case) in the summation index for successive terms. Keeping the first two terms we have

$$y_c(\kappa) = 1 + y_2\kappa^2 + O\left(\kappa^4\right) \text{ as } \kappa \to 0 \tag{2.9}$$

which can be used as an approximation valid for small κ.

3. Canonical H-Plane Bend

This case with two electrically conducting boundaries of width 2b and spacing 2d is illustrated in Fig. 3.1A. The lens region is closed at the edges of the conductors by magnetic boundaries to give the case discussed in [2 (Section 4.1)]. Here we have the fields

$$
\begin{aligned}
E_z &= E_0 \;, \quad H_\Psi = \frac{\Psi_m}{\Psi}\frac{E_0}{Z_m} \\
Z_m &= \left[\frac{\mu_0}{\varepsilon_m}\right]^{\frac{1}{2}} = Y_m^{-1}
\end{aligned}
\tag{3.1}
$$

which are integrated to give

$$
\begin{aligned}
V_0 &= -2d E_0 \\
I &= \int_{\Psi_m - b}^{\Psi_m + b} H_\Psi \, d\Psi = -\frac{\Psi_m}{Z_m} E_0 \, \ell n\left(\frac{\Psi_m + b}{\Psi_m - b}\right) \\
&= -\frac{\Psi_m}{Z_m} E_0 \, \ell n\left(\frac{1+\kappa}{1-\kappa}\right) \\
Y_c(\kappa) &= \frac{I}{V_0} = \frac{\Psi_m}{2d\, Z_m} \ell n\left(\frac{1+\kappa}{1-\kappa}\right)
\end{aligned}
\tag{3.2}
$$

For $\kappa = 0$, we have a uniform dielectric medium giving the simple result

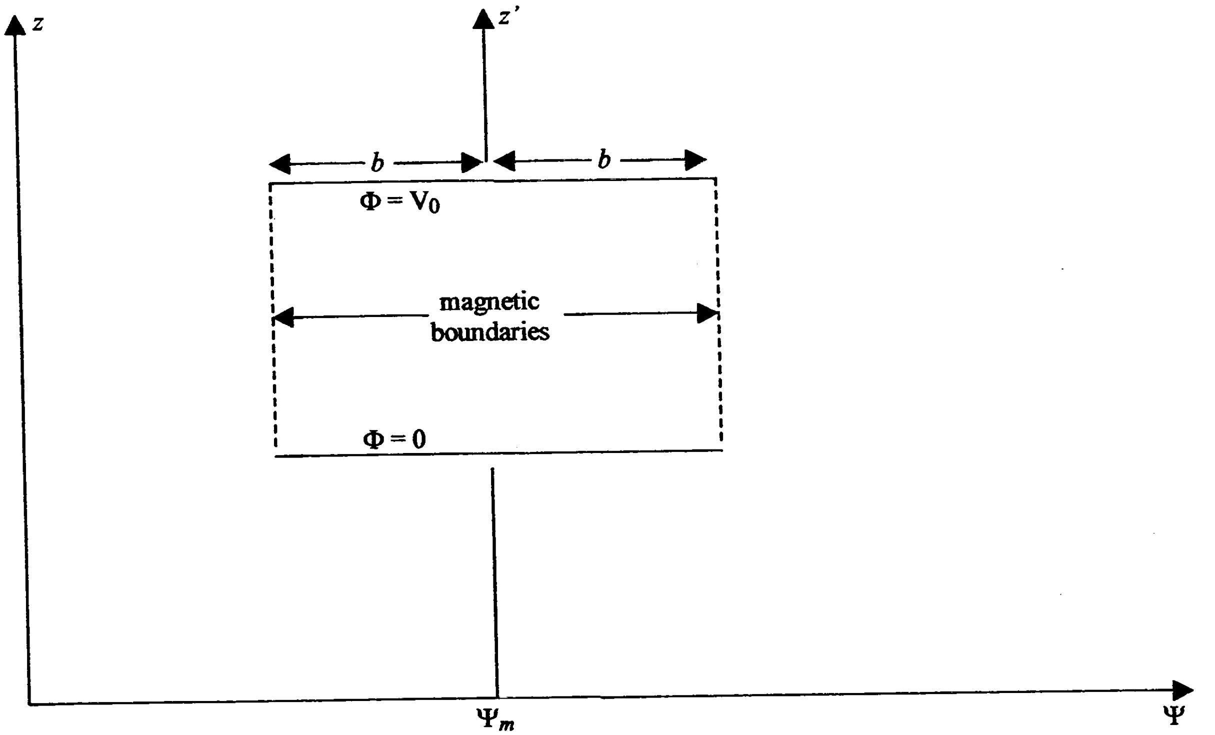
$$
Y_c(0) = \frac{b}{d} Y_m \tag{3.3}
$$

From which we find the normalized characteristic admittance

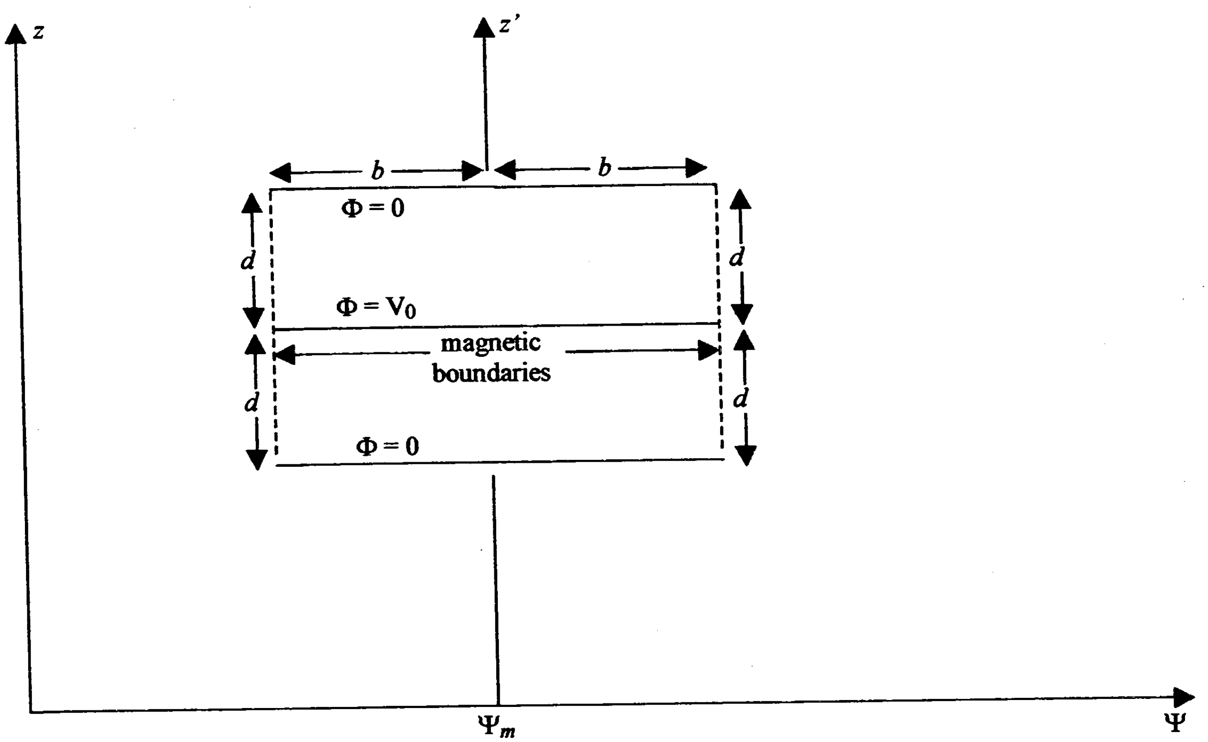
$$
y_c(\kappa) = \frac{Y_c(\kappa)}{Y_c(0)} = \frac{1}{2\kappa} \ell n\left(\frac{1+\kappa}{1-\kappa}\right) = \frac{1}{\kappa} \operatorname{arctanh}(\kappa) \tag{3.4}
$$

Expanding this for small κ we find [10]

$$
\begin{aligned}
y_c(\kappa) &= 1 + y_2 \kappa^2 + O(\kappa^4) \text{ as } \kappa \to 0 \\
y_2 &= \frac{1}{3}
\end{aligned}
\tag{3.5}
$$



A. Two conductors



B. Three conductors

Fig. 3.1. H-Plane bend: Cross Section on $\phi = 0$

Consistent with the symmetry results of Section 2.

Figure 3.1B shows the case of three conducting boundaries (coax-like) with the outer conductors at zero potential and the "center" conductor at potential V_0. With the dimensions as indicated (same overall dimensions as the previous case) the problem is readily solved as the parallel combination of two 2-conductor problems with spacing d and width $2b$. This gives $Y_c(\kappa)$ as $1/4$ of that in (3.2) and $y_c(\kappa)$ the same as in (3.4) and (3.5).

4. Canonical E-Plane Bend

Figure 4.1A shows the case of an E-plane bend where now the electric conductors are of width $2d$ with spacing $2b$. Again the lens region is closed at the edges of the conductors by magnetic boundaries to give the case discussed in [2 (Section 4.2)]. The fields are now

$$E_\Psi = \frac{\Psi}{\Psi_m} E_0 \quad , \quad H_z = -\frac{E_0}{\Psi_m Z_m} \tag{4.1}$$

which are integrated to give

$$\begin{aligned} V_0 &= -\int_{\Psi_m-b}^{\Psi_m+b} E_\Psi d\Psi = -\frac{E_0}{2\Psi_m}\left[[\Psi_m+b]^2 - [\Psi_m-b]^2\right] \\ &= -2E_0 \, \kappa \\ I &= 2d \, H_z = -2\frac{E_0}{Z_m}\frac{d}{b}\kappa \\ Y_c(\kappa) &= \frac{I}{V_0} = \frac{d}{b} Y_m \end{aligned} \tag{4.2}$$

So the characteristic admittance is *not a function of* κ, a very simple result.

The normalized characteristic admittance is simply

$$y_c(\kappa) = \frac{Y_c(\kappa)}{Y_c(0)} = 1 \tag{4.3}$$

This the case for all $|\kappa| < 1$ and the expansion for small κ gives

$$y_2 = 0 \tag{4.4}$$

with no remaining error terms. This is a special case which is still consistent with the symmetry results of Section 2.

Figure 4.1B shows the more coax-like case with three conducting boundaries with outer conductors at zero potential and the "center" conductor at potential V_0. With the dimensions as indicated the problem is solved as the parallel combination of two 2-conductor problems with spacing b and width $2d$. However, these two problems are not identical owing to the different permittivities in the two regions. Over previous results can be used by replacing

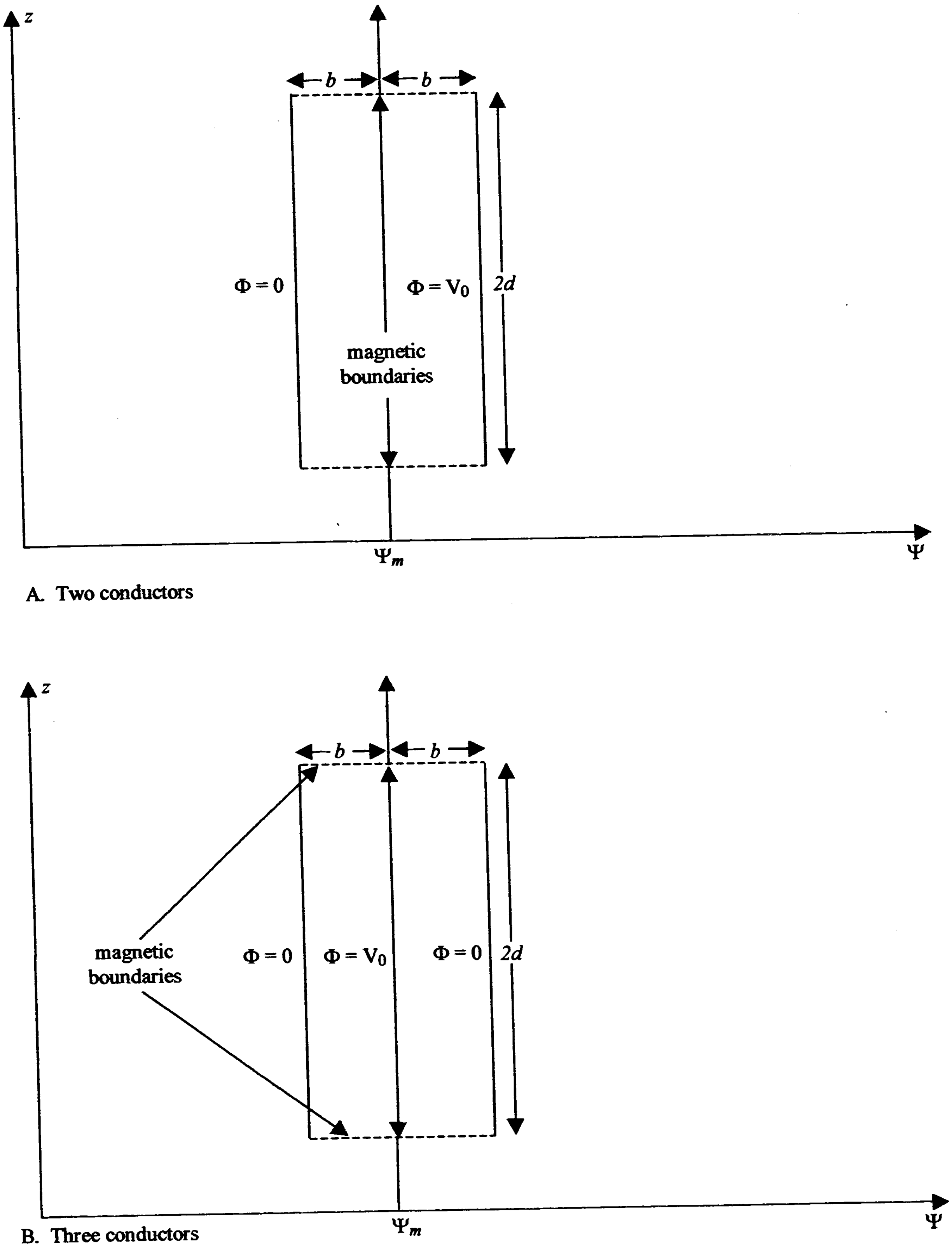


Fig. 4.1. E-plane Bend: Cross Section On $\phi e = 0$

Y_m by the value of Y (the wave admittance) at the center of each of the two regions giving $Y_m\Psi_m/[\Psi_m - b/2]$ and $Y_m\Psi_m/[\Psi_m + b/2]$ for the left and right regions respectively. Using this in conjunction with (4.2) gives

$$
\begin{aligned}
Y_c'(\kappa) &= \frac{2d}{b}\,Y_m\left[1+\frac{\kappa}{2}\right]^{-1} + \frac{2d}{b}\left[1-\frac{\kappa}{2}\right]^{-1} \\
&= 4\frac{d}{b}\,Y_m\left[1-\frac{\kappa^2}{4}\right]^{-1} = 4\frac{d}{b}\,Y_m\left[1+\frac{\kappa^2}{4} + O(\kappa^4)\right]^{-1}
\end{aligned}
\tag{4.5}
$$

Again we have the convenient result of a four-fold increase in the characteristic admittance, but now with a second order correction in κ.

5. Coaxial Bend

Now consider a bend in a circular coax as illustrated in Fig. 5.1. The inner conductor (radius a) has potential V_0, and the outer conductor (radius b) has potential 0. Note the (ξ, χ) coordinates on the $\phi = 0$ plane centered on the coax with

$$\Psi - \Psi_m = \Psi' = \xi \cos(\chi) \ , \ z = \xi \sin(\chi) \tag{5.1}$$

Here the fields are given by

$$\begin{aligned} \vec{E}(\xi,\chi) &= -\nabla_t \Phi(\xi,\chi) = -\left[\vec{1}_\xi \frac{\partial}{\partial \xi} + \vec{1}_\chi \frac{1}{\xi}\frac{\partial}{\partial \chi}\right]\Phi(\xi,\chi) \\ \vec{H}(\xi,\chi) &= Y(\xi,\chi)\,\vec{1}_\phi \times \vec{E}(\xi,\chi) \\ Y(\xi,\chi) &= Y_m \frac{\Psi_m}{\Psi_m + \xi\cos(\chi)} = Y_m\left[1 + \frac{\xi}{\Psi_m}\cos(\chi)\right]^{-1} \end{aligned} \tag{5.2}$$

The characteristic admittance is

$$\begin{aligned} Y_c(\kappa) &= \frac{I}{V_0} = -\frac{\xi}{V_0}\int_0^{2\pi} H_\chi(\xi,\chi)d\chi \\ &= \frac{\xi}{V_0} Y_m \int_0^{2\pi} E_\xi(\xi,\chi)\left[1 + \frac{\xi}{\Psi_m}\cos(\chi)\right]^{-1} d\chi \end{aligned} \tag{5.3}$$

where this applies to any ξ between a and b.

We note for later use

$$\int_a^b E_\xi(\xi,\chi)\, d\xi = V_0 \tag{5.4}$$

which applies for all χ. We also write

$$\Phi(\xi,\chi) = \Phi_0(\xi) + \Delta\Phi(\xi,\chi) \tag{5.5}$$

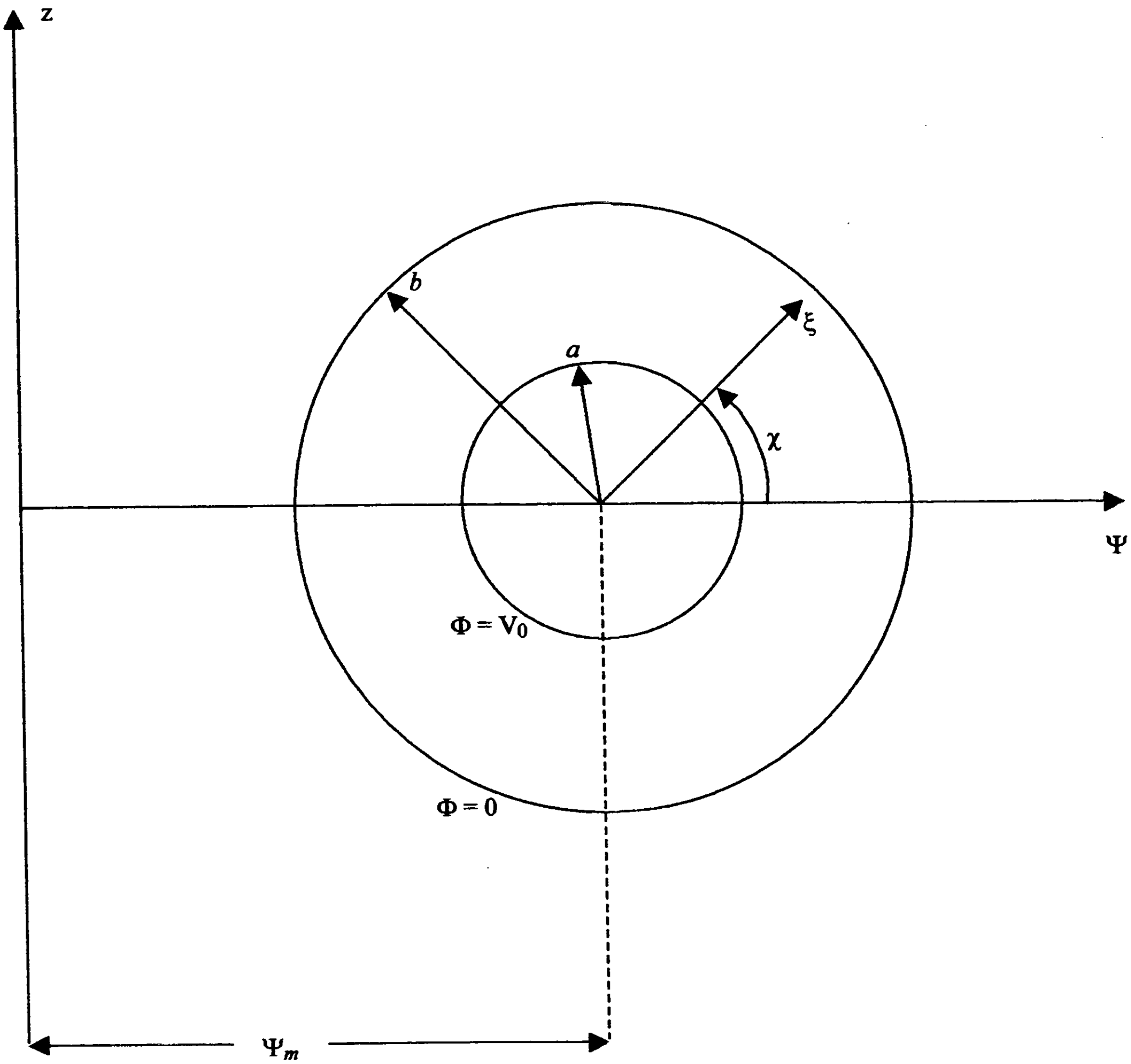


Fig. 5.1. Bent Circular Coax.

where Φ_0 applies to the case of $\kappa = 0$ and $\Delta\Phi$ has boundary conditions 0 on both conductors.

5.1 Straight coax

For $\kappa = 0$ we have the well-known case of a straight coaxial waveguide (cable) with a uniform dielectric medium. In this case we have

$$\begin{aligned}
\Phi_0(\xi) &= -V_0 \frac{\ell n\left(\frac{\xi}{b}\right)}{\ell n\left(\frac{b}{a}\right)} \\
\vec{E}_0(\xi) &= -\nabla\Phi_0(\xi) = \frac{V_0}{\xi}\,\ell_n^{-1}\left(\frac{b}{a}\right)\vec{1}_\xi = E_{0_\xi}(\xi)\,\vec{1}_\xi \\
\vec{H}_0(\xi) &= H_\chi(\xi)\,\vec{1}_\chi \;,\quad H_\chi(\xi) = -Y_m\,E_{0_\xi}(\xi) \\
I_0 &= -2\pi\xi\,H_\chi(\xi) = 2\pi\,Y_m\,\xi\,E_{0_\xi}(\xi) \\
Y_c(0) &= \frac{I_0}{V_0} = 2\pi\ell n^{-1}\left(\frac{b}{a}\right)Y_m
\end{aligned} \tag{5.6}$$

5.2 General solution through second order

From (5.3) and (5.6) we have

$$y_c(\kappa) = Y_c(\kappa)\;Y_c^{-1}(0) = \frac{1}{2\pi}\,\ell n\left(\frac{b}{a}\right)\frac{\xi}{V_0}\int_0^{2\pi} E_\xi(\xi,\chi)\left[1+\frac{\xi}{\Psi_m}\cos(\chi)\right]^{-1} d\chi \tag{5.7}$$

Divide through by ξ and integrate with respect to ξ to give

$$\begin{aligned}
y_c(\kappa)\int_a^b \frac{d\xi}{\xi} &= y_c(\kappa)\ell n\left(\frac{b}{a}\right) \\
y_c(\kappa) &= \frac{1}{2\pi V_0}\int_a^b\int_0^{2\pi} E_\xi(\xi,\chi)\left[1+\frac{\xi}{\Psi_m}\cos(\chi)\right]^{-1} d\chi d\xi
\end{aligned} \tag{5.8}$$

Now expand as a simple geometric series

$$\left[1+\frac{\xi}{\Psi_m}\cos(\xi)\right]^{-1} = \sum_{\ell=0}^{\infty}\left[-\frac{\xi}{\Psi_m}\cos(\chi)\right]^{\ell} \tag{5.9}$$

which is valid for all χ with $\xi < b < \Psi_m$, giving

$$y_c(\kappa) = \frac{1}{2\pi V_0}\int_a^b\int_0^{2\pi} E_\xi(\xi,\chi)\left[\sum_{\ell=0}^{\infty}\left[-\frac{\xi}{\Psi_m}\cos(\chi)\right]^{\ell}\right] d\chi d\xi \tag{5.10}$$

In this form we can look at the terms given by each ℓ.

Next apply symmetry for $\kappa \to -\kappa$ as in Section 2. The circular coax has cross-section symmetry on the $\phi = 0$ plane of all rotations and reflections known as $O_2 = C_{\infty a}$. This includes both the reflection with respect to the z' axis in Fig. 2.1A and the C_2 rotation with respect to $\vec{1}_a$ in Fig. 2.1B. From (2.5) we have from left bend (case 1) to right bend (case 2) as

$$\begin{aligned}
\Phi^{(1)}(\xi,\chi) &= \Phi^{(2)}(\xi,\chi-\pi) \\
E_\xi^{(1)}(\xi,\chi) &= E_\xi^{(2)}(\xi,\chi-\pi) \\
\vec{1}_\xi^{(1)} &= -\vec{1}_\xi^{(2)}
\end{aligned} \tag{5.11}$$

Both cases must give the same result for the admittance, so let us take the average giving

$$y_c(\kappa) = \frac{1}{2\pi V_0}\int_a^b\int_0^{2\pi}\frac{1}{2}\left[E_\xi(\xi,\chi) + E_\xi(\xi,\chi-\pi)\right]\left[\sum_{\ell=0}^{\infty}\left[-\frac{\xi}{\Psi_m}\cos(\chi)\right]^{\ell}\right] d\chi d\xi \tag{5.12}$$

Now change the angular variable as

$$\begin{aligned}
\chi &\to \chi' + \frac{\pi}{2} \\
\cos(\chi) &= \cos\left(\chi' + \frac{\pi}{2}\right) = -\sin(\chi')
\end{aligned} \tag{5.13}$$

Noting that any 2π interval for χ and χ' will do due to the periodicity, we then have

$$y_c(\kappa) = \frac{1}{2\pi V_0}\int_a^b \int_{-\pi}^{\pi} \frac{1}{2}\left[E_\xi\left(\xi, \chi' + \frac{\pi}{2}\right) + E_\xi\left(\xi, \chi' - \frac{\pi}{2}\right)\right]\left[\sum_{\ell=0}^{\infty}\left[\frac{\xi}{\Psi_m}\sin(\chi')\right]^\ell\right] d\chi' d\xi \tag{5.14}$$

Now we note that

$$\begin{aligned} &E_\xi\left(\xi, \chi' + \frac{\pi}{2}\right) + E_\xi\left(\xi, \chi' - \frac{\pi}{2}\right) = \text{even function of } \chi' \\ &\sin^\ell(\chi') = \begin{cases} \text{even function of } \chi' \text{ for even } \ell \\ \text{odd function of } \chi' \text{ for odd } \ell \end{cases} \end{aligned} \tag{5.15}$$

Integrating over χ' gives zero for odd functions of χ'. Then all terms for odd ℓ in (5.14) give zero, leaving only even terms. We can then write

$$y_c(\kappa) = \frac{1}{2\pi V_0}\int_a^b \int_0^{2\pi} E_\xi(\xi, \chi)\left[\sum_{\ell=0}^{\infty,2}\left[\frac{\xi}{\Psi_m}\cos(\chi)\right]^\ell\right] d\chi d\xi \tag{5.16}$$

For the $\ell = 0$ term first integrate over ξ and invoke (5.4) giving

$$\begin{aligned} &\frac{1}{2\pi V_0}\int_a^b \int_0^{2\pi} E_\xi(\xi, \chi) d\chi d\xi = \frac{1}{2\pi}\int_0^{2\pi} d\chi = 1 \\ &y_c(\kappa) = 1 + \frac{1}{2\pi V_0}\int_a^b \int_0^{2\pi} E_\xi(\xi, \chi) d\chi d\xi \left[\sum_{\ell=2}^{\infty,2}\left[\frac{\xi}{\Psi_m}\cos(\chi)\right]^\ell\right] d\chi d\xi \end{aligned} \tag{5.17}$$

The leading term 1 is now separated out and terms for $\ell \geq 2$ (even) are left as corrections.

For the $\ell = 2$ term let us first integrate by parts over ξ to give

$$\begin{aligned} \int_a^b E_\xi(\xi, \chi)\xi^2 d\xi &= -\Phi(\xi, \chi)\xi^2\Big|_a^b + 2\int_a^b \Phi(\xi, \chi)\, \xi d\xi \\ &= V_0 a^2 + 2\int_a^b \Phi(\xi, \chi)\, \xi d\xi \end{aligned}$$

$$\frac{1}{2\pi V_0}\int_a^b \int_0^{2\pi} E_\xi(\xi, \chi)\left[\frac{\xi}{\Psi_m}\right]^2 \cos^2(\chi)\, d\chi d\xi$$

$$= \frac{1}{2}\left[\frac{a}{\Psi_m}\right]^2 + \frac{1}{\pi V_0\,\Psi_m^2}\int_a^b\int_0^{2\pi}\Phi(\xi,\chi)\xi\cos^2(\chi)d\chi d\xi \tag{5.18}$$

Now write Φ as in (5.5) where

$$\begin{aligned}
\Delta\Phi(\xi,\chi) &= 0 \quad \text{on} \quad \xi = a,\, b \\
\Delta\Phi(\xi,\chi) &\to 0 \quad \text{for all} \quad \xi,\chi \quad \text{as} \quad \kappa\to 0 \\
\Delta\Phi(\xi,\chi) &= 0(1) \quad \text{as} \quad \kappa\to 0
\end{aligned} \tag{5.19}$$

We then have [10]

$$\begin{aligned}
&\int_a^b\int_0^{2\pi}\Phi(\xi,\chi)\,\xi\cos^2(\chi)\,d\chi d\xi \\
&= \int_a^b\int_0^{2\pi}\Phi_0(\xi)\,\xi\cos^2(\chi)\,d\chi d\xi + \int_a^b\int_0^{2\pi}\Delta\Phi(\xi,\chi)\,\xi\cos^2(\chi)\,d\chi d\xi \\
&= \pi\int_a^b\Phi_0(\xi)\,\xi d\xi + o(1) \\
&= -\pi\; V_0\,\ell n^{-1}\!\left(\frac{b}{a}\right)\int_a^b\xi\,\ell n\!\left(\frac{\xi}{b}\right)d\xi + o(1) \\
&= -\pi\; V_0\,b^2\ell n^{-1}\!\left(\frac{b}{a}\right)\int_{\frac{a}{b}}^{1}\xi'\,\ell n(\xi')d\xi' + o(1) \\
&= -\pi\; V_0\,b^2\ell n^{-1}\!\left(\frac{b}{a}\right)\left[\frac{\xi'^2}{2}\,\ell n(\xi') - \frac{\xi'^2}{4}\right]_{\frac{a}{b}}^{1} + o(1) \\
&= \pi\; V_0\,b^2\ell n^{-1}\!\left(\frac{b}{a}\right)\left[\frac{1}{4} + \frac{1}{2}\left[\frac{a}{b}\right]^2\ell n\!\left(\frac{a}{b}\right) - \frac{1}{4}\left[\frac{a}{b}\right]^2\right] + o(1) \\
&= \frac{\pi}{4}V_0\left[\frac{b^2-a^2}{\ell n\!\left(\frac{b}{a}\right)} - 2a^2\right] + o(1) \quad \text{as} \quad \kappa\to 0
\end{aligned} \tag{5.20}$$

Combining these results we have

$$\frac{1}{2\pi V_0}\int_a^b\int_0^{2\pi}E_\xi(\xi,\chi)\left[\frac{\xi}{\Psi_m}\right]^2\cos^2(\chi)\,d\chi d\xi$$

$$= \frac{1}{2}\left[\frac{a}{\Psi_m}\right]^2 + \frac{1}{4\Psi_m^2}\left[\frac{b^2 - a^2}{\ell n\left(\frac{b}{a}\right)} - 2a^2\right] + \Psi_m^{-2}\, o(1) \text{ as } \kappa \to 0$$

$$= \frac{1}{4}\,\frac{1-\left[\frac{a}{b}\right]^2}{\ell n\left(\frac{b}{a}\right)}\,\kappa^2 + o(\kappa^2) \text{ as } \kappa \to 0 \tag{5.21}$$

Noting the form $y_c(\kappa)$ must take as in (2.9) then the $o(\kappa^2)$ (goes to zero faster than κ^2) is combined with the $O(\kappa^4)$ to give

$$y_c(\kappa) = 1 + y_2\,\kappa^2 + O(\kappa^4) \text{ as } \kappa \to 0$$

$$y_2 = \frac{1}{4}\,\frac{1-\left[\frac{a}{b}\right]^2}{\ell n\left(\frac{b}{a}\right)} \tag{5.22}$$

5.3 Special case of small b-a

An interesting special case has

$$\frac{a}{b} = 1 - \nu \;\;,\;\; \nu \to 0_+$$

$$y_2 = \frac{1}{4}\,\frac{2\nu - \nu^2}{\nu + O(\nu^2)} = \frac{1}{2} + O(\nu) \text{ as } \nu \to 0 \tag{5.23}$$

This limit of $1/2$ can also be found from a physical problem of integrating the admittance per unit χ of two closely spaced coaxial cylinders with the CID medium between them as

$$\frac{1}{b}\,\frac{d\,Y_c}{d\chi} \simeq [b-a]^{-1}\; Y_m\left[1 + \frac{\xi}{\Psi_m}\cos(\chi)\right]^{-1}$$

$$Y_c(\kappa) \simeq \frac{b}{b-a}\; Y_m \int_0^{2\pi}\left[1 + \frac{\xi}{\Psi_m}\cos(\chi)\right]^{-1} d\chi$$

$$= \frac{b}{b-a}\; Y_m \int_0^{2\pi}\left[1 - \frac{\xi}{\Psi_m}\cos(\chi) + \left[\frac{\xi}{\Psi_m}\right]\cos(\chi) + \cdots\right] d\chi$$

$$= \frac{2\pi}{1-\frac{a}{b}}\; Y_m\left[1 + \frac{1}{2}\,\kappa^2 + \cdots\right] \tag{5.24}$$

So the general result is consistent with this simple check.

The case of small $b-a$ also gives a bound for y_2 since [10]

$$
\begin{aligned}
y_2 - \frac{1}{2} &= \frac{1}{4}\ell n^{-1}\left(\frac{b}{a}\right)\left[1-\left[\frac{a}{b}\right]^2 + 2\,\ell n\left(\frac{a}{b}\right)\right] \\
&= \frac{1}{4}\ell n^{-1}\left(\frac{b}{a}\right)\left[2\nu-\nu^2 - 2\sum_{\ell=1}^{\infty}\frac{\nu^\ell}{\ell}\right] \\
&= \frac{1}{4}\ell n^{-1}\left(\frac{b}{a}\right)\left[-2\nu^2 - 2\sum_{\ell=3}^{\infty}\frac{\nu^\ell}{\ell}\right] \\
&< 0 \text{ for } 0 < \nu < 1 \\
y_2 &\leq \frac{1}{2} \text{ for } 0 \leq \nu < 1
\end{aligned}
\tag{5.25}
$$

Note for small a/b we have $y_2 \to 0$ logarithmically. This is associated with the fact that for a small center conductor the electric field is relatively large near the center conductor where ε is close to ε_m. The permittivity near Ψ_m then dominates this case.

6. Concluding Remarks

We now have some canonical results for the characteristic admittance of a bent TEM waveguide in a CID medium. The H-plane and coax bends show an increase of the characteristic admittance over that of the straight waveguide with ε taken as ε_m, the permittivity in the center of the bent guide. However, the two-conductor version of the E-plane bend shows no change in the characteristic admittance with bending; the three-conductor version shows some increase with bending.

More general guide cross sections may also be considered. Here we have the general result that for certain symmetries in this cross section the correction to the characteristic admittance is second order in the bend curvature.

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