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Unipolarized Generalized Inhomogeneous TEM Plane Waves in Differential Geometric Lens Synthesis

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Abstract

Previous results concerning generalized inhomogeneous TEM plane waves are specialized to the case of unipolarized waves (in the formal coordinates). This results in an additional degree of freedom in choosing acceptable coordinate systems which in turn imply lens designs. Applying this to purely dielectric lenses (spatially variable permittivity, constant permeability) in the forms of bodies of revolution, several example lenses are found. Some of these are previously known, but one in particular non-obvious new lens design is found which can be called a spherical transmission line with propagation in the θ direction.

Contents

1 Introduction

A recent paper [6] has shown that one can have a generalized TEM plane wave propagating in the u_3 direction in u_1, u_2, u_3 orthogonal curvilinear coordinates. The formal fields are functions of u_1 and u_2 only and have components in both these directions. The medium is inhomogeneous but isotropic with formal propagation speed (with respect to the u_3 coordinate) a function of only u_3. In this case the u_3 surfaces can only be planes or spheres. The case of constant ϕ surfaces for u_3 gave a class of TEM waves propagating in the ϕ direction in the usual (Ψ, ϕ, z) cylindrical coordinate system, thereby giving a bending lens to change the direction of propagation of a TEM wave guided by appropriate conductors.

In the present paper the formal fields are assumed to have only one component (electric in the u_1 direction, magnetic in the u_2 direction). This removes the aforementioned restriction on the u_3 coordinate surfaces. The formal constitutive parameters μ' and ϵ' are inhomogeneous but isotropic, and the real μ is constrained to be μ_0 as before.

Specializing the dielectric lens to a body of revolution with the magnetic field in the ϕ (i.e., u_2) direction, several examples of such lenses are developed. Two of these examples correspond to well-known circular coaxial and circular conical transmission lines with uniform media. Another two of these examples are non-traditional, involving line sources or foci.

2 Summary of Previous Results

In [6] a generalized kind of TEM plane wave has been shown to satisfy the Maxwell equations. This takes the form in a (u_1, u_2, u_3) orthogonal curvilinear coordinate system as

$$\begin{aligned}
&\overrightarrow{E}' = \overrightarrow{E}_0' (u_2, u_2) f(t - \tau(u_3)) \;, \; \overrightarrow{E}_0' \cdot \overrightarrow{1}_3 = 0 \\
&\overrightarrow{H}' = \overrightarrow{H}_0' (u_1, u_2) f(t - \tau(u_3)) \;, \; \overrightarrow{H}_0' \cdot \overrightarrow{1}_3 = 0 \\
&\overrightarrow{1}_3 \times \overrightarrow{E}_0' = Z' \, \overrightarrow{H}_0' \;, \; \overrightarrow{E}_0' = -Z'^{-1} \, \overrightarrow{1}_3 \times \overrightarrow{H}_0' \\
&Z' = \left[\frac{\mu'}{\epsilon'}\right]^{\frac{1}{2}} = Z'(u_1, u_2) \text{ (a function of only } u_1 \text{ and } u_2) \\
&v' = [\mu'\epsilon']^{-\frac{1}{2}} = v'(u_3) \text{ (a function of only } u_3) \\
&\tau(u_3) = \int_0^{u_3} \frac{du_3''}{v'(u_3'')} \\
&\overrightarrow{E}_0' (u_1, u_2) = -\nabla_t' \Phi_e'(u_1, u_2) \;, \quad \nabla_t' \cdot [\epsilon' \nabla_t \Phi_e'] = 0 \\
&\overrightarrow{H}_0' (u_1, u_2) = -\nabla_t' \Phi_h'(u_1, u_2) \;, \quad \nabla_t' \cdot [\mu' \nabla_t' \Phi_h'] = 0.
\end{aligned} \tag{2.1}$$

Here the magnetic scalar potential Φ_h' may be multiple valued (as in a conformal transformation) as one goes around a conductor. The functional forms of Z' and v' can be combined to give

$$\begin{aligned}
\mu'(u_1, u_2, u_3) &= \frac{Z'(u_1, u_2)}{v'(u_3)} = \begin{array}{c}\text{function of } u_1 \text{ and } u_2 \text{ times} \\ \text{function of } u_3 \text{ (partly factored)}\end{array} \\
\epsilon'(u_1, u_2, u_3) &= [v'(u_3) Z'(u_1, u_2)]^{-1} = \begin{array}{c}\text{function of } u_1 \text{ and } u_2 \text{ times} \\ \text{function of } u_3 \text{ (partly factored)}\end{array}
\end{aligned} \tag{2.2}$$

Note that μ' and ϵ' are taken as scalars. Since there are no field components in the $\overrightarrow{1}_3$ direction, then these formal constitutive parameters need not have the 3, 3 components (as in a diagonal tensor form) specified. The formal fields, parameters, and operators above (indicated by a prime $'$ on the fields, potentials, and transverse (to $\overrightarrow{1}_3$) operators ∇_t' and $\nabla_t' \cdot$) take the form as though (u_1, u_2, u_3) were the same as (x, y, z) (Cartesian coordinates). These are to be distinguished from the real or physical fields, parameters, and operators which, when cast in terms of general orthogonal curvilinear coordinates (u_1, u_2, u_3), take on a more complicated form.

The transformation between the real and formal quantities is given by [13]

$$\vec{E}' = (\alpha_{n,m}) \cdot \vec{E} \,, \quad \vec{H}' = (\alpha_{n,m}) \cdot \vec{H}$$

$$(\epsilon'_{n,m}) = (\delta_{n,m}) \cdot (\epsilon_{n,m}) \,, \quad (\mu'_{n,m}) = (\delta_{n,m}) \cdot (\mu_{n,m})$$

$$(\alpha_{n,m}) = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \,, \quad (\gamma_{n,m}) = \begin{pmatrix} \frac{h_2 h_3}{h_1} & 0 & 0 \\ 0 & \frac{h_3 h_1}{h_2} & 0 \\ 0 & 0 & \frac{h_1 h_2}{h_3} \end{pmatrix} \tag{2.3}$$

$$h_n^2 = \left[\frac{\partial x}{\partial u_n}\right]^2 + \left[\frac{\partial y}{\partial u_n}\right]^2 + \left[\frac{\partial z}{\partial u_n}\right]^2 \quad \text{(scale factors)}$$

$$[d\ell]^2 = \sum_{n=1}^{3} h_n^2 \, [du_n]^2 \quad \text{(line element)}$$

where the constitutive-parameter tensors have been assumed diagonal in the u_n coordinate system.

Having assumed that the first two diagonal elements of the constitutive-parameter dyadics are equal (isotropic medium), this led to

$$h_1 = h_2 \tag{2.4}$$

which required that surfaces of constant u_3 can only be planes or spheres (a restriction one might wish to relax). Furthermore there was

$$\mu' = h_3 \mu \,, \quad \epsilon' = h_3 \epsilon$$

$$Z'(u_1, u_2) = \left[\frac{\mu'}{\epsilon'}\right]^{\frac{1}{2}} = \left[\frac{\mu}{\epsilon}\right]^{\frac{1}{2}} = Z(u_1, u_2) \quad \text{(a function of only } u_1 \text{ and } u_2\text{)} \tag{2.5}$$

$$v \equiv [\mu\epsilon]^{-\frac{1}{2}} = h_3 \, [\mu'\epsilon']^{-\frac{1}{2}} = h_3 v'(u_3)$$

The imposition of

$$\mu = \mu_0 \quad \text{(nonpermeable medium)} \tag{2.6}$$

then led to

$$\epsilon = \epsilon(u_1, u_2) \quad \text{(function of } u_1 \text{ and } u_2 \text{ only)}$$

$$v = [\mu_0 \epsilon]^{-\frac{1}{2}} = v(u_1, u_2) \quad \text{(function of } u_1 \text{ and } u_2 \text{ only)} \tag{2.7}$$

$$h_3 = \frac{v(u_1, u_2)}{v'(u_3)} \quad \text{(partly factored dependence on the } u_n\text{)}$$

3 Unipolarized Generalized Inhomogeneous TEM Plane Waves in Inhomogeneous but Isotropic Media

Now modify the form of the plane wave in (2.1) by restricting to a single polarization in the u_n coordinate system. Specifically restrict the electric field to the $\vec{1}_1$ direction, or equivalently the magnetic field to the $\vec{1}_2$ direction. With no other components then the gradients of the potentials imply

$$
\begin{aligned}
\vec{E'}_0\,(u_1) &= -\,\vec{1}_1\,\frac{d}{du_1}\Phi'_e(u_1) \quad \text{(functions of } u_1 \text{ only)} \\
\vec{H'}_0\,(u_1) &= -\,\vec{1}_2\,\frac{d}{du_2}\Phi'_h(u_2) \quad \text{(functions of } u_2 \text{ only)}
\end{aligned}
\tag{3.1}
$$

Then (2.1) simplifies to

$$
\begin{aligned}
&\vec{E'} = E'_{0_1}(u_1)\,\vec{1}_1\,f(t-\tau(u_3)) \\
&\vec{H'} = H'_{0_2}(u_2)\,\vec{1}_2\,f(t-\tau(u_3)) \\
&Z'(u_1,u_2) = \frac{E'_{0_1}(u_1)}{H'_{0_2}(u_2)} = \left[\frac{\mu'}{\epsilon'}\right]^{\frac{1}{2}} = \quad \text{product of function of } u_1 \text{ and function of } u_2 \\
&v'(u_3) = [\mu'\epsilon']^{-\frac{1}{2}} = \text{ function of } u_3 \text{ only} \\
&\tau(u_3) = \int_0^{u_3} \frac{du''_3}{v'(u''_3)} \\
&E'_{0_1}(u_1) = -\frac{d}{du_1}\Phi'_e(u_1)\,, \quad \frac{d}{du_1}\left[\epsilon'\frac{d}{du_1}\Phi'_e(u_1)\right] = 0 \\
&H'_{0_2}(u_2) = -\frac{d}{du_2}\Phi'_h(u_2)\,, \quad \frac{d}{du_2}\left[\mu'\frac{d}{du_2}\Phi'_h(u_2)\right] = 0
\end{aligned}
\tag{3.2}
$$

The functional forms of Z' and v' can be combined to give

$$
\begin{aligned}
\mu' &= \frac{Z'(u_1,u_2)}{v'(u_3)} = \frac{E'_{0_1}(u_1)}{H'_{0_2}(u_2)v'(u_3)} \\
&= \mu'(u_1,u_2,u_3) = \text{ product of functions each} \\
&\qquad \text{depending on only one of the } u_n \text{ (totally factored)} \\
\epsilon' &= [v'(u_3)Z'(u_1,u_2)]^{-1} = \frac{H'_{0_2}(u_2)}{E'_{0_1}(u_1)v'(u_3)} \\
&= \epsilon'(u_1,u_2,u_3) = \text{ product of functions each} \\
&\qquad \text{depending on only one of the } u_n \text{ (totally factored)}
\end{aligned}
\tag{3.3}
$$

Reflecting back on [6 (Section 4)] this unipolarized case fits the examples of the parallel and series media given there.

Equations (3.3) are the only acceptable forms for a medium with a unipolarized wave. Note that these equations imply

$$\epsilon' \left[E'_{0_1}(u_1)\right]^2 = \mu' \left[H'_{0_2}(u_2)\right]^2 \tag{3.4}$$

which expresses the fact that locally the electric and magnetic energy densities are equal. This result can also be obtained from (2.1), since

$$(\vec{1}_3 \times \vec{E'}_0) \cdot (\vec{1}_3 \times \vec{E'}_0) = (Z')^2 \, \vec{H'}_0 \cdot \vec{H'}_0 = \vec{E'}_0 \cdot \vec{E'}_0 = \frac{\mu'}{\epsilon'} \, \vec{H'}_0 \cdot \vec{H'}_0 \tag{3.5}$$

4 Scaling Unipolarized Generalized Inhomogeneous TEM Plane Waves

Now relate the formal fields and constitutive parameters to the real or physical quantities represented in the u_n coordinates. For the fields we simply have

$$E'_{0_1}(u_1) = h_1 E_{0_1}\ , \quad H'_{0_2}(u_2) = h_2 H_{0_2} \tag{4.1}$$

Since the scale factors may be general functions of the coordinates at this point, then similarly general are E_{0_1} and H_{0_2}.

Noting that since there is only an E_{0_1} component of the electric field, then only the 1, 1 element in the matrices in (2.3) is needed for electric parameters. Similarly only the 2, 2 element in these matrices is needed for magnetic parameters. This then gives

$$\epsilon' = \frac{h_2 h_3}{h_1}\epsilon\ , \quad \mu' = \frac{h_3 h_1}{h_2}\mu \tag{4.2}$$

In particular we do not at this point have the requirement of equal h_1 and h_2 as in (2.4), thereby giving us some additional freedom. From (4.2) we also have

$$\begin{aligned} v &= [\mu\epsilon]^{-\frac{1}{2}} = h_3\left[\mu'\epsilon'\right]^{-\frac{1}{2}} = h_3 v'(u_3) \\ Z &= \left[\frac{\mu}{\epsilon}\right]^{\frac{1}{2}} = \frac{h_2}{h_1}\left[\frac{\mu'}{\epsilon'}\right]^{\frac{1}{2}} = \frac{h_2}{h_1}Z'(u_1,u_2) \\ &= \frac{h_2}{h_1}\frac{E'_{0_1}(u_1)}{H'_{0_2}(u_2)} = \frac{E_{0_1}}{H_{0_2}} \end{aligned} \tag{4.3}$$

Thus v in general can be a function of all of the u_n, while v' is only a function of u_3. If (3.3) and (4.2) are combined we obtain

$$\begin{aligned} \mu &= \frac{h_2}{h_1 h_3}\left[\frac{E'_{0_1}(u_1)}{H'_{0_2}(u_2)}\right]\frac{1}{v'(u_3)} \\ \epsilon &= \frac{h_1}{h_2 h_3}\left[\frac{H'_{0_2}(u_2)}{E'_{0_1}(u_1)}\right]\frac{1}{v'(u_3)} \end{aligned} \tag{4.4}$$

Thus we have expressions for the medium parameters in terms of the scale factors, formal fields, and formal wave speed. Hence a choice of a coordinate system (u_1, u_2, u_3) will determine the scale factors h_n, which in turn will determine the real fields as well as $\mu\epsilon$ and μ/ϵ.

5 Uniform Permeability Lenses

Now add the requirement that

$$\mu = \mu_0 \tag{5.1}$$

i.e., the permeability is uniform, typically the permeability of free space. In addition we require that in the lens region.

$$\epsilon \geq \epsilon_{\text{min}} \geq \epsilon_0 \tag{5.2}$$

i.e., a permittivity bounded below by that of free space. The wave speed and impedance are bounded by

$$\begin{aligned} v &= [\mu_0 \epsilon]^{-\frac{1}{2}} \leq [\mu_0 \epsilon_0]^{-\frac{1}{2}} \equiv c \\ Z &= \left[\frac{\mu_0}{\epsilon}\right]^{\frac{1}{2}} \leq \left[\frac{\mu_0}{\epsilon_0}\right]^{\frac{1}{2}} \equiv Z_0 \end{aligned} \tag{5.3}$$

From (4.2) and (3.3) we have

$$\begin{aligned} \frac{h_3 h_1}{h_2} &= \frac{\mu'}{\mu_0} = \frac{1}{\mu_0} \frac{E'_{0_1}(u_1)}{H'_{0_2}(u_2) v'(u_3)} \\ &= \text{totally factored function of the } u_n \end{aligned} \tag{5.4}$$

which gives a constraint on the h_n. We also obtain

$$\frac{\epsilon}{\epsilon_0} = \frac{\epsilon}{\epsilon'} \frac{\epsilon'}{\epsilon_0} = \frac{h_1}{h_2 h_3} \frac{1}{\epsilon_0} \frac{H'_{0_2}(u_2)}{E'_{0_1}(u_1) v'(u_3)} \tag{5.5}$$

which can be combined with (5.4) to give

$$\begin{aligned} \frac{\epsilon}{\epsilon_0} &= \left[\frac{c}{h_3 v'(u_3)}\right]^2 \\ &= \left[\frac{h_1}{h_2} Z_0 \frac{H'_{0_2}(u_2)}{E'_{0_1}(u_1)}\right]^2 = \left[\frac{h_1}{h_2} \frac{Z_0}{Z'(u_1, u_2)}\right]^2 \end{aligned} \tag{5.6}$$

6 Case of Translation Symmetry in One Transverse Dimension in u_n Coordinates

As a special case we can assume that ϵ' and μ' are independent of u_2, giving

$$\begin{aligned} \mu'(u_1, u_3) &= \frac{Z'(u_1)}{v'(u_3)} \\ \epsilon'(u_1, u_3) &= \left[v'(u_3) Z'(u_1)\right]^{-1} \end{aligned} \tag{6.1}$$

From (3.3) this implies

$$H'_{0_2} = \text{ constant} \quad (\text{not a function of any } u_n) \tag{6.2}$$

Furthermore, (3.2) gives

$$\Phi'(u_2) = -u_2 H'_{0_2} \tag{6.3}$$

where an additive constant is suppressed. This is a translation symmetry of the formal parameters with respect to u_2.

The scaling (Section 4) leads to the constraint

$$h_2 H_{0_2} = H'_{0_2} = \text{ constant} \tag{6.4}$$

Adding the uniform permeability constraint (Section 5) leads to

$$\begin{aligned} \frac{h_3 h_1}{h_2} = \frac{\mu'}{\mu_0} &= \frac{1}{\mu_0} \frac{E'_{0_1}(u_1)}{H'_{0_2} v'(u_3)} \\ &= \text{ factored function of } u_1 \text{ and } u_3 \end{aligned} \tag{6.5}$$

In this last result we see that any dependence of the scale factors on u_2 must be cancelled among the scale factors in the above combination. The result that H'_{0_2} is a constant can also be obtained directly from (3.4) in the case that ϵ' and μ' are independent of u_2.

7 Dielectric Body of Revolution for Generalized Coax

Now let us constrain the general form that the u_n coordinates take in real Euclidean space to be a body of revolution (BOR). The usual cylindrical coordinates (Ψ, ϕ, z) are related to Cartesian coordinates (x, y, z) via

$$x = \Psi \cos(\phi) \ , \quad y = \Psi \sin(\phi) \tag{7.1}$$

Unlike [6], where propagation (u_3) was taken in the azimuthal (ϕ) direction, here it (u_3) is taken in a generalized axial direction (combination of axial z and radial Ψ directions). This gives what can be thought of as a generalized coax, the conducting boundaries of which are bodies of revolution with respect to the z axis ($C_{\infty a}$ symmetry, continuous rotation with respect to the z axis plus axial symmetry planes [3]).

Now choose

$$\begin{aligned} u_2 &= \Psi_m \phi \ , \quad \vec{1}_2 = \vec{1}_\phi \\ \Psi_m &= \text{ positive constant (dimension meters)} \end{aligned} \tag{7.2}$$

so that the magnetic field is in the $\vec{1}_\phi$ direction and the electric field is in some generalized radial direction ($\vec{1}_1$). This immediately gives (from (2.3)) the scale factor

$$h_2 = \frac{\Psi}{\Psi_m} \tag{7.3}$$

where we note that both z and Ψ may be functions of u_1 and u_3. Note that, consistent with Section 6, all fields and constitutive parameters are now independent of u_2 (or ϕ) when expressed in terms of the u_n coordinates.

Now the real magnetic field is given from (6.4) by

$$H_{0_2} = \frac{\Psi_m}{\Psi} H'_{0_2} = H_{0_2}(u_1, u_3) \tag{7.4}$$

This result shows that for our generalized coax the magnetic field $H_{0_2} f(t - \tau(u_3))$ has the simple result that its amplitude varies as Ψ^{-1}, no matter what acceptable u_1 and u_3 coordinates we may choose.

The constraint (6.5) on the scale factors now becomes

$$h_3 h_1 \frac{\Psi_m}{\Psi} = \frac{\mu'}{\mu_0} = \frac{1}{\mu_0} \frac{E'_{0_1}(u_1)}{H'_{0_2} v'(u_3)} \tag{7.5}$$

$$= \text{factored function of } u_1 \text{ and } u_3$$

so that we need to consider the functional dependence of h_1 and h_3 on u_1 and u_3.

8 Construction of u_1 and u_3 Coordinates Via Conformal Transformation

The u_1 and u_3 coordinates are, with our previous constraints, already orthogonal to u_2. Being functions of only z and Ψ, then we can consider u_1 and u_3 on a plane of constant ϕ. One can construct orthogonal coordinates on such a plane via a conformal transformation of the form

$$\begin{aligned} w(\zeta) &= w_1(\zeta) + jw_3(\zeta) \\ \zeta &= z + j\Psi \quad \text{(complex Cartesian coordinates)} \end{aligned} \tag{8.1}$$

such as done in [5]. With w as an analytic function of ζ, then w_1 and w_3 give an orthogonal curvilinear system on the plane. We can then construct a more general orthogonal system of the form.

$$\begin{aligned} u_1(z, \Psi) &= u_1(w_1) \\ u_3(z, \Psi) &= u_3(w_3) \end{aligned} \tag{8.2}$$

So u_1 is a scale of w_1 only, and similarly for u_3. Instead of curvilinear squares given by equal decrements of w_1 and w_3, we have curvilinear rectangles given by equal decrements of u_1 and u_3.

There is a scale factor for the w coordinates as

$$h_w = \left|\frac{d\zeta}{dw}\right| = \left|\frac{dw}{d\zeta}\right|^{-1} = h_{w_1} = h_{w_3} \tag{8.3}$$

As an analytic function the derivative of w is independent of direction in the ζ plane. As a conformal transformation (curvilinear squares) the scale factors for w_1 and w_3 are the same. The scale factors for u_1 and u_3 can now be expressed as

$$h_n = \left|\frac{dw_n}{du_n}\right| h_w \quad \text{for } n = 1, 3 \tag{8.4}$$

where

$$\frac{dw_n}{du_n} = \quad \text{function of } u_n \text{ (or } w_n\text{) only} \tag{8.5}$$

The constraint (7.5) on the scale factors now becomes

$$\begin{aligned} X \equiv \frac{h_w^2}{\Psi} &= \frac{1}{\mu_0 \Psi_m} \left| \frac{du_1}{dw_1} \right| \left| \frac{du_3}{dw_3} \right| \frac{E'_{0_1}(u_1)}{H'_{0_2} v'(u_3)} \\ &= \text{factored function of } w_1 \text{ and } w_3 \\ &\quad \text{(or } u_1 \text{ and } u_3\text{)} \end{aligned} \tag{8.6}$$

So X has to factor as a function of w_1 and w_2, restricting the allowable conformal transformations $w(\zeta)$.

9 Simple Canonical Examples

As a first procedure to construct generalized coax examples let us expand $\zeta(w)$ in a power (Taylor) series as

$$\zeta(w) = a_0 + a_1 w + a_2 w^2 + \cdots \tag{9.1}$$

The constant term is uninteresting and can be absorbed into the $a_1 w$ term by a simple shift. The first-power term gives

$$\begin{aligned}
\zeta(w) &= z + j\Psi = a_1 [w_1 + jw_3] \\
h_w &= \left|\frac{d\zeta}{dw}\right| = |a_1| \\
X = \frac{h_w^2}{\Psi} &= \mathrm{Im}^{-1}[a_1 w]\,|a_1|^2 \\
&= |a_1|^2 [\mathrm{Im}[a_1]\,w_1 + \mathrm{Re}[a_1]\,w_3] \\
&= \text{factored function of } w_1 \text{ and } w_3 \\
&\quad\ \text{provided } a_1 \text{ is pure real or pure imaginary}
\end{aligned} \tag{9.2}$$

which provides two acceptable cases. The quadratic term gives

$$\begin{aligned}
\zeta(w) &= z + j\Psi = a_2 w^2 \\
\Psi &= \mathrm{Im}[a_2]\left[w_1^2 - w_3^2\right] + \mathrm{Re}[a_2]\,2w_1 w_3 \\
h_w &= \left|\frac{d\zeta}{dw}\right| = |2a_2 w| = 2\,|a_2|\left[w_1^2 + w_3^2\right]^{\frac{1}{2}} \\
X &= \frac{h_w^2}{\Psi} \neq \text{factored function of } w_1 \text{ and } w_2 \\
&\quad\ (\text{except in trivial case of } a_2 = 0)
\end{aligned} \tag{9.3}$$

As one can see higher powers of w only increase the complexity of the above terms in unfactorable forms.

So let us look at the two acceptable cases in (9.2).

9.1 Usual coax: propagation in the z direction

Letting a_1 be imaginary we can choose (for simplicity)

$$\begin{aligned}
a_1 = -j\,, \quad \zeta(w) = -jw\,, \quad h_w = 1 \\
z = w_3\,, \quad \Psi = w_1
\end{aligned} \tag{9.4}$$

so that propagation is in the z direction. Surfaces of constant u_3 are planes, and surfaces of constant u_1 are circular cylinders on which conductors can be placed to form the usual coaxial waveguide (cable).

To fill in the details we have

$$
\begin{aligned}
v = [\mu_0\epsilon]^{-\frac{1}{2}} = h_3\,[\mu'\epsilon']^{-\frac{1}{2}} &= h_w \left|\frac{dw_3}{du_3}\right| v'(u_3) \\
= \left|\frac{dw_3}{du_3}\right| v'(u_3) &= \text{function of } u_3 \text{ (or } w_3\text{) only} \\
&\quad \text{(i.e. not a function of } u_1\text{)}
\end{aligned}
\tag{9.5}
$$

$$\Rightarrow \epsilon \neq \text{function of } u_1 \text{ (or } w_1\text{)}$$

and also

$$
\begin{aligned}
Z = \left[\frac{\mu_0}{\epsilon}\right]^{\frac{1}{2}} = \frac{h_2}{h_1}\left[\frac{\mu'}{\epsilon'}\right]^{\frac{1}{2}} &= \frac{\Psi}{\Psi_m h_w}\left|\frac{du_1}{dw_1}\right| Z'(u_1) \\
= \frac{w_1}{\Psi_m}\left|\frac{du_1}{dw_1}\right| Z'(u_1) &= \text{function of } u_1 \text{ (or } w_1\text{) only} \\
&\quad \text{(i.e. not a function of } u_3\text{)}
\end{aligned}
\tag{9.6}
$$

$$\Rightarrow \epsilon \neq \text{function of } u_3 \text{ (or } w_3\text{)}$$

Combining these results we have

$$
\begin{aligned}
\epsilon &= \text{constant} \\
v &= \text{constant} \\
Z &= \text{constant}
\end{aligned}
\tag{9.7}
$$

So our coax has a uniform dielectric medium, as we should expect. From the wave impedance we next infer

$$
\begin{aligned}
H_{0_2}(u_1) &= \frac{\Psi_m}{\Psi} H'_{0_2} = \frac{\Psi_m}{w_1} H'_{0_2} \\
E_{0_2}(u_1) &= Z H_{0_2}(u_1) = Z\frac{\Psi_m}{\Psi} H'_{0_2} = Z\frac{\Psi_m}{w_1} H'_{0_2}
\end{aligned}
\tag{9.8}
$$

So both E and H fall off as Ψ^{-1} as is well known for a coax.

At this point we can note that there is some flexibility in our choice of $v'(u_3)$. As in (3.2) we integrate to obtain $\tau(u_3)$ which will just give u_3/v, the delay in our coax. For simplicity let us select

$$v' = [\mu'\epsilon']^{-\frac{1}{2}} \equiv \text{constant (i.e. not a function of } u_3\text{)}. \tag{9.9}$$

Then we also have

$$h_3 = \frac{v}{v'} = \left[\frac{\mu'\epsilon'}{\mu_0\epsilon}\right]^{\frac{1}{2}} = \text{ constant.} \tag{9.10}$$

Choosing this constant as 1 we have

$$\begin{aligned} & v = v' \\ & h_3 = 1 = h_w \left|\frac{dw_3}{du_3}\right| = \left|\frac{dw_3}{du_3}\right| \\ & u_3 = w_3 \ \ (+\text{ sign chosen and integration constant suppressed}) \end{aligned} \tag{9.11}$$

Concerning the u_1 coordinate there are various possibilities. One that admits constant ϵ' and μ' comes from

$$\begin{aligned} \epsilon' &= \text{ constant}, \quad \mu' = \text{ constant} \\ Z' &= \text{ constant}, \quad E'_{0_1} = Z'H'_{0_2} = \text{ constant} \\ \frac{h_1}{h_2} &= \frac{Z'}{Z} = \text{ constant} \end{aligned} \tag{9.12}$$

Setting this constant to 1 gives

$$\begin{aligned} Z' &= Z \\ \mu' &= \frac{Z'}{v'} = \frac{Z}{v} = \mu_0 \\ \epsilon' &= [v'Z']^{-1} = [vZ]^{-1} = \epsilon \\ h_1 &= h_2 = \frac{\Psi}{\Psi_m} = \frac{w_1}{\Psi_m} = h_w \left|\frac{dw_1}{du_1}\right| = \left|\frac{dw_1}{du_1}\right| \\ u_1 - u_1^{(0)} &= \Psi_m \int_{w_1^{(0)}}^{w_1} \frac{dw_1''}{w_1''} = \Psi_m \ell n\left(\frac{w_1}{w_1^{(0)}}\right) = \Psi_m \ell n\left(\frac{\Psi}{\Psi_m}\right) \end{aligned} \tag{9.13}$$

with integration constants and signs chosen for convenience (e.g. zero voltage on one of the conductors). Here u_1 is proportional to the electric potential.

9.2 Radial transmission line

Letting a_1 be real we can choose (for simplicity)

$$\begin{aligned} & a_1 = 1, \quad \zeta(w) = w, \quad h_w = 1 \\ & z = w_1, \quad \Psi = w_3 \end{aligned} \tag{9.14}$$

so that propagation is now in Ψ direction. Surfaces of constant u_3 are circular cylinders, and surfaces of constant u_1 are planes on which conductors can be placed to form a radial transmission line.

Now we have

$$\begin{aligned}\Psi^{-1}v &= \Psi^{-1}\left[\mu_0\epsilon\right]^{-\frac{1}{2}} = \Psi^{-1}h_3\left[\mu'\epsilon'\right]^{-\frac{1}{2}} = \frac{h_w}{\Psi}\left|\frac{dw_3}{du_3}\right|v'(u_3)\\ &= \frac{1}{w_3}\left|\frac{dw_3}{du_3}\right|v'(u_3) = \text{ function of } u_3 \text{ (or } w_3\text{) only}\\ &\Rightarrow \Psi^2\epsilon \neq \text{ function of } u_1 \text{ (or } w_1\text{)}\end{aligned} \tag{9.15}$$

and also

$$\begin{aligned}\Psi^{-1}Z &= \Psi^{-1}\left[\frac{\mu_0}{\epsilon}\right]^{\frac{1}{2}} = \frac{h_2}{\Psi h_1}\left[\frac{\mu'}{\epsilon'}\right]^{\frac{1}{2}} = \frac{1}{\Psi_m h_w}\left|\frac{du_1}{dw_1}\right|Z'(u_3)\\ &= \frac{1}{\Psi_m}\left|\frac{du_1}{dw_1}\right|Z'(u_1) = \text{ function of } u_1 \text{ (or } w_1\text{) only}\\ &\Rightarrow \Psi^2\epsilon \neq \text{ function of } u_3 \text{ (or } w_3\text{)}\end{aligned} \tag{9.16}$$

Combining these results gives

$$\begin{aligned}&\Psi^2\epsilon = \text{ constant}\\ &\frac{\epsilon}{\epsilon_{\text{min}}} = \left[\frac{\Psi_m}{\Psi}\right]^2\\ &\epsilon_{\text{min}} \geq \epsilon_0\\ &\Psi_m \equiv \text{ maximum } \Psi \text{ of interest}\end{aligned} \tag{9.17}$$

Then we also have

$$\begin{aligned}v &= \left[\mu_0\epsilon\right]^{-\frac{1}{2}} = v_{\text{max}}\frac{\Psi}{\Psi_m}\\ v_{\text{max}} &= \left[\mu_0\epsilon_{\text{min}}\right]^{-\frac{1}{2}} \leq c\\ Z &= \left[\frac{\mu_0}{\epsilon}\right]^{\frac{1}{2}} = Z_{\text{max}}\frac{\Psi}{\Psi_m}\\ Z_{\text{max}} &= \left[\frac{\mu_0}{\epsilon_{\text{min}}}\right]^{\frac{1}{2}} \leq Z_0\end{aligned} \tag{9.18}$$

So now we have a nonuniform dielectric medium, but one of simple functional dependence. This result can also be derived by the technique in [13 (Appendix F)]. While the wave impedance varies with position, the characteristic impedance Z_c is constant as can be seen from

$$Z_c = Z\frac{d}{2\pi\Psi} = Z_{\max}\frac{d}{2\pi\Psi_m}$$
$$d \equiv \text{distance between conducting sheets (on planes of constant } z \text{ (or } u_1)) \tag{9.19}$$

We next infer

$$H_{0_2}(u_1) = \frac{\Psi_m}{\Psi}H'_{0_2} = \frac{\Psi_m}{w_3}H'_{0_2}$$
$$E_{0_2}(u_1) = ZH_{0_2}(u_1) = Z_{\max}H'_{0_2} = E_{0_2} = \text{constant} \tag{9.20}$$

which should not be surprising.

Concerning the u_1 coordinate, there are various possibilities. Noting the constant electric field one can make u_1 proportional to w_1 and for convenience

$$u_1 = w_1 = z\,, \quad \left|\frac{dw_1}{du_1}\right| = 1\,, \quad h_1 = 1 \tag{9.21}$$

This in turn implies

$$Z' = \frac{h_1}{h_2}Z = \frac{\Psi_m}{\Psi}Z_{\max}\frac{\Psi}{\Psi_m} = Z_{\max} = \text{constant}$$
$$E'_{0_1} = Z'H'_{0_2} = Z_{\max}H'_{0_2} = \text{constant} \tag{9.22}$$

Considering u_3 there is again some flexibility. Choosing

$$v' = [\mu'\epsilon']^{-\frac{1}{2}} \equiv \text{constant (not a function of } u_3) \tag{9.23}$$

this gives

$$h_3 = \frac{v}{v'} = \frac{v_{\max}}{v'}\frac{\Psi}{\Psi_m} = \frac{v_{\max}}{v'}\frac{w_3}{\Psi_m}. \tag{9.24}$$

Combining with

$$h_3 = h_w\left|\frac{dw_3}{du_3}\right| = \left|\frac{dw_3}{du_3}\right| \tag{9.25}$$

this gives

$$u_3 - u_3^{(0)} = \frac{v'\Psi_m}{v_{\max}}\int_{w_3^{(0)}}^{w_3}\frac{dw_3''}{w_3''} = \frac{v'\Psi_{\max}}{v_{\max}}\ell n\left(\frac{w_3}{w_3^{(0)}}\right) = \frac{v'\Psi_{\max}}{v_{\max}}\ell n\left(\frac{\Psi}{\Psi_m}\right) \tag{9.26}$$

with integration constants and signs chosen for convenience. At this point it is convenient to choose

$$v' \equiv v_{\max} \tag{9.27}$$

giving

$$\begin{aligned} \mu' &= \frac{Z'}{v'} = \frac{Z_{\max}}{v_{\max}} = \mu_0 = \text{constant} \\ \epsilon' &= [v'Z']^{-1} = [v_{\max}Z_{\max}]^{-1} = \epsilon_{\min} = \text{constant} \end{aligned} \tag{9.28}$$

as well as

$$\begin{aligned} h_3 = h_2 &= \frac{\Psi}{\Psi_m} \\ u_3 - u_3^{(0)} &= \Psi_m \ell n \left(\frac{\Psi}{\Psi_m}\right). \end{aligned} \tag{9.29}$$

10 Additional Examples

Another procedure to construct generalized coax examples involves expanding $\zeta(w)$ in terms of the exponential of a power series as

$$\zeta(w) = \zeta_0 e^{b_0 + b_1 w + b_2 w^2 + \cdots} \tag{10.1}$$

Noting the factoring requirement in (8.6), the motivation behind such a choice comes from the fact that the exponential of a sum is a product of exponentials. The constant term is uninteresting and can be absorbed into the $b_1 w$ term by a simple shift. The first-power term gives

$$\begin{aligned}
\zeta(w) &= z + j\Psi = \zeta_0 e^{b_1 w} \\
h_w &= \left|\frac{d\zeta}{dw}\right| = \left|\zeta_0 b_1 e^{b_1 w}\right| = \left|b_1 \zeta(w)\right| \\
&= \left|\zeta_0 b_1\right| e^{\mathrm{Re}[b_1 w]} \\
\Psi &= \mathrm{Im}\left[\zeta_0 e^{b_1 w}\right] = e^{\mathrm{Re}[b_1 w]} \mathrm{Im}\left[\zeta_0 e^{j\mathrm{Im}[b_1 w]}\right] \\
X &= \frac{h_w^2}{\Psi} = \frac{\left|\zeta_0 b_1\right|^2 e^{\mathrm{Re}[b_1 w]}}{\mathrm{Im}\left[\zeta_0 e^{j\mathrm{Im}[b_1 w]}\right]}
\end{aligned} \tag{10.2}$$

which provides two acceptable cases. The quadratic term gives

$$\begin{aligned}
\zeta(w) &= \zeta_0 e^{b_2 w^2} \\
h_w &= \left|\frac{d\zeta}{dw}\right| = \left|2\zeta_0 b_2 w e^{b_2 w^2}\right| = \left|2 b_2 w\right| \left|\zeta(w)\right| \\
&= \left|2 b_2 w\right| e^{\mathrm{Re}[b_2 w^2]} \\
\Psi &= \mathrm{Im}\left[\zeta_0 e^{b_2 w^2}\right] \\
X &= \frac{h_w^2}{\Psi} \neq \text{factored function of } w_1 \text{ and } w_2 \\
&\qquad \text{(except in trivial case of } \zeta_0 = 0 \text{ or } b_2 = 0)
\end{aligned} \tag{10.3}$$

Higher powers of w also give unfactorable forms.

Looking at the cases in (10.2), the separability is limited by the denominator

$$\begin{aligned}
\mathrm{Im}\left[\zeta_0 e^{j\mathrm{Im}[b_1 w]}\right] &= \left|\zeta_0\right| \mathrm{Im}\left[e^{j\mathrm{Im}[b_1 w] + j\arg(\zeta_0)}\right] \\
&= \left|\zeta_0\right| \sin\left(\mathrm{Im}[b_1 w] + \arg(\zeta_0)\right) \\
&= \left|\zeta_0\right| \sin\left(\mathrm{Im}[b_1] w_1 + \mathrm{Re}[b_1] w_3 + \arg(\zeta_0)\right)
\end{aligned} \tag{10.4}$$

This separates provided b_1 is pure real or pure imaginary. The numerator does not have this separability problem. Here ζ_0 is merely a convenient scaling constant.

For these cases it is convenient to introduce the usual spherical coordinates (r, θ, ϕ) with

$$z = r\cos(\theta)\,, \quad \Psi = r\sin(\theta) \tag{10.5}$$

This gives for our conformal transformation

$$\begin{aligned} \zeta(w) &= re^{j\theta} = \zeta_0 e^{b_1 w} \\ w &= \frac{1}{b_1}\left[\ell n\left(\frac{r}{\zeta_0}\right) + j\theta\right] = \frac{1}{b_1}\left[\ell n\left(\frac{r}{|\zeta_0|}\right) + j\left[\theta - \arg(\zeta_0)\right]\right] \end{aligned} \tag{10.6}$$

with the related factors

$$\begin{aligned} h_w &= |b_1| r \\ X &= \frac{|b_1|^2 r}{\sin(\theta)} \end{aligned} \tag{10.7}$$

So we can see that w_1 and w_2 need to be functions of r and θ alone, separately, and in either order.

10.1 Circular-conical transmission line: propagation in the r direction

Letting b_1 be imaginary we can choose (for simplicity)

$$\begin{aligned} & b_1 = -j\,, \quad \zeta_0 = \Psi_m > 0\,, \quad \zeta(w) = \Psi_m e^{-jw} \\ & h_w = \Psi_m e^{w_3}\,, \quad w_1 = -\theta\,, \quad w_3 = \ell n\left(\frac{r}{\Psi_m}\right) \\ & h_2 = \frac{\Psi}{\Psi_m} = \mathrm{Im}\left[e^{-jw}\right] = -e^{w_3}\sin(w_1) = \frac{r}{\Psi_m}\sin(\theta) \end{aligned} \tag{10.8}$$

so that propagation is in the r direction. Surfaces of constant u_3 are spheres, and surfaces of constant u_1 are circular cones.

To fill in the details we have

$$
\begin{aligned}
v = [\mu_0\epsilon]^{-\frac{1}{2}} &= h_3\,[\mu'\epsilon']^{-\frac{1}{2}} = h_w \left|\frac{dw_3}{du_3}\right| v'(u_3) \\
&= \Psi_m e^{w_3}\left|\frac{dw_3}{du_3}\right| v'(u_3) = \text{function of } u_3 \text{ (or } w_3\text{) only} \\
&\qquad\qquad\qquad\qquad \text{(i.e. not a function of } u_1\text{)}
\end{aligned}
\tag{10.9}
$$

$$\Rightarrow \epsilon \neq \text{function of } u_1 \text{ (or } w_1\text{)}$$

and also

$$
\begin{aligned}
Z = \left[\frac{\mu_0}{\epsilon}\right]^{\frac{1}{2}} &= \frac{h_2}{h_1}\left[\frac{\mu'}{\epsilon'}\right]^{\frac{1}{2}} = \frac{\Psi}{\Psi_m h_w}\left|\frac{du_1}{dw_1}\right| Z'(u_1) \\
&= -\frac{\sin(w_1)}{\Psi_m}\left|\frac{du_1}{dw_1}\right| Z'(u_1) = \text{function of } u_1 \text{ (or } w_1\text{) only} \\
&\qquad\qquad\qquad\qquad \text{(i.e. not a function of } u_3\text{)}
\end{aligned}
\tag{10.10}
$$

$$\Rightarrow \epsilon \neq \text{function of } u_3 \text{ (or } w_3\text{)}$$

Combining these results we have

$$
\begin{aligned}
\epsilon &= \text{constant} \\
v &= \text{constant} \\
Z &= \text{constant}
\end{aligned}
\tag{10.11}
$$

So our circular-conical transmission line has a uniform dielectric medium. From the wave impedance we next infer

$$
\begin{aligned}
H_{0_2}(u_1,u_3) &= \frac{\Psi_m}{\Psi}H'_{0_2} = \frac{-e^{-w_3}}{\sin(w_1)}H'_{0_2} = \frac{\Psi_m}{r\sin(\theta)}H'_{0_2} \\
E_{0_2}(u_1,u_3) &= ZH_{0_2}(u_1) = Z\frac{\Psi_m}{\Psi}H'_{0_2} = -Z\frac{e^{w_3}}{\sin(w_1)} = \frac{Z\Psi_m}{r\sin(\theta)}H'_{0_2}
\end{aligned}
\tag{10.12}
$$

So both E and H fall off as $r\sin(\theta)$ which is well known for such structures.

Again for convenience choose

$$v' = [\mu'\epsilon']^{-\frac{1}{2}} = \text{constant (i.e. not a function of } u_3\text{)} \tag{10.13}$$

Then we have

$$h_3 = \frac{v}{v'} = \left[\frac{\mu'\epsilon'}{\mu_0\epsilon_0}\right]^{\frac{1}{2}} = \text{constant} \tag{10.14}$$

Choosing this constant as 1 we have

$$h_3 = 1 = h_w \left| \frac{dw_3}{du_3} \right| = \Psi_m e^{w_3} \left| \frac{dw_3}{du_3} \right|$$

$$u_3 = \Psi_m e^{w_3} = r \text{ (+ sign chosen and integration constant suppressed)} \tag{10.15}$$

For the u_1 coordinate let us for convenience choose constant μ' and ϵ' as

$$\begin{aligned} \epsilon' &= \text{constant}, \quad \mu' = \text{constant} \\ Z' &= \text{constant}, \quad E'_{0_1} = Z' H'_{0_2} = \text{constant} \\ \frac{h_1}{h_2} &= \frac{Z'}{Z} = \text{constant} \end{aligned} \tag{10.16}$$

Setting this constant to 1 we have

$$\begin{aligned} Z' &= Z \\ \mu' &= \frac{Z'}{v'} = \frac{Z}{v} = \mu_0 \\ \epsilon' &= [v'Z']^{-\frac{1}{2}} = [vZ]^{-\frac{1}{2}} = \epsilon \\ h_1 = h_2 &= \frac{\Psi}{\Psi_m} = -e^{w_3} \sin(w_1) = h_w \left| \frac{dw_1}{du_1} \right| = \Psi_m e^{w_3} \left| \frac{dw_1}{du_1} \right| \\ u_1 - u_1^{(0)} &= -\Psi_m \int_{w_1^{(0)}}^{w_1} \frac{dw_1''}{\sin(w_1'')} = \Psi_m \ell n \left(\frac{\tan\left(\frac{w_1^{(0)}}{2}\right)}{\tan\left(\frac{w_1}{2}\right)} \right) \\ &= \Psi_m \ell n \left(\frac{\cot\left(\frac{\theta}{2}\right)}{\cot\left(\frac{\theta_0}{2}\right)} \right) \end{aligned} \tag{10.17}$$

with integration constants and signs chosen for convenience (e.g. zero voltage on the "inner" conductor ($\theta = \theta_0 = -w_1^{(0)}$)). Here u_1 is proportional to the electric potential which takes the well-known form.

10.2 Spherical transmission line: propagation in the θ direction

Letting b_1 be a real we can choose (for simplicity)

$$\begin{aligned} & b_1 = 1\,, \quad \zeta_0 = \Psi_m > 0\,, \quad \zeta(w) = \Psi_m e^w \\ & h_w = \Psi_m e^{w_1}\,, \quad w_1 = \ell n \left(\frac{r}{\Psi_m} \right)\,, \; w_3 = \theta \\ & h_2 = \frac{\Psi}{\Psi_m} = \text{Im}\left[e^w\right] = e^{w_1} \sin(w_3) = \frac{r}{\Psi_m} \sin(\theta) \end{aligned} \tag{10.18}$$

so that propagation is in the θ direction. Surfaces of constant u_3 are circular cones, and surfaces of constant u_1 are spheres.

Now we have

$$\begin{aligned}\Psi^{-1}v &= \Psi^{-1}[\mu_0\epsilon]^{-\frac{1}{2}} = \Psi^{-1}h_3[\mu'\epsilon']^{-\frac{1}{2}} = \frac{h_w}{\Psi}\left|\frac{dw_3}{du_3}\right| v'(u_3) \\ &= \frac{1}{\sin(w_3)}\left|\frac{dw_3}{du_3}\right| v'(u_3) = \text{function of } u_3 \text{ (or } w_3\text{) only} \qquad (10.19)\\ &\Rightarrow \Psi^2\epsilon \neq \text{function of } u_1 \text{ (or } w_1\text{)}\end{aligned}$$

and also

$$\begin{aligned}\Psi^{-1}Z &= \Psi^{-1}\left[\frac{\mu_0}{\epsilon}\right]^{\frac{1}{2}} = \frac{h_2}{\Psi h_1}\left[\frac{\mu'}{\epsilon'}\right]^{\frac{1}{2}} = \frac{1}{\Psi_m h_w}\left|\frac{du_1}{dw_1}\right| Z'(u_1) \\ &= \frac{e^{-w_1}}{\Psi_m^2}\left|\frac{du_1}{dw_1}\right| Z'(u_1) = \text{function of } u_1 \text{ (or } w_1\text{) only} \qquad (10.20)\\ &\qquad \text{(i.e. not a function of } u_3\text{)} \\ &\Rightarrow \Psi^2\epsilon \neq \text{function of } u_3 \text{ (or } w_3\text{)}\end{aligned}$$

Combining these results gives

$$\begin{aligned}&\Psi^2\epsilon = \text{constant} \\ &\frac{\epsilon}{\epsilon_{\min}} = \left[\frac{\Psi_m}{\Psi}\right]^2 = \left[\frac{\Psi_m}{r\sin(\theta)}\right]^2 = \frac{e^{-2w_1}}{\sin^2(w_3)} \qquad (10.21)\\ &\epsilon_{\min} \geq \epsilon_0 \\ &\Psi_m \equiv \text{maximum } \Psi \text{ of interest}\end{aligned}$$

Then we also have

$$\begin{aligned}v &= [\mu_0\epsilon]^{-\frac{1}{2}} = v_{\max}\frac{\Psi}{\Psi_m} \\ v_{\max} &= [\mu_0\epsilon_{\min}]^{-\frac{1}{2}} \leq c \\ Z &= \left[\frac{\mu_0}{\epsilon}\right]^{\frac{1}{2}} = Z_{\max}\frac{\Psi}{\Psi_m} \qquad (10.22)\\ Z_{\max} &= \left[\frac{\mu_0}{\epsilon_{\min}}\right]^{\frac{1}{2}} \leq Z_0\end{aligned}$$

We next infer

$$H_{0_2}(u_1,u_3) = \frac{\Psi_m}{\Psi} H'_{0_2} = \frac{e^{-w_1}}{\sin(w_3)} H'_{0_2} = \frac{\Psi_m}{r\sin(\theta)} H'_{0_2}$$

$$\begin{aligned} E_{0_2}(u_1,u_3) &= Z H_{0_2}(u_1,u_3) = Z_{\max}\frac{\Psi}{\Psi_m} H'_{0_2} = Z_{\max} e^{w_1}\sin(w_3) H'_{0_2} \\ &= Z_{\max}\frac{r\sin(\theta)}{\Psi_m} H'_{0_2} \end{aligned} \tag{10.23}$$

So E and H have opposite dependencies on $r\sin(\theta)$.

For convenience choose

$$v' = [\mu'\epsilon']^{-\frac{1}{2}} = \text{constant (i.e. not a function of } u_3) \tag{10.24}$$

Then we have

$$h_3 = \frac{v}{v'} = \left[\frac{\mu'\epsilon'}{\mu_0\epsilon}\right]^{\frac{1}{2}} = \frac{v_{\max}}{v'}\left[\frac{\epsilon_{\min}}{\epsilon}\right]^{\frac{1}{2}} = \frac{v_{\max}}{v'}\frac{\Psi}{\Psi_m} \tag{10.25}$$

Choosing

$$v' = v_{\max} \tag{10.26}$$

we have

$$h_3 = \frac{\Psi}{\Psi_m} = h_2 = e^{w_1}\sin(w_3) = \frac{r\sin(\theta)}{\Psi_m} \tag{10.27}$$

Combining with

$$h_3 = h_w\left|\frac{dw_3}{du_3}\right| = \Psi_m e^{w_1}\left|\frac{dw_3}{du_3}\right| \tag{10.28}$$

this gives

$$\begin{aligned} u_3 - u_3^{(0)} &= \Psi_m \int_{w_3^{(0)}}^{w_3} \frac{dw''_3}{\sin(w''_3)} = \Psi_m \ell n\left(\frac{\tan\left(\frac{w_3}{2}\right)}{\tan\left(\frac{w_3^{(0)}}{2}\right)}\right) \\ &= \Psi_m \ell n\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta_0}{2}\right)}\right) \end{aligned} \tag{10.29}$$

with integration constants and signs chosen for convenience (e.g. $\theta_0 = 0$ being some reference cone for the wave propagation).

For the u_1 coordinate let us for convenience choose constant μ' and ϵ' as

$$\begin{aligned} &\epsilon' = \text{constant}\,, \quad \mu' = \text{constant} \\ &Z' = \text{constant}\,, \quad E'_{0_1} = Z' H'_{0_2} = \text{constant} \\ &\frac{h_1}{h_2} = \frac{Z'}{Z} = \frac{Z'}{Z_{\max}}\left[\frac{\epsilon}{\epsilon_{\min}}\right]^{\frac{1}{2}} = \frac{Z'}{Z_{\max}}\frac{\Psi_m}{\Psi} \end{aligned} \tag{10.30}$$

Choosing

$$Z' = Z_{\max} \tag{10.31}$$

we have

$$\begin{aligned} \mu' &= \frac{Z'}{v'} = \frac{Z}{v} = \mu_0 \\ \epsilon' &= [v'Z']^{-\frac{1}{2}} = [vZ]^{-\frac{1}{2}} = \epsilon \end{aligned} \tag{10.32}$$

This gives

$$\begin{aligned} h_1 &= \frac{\Psi_m}{\Psi} h_2 = 1 = h_w \left|\frac{dw_1}{du_1}\right| = \Psi_m e^{w_1}\left|\frac{dw_1}{du_1}\right| \\ u_1 - u_2^{(0)} &= \Psi_m \int_{w_1^{(0)}}^{w_1} e^{w_1''} dw_1'' = \Psi_m\left[e^{w_1} - e^{w_1^{(0)}}\right] \\ u_1 &= \Psi_m e^{w_1} = r \end{aligned} \tag{10.33}$$

with integration constants and signs chosen for convenience.

Since this gives a new type of transient lens let us consider an illustration of it as in fig. 10.1. This is taken as a cross section on any (z, Ψ) plane (constant ϕ and $\phi+\pi$). One possible application is a connection to a coax at the $z = 0$ plane, where the propagation is parallel to the z axis. This would allow launching a wave from a near-line source (small θ_0) onto a coax, or receiving a wave from a coax onto the lens which brings it to a line focus. With Ψ_m defining the radius of the outer spherical conductor, and Ψ_1 defining the radius of the inner one, these same dimensions can be used to define the radii of the circular cylindrical conductors of the coax. With permeability μ_0, the permittivity ϵ_c (uniform) of the coax can be chosen as some average of the $\epsilon(\Psi)$ in the lens at the $z = 0$ plane. A good choice would be to have the transmission-line characteristic impedances match at this interface. From

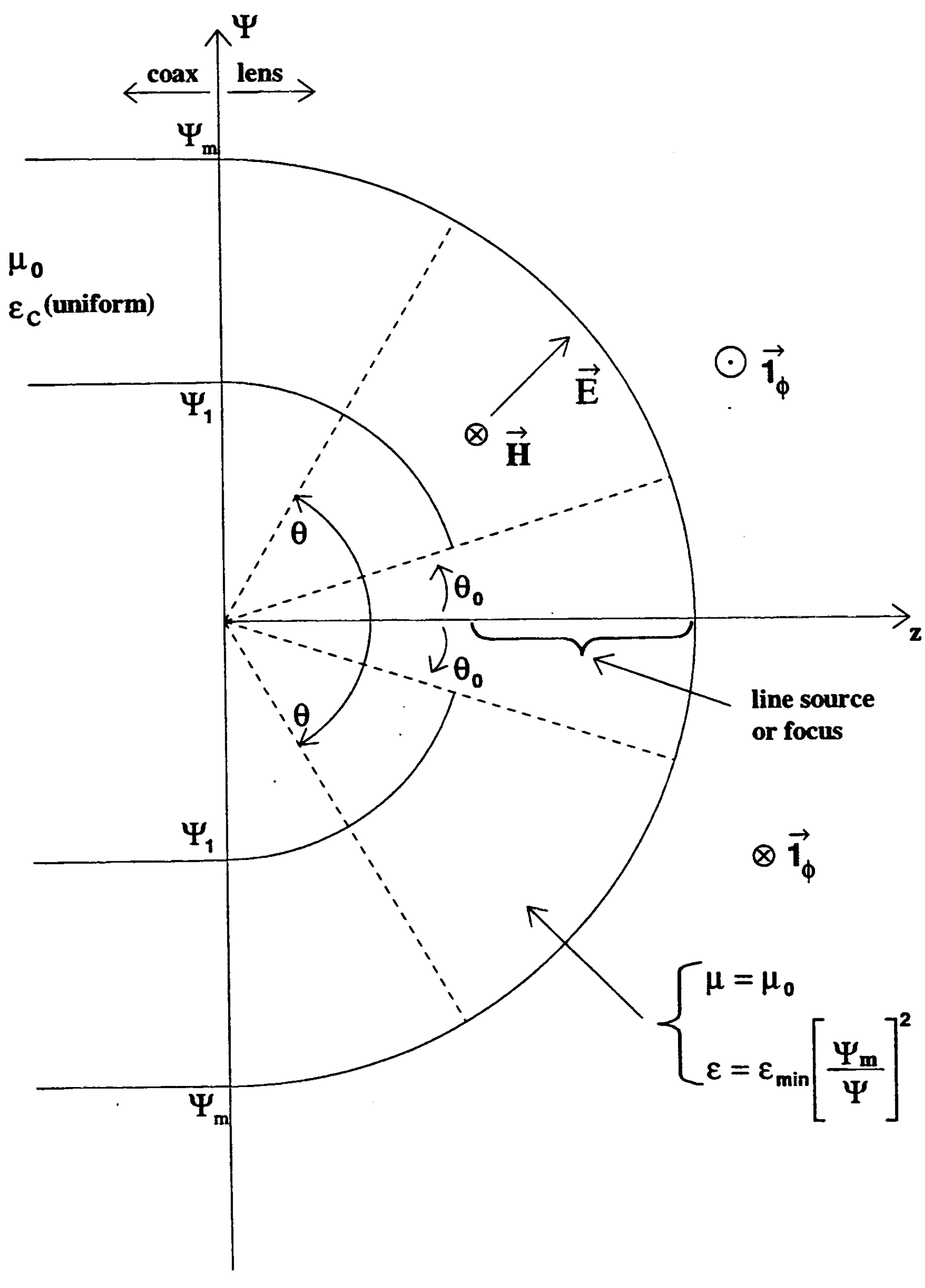

Figure 10.1 Spherical Transmission Line

(10.23) we have for the lens

$$Z_{c_{\text{lens}}} = \frac{\frac{Z_{\max}}{\Psi_m}\int\limits_{\Psi_1}^{\Psi_m} \Psi d\Psi}{\Psi_m \int\limits_0^{2\pi} d\phi} = \frac{Z_{\max}}{2\pi}\left[1 - \left[\frac{\Psi_1}{\Psi_m}\right]^2\right] \tag{10.34}$$

$$Z_{\max} = \left[\frac{\mu_0}{\epsilon_{\min}}\right]^{\frac{1}{2}}$$

For the coax we have

$$Z_{c_{\text{coax}}} = \left[\frac{\mu_0}{\epsilon_c}\right]^{\frac{1}{2}} \frac{1}{2\pi} \ell n\left(\frac{\Psi_m}{\Psi_1}\right) \tag{10.35}$$

Equating these gives

$$\frac{\epsilon_c}{\epsilon_{\min}} = \left[1 - \left[\frac{\Psi_1}{\Psi_m}\right]^2\right]^{-1} \ell n\left(\frac{\Psi_m}{\Psi_1}\right) \tag{10.36}$$

The reader can compare the imperfect matching to coax in this example to the matching problem for a coaxial bend to coax [3, 4].

11 Concluding Remarks

The generalized unipolarized TEM plane wave, when applied to a body of revolution with generalized axial propagation, has led to a set of lenses with interesting properties. The permittivity μ_0 is constant but the permittivity ϵ is allowed to vary as a function of position. Two of the examples (coax and circular-conical transmission line) have a uniform ϵ and are well-known classical examples. The remaining two examples have ϵ varying as Ψ^{-2}, like the cases of azimuthal propagation in [1-4,6]. Here we have examples of cylindrical radial propagation and θ propagation between spherical conducting surfaces, the latter being quite unusual.

Appendix A. Orthogonal Curvilinear Coordinates on a Plane

In Section 8 orthogonal curvilinear coordinates are constructed based on a conformal transformation of the form

$$\begin{aligned} w(\zeta) &= w_1(\zeta) + j w_3(\zeta) \\ \zeta &= x + jy \quad \text{(complex Cartesian coordinates)} \end{aligned} \tag{A.1}$$

The real and imaginary components of w are then separately scaled to give the orthogonal curvilinear coordinates of interest. Here we show that this is a very general approach, covering all the possibilities.

Note that general two-dimensional coordinates on a surface S can be written with line element as [10]

$$\begin{aligned} [d\ell]^2 &= \sum_{n=1}^{2} \sum_{m=1}^{2} g_{n,m} dU_n dU_m = \text{positive-definite quadratic form} \\ g_{2,1} &= g_{1,2} \\ g_{n,m} &= g_{n,m}(U_1, U_2) \\ [d\ell]^2 &= g_{1,1}[dU_1]^2 + 2 g_{1,2} dU_1 dU_2 + g_{2,2}[dU_2]^2 \end{aligned} \tag{A.2}$$

This is also written in the form

$$\begin{aligned} [d\ell]^2 = &\left[g_{1,1}^{\frac{1}{2}} dU_1 + \frac{g_{1,2} + j\left[g_{1,1}g_{2,2} - g_{1,2}^2\right]^{\frac{1}{2}}}{g_{1,1}^{\frac{1}{2}}} dU_2 \right] \\ &\left[g_{1,1}^{\frac{1}{2}} dU_1 + \frac{g_{1,2} - j\left[g_{1,1}g_{2,2} - g_{1,2}^2\right]^{\frac{1}{2}}}{g_{1,1}^{\frac{1}{2}}} dU_2 \right] \end{aligned} \tag{A.3}$$

$$\left[g_{1,1}g_{2,2} - g_{1,2}^2\right]^{\frac{1}{2}} = \text{real and positive for real surfaces with real coordinate lines}$$

At this point we can note that if (U_1, U_2) are orthogonal curvilinear coordinates this reduces to

$$[d\ell]^2 = \left[g_{1,1}^{\frac{1}{2}} dU_1 + j g_{2,2}^{\frac{1}{2}} dU_2\right] \left[g_{1,1}^{\frac{1}{2}} dU_1 - j g_{2,2}^{\frac{1}{2}} dU_2\right] = g_{1,1}[dU_1]^2 + g_{2,2}[dU_2]^2 \tag{A.4}$$

$$g_{1,2} = 0$$

Following [12] we can write (A.3) as

$$\chi_1 = g_{1,1}^{\frac{1}{2}}, \quad \chi_2 = \frac{g_{1,2} + j\left[g_{1,1}g_{2,2} - g1,2^2\right]^{\frac{1}{2}}}{g_{1,1}^{\frac{1}{2}}}$$

$$[d\ell]^2 = [(\chi_n)\cdot(dU_n)]\,[(\chi_n^*)\cdot(dU_n)] \tag{A.5}$$

$$\chi_n = \chi_n(U_1, U_2)$$

Let

$$\psi = \psi(U_1,U_2) = \psi_1 + j\psi_2 = \text{non-zero integrating factor of } (\chi_n)\cdot(dU_n) = 0 \tag{A.6}$$

i.e.

$$d\xi = \psi(\chi_n)\cdot(dU_n) = \psi\left[\chi_1 dU_1 + \chi_2 dU_2\right]$$

$$\xi = \xi(U_1,U_2) = \text{complex valued function} \tag{A.7}$$

The existence of such an integrating factor (actually an infinite number of them) is basic to the theory of first-order differential equations [9]. So we can always construct things this way, subject to appropriate differentiability conditions (smoothness) of S and the defining function for the coordinates [7, 8].

Then we have

$$[d\xi][d\xi]^* = \psi\psi^*\,[(\chi_n)\cdot(dU_n)]\,[(\chi_n^*)\cdot(dU_n)] = \psi\psi^*[d\ell]^2$$

$$\xi = \xi_1 + j\xi_2, \quad \xi_n = \text{real}$$

$$[d\xi][d\xi]^* = [d\xi_1]^2 + [d\xi_2]^2 \tag{A.8}$$

$$[d\ell]^2 = \frac{1}{\psi\psi^*}\left[[d\xi_1]^2 + [d\xi_2]^2\right] = \left[\frac{d\xi}{\psi}\right]\left[\frac{d\xi}{\psi}\right]^*$$

$$\psi\psi^* > 0$$

so (ξ_1, ξ_2) are a set of orthogonal curvilinear coordinates. These coordinates (ξ_1, ξ_2) are called *isometric* or *isothermal* coordinates. Here we see that general coordinates (U_1, U_2) can be transformed to isometric coordinates. For orthogonal curvilinear coordinates we have

$$\chi_1 = g_{1,1}^{\frac{1}{2}} \equiv h_1 \,, \quad \chi_2 = j g_{2,2}^{\frac{1}{2}} \equiv j h_2 \tag{A.9}$$

where the h_n are the customary scale factors for orthogonal curvilinear coordinates (U_1, U_2).

Note that since U_1 and U_2 are each functions of the usual three-dimensional Cartesian coordinates

$$\vec{r} = x\vec{1_x} + y\vec{1_y} + z\vec{1_z} \tag{A.10}$$

then we can also regard $\xi, \psi, g_{n,m}, h_n$, etc. as functions of r, as well as functions of (U_1, U_2).

Citing some results [9] we have

$$\chi_1 \frac{\partial \xi}{\partial U_2} - \chi_2 \frac{\partial \xi}{\partial U_1} = 0 \tag{A.11}$$

is the necessary and sufficient condition that ξ (nonconstant) be the general solution. The integrating factor satisfies

$$\psi = \frac{1}{\chi_1} \frac{\partial \xi}{\partial U_1} = \frac{1}{\chi_2} \frac{\partial \xi}{\partial U_2} \tag{A.12}$$

It is also known that the most general integrating factor has the form $\psi \Xi(\xi)$. Furthermore [10], given isometric parameters (ξ_1, ξ_2) for a surface every other pair is given by an analytic conformal transformation of ξ. The integrating factor satisfies a differential equation. From the requirement that (A.7) be exact we have [9]

$$\frac{\partial}{\partial U_2}(\psi \chi_1) = \frac{\partial}{\partial U_1}(\psi \chi_2) \tag{A.13}$$

which leads to

$$\begin{aligned} \frac{\partial \chi_1}{\partial U_2} - \frac{\partial \chi_2}{\partial U_1} &= \frac{1}{\psi} \left[\chi_2 \frac{\partial \psi}{\partial U_1} - \chi_1 \frac{\partial \psi}{\partial U_2} \right] \\ &= \chi_2 \frac{\partial \ell n(\psi)}{\partial U_1} - \chi_1 \frac{\partial \ell n(\psi)}{\partial U_2} \end{aligned} \tag{A.14}$$

Now (A.8) has the form

$$[d\ell]^2 = h_\xi^2 \left[[d\xi_1]^2 + [d\xi_2]^2\right] \qquad \text{(A.15)}$$
$$h_\xi = |\Psi|^{-1}$$

It is shown in [13 (Appendix A.5)] that if S is Euclidean (a plane being a special case of this), then (A.8) implies that $\xi(\zeta)$ is an analytic conformal transformation where

$$\zeta = x + jy = \text{Cartesian coordinates of plane or other Euclidean surface}$$
$$h_\xi = \left|\frac{d\zeta}{d\xi}\right| = \left|\frac{d\xi}{d\zeta}\right|^{-1} \qquad \text{(A.16)}$$
$$h_\xi^2 = h_{\xi_n}^2 = \left[\frac{\partial x}{\partial \xi_n}\right]^2 + \left[\frac{\partial y}{\partial \xi_n}\right]^2 \text{ for } n = 1, 2$$

So now we see that the integrating factor Ψ is simply related to the scale factor h_ξ of this last transformation. Thereby a general set of coordinates (U_1, U_2) on a plane can be transformed to a conformal transformation with an analytic complex function.

Restricting S to be an x, y plane (or other Euclidean surface with these coordinates) we have

$$g_{1,1} = \left[\frac{\partial x}{\partial U_1}\right]^2 + \left[\frac{\partial y}{\partial U_1}\right]^2$$
$$g_{2,2} = \left[\frac{\partial x}{\partial U_2}\right]^2 + \left[\frac{\partial y}{\partial U_2}\right]^2 \qquad \text{(A.17)}$$
$$g_{1,2} = g_{2,1} = \frac{\partial x}{\partial U_1}\frac{\partial x}{\partial U_2} + \frac{\partial y}{\partial U_1}\frac{\partial y}{\partial U_2}$$

Further restricting the coordinates to be orthogonal we have

$$g_{1,2} = g_{2,1} = 0 = \frac{\partial x}{\partial U_1}\frac{\partial x}{\partial U_2} + \frac{\partial y}{\partial U_1}\frac{\partial y}{\partial U_2}$$
$$\chi_1 = g_{1,1}^{\frac{1}{2}} = h_1\,, \quad h_1 > 0$$
$$\chi_2 = j g_{2,2}^{\frac{1}{2}} = j h_2\,, \quad h_2 > 0 \qquad \text{(A.18)}$$
$$g_{1,1} = \left|\frac{\partial \xi}{\partial U_1}\right|^2 = h_1^2$$
$$g_{2,2} = \left|\frac{\partial \xi}{\partial U_2}\right|^2 = h_2^2$$

Some previous results become

$$
\begin{aligned}
h_1\frac{\partial \xi}{\partial U_2} - jh_2\frac{\partial \xi}{\partial U_1} &= 0 \\
\psi = \frac{1}{h_1}\frac{\partial \xi}{\partial U_1} &= -\frac{j}{h_2}\frac{\partial \xi}{\partial U_2} \\
\frac{\partial}{\partial U_2}(\psi h_1) &= j\frac{\partial}{\partial U_1}(\psi h_2) \\
\frac{\partial h_1}{\partial U_2} - j\frac{\partial h_2}{\partial U_1} &= \frac{1}{\psi}\left[jh_2\frac{\partial \psi}{\partial U_1} - h_1\frac{\partial \psi}{\partial U_2}\right] \\
&= jh_2\frac{\partial \ell n(\psi)}{\partial U_1} - h_1\frac{\partial \ell n(\psi)}{\partial U_2} \\
d\xi &= \psi\left[h_1 dU_1 + jh_2 dU_2\right] \\
[d\ell]^2 = |d\zeta|^2 &= h_1^2\,[dU_1]^2 + h_2^2\,[dU_2]^2 \\
&= |\psi|^{-2}\,|d\xi|^2 = h_\xi^2\,|d\xi|^2
\end{aligned}
\tag{A.19}
$$

In (A.19) we have

$$d\xi = \psi\left[h_1 dU_1 + jh_2 dU_2\right] \tag{A.20}$$

An incremental length in the U_1 direction is $h_1 dU_1$. If ψ is real this is just dx. In the U_2 direction the incremental length is $h_2 dU_2$, but $jh_2 dU_2$ is just jdy. This gives

$$d\xi = \psi\left[dx + jdy\right] = \psi d\zeta \tag{A.21}$$

More generally (complex ψ) we have

$$
\begin{aligned}
d\xi &= |\psi| e^{j\arg(\psi)} d\zeta \\
e^{j\arg(\psi)} d\zeta &= \left[\cos(\arg(\psi))dx - \sin(\arg(\psi))dy\right] \\
&\quad + j\left[\sin(\arg(\psi))dx + \cos(\arg(\psi))dy\right] \\
&\equiv \text{rotation of } d\zeta \text{ by } \arg(\psi)
\end{aligned}
\tag{A.22}
$$

showing a local rotation of the (U_1, U_2) coordinates with respect to the (x, y) coordinates.

So now we can identify

$$\frac{d\xi}{d\zeta} = \psi \equiv \text{integrating factor} \tag{A.23}$$

Since ξ is an analytic function of ζ, then so is ψ. The most general integrating factor is $\psi\Xi(\xi)$ which is now analytic in ζ and ξ. Se we can write

$$\frac{d\xi}{d\zeta} = \psi(\xi)\Xi(\xi) \tag{A.24}$$

If we choose

$$\Xi(\xi) = \psi^{-1}(\xi) \tag{A.25}$$

then we have 1 as an integrating factor and

$$\begin{aligned} \frac{d\xi}{d\zeta} &= 1 \\ \xi &= \zeta + \text{constant} \end{aligned} \tag{A.26}$$

So what we have shown is that for U_1, U_2 as orthogonal curvilinear coordinates on a plane (Euclidean surface), then (A.7) is *exact without an integrating factor.*

Returning to (A.19) we can now write

$$d\xi = h_1 dU_1 + j h_2 dU_2 \tag{A.27}$$

as an *exact* differential equation. Furthermore we have from (A.19)

$$\frac{\partial h_1}{\partial U_2} = j \frac{\partial h_2}{\partial U_1} \tag{A.28}$$

Taking real and imaginary parts we have

$$\frac{\partial h_1}{\partial U_2} = 0\,, \quad \frac{\partial h_2}{\partial U_1} = 0 \tag{A.29}$$

from which we conclude

$$\begin{aligned} h_1 &= h_1(U_1) \quad \text{(not a function of } U_2\text{)} \\ h_2 &= h_2(U_2) \quad \text{(not a function of } U_1\text{)} \end{aligned} \tag{A.30}$$

By a change of variable

$$\begin{aligned} h_1(U_1)dU_1 &= du_1\,, \quad u_1 = \int h_1(U_1)dU_1 \\ h_2(U_2)dU_2 &= du_2\,, \quad u_2 = \int h_2(U_2)dU_2 \end{aligned} \tag{A.31}$$

we have

$$d\xi = du \equiv du_1 + jdu_2$$
$$u = u_1 + ju_2 \quad \text{(A.32)}$$
$$\xi = u + \text{constant}$$

where we can set this constant to zero for convenience.

Summarizing, for a plane (Euclidean surface) an *arbitrary* orthogonal curvilinear coordinate system (U_1, U_2) is constructed in the form

$$\xi(\zeta) = \xi_1(\zeta) + j\xi_2(\zeta) = u(\zeta) = u_1(\zeta) + ju_2(\zeta) \quad \text{(analytic)}$$
$$U_1 = U_1(u_1) \quad (\text{not a function of } u_2)$$
$$U_2 = U_2(u_2) \quad (\text{not a function of } u_1)$$
$$U_1 = \int h_1'^{-1}(u_1)du_1 \quad \text{(A.33)}$$
$$U_2 = \int h_2'^{-2}(u_2)du_2$$
$$h_1(U_1) = h_1'(u_1) = \frac{du_1}{dU_1}$$
$$h_2(U_2) = h_2'(u_2) = \frac{du_2}{dU_2}$$

all such coordinate systems being representable this way.

We note that integrating factors always exist for differential forms in the plane. In vector analysis terms a vector field $\overrightarrow{F}$ can be expressed as a gradient of a differentiable function f (i.e., $\overrightarrow{F} = \nabla f$) iff $\nabla \times \overrightarrow{F} = \overrightarrow{0}$, modulo topological restrictions on the domain. The corresponding result in the formalism of differential forms is that a 1-form α is expressible as $\alpha = df$ iff $d\alpha = 0$, which is a special case of the Poincare Lemma [11]. We may also ask under what conditions are there functions f and g satisfying $\alpha = gdf$? That is, when is there an integrating factor for the differential equation $\alpha = 0$? The Frobenius Theorem [11] provides an answer. Certainly if $\alpha = gdf$ then the exterior derivative $d\alpha$ is given by

$$d\alpha = dg_\wedge df = dg_\wedge g^{-1}\alpha \quad \text{(A.34)}$$
$$d\alpha = \theta_\wedge \alpha$$

where $\theta = g^{-1}dg = d(\ell n|g|)$, and so

$$\alpha_\wedge d\alpha = \alpha_\wedge \theta_\wedge \alpha = 0.$$

The symbol "$\wedge$" is the wedge or exterior product. For a 1-form $\alpha = Pdx + Qdy + Rdz$ in $\mathbb{R}^3$ this is the condition

$$P(R_y - Q_z) + Q(P_z - R_x) + R(Q_x - P_y) = 0\,. \tag{A.35}$$

Thus if $\overrightarrow{F}$ is not expressible as a gradient then $\overrightarrow{F} = g\nabla f$ for some differentiable functions f and g iff $\overrightarrow{F} \cdot \nabla \times \overrightarrow{F} = 0$. Hence if $\vec{F}$ is a differentiable vector field in the plane, then $\overrightarrow{F} \cdot \nabla \times \overrightarrow{F} = 0$ is always satisfied and the existence of an integrating factor g is guaranteed.

The general differential geometric result is that if α is a non-vanishing 1-form in a neighborhood of the origin in $\mathbb{R}^n$, then $\alpha_\wedge d\alpha = 0$ iff $d\alpha = \alpha_\wedge \theta$, for some 1-form θ. This result is a special case of the Frobenius Integration Theorem [11].

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