Sensor and Simulation Notes

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Transmission-Line Solution for Propagation on Periodic Array of Wave Launchers, All in Phase

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Abstract

This paper develops a general transmission-line solution for a unit cell in an infinite array of coupled wave launchers based on nonuniform-multiconductor-transmission-line theory. Each unit cell takes the form of two conductors expanding from a point source (a quasi TEM horn) to meet the conductors from adjacent cells at cell boundaries. The general solution is formulated in terms of product integrals, and an interesting special case is explicitely solved in closed form in terms of Bessel functions (in complex frequency domain), under the restriction that all unit cells are identically excited (including the same phase) so that the array is radiating in a boresight sense.

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2. Introduction

Now it is time to revisit a decade-old problem [1–4]. These references consider the transmission-line model for a unit cell of a wave-launcher array such as illustrated in figs. 1.1 and 1.2. Such a model presumes that the cross-section dimensions ($2a$ and $2b$) are small compared to wavelength. This limits the extent to which the results of such a model can be applied at high frequencies. A previous paper [7] has considered the high frequency form of the transmission-line equations. This has been used to obtain the high-frequency or early-time propagation on such a unit cell, obtaining thereby an estimate of the wave launched from the array for $z > \ell$ in fig. 1 for early times from a fast-rising source $V(t)$ at $z = 0$ [3, 4].

Fundamental to this analysis is the assumption that all array elements (extending periodically in both $\pm x$ and $\pm y$ directions) have the same $V(t)$ excitation (thereby all being in phase, or perhaps better expressed as simultaneous or "in same time"). This allows one to replace the other unit cells in the array by boundary conditions on the cell boundaries (a special case of a Floquet expansion). As indicated in fig. 1.1, the boundaries at $y = \pm a$ are electric walls (zero tangential electric field) which can be thought of perfectly conducting sheets (extending to large $\pm x$ for experimental purposes, if desired). The boundaries at $x = \pm b$ are magnetic walls (zero tangential magnetic field) which is fine for calculations, but difficult to implement experimentally. The symmetry of the unit cell (two symmetry planes, $x = 0$ and $y = 0$) allows one to define two voltages V_1 and V_2 (differential for $y = 0$ as the zero-voltage reference), and two currents I_1 and I_3, all as functions of z (as in fig. 1.2) for establishing the transmission-line equations.

In recent years the analysis techniques for nonuniform multiconductor transmission lines (NMTLs) have been extended considerably [8, 9, 11–15, 17]. This allows for closed-form solutions for all frequencies in various special cases. In particular, one can use the symmetrical interpolation formulas for the characteristic-impedance matrix (maintaining exact reciprocity) in [15] to define analytically soluable cases, such as will be studied in this paper.

The reader should note that transmission-line theory is an approximation to the real physical problem. Wavelengths are assumed large compared to the cross-section dimensions so that all propagation is one dimensional along the length of the transmission line (which can be many wavelengths long if the cross-section dimensions are sufficiently small compared to the length). So, while the model gives results for frequencies arbitrarily large (or equivalently arbitrarily small time steps near the wavefront), one should realize that the model is not necessarily accurate in such cases.

The present approach is not the only way to treat such arrays. One can perform a more general electromagnetic analysis (including high-and low-frequency approximations) as in [5], including references therein. This can also be approached numerically [6]. However, the present NMTL approach can shed important insight into the array performance and allow analytic parameter variation.

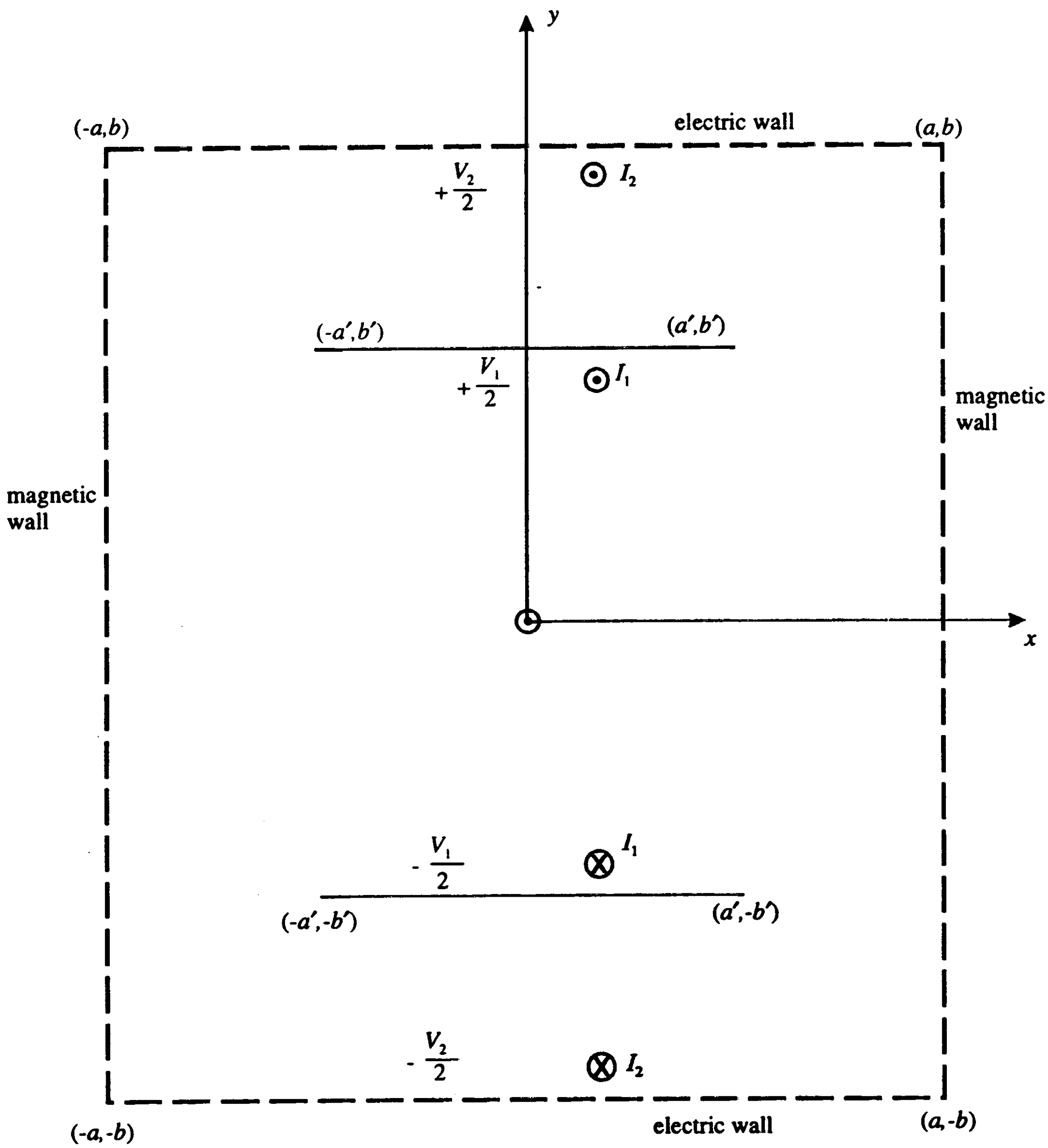

Fig. 1.1. Cross Section of a Unit Cell of the Array.

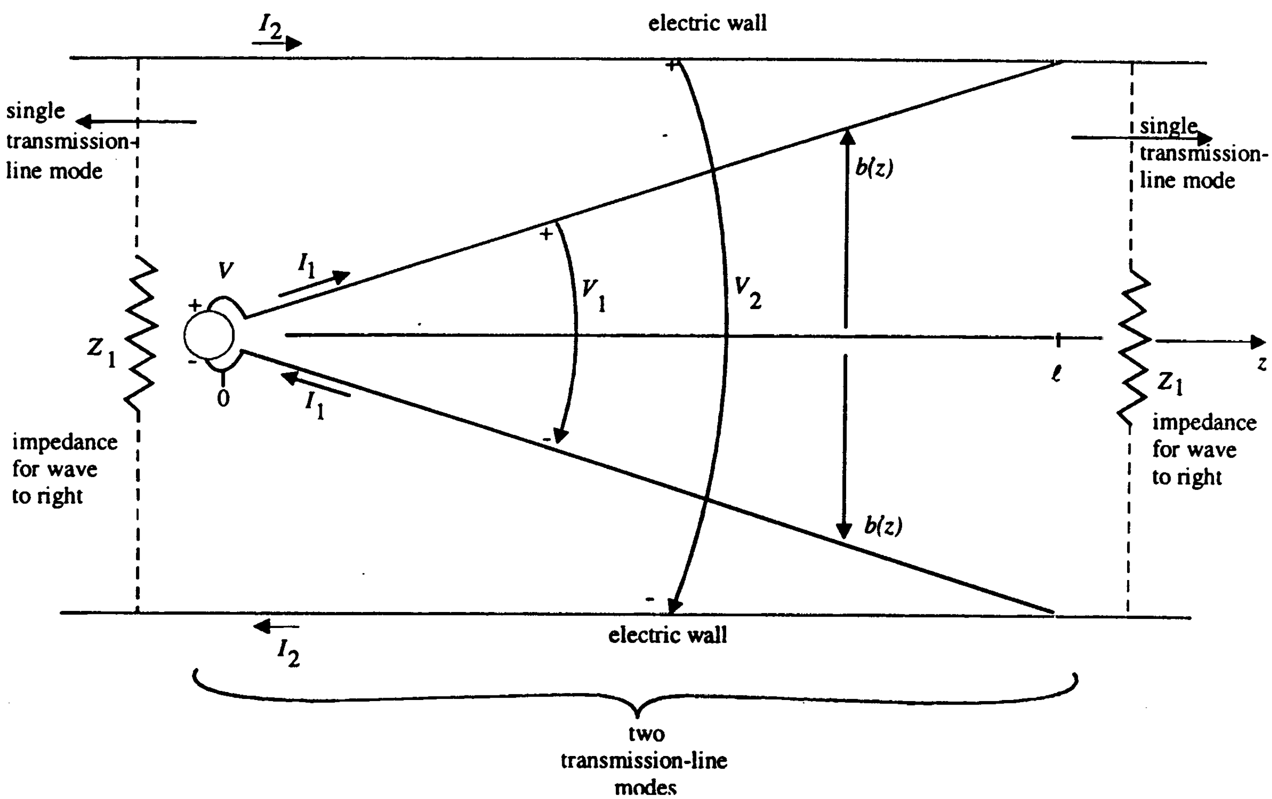

Fig. 1.2. Side View of a Unit Cell of the Array.

2. Per-Unit-Length Parameters for Unit Cell of Wave Launcher

Summarizing from [1] we have per-unit-length inductance and capacitance matrices as

$$
\begin{aligned}
\left(L'_{n,m}(z)\right) &= \mu_0 \left(f_{g_{n,m}}(z)\right) \\
\left(C'_{n,m}(z)\right) &= \varepsilon_0 \left(f_{g_{n,m}}(z)\right)^{-1} \\
\left(f_{g_{n,m}}(z)\right) &= \left(f_{g_{n,m}}(z)\right)^{T} \quad \text{(reciprocity)} \\
&\equiv \text{geometric-factor matrix (dimensionless)} \\
\mu_0 &\equiv \text{permeability of free space} \\
\varepsilon_0 &\equiv \text{permitivity of free space}
\end{aligned}
\tag{2.1}
$$

where these matrices are 2×2. These are used to give

$$
\begin{aligned}
\left(\tilde{Z}'_{n,m}(z,s)\right) &= s\left(L'_{n,m}(z)\right) \equiv \text{per-unit-length impedance matrix} \\
\left(Y'_{n,m}(z,s)\right) &= s\left(C'_{n,m}(z)\right) \equiv \text{per-unit-length admittance matrix} \\
\left(\tilde{\gamma}_{n,m}(z,s)\right) &= \left[\left(Z'_{n,m}(z,s)\right) \cdot \left(Y'_{n,m}(z,s)\right)\right]^{\frac{1}{2}} = \frac{s}{c}\left(1_{n,m}\right) \equiv \tilde{\gamma}(s)\left(1_{n,m}\right) \equiv \text{propagation matrix} \qquad (2.2) \\
c &= \left[\mu_0\, \varepsilon_0\right]^{-\frac{1}{2}} \equiv \text{speed of light} \\
\tilde{\gamma}(s) &= \frac{s}{c} \equiv \text{propagation constant}
\end{aligned}
$$

This geometric-factor matrix (or geometric-impedance-factor matrix) takes a simple form for our problem as

$$
\left(f_{g_{n,m}}(z)\right) = \begin{pmatrix} f_{g_{1,1}}(z) & \dfrac{b'(z)}{a} \\ \dfrac{b'(z)}{a} & \dfrac{b}{a} \end{pmatrix}
\tag{2.3}
$$

This is easily seen in fig. 1.1 and 1.2 by setting $I_2 = 0$ and observing that at each cross section

$$
\begin{aligned}
V_2 &= Z_1\, I_2 \quad , \quad Z_1 = Z_0 \frac{b}{a} = Y_1^{-1} \\
V_1 &= \frac{b'}{b} V_2 = \frac{b'}{b} Z_1\, I_2 = Z_0 \frac{b'}{a}\, I_2
\end{aligned}
\tag{2.4}
$$

Important for this result is the constraint, as in the illustrations that for each z the conducting sheets of width $2a'$ are on lines of constant y'. This plate then does not perturb the magnetic field which has no y component in the above gedankenexperiment. Also $b'(z)$ is assumed to be slowly varying with respect to z so that there is no significant z component to the electric field. With the wave launcher extending over $0 \leq z \leq \ell$ we then require

$$\ell >> a, b \tag{2.5}$$

for the transmission-line approximation to be valid. The form of $f_{g_{1,1}}(z)$ is more complicated, but can be computed numerically [2].

There are physical constraints on the geometric-factor matrix. As a real symmetric matrix (special case of Hermitian) it has real eigenvalues and real eigenvectors. As representing a passive system it is positive semi-definite (non-negative eigenvalues) implying

$$\det\Big(\big(f_{g_{n,m}}(z)\big)\Big) = \frac{b}{c}\, f_{g_{1,1}}(z) - \left[\frac{b'(z)}{a}\right]^2 \geq 0 \tag{2.6}$$

If we choose a particular profile for $b'(z)$ (as later we will choose it proportional to z) this gives a constraint on $f_{g_{1,1}}$. Except at $z = \ell$ (where the wave-launcher conductors reach the cell boundaries at $y = \pm a$) both eigenvalues will be taken as positive. The wave-launcher end will need to be considered as a special limiting case.

Together with the impedance matrix for the wave launcher we need the impedance driven by each wavelauncher to both left and right of the array as

$$Z_1 = \frac{b}{a} Z_0 = Y_1^{-1} \tag{2.7}$$

as indicated in fig. 1.2. This indicates that b/a is a common factor that can be conveniently removed from the calculations.

For convenience we have a normalized geometric-factor matrix

$$\left(F_{n,m}(z)\right) \equiv \frac{a}{b}\left(f_{g_{n,m}}(z)\right) = \begin{pmatrix} F_{1,1}(z) & F_{1,2}(z) \\ F_{1,2}(z) & 1 \end{pmatrix} = \begin{pmatrix} F_{1,1}(z) & \frac{b'(z)}{b} \\ \frac{b'(z)}{b} & 1 \end{pmatrix}$$

$$F_{1,1}(z) \geq F_{1,2}^2(z) \quad \text{(positive semidefinite)} \tag{2.8}$$

$$\left(F_{n,m}(z)\right)^{-1} = \left[F_{1,1}(z) - F_{1,2}^2(z)\right]^{-1} \begin{pmatrix} 1 & -F_{1,2}(z) \\ -F_{1,2}(z) & .F_{1,1}(z) \end{pmatrix}$$

This has constraints

$$\left(F_{n,m}(0)\right) = \begin{pmatrix} F_{1,1} & 0 \\ 0 & 1 \end{pmatrix}$$
$$\left(F_{n,m}(\ell)\right) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.9}$$

For later use we need

$$\left(F_{n,m}(0)\right)^{\pm\frac{1}{2}} = \begin{pmatrix} F_{1,1}^{\pm\frac{1}{2}}(0) & 0 \\ 0 & 1 \end{pmatrix}$$

$$0 < F_{1,1}(0) \quad \left(\text{for non-singular } \left(F_{n,m}(0)\right)\right) \tag{2.10}$$

with the square root taken in the positive sense. Since $(F_{n,m}(0))$ is later used for a similarity transformation along the entire $0 \leq z \leq \ell$ we constrain it to be non singular.

3. Transmission-Line Equations

In the wave-launcher ($0 \le z \le \ell$) we have the transmission-line equations (telegrapher equations)

$$\begin{aligned}\frac{\partial}{\partial z}\left(\tilde{V}_n(z,s)\right) &= -\left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{I}_n(z,s)\right) \\ \frac{\partial}{\partial z}\left(\tilde{I}_n(z,s)\right) &= -\left(\tilde{Y}'_{n,m}(z,s)\right) \cdot \left(\tilde{V}_n(z,s)\right)\end{aligned} \tag{3.1}$$

where no sources are included for present purposes. The above equations with 2×2 matrices can be combined as

$$\begin{aligned}\frac{\partial}{\partial z}\begin{pmatrix}\left(\tilde{V}_n(z,s)\right) \\ Z_1\left(\tilde{I}_n(z,s)\right)\end{pmatrix} &= \left(\left(\tilde{\phi}_{n,m}(z,s)\right)_{u,v}\right) \odot \begin{pmatrix}\left(\tilde{V}_n(z,s)\right) \\ Z_1\left(\tilde{I}_n(z,s)\right)\end{pmatrix} \\ \left(\left(\tilde{\phi}_{n,m}(z,s)\right)_{u,v}\right) &= \begin{pmatrix}\left(0_{n,m}\right) & -Y_1\left(\tilde{Z}'_{n,m}(z,s)\right) \\ -Z_1\left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right)\end{pmatrix} \\ &= -\,\tilde{\gamma}(s)\begin{pmatrix}\left(0_{n,m}\right) & \left(F_{n,m}(z)\right) \\ \left(F_{n,m}(z)\right)^{-1} & \left(0_{n,m}\right)\end{pmatrix}\end{aligned} \tag{3.2}$$

where Z_1 has been included as a convenient normalizing impedance. This kind of equation is solved via a matrizant which is expressed as a product integral as

$$\begin{aligned}\frac{\partial}{\partial z}\left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{u,v}\right) &= \left(\left(\tilde{\Phi}_{n,m}(z,s)\right)_{u,v}\right) \odot \left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{u,v}\right) \\ \left(\left(\tilde{\Phi}_{n,m}(z_0,z_0;s)\right)_{u,v}\right) &= \left(\left(1_{n,m}\right)_{u,v}\right) = \left(1_{n,m}\right) \otimes \left(1_{u,v}\right) \\ \left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{u,v}\right)^{-1} &= \left(\left(\tilde{\Phi}_{n,m}(z_0,z;s)\right)_{u,v}\right) \\ \left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{u,v}\right) &= \prod_{z_0}^{z} e^{\left(\left(\tilde{\Phi}_{n,m}(z',s)\right)_{u,v}\right)dz'}\end{aligned} \tag{3.3}$$

where these are discussed in further detail in [15]. The solution to (3.2) is then given by

$$\begin{pmatrix}\left(\tilde{V}_n(z,s)\right) \\ Z_1\left(\tilde{I}_n(z,s)\right)\end{pmatrix} = \left(\left(\tilde{\Phi}_{n,m}(z,0)\right)_{u,v}\right) \odot \begin{pmatrix}\left(\tilde{V}_n(0,s)\right) \\ Z_1\left(\tilde{I}_n(0_s,s)\right)\end{pmatrix} \tag{3.4}$$

in terms of the boundary conditions at $z = 0$. This can equally well be expressed in terms of the boundary conditions at $z = \ell$ as

$$\begin{pmatrix} (\tilde{V}_n(z,s)) \\ Z_1(\tilde{I}_n(z,s)) \end{pmatrix} = \left((\tilde{\Phi}_{n,m}(z,\ell))_{u,v} \right) \odot \begin{pmatrix} (\tilde{V}_n(\ell,s)) \\ Z_1(\tilde{I}_n(\ell,s)) \end{pmatrix} \tag{3.5}$$

In [15] there is a symmetrical renormalization for the voltage and current variable which we take in terms of the normalized geometric-factor marix as

$$\begin{pmatrix} (\tilde{v}_n(z,s)) \\ (\tilde{i}_n(z,s)) \end{pmatrix} \equiv \begin{pmatrix} (F_{n,m}(0))^{-\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{\frac{1}{2}} \end{pmatrix} \odot \begin{pmatrix} (\tilde{V}_n(z,s)) \\ Z_1(\tilde{I}_n(z,s)) \end{pmatrix} \tag{3.6}$$

where the normalization is of particularly simple form in (2.10). In terms of the new variables we have the NMTL equations

$$\begin{aligned}
\frac{\partial}{\partial z} \begin{pmatrix} (\tilde{v}_n(z,s)) \\ (\tilde{i}_n(z,s)) \end{pmatrix} &= \left((\tilde{\xi}_{n,m}(z,s))_{u,v} \right) \odot \begin{pmatrix} (\tilde{v}_n(z,s)) \\ (\tilde{i}_n(\ell,s)) \end{pmatrix} \\
\left((\tilde{\xi}_{n,m}(z,s))_{u,v} \right) &= -\tilde{\gamma}(s) \begin{pmatrix} (0_{n,m}) & (X_{n,m}(z)) \\ (X_{n,m}(z))^{-1} & (0_{n,m}) \end{pmatrix} \\
(X_{n,m}(z)) &= (F_{n,m}(0))^{-\frac{1}{2}} \cdot (F_{n,m}(z)) \cdot (F_{n,m}(z))^{-\frac{1}{2}} \\
&= (X_{n,m}(z))^T
\end{aligned} \tag{3.7}$$

This has a matrizant solution

$$\begin{aligned}
\frac{\partial}{\partial z} \left((\tilde{\Xi}_{n,m}(z,z_0;s))_{u,v} \right) &= \left((\tilde{\xi}_{n,m}(z,s))_{u,v} \right) \odot \left((\tilde{\Xi}_{n,m}(z,z_0;s))_{u,v} \right) \\
\left((\tilde{\Xi}_{n,m}(z_0,z_0;s))_{u,v} \right) &= \left((1_{n,m})_{u,v} \right) = (1_{n,m}) \otimes (1_{u,v}) \\
\left((\tilde{\Xi}_{n,m}(z,z_0;s))_{u,v} \right) &= \prod_{z_0}^{z} e^{\left((\tilde{\xi}_{n,m}(z',s))_{u,v} \right) dz'}
\end{aligned}$$

$$
\begin{pmatrix} (\tilde{v}_n(z,s)) \\ (\tilde{i}_n(z,s)) \end{pmatrix} = \left((\tilde{\Xi}_{n,m}(z,0;s))_{u,v} \right) \odot \begin{pmatrix} (\tilde{v}_n(0,s)) \\ (\tilde{i}_n(0,s)) \end{pmatrix}
= \left((\tilde{\Xi}_{n,m}(z,\ell;s))_{u,v} \right) \odot \begin{pmatrix} (\tilde{v}_n(\ell,s)) \\ (\tilde{i}_n(\ell,s)) \end{pmatrix} \tag{3.8}
$$

The original voltage and current variables are recovered via

$$
\begin{aligned}
&\left((\tilde{\Phi}_{n,m}(z,z_0;s))_{u,v} \right) \\
&= \begin{pmatrix} (F_{n,m}(0))^{\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{-\frac{1}{2}} \end{pmatrix} \odot \left((\tilde{\Xi}(z,z_0;s))_{u,v} \right) \odot \begin{pmatrix} (F_{n,m}(0))^{-\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{\frac{1}{2}} \end{pmatrix} \\
&= \begin{pmatrix} (F_{n,m}(0))^{\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{-\frac{1}{2}} \end{pmatrix} \odot \left[\prod_{z_0}^{z} e^{\left((\tilde{\xi}_{n,m}(z',s))_{u,v} \right) dz'} \right] \odot \begin{pmatrix} (F_{n,m}(0))^{-\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{\frac{1}{2}} \end{pmatrix} \\
&= \prod_{z_0}^{z} \exp\left(\begin{pmatrix} (F_{n,m}(0))^{\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{-\frac{1}{2}} \end{pmatrix} \odot \left((\tilde{\xi}_{n,m}(z',s))_{u,v} \right) \odot \begin{pmatrix} (F_{n,m}(0))^{-\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{\frac{1}{2}} \end{pmatrix} dz' \right)
\end{aligned} \tag{3.9}
$$

This last result from the similarity rule of the product integral is consistent with

$$
\begin{aligned}
&\left((\tilde{\phi}_{n,m}(z,s))_{u,v} \right) \\
&= \begin{pmatrix} (F_{n,m}(0))^{\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{-\frac{1}{2}} \end{pmatrix} \odot \left((\tilde{\xi}_{n,m}(z,s))_{u,v} \right) \odot \begin{pmatrix} (F_{n,m}(0))^{-\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{\frac{1}{2}} \end{pmatrix}
\end{aligned} \tag{3.10}
$$

It is in terms of $(X_{n,m}(z))$ and the related matrizant that we proceed to construct our solution.

4. Constraints on $(X_{n,m}(z))$

From (2.9) and (3.7) we have

$$
\begin{aligned}
(X_{n,m}(0)) &= (1_{n,m}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
(X_{n,m}(\ell)) &= \begin{pmatrix} F_{1,1}^{-\frac{1}{2}}(0) & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} F_{1,1}^{-\frac{1}{2}}(0) & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} F_{1,1}^{-1}(0) & F_{1,1}^{-\frac{1}{2}}(0) \\ F_{1,1}^{-\frac{1}{2}}(0) & 1 \end{pmatrix} \\
F_{1,1}(0) &> 0 \quad \text{(constraint)}
\end{aligned}
\tag{4.1}
$$

Following [15] let us diagonalize $(X_{n,m}(\ell))$. The eigenvalues are given by

$$
\begin{aligned}
&\det\Big((X_{n,m}(\ell)) - X_\beta(\ell)(1_{n,m})\Big) = 0 \\
&\Big[F_{1,1}^{-1}(0) - X_\beta(\ell)\Big]\Big[1 - X_\beta(\ell)\Big] - F_{1,1}^{-1}(0) = 0 \\
&X_\beta(\ell) = \begin{cases} 1 + F_{1,1}^{-1}(0) & \text{for } \beta = 1 \\ 0 & \text{for } \beta = 2 \end{cases}
\end{aligned}
\tag{4.2}
$$

The eigenvectors are given by (left and right forms being the same by symmetry)

$$
\Big[(X_{n,m}(\ell)) - X_\beta(\ell)(1_{n,m})\Big] \cdot (x_n)_\beta = (0_n)
$$

$$
\begin{pmatrix} -1 & F_{1,1}^{-\frac{1}{2}}(0) \\ F_{1,1}^{-\frac{1}{2}}(0) & F_{1,1}^{-1}(0) \end{pmatrix} \cdot (x_n)_1 = (0_n)
$$

$$
(x_n)_1 = \big[1 + F_{1,1}(0)\big]^{-\frac{1}{2}} \begin{pmatrix} 1 \\ \dfrac{1}{F_{1,1}^{\frac{1}{2}}(0)} \end{pmatrix}
$$

$$
\begin{pmatrix} F_{1,1}^{-1}(0) & F_{1,1}^{-\frac{1}{2}}(0) \\ F_{1,1}^{-\frac{1}{2}}(0) & 1 \end{pmatrix} \cdot (x_n)_2 = (0_n)
$$

$$(x_n)_2 = \left[1 + F_{1,1}(0)\right]^{-\frac{1}{2}} \begin{pmatrix} -F_{1,1}^{\frac{1}{2}}(0) \\ 1 \end{pmatrix}$$

$$(X_n)_{\beta_1} \cdot (X_n)_{\beta_2} = 1_{\beta_1,\beta_2} = \begin{cases} 1 \text{ for } \beta_1 = \beta_2 \\ 0 \text{ for } \beta_1 \neq \beta_2 \end{cases} \tag{4.3}$$

(orthonormal)

Then write in dyadic form

$$\left(X_{n,m}(1)\right) = \left[1 + F_{1,1}(0)\right]^{-1} \left[X_1(\ell) \begin{pmatrix} 1 \\ F_{1,1}^{\frac{1}{2}}(0) \end{pmatrix} \begin{pmatrix} 1 \\ F_{1,1}^{\frac{1}{2}}(0) \end{pmatrix} + X_2(\ell) \begin{pmatrix} -F_{1,1}^{\frac{1}{2}}(0) \\ 1 \end{pmatrix} \begin{pmatrix} -F_{1,1}^{\frac{1}{2}}(0) \\ 1 \end{pmatrix} \right] \tag{4.4}$$

Next note that

$$X_\beta(0) = 1 \text{ for } \beta = 1, 2$$

$$\left(X_{n,m}(0)\right) = \left(1_{n,m}\right)$$

$$= \left[1 + F_{1,1}(0)\right]^{-1} \left[X_1(0) \begin{pmatrix} 1 \\ F_{1,1}^{\frac{1}{2}}(0) \end{pmatrix} \begin{pmatrix} 1 \\ F_{1,1}^{\frac{1}{2}}(0) \end{pmatrix} + X_1(0) \begin{pmatrix} -F_{1,1}^{\frac{1}{2}}(0) \\ 1 \end{pmatrix} \begin{pmatrix} -F_{1,1}^{\frac{1}{2}}(0) \\ 1 \end{pmatrix} \right] \tag{4.5}$$

since any complete set of orthonormal eigenvectors can represent the identity as

$$\left(1_{n,m}\right) = \sum_{\beta=1}^{2} (x_n)_\beta (x_n)_\beta \tag{4.6}$$

Then *construct* $(X_{n,m}(z))$ for $0 \le z \le \ell$ as

$$\left(X_{n,m}(z)\right) \equiv \sum_{\beta=1}^{2} X_\beta(z)\ (x_n)_\beta\, (x_n)_\beta \tag{4.7}$$

This matches the required matrix values at $z = 0, \ell$ and interpolates between these with eigenvectors which are *not functions of* z. Furthermore, this matrix is symmetric for all z in the interval. It is not an arbitrary symmetric matrix but one we can use to *define* a class of useful matrices for our purposes.

The normalized geometric-factor matrix can be reconstructed for such a case as

$$
\begin{aligned}
\left(F_{n,m}(z)\right) &= \left(F_{n,m}(0)\right)^{\frac{1}{2}} \cdot \left(X_{n,m}(z)\right) \cdot \left(F_{n,m}(0)\right)^{\frac{1}{2}} \\
&= \left[1+F_{1,1}(0)\right]^{-1} \begin{pmatrix} F_{1,1}^{\frac{1}{2}}(0) & 0 \\ 0 & 1 \end{pmatrix} \cdot \left[X_1(2) \begin{pmatrix} 1 \\ F_{1,1}^{\frac{1}{2}}(0) \end{pmatrix} \begin{pmatrix} 1 \\ F_{1,1}^{\frac{1}{2}}(0) \end{pmatrix} + X_2(z) \begin{pmatrix} -F_{1,1}^{\frac{1}{2}}(0) \\ 1 \end{pmatrix} \begin{pmatrix} -F_{1,1}^{\frac{1}{2}}(0) \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} F_{1,1}^{\frac{1}{2}}(0) & 0 \\ 0 & 1 \end{pmatrix} \\
&= \left[1+F_{1,1}(0)\right]^{-1} \left[X_1(2)\ F_{1,1}(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + X_2(z) \begin{pmatrix} -F_{1,1}(0) \\ 1 \end{pmatrix} \begin{pmatrix} -F_{1,1}(0) \\ 1 \end{pmatrix} \right] \\
&= \left[1+F_{1,1}(0)\right]^{-1} \left[X_1(2)\, F_{1,1}(0) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + X_2(z) \begin{pmatrix} F_{1,1}^{2}(0) & -F_{1,1}(0) \\ -F_{1,1}(0) & 1 \end{pmatrix} \right] \qquad (4.8) \\
&= \left[1+F_{1,1}(0)\right]^{-1} \begin{pmatrix} F_{1,1}(0)\, X_1(z) + F_{1,1}^{2}(0)\, X_2(z) & F_{1,1}(0)\left[X_1(z) - X_2(z)\right] \\ F_{1,1}(0)\left[X_1(z) - X_2(z)\right] & F_{1,1}(0) X_1(z) + X_2(z) \end{pmatrix}
\end{aligned}
$$

which is symmetric, satisfying reciprocity. There are, however, additional constraints on $(F_{n,m}(z))$ which limit the acceptable choices for the $X_\beta(z)$.

Recalling from (2.8)

$$
\left(F_{n,m}(z)\right) = \begin{pmatrix} F_{1,1}(z) & F_{1,2}(z) \\ F_{1,2}(z) & 1 \end{pmatrix} = \begin{pmatrix} F_{1,1}(z) & \frac{b'(z)}{b} \\ \frac{b'(z)}{b} & 1 \end{pmatrix}
$$

$$
F_{1,1}(z) \ge F_{1,2}^{2}(z) \quad \text{(positive semidefinite)} \qquad (4.9)
$$

we also have the physical constraint

$$
0 \le F_{1,2}(z) \le 1 \qquad (4.10)
$$

This implies for $F_{1,1}(z) > 0$ that

$$
0 \le X_1(z) - X_2(z) \le \frac{1+F_{1,1}(0)}{F_{1,1}(0)} = 1 + F_{1,1}^{-1}(0) \qquad (4.11)
$$

From the positive-semidefinite requirement we have

$$\left[1+F_{1,1}(0)\right]^{-1}\left[F_{1,1}(0)\,X_1(z)+F_{1,1}^2(0)\,X_2(z)\right]$$
$$\geq \left[1+F_{1,1}(0)\right]^{-2} F_{1,1}^2(0)\left[X_1(z)-X_2{}^2(z)\right] \tag{4.12}$$
$$\left[1+F_{1,1}(0)\right]\left[X_1(z)+F_{1,1}(0)\,X_2(z)\right] \geq \left[X_1(z)-X_2(z)\right]^2$$

Furthermore, we have the constraint

$$F_{2,2}(z) = 1 = \left[1+F_{1,1}(0)\right]^{-1}\left[F_{1,1}(0)\,X_1(z)+X_2(z)\right]$$
$$1+F_{1,1}(0) = F_{1,1}(0)\;X_1(z)+X_2(z) \tag{4.13}$$

so that we can only choose one eigenvalue independently.

These constraints can be combined for more convenient results. Substituting for $X_2(z)$ from (4.13) in (4.11) we have

$$0 \leq \left[1+F_{1,1}(0)\right]\left[X_1(z)-1\right] \leq \frac{1+F_{1,1}(0)}{F_{1,1}(0)}$$
$$1 \leq X_1(z) \leq 1 + F_{1,1}^{-1}(0) \tag{4.14}$$

These bounds correspond to the values of X_1 at $z = 0$ and ℓ, respectively, indicating that a monotonic variation of X_1 between these values might be appropriate. Substituting for $X_1(z)$ similarly gives

$$0 \leq \left[1+F_{1,1}^{-1}(0)\right]\left[1-X_2(z)\right] \leq 1+F_{1,1}^{-1}(0)$$
$$0 \leq X_2(z) \leq 1 \tag{4.15}$$

These bounds correspond to the value of X_2 at $z = \ell$ and 0 respectively, also allowing for a monotonic variation of X_2 between these values. One can also substitute for $X_1(z)$ and $X_2(z)$ in (4.12) to obtain similar constraints.

Now let us take a specific example by choosing

$$X_1(z) = 1 + \frac{z}{F_{1,1}(0)\,\ell}$$
$$X_2(z) = 1 - \frac{z}{\ell} \tag{4.16}$$

This satisfies (4.13) and the boundary conditions in (4.2) and (4.5). Forming

$$X_1(z) - X_2(z) = \left[1 + F_{1,1}^{-1}(0)\right]\frac{z}{\ell} \tag{4.17}$$

we see that the constraint of (4.11) on $F_{1,2}(z)$ is satisfied for all $0 \le z \le \ell$. Forming

$$\begin{aligned} X_1(z) + F_{1,1}(0)\,X_2(z) &= 1 + F_{1,1}(0) + \left[F_{1,1}^{-1}(0) - F_{1,1}(0)\right]\frac{z}{\ell} \\ &= F_{1,1}(0)\left[1 + F_{1,1}^{-1}(0)\right]\left[1 + \left[F_{1,1}^{-1}(0) - 1\right]\frac{z}{\ell}\right] \end{aligned} \tag{4.18}$$

the positive-semidefinite requirement in (4.12) becomes

$$\begin{aligned} F_{1,1}(0)\left[1 + \left[F_{1,1}^{-1}(0) - 1\right]\frac{z}{\ell}\right] &\ge \left[\frac{z}{\ell}\right]^2 \\ F_{1,1}(0) &\ge \left[\frac{z}{\ell}\right]^2 - \left[1 - F_{1,1}(0)\right]\frac{z}{\ell} \\ F_{1,1}(0) &\ge \left[\frac{z}{\ell} - \frac{1 - F_{1,1}(0)}{2}\right]^2 - \left[\frac{1 - F_{1,1}(0)}{2}\right]^2 \end{aligned} \tag{4.19}$$

In the last form we see that the right-hand side is maximized at one or both end points of the interval $0 \le z \le \ell$. In these cases we have

$$F_{1,1}(0) \ge \begin{cases} \left[\dfrac{1 + F_{1,1}(0)}{2}\right]^2 - \left[\dfrac{1 - F_{1,1}(0)}{2}\right]^2 = F_{1,1}(0) \text{ for } z = \ell \\ 0 \text{ for } z = 0 \end{cases} \tag{4.20}$$

which merely establishes a lower limit on $F_{1,1}(0)$ of 0. So we have a simple constraint

$$0 < F_{1,1}(0) < \infty \tag{4.21}$$

where this needs to be strictly positive so that $(F_{n,m}(0))^{-1/2}$ is not singular.

For the example in (4.16) the normalized geometric-factor matrix can be constructed from (4.8) as

$$\left(F_{n,m}(z)\right) = \begin{pmatrix} F_{1,1}(0)+\left[1-F_{1,1}(0)\right]\frac{z}{\ell} & \frac{z}{\ell} \\ \frac{z}{\ell} & 1 \end{pmatrix} \tag{4.22}$$

This has a particularly simple form with each element of the form of a constant plus a constant times z. The off-diagonal terms, being just $b'(z)/b$ show that from the side (fig. 1.2) the wave-launcher plates are flat. The plate width $a'(z)$, however, is not simply proportional to z, but can be calculated numerically [2].

For this example, we also have

$$\begin{aligned}
\left(X_{n,m}(z)\right) &= \left[1+F_{1,1}(0)\right]^{-1}\left[X_1(z)\begin{pmatrix} 1 & F_{1,1}^{\frac{1}{2}}(0) \\ F_{1,1}^{\frac{1}{2}}(0) & F_{1,1}(0) \end{pmatrix} X_2(z)\begin{pmatrix} F_{1,1}(0) & -F_{1,1}^{\frac{1}{2}}(0) \\ -F_{1,1}^{\frac{1}{2}}(0) & 1 \end{pmatrix}\right] \\
&= \left[1+F_{1,1}(0)\right]^{-1}\begin{pmatrix} X_1(z)+F_{1,1}(0)\,X_2(z) & F_{1,1}^{\frac{1}{2}}(0)\left[X_1(z)-X_2(z)\right] \\ F_{1,1}^{\frac{1}{2}}(0)\left[X_1(z)-X_2(z)\right] & F_{1,1}(0)X_1(z)+X_2(z) \end{pmatrix} \\
&= \begin{pmatrix} 1+\left[F_{1,1}^{-1}(0)-1\right]\frac{z}{\ell} & F_{1,1}^{-\frac{1}{2}}(0)\frac{z}{\ell} \\ F_{1,1}^{-\frac{1}{2}}(0)\frac{z}{\ell} & 1 \end{pmatrix}
\end{aligned} \tag{4.23}$$

$$tr\left(\left(X_{n,m}(z)\right)\right) = 2+\left[F_{1,1}^{-1}(0)-1\right]\frac{z}{\ell}\;X_1(z)+X_2(z)$$

$$\det\left(\left(X_{n,m}(z)\right)\right) = 1+\left[F_{1,1}^{-1}(0)-1\right]\frac{z}{\ell} - F_{1,1}^{-1}(0)\left[\frac{z}{\ell}\right]^2 = X_1(z)X_2(z)$$

5. Solution of Matrizants

In Section 3 the solution of the NMTL equations is given in terms of a matrizant which is a product integral over $0 \le z \le \ell$. With our special acceptable choice for the eigenvalues of $(X_{n,m}(z))$ in (4.16) we are now in a position to solve this product integral. These eigenvalues have the general form

$$X_\beta(z) = c_\beta\left[z + \ell_\beta\right] \text{ for } \beta = 1, 2 \tag{5.1}$$

From Section 3 we need the product integral

$$\begin{aligned}
\left(\left(\tilde{\Xi}_{n,m}(z, z_0; s)\right)_{u,v}\right) &= \prod_{z_0}^{z} e^{\left(\left(\tilde{\xi}_{n,m}(z', s)\right)_{u,v}\right)dz'} \\
\left(\left(\tilde{\xi}_{n,m}(z', s)\right)_{u,v}\right) &= -\tilde{\gamma}(s)\begin{pmatrix} (0_{n,m}) & (X_{n,m}(z)) \\ (X_{n,m}(z))^{-1} & (0_{n,m}) \end{pmatrix} \\
&= -\tilde{\gamma}(s)\begin{pmatrix} (0_{n,m}) & \sum\limits_{\beta=1}^{2} X_\beta(z)\,(x_n)_\beta (x_n)_\beta \\ \sum\limits_{\beta=1}^{2} X_\beta^{-1}(z)(x_n)_\beta (x_n)_\beta & (0_{n,m}) \end{pmatrix} \\
&= -\tilde{\gamma}(s)\sum_{\beta=1}^{2}\left[(x_n)_\beta (x_n)_\beta\right] \otimes \begin{pmatrix} 0 & X_\beta(z) \\ X_\beta^{-1}(z) & 0 \end{pmatrix} \\
\left(\left(\Xi_{n,m}(z, z_0; s)\right)_{u,v}\right) &\equiv \sum_{\beta=1}^{2}\left(\left(\tilde{\Xi}_{n,m}(z, z_0; s)\right)_{u,v}\right)_\beta \\
&= \sum_{\beta=1}^{2}\left[(x_n)_\beta (x_n)_\beta\right] \otimes \left[\prod_{z_0}^{z} \exp\left(-\tilde{\gamma}(s)\begin{pmatrix} 0 & X_\beta(z') \\ X_\beta^{-1}(z') & 0 \end{pmatrix}dz'\right)\right]
\end{aligned} \tag{5.2}$$

using the results of [15] where the direct product $\otimes$ for constructing supermatrices is also discussed.

Our problem is reduced to the solution of a 2 x 2 product integral

$$
\begin{aligned}
\left(\tilde{\psi}_{u,v}(z,z_0;s)\right)_\beta &= \prod_{z_0}^{z} \exp\left(-\tilde{\gamma}(s)\begin{pmatrix} 0 & X_\beta(z') \\ X_\beta^{-1}(z') & 0 \end{pmatrix} dz'\right) \\
\frac{d}{dz}\left(\tilde{\psi}_{u,v}(z,z_0;s)\right)_\beta &= -\tilde{\gamma}(s)\begin{pmatrix} 0 & X_\beta(z) \\ X_\beta^{-1}(z) & 0 \end{pmatrix} \cdot \left(\tilde{\psi}_{u,v}(z,z_0;s)\right)_\beta \\
\left(\tilde{\psi}_{u,v}(z,z_0;s)\right)_\beta &= \left(1_{u,v}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
\tag{5.3}
$$

This is solved from the second-order linear differential equations

$$
\begin{aligned}
&\left[\frac{\partial^2}{\partial z^2} - \left[\frac{\partial}{\partial z}\ell n\left(X_\beta(z)\right)\right]\frac{\partial}{\partial z} - \tilde{\gamma}^2(s)\right]\tilde{\psi}_{1,v;\beta}(z,z_0;s) = 0 \qquad \text{(voltage like)} \\
&\left[\frac{\partial^2}{\partial z^2} + \left[\frac{\partial}{\partial z}\ell n\left(X_\beta(z)\right)\right]\frac{\partial}{\partial z} - \tilde{\gamma}^2(s)\right]\tilde{\psi}_{2,v;\beta}(z,z_0;s) = 0 \qquad \text{(current like)} \\
&\qquad v = 1,2
\end{aligned}
\tag{5.4}
$$

Each of these two differential equations has two independent solutions

$$
\begin{aligned}
\tilde{v}_\beta^{(\delta)}(z,s) &\equiv \text{voltage modes} \\
\tilde{i}_\beta^{(\delta)}(z,s) &\equiv \text{current modes} \\
\delta &= 1,2
\end{aligned}
\tag{5.5}
$$

respectively. These are related by

$$
\begin{aligned}
\frac{\partial}{\partial z}\tilde{v}_\beta^{(\delta)}(z,s) &= -\tilde{\gamma}(s)\ X_\beta(z)\,\tilde{i}_\beta^{(\delta)}(z,s) \\
\frac{\partial}{\partial z}\tilde{i}_\beta^{(\delta)}(z,s) &= -\tilde{\gamma}(s)\ X_\beta^{-1}(z)\,\tilde{v}_\beta^{(\delta)}(z,s)
\end{aligned}
\tag{5.6}
$$

Including the boundary conditions at $z = z_0$ we have the solution in the form

$$
\left(\tilde{\Psi}_{u,v}(z,z_0;s)\right)_\beta
= \tilde{\Delta}_\beta(s)\begin{pmatrix}
\left[\tilde{i}_\beta^{(2)}(z_0,s)\tilde{v}_\beta^{(1)}(z,s) - \tilde{i}_\beta^{(1)}(z_0,s)\tilde{v}_\beta^{(2)}(z,s)\right] & \left[-\tilde{v}_\beta^{(2)}(z_0,s)\tilde{v}_\beta^{(1)}(z,s) + \tilde{v}_\beta^{(1)}(z_0,s)\tilde{v}_\beta^{(2)}(z,s)\right] \\
\left[\tilde{i}_\beta^{(2)}(z_0,s)\tilde{i}_\beta^{(1)}(z,s) - \tilde{i}_\beta^{(1)}(z_0,s)\tilde{i}_\beta^{(2)}(z,s)\right] & \left[-\tilde{v}_\beta^{(2)}(z_0,s)\tilde{i}_\beta^{(1)}(z,s) + \tilde{v}_\beta^{(1)}(z_0,s)\tilde{i}_\beta^{(2)}(z,s)\right]
\end{pmatrix}
$$

$$
\begin{aligned}
\tilde{\Delta}_\beta^{-1}(s) &= \tilde{v}_\beta^{(1)}(z_0,s)\tilde{i}_\beta^{(2)}(z_0,s) - \tilde{v}_\beta^{(2)}(z_0,s)\,\tilde{i}_\beta^{(1)}(z_0,s) \\
&= -\tilde{\gamma}^{-1}(s)\left[\tilde{v}_\beta^{(1)}(z_0,s)\frac{\partial}{\partial z}\tilde{v}_\beta^{(2)}(z,s)\bigg|_{z=z_0} - \tilde{v}_\beta^{(2)}(z_0,s)\frac{\partial}{\partial z}\tilde{v}_\beta^{(1)}(z,s)\bigg|_{z=z_0}\right] \\
&= \tilde{\gamma}^{-1}(s)\left[\tilde{i}_\beta^{(1)}(z_0,s)\frac{\partial}{\partial z}\tilde{i}_\beta^{(2)}(z,s)\bigg|_{z=z_0} - \tilde{i}_\beta^{(2)}(z_0,s)\frac{\partial}{\partial z}\tilde{i}_\beta^{(1)}(z,s)\bigg|_{z=z_0}\right]
\end{aligned}
\tag{5.7}
$$

Now apply these results (from [15]) to our eigenvalues in (5.1). Writing out our second-order differential equation for the current modes as

$$
\begin{gathered}
\left[\frac{\partial^2}{\partial z^2} + \left[z+\ell_\beta\right]^{-1}\frac{\partial}{\partial z} - \tilde{\gamma}^2(s)\right]\tilde{i}_\beta^{(\delta)}(z,s) = 0 \\
\left[\frac{\partial^2}{\partial(\pm\tilde{\gamma}(s)z)^2} + \left[\pm\tilde{\gamma}(s)\left[z+\ell_\beta\right]\right]^{-1}\frac{\partial}{\partial(\pm\tilde{\gamma}(s)z)} - 1\right]\tilde{i}_\beta^{(\delta)}(z,s) = 0
\end{gathered}
\tag{5.8}
$$

Here we recognize the Bessel equation of order zero for the modified Bessel functions [9, 16]. Furthermore, noting that this equation can be written in terms of $\pm\tilde{\gamma}(s)[z+\ell_\beta]$, we can take solutions as functions of either sign to kep the Bessel functions analytic in the right half s plane. This depends on the sign of $z+\ell_\beta$ in the interval $0 \le z \le \ell$. So let us take

$$
\begin{aligned}
\tilde{i}_\beta^{(1)}(z,s) &= \mathrm{K}_0\left(\pm\tilde{\gamma}(s)\left[z+\ell_\beta\right]\right) \\
\tilde{i}_\beta^{(2)}(z,s) &= \mathrm{I}_0\left(\pm\tilde{\gamma}(s)\left[z+\ell_\beta\right]\right)
\end{aligned}
\tag{5.9}
$$

with the associated voltage modes

$$\begin{aligned}\tilde{v}_{\beta}^{(1)}(z,s) &= -\tilde{\gamma}^{-1}(s)\; c_{\beta}\left[z+\ell_{\beta}\right]\frac{\partial}{\partial z}\mathrm{K}_0\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right)\\ &= \pm\; c_{\beta}\left[z+\ell_{\beta}\right]\mathrm{K}_1\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right)\\ \tilde{v}_{\beta}^{(2)}(z,s) &= -\tilde{\gamma}^{-1}(s)\; c_{\beta}\left[z+\ell_{\beta}\right]\frac{\partial}{\partial z}\mathrm{I}_0\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right)\\ &= \mp\; c_{\beta}\left[z+\ell_{\beta}\right]\mathrm{I}_1\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right)\end{aligned} \tag{5.10}$$

From the Wronskian relationship [16]

$$\begin{aligned}W\{\mathrm{K}_0(\Gamma),\mathrm{I}_0(\Gamma)\} &= \mathrm{K}_0(\Gamma)\frac{d}{d\Gamma}\mathrm{I}_0(\Gamma) - \mathrm{I}_0(\Gamma)\frac{d}{d\Gamma}\mathrm{K}_0(\Gamma)\\ &= \mathrm{K}_0(\Gamma)\,\mathrm{I}_1(\Gamma) + \mathrm{I}_0(\Gamma)\,\mathrm{K}_1(\Gamma)\\ &= \Gamma^{-1}\end{aligned} \tag{5.11}$$

we have

$$\begin{aligned}\tilde{\Delta}_{\beta}^{-1}(s) &= \left[\pm c_{\beta}\left[z+\ell_{\beta}\right]\;\left[\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right]^{-1}\right]_{z=z_0}\\ &= \frac{c_{\beta}}{\tilde{\gamma}(s)}\end{aligned} \tag{5.12}$$

which is conveniently quite simple, independent of both z and z_0. The matrizant then becomes

$$\begin{aligned}&\left(\tilde{\psi}_{u,v}(z,z_0;s)\right)_{\beta}\\ &= \frac{\tilde{\gamma}(s)}{c_{\beta}}\begin{pmatrix} \pm c_{\beta}\left[z+\ell_{\beta}\right]\left[\mathrm{I}_0\left(\pm\tilde{\gamma}(s)\left[z_0+\ell_{\beta}\right]\right)\mathrm{K}_1\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right) + \mathrm{K}_0\left(\pm\tilde{\gamma}(s)\left[z_0+\ell_{\beta}\right]\right)I_1\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right)\right] & c_{\beta}^2\left[z_0+\ell_{\beta}\right]\left[z+\ell_{\beta}\right]\left[\mathrm{I}_1\left(\pm\tilde{\gamma}(s)\left[z_0+\ell_{\beta}\right]\right)\mathrm{K}_1\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right) - \mathrm{K}_1\left(\pm\tilde{\gamma}(s)\left[z_0+\ell_{\beta}\right]\right)I_1\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right)\right]\\ \left[\mathrm{I}_0\left(\pm\tilde{\gamma}(s)\left[z_0+\ell_{\beta}\right]\right)\mathrm{K}_0\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right) - \mathrm{K}_0\left(\pm\tilde{\gamma}(s)\left[z_0+\ell_{\beta}\right]\right)I_0\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right)\right] & \pm c_{\beta}\left[z_0+\ell_{\beta}\right]\left[\mathrm{I}_1\left(\pm\tilde{\gamma}(s)\left[z_0+\ell_{\beta}\right]\right)\mathrm{K}_0\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right) + \mathrm{K}_1\left(\pm\tilde{\gamma}(s)\left[z_0+\ell_{\beta}\right]\right)I_0\left(\pm\tilde{\gamma}(s)\left[z+\ell_{\beta}\right]\right)\right]\end{pmatrix}\\ &= \left(1_{n,m}\right)\ \text{for}\ z = z_0\end{aligned} \tag{5.13}$$

At this point note that the eigenvalues in (4.17) and (5.1) have parameters

$$\begin{aligned}\ell_1 &= F_{1,1}(0)\ell \quad , \quad c_1 = \left[F_{1,1}(0)\ell\right]^{-1} = \ell_1^{-1}\\ \ell_2 &= -\ell \quad , \quad c_2 = -\ell^{-1} = \ell_2^{-1}\end{aligned} \tag{5.14}$$

Thus on the interval $0 \le z \le \ell$ we have

$$
\begin{aligned}
&X_1(z) = 1 + \frac{z}{F_{1,1}(0)\ell} = c_1[z+\ell_1] > 0 \\
&z + \ell_1 > 0 \quad \text{(Use + (upper) sign in (5.13).)} \\
&X_2(z) = 1 - \frac{z}{\ell} = c_2[z+\ell_2] \geq 0 \\
&z + \ell_2 \leq 0 \quad \text{(Use − (lower) sign in (5.13).)}
\end{aligned}
\tag{5.15}
$$

Using the indicated signs keeps the Bessel functions analytic for s in the right half plane.

This is all put together as

$$
\begin{pmatrix} (\tilde{V}_n(z,s)) \\ Z_1(\mathrm{I}_n(z,s)) \end{pmatrix} = \left((\tilde{\Phi}_{n,m}(z,z_0;s))_{u,v} \right) \odot \begin{pmatrix} (V_n(z_0,s)) \\ Z_1(\mathrm{I}_n(z_0,s)) \end{pmatrix}
$$

$$
\left((\tilde{\Phi}_{n,m}(z,z_0;s))_{u,v} \right)
= \begin{pmatrix} (F_{n,m}(0))^{\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{-\frac{1}{2}} \end{pmatrix} \odot \left((\Xi_{n,m}(z,z_0;s))_{u,v} \right) \odot \begin{pmatrix} (F_{n,m}(0))^{-\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{\frac{1}{2}} \end{pmatrix}
$$

$$
(F_{n,m}(0))^{\pm\frac{1}{2}} = \begin{pmatrix} F_{1,1}^{\pm\frac{1}{2}}(0) & 0 \\ 0 & 1 \end{pmatrix}
$$

$$
\left((\Xi_{n,m}(z,z_0;s))_{u,v} \right) = \sum_{\beta=1}^{2} \left((\Xi_{n,m}(z,z_0;s))_{u,v} \right)_\beta
$$

$$
\left((\Xi_{n,m}(z,z_0;s))_{u,v} \right)_\beta = \left[(x_n)_\beta (x_n)_\beta \right] \otimes (\tilde{\psi}_{u,v}(z,z_0;s))_\beta
$$

$$
(x_n)_1 = \left[1 + F_{1,1}(0) \right]^{-\frac{1}{2}} \begin{pmatrix} 1 \\ \frac{1}{F_{1,1}^{\frac{1}{2}}(0)} \end{pmatrix}, \quad (x_n)_2 = \left[1 + F_{1,1}(0) \right]^{-\frac{1}{2}} \begin{pmatrix} -F_{1,1}^{\frac{1}{2}}(0) \\ 1 \end{pmatrix}
$$

$$
0 < F_{1,1} < \infty
$$

$$
(F_{n,m}(0))^{\frac{1}{2}} \cdot (x_n)_1 = (x_n)_1 \cdot (F_{n,m}(0))^{\frac{1}{2}} = F_{1,1}^{\frac{1}{2}}(0)\left[1 + F_{1,1}(0)\right]^{-\frac{1}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$

$$
(F_{n,m}(0))^{\frac{1}{2}} \cdot (x_n)_2 = (x_n)_2 \cdot (F_{n,m}(0))^{\frac{1}{2}} = \left[1 + F_{1,1}(0)\right]^{-\frac{1}{2}} \begin{pmatrix} -F_{1,1}(0) \\ 1 \end{pmatrix}
$$

$$
(F_{n,m}(0))^{-\frac{1}{2}} \cdot (x_n)_1 = (x_n)_1 \cdot (F_{n,m}(0))^{-\frac{1}{2}} = \left[1 + F_{1,1}(0)\right]^{-\frac{1}{2}} \begin{pmatrix} F_{1,1}^{-\frac{1}{2}}(0) \\ \frac{1}{F_{1,1}^{\frac{1}{2}}(0)} \end{pmatrix}
$$

$$
(F_{n,m}(0))^{-\frac{1}{2}} \cdot (x_n)_2 = (x_n)_2 \cdot (F_{n,m}(0))^{-\frac{1}{2}} = \left[1 + F_{1,1}(0)\right]^{-\frac{1}{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}
$$

$$
\left((\tilde{\Phi}_{n,m}(z,z_0;s))_{u,v} \right)
$$

$$\left(\left(\tilde{\Phi}_{n,m}(z,z_0;s)\right)_{u,v}\right)$$

$$= \sum_{\beta=1}^{2}\left[\left(F_{n,m}(0)\right)^{\frac{1}{2}} \cdot (x_n)_\beta (x_n)_\beta \cdot \left(F_{n,m}(0)\right)^{-\frac{1}{2}}\right] \otimes \begin{pmatrix} \tilde{\psi}_{1,1;\beta}(z,z_0;s) & 0 \\ 0 & 0 \end{pmatrix}$$
$$+ \left[\left(F_{n,m}(0)\right)^{\frac{1}{2}} \cdot (x_n)_\beta (x_n)_\beta \cdot \left(F_{n,m}(0)\right)^{\frac{1}{2}}\right] \otimes \begin{pmatrix} 0 & \tilde{\psi}_{1,2;\beta}(z,z_0;s) \\ 0 & 0 \end{pmatrix}$$
$$+ \left[\left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot (x_n)_\beta (x_n)_\beta \cdot \left(F_{n,m}(0)\right)^{-\frac{1}{2}}\right] \otimes \begin{pmatrix} 0 & 0 \\ \tilde{\psi}_{2,1;\beta}(z,z_0;s) & 0 \end{pmatrix}$$
$$+ \left[\left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot (x_n)_\beta (x_n)_\beta \cdot \left(F_{n,m}(0)\right)^{\frac{1}{2}}\right] \otimes \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\psi}_{2,2;\beta}(z,z_0;s) \end{pmatrix} \tag{5.16}$$

with $(\tilde{\psi}_{u,v})_\beta$ as in (5.13). With z_0 taken as 0 or ℓ (with proper limiting considerations) this characterizes the wave-launcher section. There are still boundary conditions to impose at $z = 0, \ell$.

An alternate approach utilizes the different variables from Section 3 as

$$\begin{pmatrix} (\tilde{v}_n(z,s)) \\ (\tilde{i}_n(z,s)) \end{pmatrix} = \begin{pmatrix} \left(F_{n,m}(0)\right)^{-\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & \left(F_{n,m}(0)\right)^{\frac{1}{2}} \end{pmatrix} \odot \begin{pmatrix} (\tilde{V}_n(z,s)) \\ Z_1\left(\tilde{I}_n(z,s)\right) \end{pmatrix}$$
$$= (\tilde{v}_n(z,s)) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (\tilde{i}_n(z,s)) \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.17}$$

In this form the matrizant solution is written as

$$\begin{pmatrix} (\tilde{v}_n(z,s)) \\ (\tilde{i}_n(z,s)) \end{pmatrix} = \left(\left(\Xi_{n,m}(z,z_0;s)\right)_{u,v}\right) \odot \begin{pmatrix} (\tilde{v}_n(z_0,s)) \\ (\tilde{i}_n(z_0,s)) \end{pmatrix}$$
$$= \left[\sum_{\beta=1}^{2}\left[(x_n)_\beta (x_n)_\beta\right] \otimes \left(\tilde{\psi}_{u,v}(z,z_0;s)\right)_\beta\right] \odot \begin{pmatrix} (\tilde{v}_n(z_0,s)) \\ (\tilde{i}_n(z_0,s)) \end{pmatrix}$$
$$= \sum_{\beta=1}^{2}\left[(x_n)_\beta (x_n)_\beta\right](x_n)_\beta \otimes \left[\left(\tilde{\psi}_{u,v}(z,z_0;s)\right)_\beta \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$
$$+ \left[(x_n)_\beta \cdot \left(\tilde{i}_n(z_0,s)\right)_\beta\right](x_n)_\beta \otimes \left[\left(\tilde{\psi}_{u,v}(z,z_0;s)\right)_\beta \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \tag{5.18}$$

6. Application of Boundary Condition at $z = \ell$

At $z = \ell$ where the conductors join to give a single mode of propagation for $z > \ell$ (with impedance Z_1) we have

$$\tilde{V}_1(\ell,s) = \tilde{V}_2(\ell,s) = Z_1\left[\tilde{I}_1(\ell,s) + \tilde{I}_2(\ell,s)\right]$$
$$\left(\tilde{V}_n(\ell,s)\right) = Z_1\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \cdot \left(\tilde{I}_n(\ell,s)\right) \tag{6.1}$$

from which we identify an impedance matrix

$$\left(Z_{n,m}\right)_\ell = Z_1\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = Z_0\frac{a}{b}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \tag{6.2}$$

In terms of our normalized variables we have

$$\begin{aligned}
\left(\tilde{v}_n(\ell,s)\right) &= \left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot \left(\tilde{V}_n(\ell,s)\right)\\
&= Z_1\left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\left(\tilde{I}_n(\ell,s)\right)\\
&= \left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \cdot \left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot \left(\tilde{i}_n(\ell,s)\right)\\
&= \left(F_{n,m}^{(\ell)}\right) \cdot \left(\tilde{i}_n(\ell,s)\right)\\
\left(F_{n,m}^{(\ell)}\right) &= \left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \cdot \left(F_{n,m}(0)\right)^{-\frac{1}{2}}\\
&= \left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \cdot \left(F_{n,m}(0)\right)^{-\frac{1}{2}} = \begin{pmatrix}F_{1,1}^{-\frac{1}{2}}(0)\\ 1\end{pmatrix}\begin{pmatrix}F_{1,1}^{-\frac{1}{2}}(0)\\ 1\end{pmatrix}\\
&= F_{1,1}^{-1}(0)\begin{pmatrix}1\\ F_{1,1}^{-\frac{1}{2}}(0)\end{pmatrix}\begin{pmatrix}1\\ F_{1,1}^{-\frac{1}{2}}(0)\end{pmatrix} = \left[1 + F_{1,1}^{-1}(0)\right](x_n)_1\,(x_n)_1\\
&= \begin{pmatrix}F_{1,1}^{-1}(0) & F_{1,1}^{-\frac{1}{2}}(0)\\ F_{1,1}^{-\frac{1}{2}}(0) & 1\end{pmatrix}
\end{aligned} \tag{6.3}$$

Note that $(F_{n,m}^{(\ell)})$ does not have an inverse (being singular).

From the boundary conditions at $z = \ell$ the voltages and currents for smaller z can be related as

$$
\begin{pmatrix} (\tilde{v}_n(z,s)) \\ (\tilde{i}_n(z,s)) \end{pmatrix} = \left((\Xi(z,\ell_-;s))_{u,v} \right) \odot \begin{pmatrix} (\tilde{v}_n(\ell_-,s)) \\ (\tilde{i}_n(\ell_-,s)) \end{pmatrix}
$$

$$
\begin{aligned}
(\tilde{v}_n(z,s)) &= \left(\tilde{\Xi}(z,\ell_-;s)\right)_{1,1} \cdot (\tilde{v}_n(\ell_-,s)) + \left(\tilde{\Xi}_{n,m}(z,\ell_-;s)\right)_{1,2} \cdot (\tilde{i}_n(\ell_-,s)) \\
&= \left[\left(\tilde{\Xi}_{n,m}(z,\ell_-;s)\right)_{1,1} \cdot \left(F_{n,m}^{(\ell)}\right) + \left(\tilde{\Xi}_{n,m}(z,\ell_-;s)\right)_{1,2}\right] \cdot (\tilde{i}_n(\ell_-,s)) \\
(\tilde{i}_n(z,s)) &= \left(\tilde{\Xi}_{n,m}(z,\ell_-;s)\right)_{2,1} \cdot (\tilde{v}_n(\ell_-,s)) + \left(\tilde{\Xi}_{n,m}(z,\ell_-;s)\right)_{2,2} \cdot (\tilde{i}_n(\ell_-,s)) \\
&= \left[\left(\tilde{\Xi}_{n,m}(z,\ell_-;s)\right)_{2,1} \cdot \left(F_{n,m}^{(\ell)}\right) + \left(\tilde{\Xi}_{n,m}(z,\ell_-;s)\right)_{2,2}\right] \cdot (\tilde{i}_n(\ell_-,s))
\end{aligned}
\tag{6.4}
$$

where ℓ is interpreted in the sense of $z_0 \to \ell$ with $z_0 < \ell$. At this point it is convenient to decompose the above vector variables according to the eigenvectors $(x_n)_\beta$ for which we define

$$
\begin{aligned}
&\tilde{v}^{(\beta)}(z,s) = (x_n)_\beta \cdot (\tilde{v}_n(z,s)) \quad , \quad \tilde{i}^{(\beta)}(z,s) = (x_n)_\beta \cdot (\tilde{i}_n(z,s)) \\
&(\tilde{v}_n(z,s)) = \sum_{\beta=1}^{2} \tilde{v}^{(\beta)}(z,s)(x_n)_\beta \quad , \quad (\tilde{i}_n(z,s)) = \sum_{\beta=1}^{2} \tilde{i}^{(\beta)}(z,s)(x_n)_\beta
\end{aligned}
\tag{6.5}
$$

Applying these to (6.4) by dot multiplying by $(x_n)_\beta$ on the left gives

$$
\begin{aligned}
\tilde{v}^{(\beta)}(z,s) &= \tilde{\psi}_{1,1;\beta}(z,\ell_-;s)\,\tilde{v}^{(\beta)}(\ell_-,s) + \tilde{\psi}_{1,2;\beta}(z,\ell_-;s)\,\tilde{i}^{(\beta)}(\ell_-,s) \\
\tilde{i}^{(\beta)}(z,s) &= \tilde{\psi}_{2,1;\beta}(z,\ell_-;s)\,\tilde{v}^{(\beta)}(\ell_-,s) + \tilde{\psi}_{2,2;\beta}(z,\ell_-;s)\,\tilde{i}^{(\beta)}(\ell_-,s)
\end{aligned}
\tag{6.6}
$$

The boundary condition at $z = \ell$ in (6.3) becomes

$$
\begin{aligned}
\sum_{\beta=1}^{2} \tilde{v}^{(\beta)}(\ell_-,s)(x_n)_\beta &= \left(F_{n,m}^{(\ell)}(s)\right) \cdot \sum_{\beta=1}^{2} \tilde{i}^{(\beta)}(\ell_-,s)(x_n)_\beta \\
&= \left[1 + F_{1,1}^{-1}(0)\right](x_n)_1 (x_n)_1 \cdot \sum_{\beta=1}^{2} \tilde{i}^{(\beta)}(\ell_-,s)(x_n)_\beta \\
&= \left[1 + F_{1,1}^{-1}(0)\right] \tilde{i}^{(1)}(\ell_-,s)(x_n)_1 \\
\tilde{v}^{(\beta)}(\ell_-,s) &= \begin{cases} \left[1 + F_{1,1}^{-1}(0)\right] \tilde{i}^{(1)}(\ell_1,s) & \text{for } \beta = 1 \\ 0 & \text{for } \beta = 2 \end{cases}
\end{aligned}
\tag{6.7}
$$

Now we can define modal impedance variables along the wave launcher via

$$
\begin{aligned}
\left(\tilde{\mathrm{v}}_n(z,s)\right) &= \left(\tilde{\zeta}_{n,m}(z,s)\right) \cdot \left(\tilde{i}_n(z,s)\right) \\
\left(\tilde{\zeta}_{n,m}(z,s)\right) &= \sum_{\beta=1}^{2} \tilde{\zeta}_{\beta}(z,s)\,(x_n)_{\beta}(x_n)_{\beta}
\end{aligned}
\tag{6.8}
$$

From (6.6) and (6.7) we have

$$
\begin{aligned}
\tilde{\zeta}_1(z,s) &= \frac{\tilde{\mathrm{v}}^{(1)}(z,s)}{\tilde{i}^{(1)}(z,s)} = \frac{\left[1+F_{1,1}^{-1}(0)\right]\tilde{\psi}_{1,1;1}(z,\ell_-;s) + \tilde{\psi}_{1,2;1}(z,\ell_-;s)}{\left[1+F_{1,1}^{-1}(0)\right]\tilde{\psi}_{2,1;1}(z,\ell_-;s) + \tilde{\psi}_{2,2;1}(z,\ell_-;s)} \\
\tilde{\zeta}_2(z,s) &= \frac{\tilde{\mathrm{v}}^{(2)}(z,s)}{\tilde{i}^{(2)}(z,s)} = \frac{\tilde{\psi}_{1,2;2}(z,\ell_-;s)}{\tilde{\psi}_{2,2;2}(z,\ell_-;s)}
\end{aligned}
\tag{6.9}
$$

As it turns out for $\tilde{\zeta}_1$ we can set $\ell_- = \ell$ directly, but for $\tilde{\zeta}_2$ we need to consider the limit.

To understand this impedance in more detail let us look at what happens when $z_0 \to \ell$. For $\beta = 1$ things are well behaved as

$$
\begin{aligned}
&X_1(2) = 1 + \frac{z}{F_{1,1}(0)\ell} = c_1\left[z+\ell_1\right] \\
&\quad z+\ell_1 > 0 \text{ (Use upper sign in (5.13)} \\
&\quad \ell_1 \;\; F_{1,1}(0)\ell\,, \;\; c_1 = \ell_1^{-1} \\
&\quad \ell+\ell_1 = \ell\left[1+F_{1,1}(0)\right] > 0 \\
&\left(\tilde{\psi}_{u,\mathrm{v}}(z,\ell;s)\right) \\
&= \tilde{\gamma}(s)\ell_1 \begin{pmatrix}
\left[\frac{z}{\ell_1}+1\right]\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1])\mathrm{K}_1(\tilde{\gamma}(s)[z+\ell_1]) + \mathrm{K}_0(\tilde{\gamma}(s)[\ell+\ell_1])I_1(\tilde{\gamma}(s)[z+\ell_1])\right] & \left[\frac{\ell}{\ell_1}+1\right]\left[\frac{z}{\ell_1}+1\right]\left[I_1(\tilde{\gamma}(s)[\ell+\ell_1])\mathrm{K}_1(\tilde{\gamma}(s)[z+\ell_1]) - \mathrm{K}_1(\tilde{\gamma}(s)[\ell+\ell_1])I_1(\tilde{\gamma}(s)[z+\ell_1])\right] \\
\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1])\mathrm{K}_0(\tilde{\gamma}(s)[z+\ell_1]) - \mathrm{K}_0(\tilde{\gamma}(s)[\ell+\ell_1])I_0(\tilde{\gamma}(s)[z+\ell_1])\right] & \left[\frac{\ell}{\ell_1}+1\right]\left[I_1(\tilde{\gamma}(s)[\ell+\ell_1])\mathrm{K}_0(\tilde{\gamma}(s)[z+\ell_1]) + \mathrm{K}_1(\tilde{\gamma}(s)[\ell+\ell_1])I_0(\tilde{\gamma}(s)[z+\ell_1])\right]
\end{pmatrix}
\end{aligned}
\tag{6.10}
$$

from which $\tilde{\zeta}_1$ is readily constructed.

However, for $\beta = 2$, things are more delicate, where we are concerned with $\ell_- \to \ell$, and we have

$$
\begin{aligned}
X_2(z) &= 1-\frac{z}{\ell}=c_2\left[z+\ell_2\right] \\
z+\ell_2 &< 0 \quad \text{(Use lower sign in (5.13).)} \\
\ell_2 &= -\ell \quad , \quad c_2 = -\ell^{-1} = \ell_2^{-1} \\
\Gamma &\equiv -\tilde{\gamma}(s)\left[\ell_- + \ell_2\right] = \tilde{\gamma}(s)\left[\ell-\ell_-\right] \to 0
\end{aligned} \tag{6.11}
$$

next we need the modified Bessel functions for small argument [16] as

$$
\begin{aligned}
I_0(\Gamma) &= 1+O\left(\Gamma^2\right) \quad , \quad I_1(\Gamma) = \frac{\Gamma}{2} + O\left(\Gamma^3\right) \\
K_0(\Gamma) &= -\left[\ell n\left(\frac{\Gamma}{2}\right)+\Gamma_0\right]\left[1+O\left(\Gamma^2\right)\right] \\
K_0(\Gamma) &= \Gamma^{-1} + O\left(\Gamma \, \ell n\left(\frac{\Gamma}{2}\right)\right) \\
\Gamma_0 &= 057721\cdots
\end{aligned} \tag{6.12}
$$

From (5.13) we can note that $\tilde{\psi}_{1,1;2}$ and $\tilde{\psi}_{2,1;2}$ are singular as $\Gamma \to 0$. Fortunately, these two terms are not present in (6.9). So we need

$$
\begin{aligned}
\tilde{\psi}_{1,2;2}(z,\ell;s) &= c_2\left[z+\ell_2\right] \, I_1\left(-\tilde{\gamma}(s)\left[z+\ell_2\right]\right) \\
&= \left[1-\frac{z}{\ell}\right] I_1\left(\tilde{\gamma}(s)\left[\ell-z\right]\right) \\
\tilde{\psi}_{2,2;2}(z,\ell;s) &= I_0\left(-\tilde{\gamma}(s)\left[z+\ell_2\right]\right) = I_0\left(\tilde{\gamma}(s)\left[\ell-z\right]\right)
\end{aligned} \tag{6.13}
$$

with limit taken as $\Gamma \to 0$ (or $\ell_- \to \ell$). From this we have

$$
\tilde{\zeta}_2(z,s) = \left[1-\frac{z}{\ell}\right] \frac{I_1\left(\tilde{\gamma}(s)\left[\ell-z\right]\right)}{I_0\left(\tilde{\gamma}(s)\left[\ell-z\right]\right)} \tag{6.14}
$$

which is fortunately relatively simple.

For the case that $z = 0$ things further simplify as

$$\left(\tilde{\psi}_{u,v}(\ell,\ell;s)\right)_1$$

$$= \tilde{\gamma}(s)\ell_1 \begin{pmatrix} \left[I_0(\tilde{\gamma}(s)[\ell+\ell_1])K_1(\tilde{\gamma}(s)\ell_1)+K_0(\tilde{\gamma}(s)[\ell+\ell_1])I_1(\tilde{\gamma}(s)\ell_1)\right] & \left[\frac{\ell}{\ell_1}+1\right]\left[I_1(\tilde{\gamma}(s)[\ell+\ell_1])K_1(\tilde{\gamma}(s)\ell_1)-K_1(\tilde{\gamma}(s)[\ell+\ell_1])I_1(\tilde{\gamma}(s)\ell_1)\right] \\ \left[I_0(\tilde{\gamma}(s)[\ell+\ell_1])K_0(\tilde{\gamma}(s)\ell_1)-K_0(\tilde{\gamma}(s)[\ell+\ell_1])I_0(\tilde{\gamma}(s)\ell_1)\right] & \left[\frac{\ell}{\ell_1}+1\right]\left[I_1(\tilde{\gamma}(s)[\ell+\ell_1])K_0(\tilde{\gamma}(s)\ell_1)+K_1(\tilde{\gamma}(s)[\ell+\ell_1])I_0(\tilde{\gamma}(s)\ell_1)\right] \end{pmatrix}$$

$$\tilde{\zeta}_1(0,s) = \frac{\left[1+F_{1,1}^{-1}(0)\right]\tilde{\psi}_{1,1;1}(z,\ell;s) + \tilde{\psi}_{1,2;1}(z,\ell;s)}{\left[1+F_{1,1}^{-1}(0)\right]\tilde{\psi}_{2,1;1}(z,\ell;s) + \tilde{\psi}_{2,2;1}(z,\ell;s)}$$

$$\tilde{\zeta}_2(0,s) = \frac{I_1(\tilde{\gamma}(s)\ell)}{I_0(\tilde{\gamma}(s)\ell)} \tag{6.15}$$

$$1 + F_{1,1}^{-1}(0) = \frac{\ell}{\ell_1} + 1$$

We can further write out $\tilde{\zeta}_1(0,s)$ as

$$\tilde{\zeta}_1(0,s) = \frac{\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1])+I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]K_1(\tilde{\gamma}(s)\ell_1)+\left[K_0(\tilde{\gamma}(s)[\ell+\ell_1])-K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]I_1(\tilde{\gamma}(s)\ell_1)}{\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1])+I_1(\gamma[\ell+\ell_1])\right]K_0(\tilde{\gamma}(s)\ell_1)-\left[K_0(\tilde{\gamma}(s)[\ell+\ell_1])-K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]I_0(\tilde{\gamma}(s)\ell_1)} \tag{6.16}$$

Changing back to the voltage and current variables we have

$$\left(\tilde{V}_n(z,s)\right) = \left(\tilde{Z}_{n,m}(z,s)\right) \cdot \left(\tilde{I}_n(z,s)\right)$$

$$\left(\tilde{Z}_{n,m}(z,s)\right) = Z_1\left(F_{n,m}(0)\right)^{\frac{1}{2}} \cdot \left(\tilde{\zeta}_{n,m}(z,s)\right) \cdot \left(F_{n,m}(0)\right)^{\frac{1}{2}}$$

$$= Z_1 \sum_{\beta=1}^{2} \tilde{\zeta}_\beta(z,s)\left[\left(F_{n,m}(0)\right)^{\frac{1}{2}} \cdot (x_n)_\beta\right]\left[(x_n)_\beta \cdot \left(F_{n,m}(0)\right)^{\frac{1}{2}}\right] \tag{6.17}$$

$$= Z_1\left[\tilde{\zeta}_1(z,s)\frac{F_{1,1}(0)}{1+F_{1,1}(0)}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} + \tilde{\zeta}_2(z,s)\left[1+F_{1,1}(0)\right]^{-1}\begin{pmatrix}-F_{1,1}(0)\\1\end{pmatrix}\begin{pmatrix}-F_{1,1}(0)\\1\end{pmatrix}\right]$$

This allows us to obtain the input impedance for waves propagating to the right (positive z) for the wave launcher as $(\tilde{Z}_{n,m}(0,s))$. For later use we also have the admittance

$$
\begin{aligned}
\left(\tilde{Y}_{n,m}(z,s)\right) &= \left(\tilde{Z}_{n,m}(z,s)\right)^{-1} \\
&= Y_1\left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot \left(\zeta_{n,m}(z,s)\right)^{-1} \cdot \left(F_{n,m}(0)\right)^{-\frac{1}{2}} \\
&= Y_1 \sum_{\beta=1}^{2} \zeta_{\beta}^{-1}(z,s)\left[\left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot (x_n)_\beta\right]\left[(x_n)_\beta \cdot \left(F_{n,m}(0)\right)^{-\frac{1}{2}}\right] \\
&= Y_1\left[\tilde{\zeta}_1^{-1}(z,s)\ \frac{F_{1,1}(0)}{1+F_{1,1}(0)}\begin{pmatrix} F_{1,1}^{-1}(0) \\ 1 \end{pmatrix}\begin{pmatrix} F_{1,1}^{-1}(0) \\ 1 \end{pmatrix}\right. \\
&\qquad \left. + \tilde{\zeta}_2^{-1}(z,s)\left[1+F_{1,1}(0)\right]^{-1}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right] \\
Y_1 &\equiv Z_1^{-1}
\end{aligned}
\qquad (6.18)
$$

which will be evaluated at $z = 0$.

7. Application of Boundary Condition at $z = 0$.

At $z = 0$, there is an impedance Z_1 (as in fig. 1.2) associated with the wave propagating to the left (negative z). This is associated only with the voltage V_2 and current I_2. So we have the boundary condition

$$\begin{aligned} \tilde{V}_2(0,s) &= Z_1 \tilde{I}_2(0,s) \\ \tilde{V}_1(0,s) &= \tilde{V}(s) \end{aligned} \tag{7.1}$$

From the admittance matrix for the right (+) propagating waves in (6.18) we have

$$\begin{aligned} \tilde{V}_2(0,s) &= -Z_1\left[\tilde{Y}_{2,1}(0,s)\tilde{V}_1(0,s) + \tilde{Y}_{2,2}(0,s)\tilde{V}_2(0,s)\right] \\ \tilde{V}_2(0,s) &= -\left[1 + Z_1 \tilde{Y}_{2,2}(0,s)\right]^{-1} Z_1 \tilde{Y}_{2,1}(0,s)\tilde{V}(s) \end{aligned} \tag{7.2}$$

Thus the input to the wave launcher is given by the voltage vector

$$\begin{aligned} \left(\tilde{V}_n(0,s)\right) &= \tilde{V}(s)\begin{pmatrix} 1 \\ -\left[1 + Z_1\tilde{Y}_{2,2}(0,s)\right]^{-1} Z_1\tilde{Y}_{2,1}(0,s) \end{pmatrix} \\ &= \tilde{V}(s)\begin{pmatrix} 1 \\ -\left[Y_1 + \tilde{Y}_{2,2}(0,s)\right]^{-1} \tilde{Y}_{2,1}(0,s) \end{pmatrix} \\ \frac{\tilde{Y}_{2,1}(0,s)}{Y_1 + \tilde{Y}_{2,2}(0,s)} &= \frac{\left[1 + F_{1,1}(0)\right]^{-1}\left[\tilde{\zeta}_1^{-1}(0,s) - \tilde{\zeta}_2^{-1}(0,s)\right]}{1 + \left[1 + F_{1,1}(0)\right]^{-1}\left[F_{1,1}(0)\,\tilde{\zeta}_1^{-1}(0,s) + \tilde{\zeta}_2^{-1}(0,s)\right]} \\ &= \frac{\tilde{\zeta}_1^{-1}(0,s) - \tilde{\zeta}_2^{-1}(0,s)}{1 + F_{1,1}(0) + F_{1,1}(0)\,\tilde{\zeta}_1^{-1}(0,s) + \tilde{\zeta}_2^{-1}(0,s)} \\ &= \frac{\left[\tilde{\zeta}_1^{-1}(0,s) + 1\right] - \left[\tilde{\zeta}_2^{-1}(0,s) + 1\right]}{F_{1,1}(0)\left[\tilde{\zeta}_1^{-1}(0,s) + 1\right] + \left[\tilde{\zeta}_2^{-1}(0,s) + 1\right]} \end{aligned} \tag{7.3}$$

In terms of our normalized variables we have

$$\left(\tilde{v}_n(0,s)\right) = \tilde{V}(s)\begin{pmatrix} F_{1,1}^{-\frac{1}{2}} \\ -\left[Y_1 + \tilde{Y}_{2,2}(0,s)\right]^{-1} \tilde{Y}_{2,1}(0,s) \end{pmatrix} \tag{7.4}$$

We also need the currents into the wave launcher as

$$\begin{aligned}
\left(\tilde{I}_n(0,s)\right) &= \left(\tilde{Y}_{n,m}(0,s)\right) \cdot \left(\tilde{V}_n(0,s)\right) \\
\left(\tilde{i}_n(0,s)\right) &= \left(\tilde{\zeta}_{n,m}(0,s)\right)^{-1} \cdot \left(\tilde{v}_n(0,s)\right) \\
\left(\tilde{\zeta}_{n,m}(0,s)\right)^{-1} &= \sum_{\beta=1}^{2} \tilde{\zeta}_{\beta}^{-1}(0,s)\,(x_n)_\beta (x_n)_\beta
\end{aligned} \tag{7.5}$$

The input admittance driven by the source is

$$\begin{aligned}
\tilde{Y}_{in}(s) &= \frac{\tilde{I}_1(0,s)}{\tilde{V}_1(0,s)} = \frac{\tilde{I}_1(0,s)}{\tilde{V}(s)} = \tilde{Z}_{in}^{-1}(s) \\
\tilde{I}_1(0,s) &= \tilde{Y}_{1,1}(0,s)\,\tilde{V}_1(0,s) - \tilde{Y}_{1,2}(0,s)\,\tilde{V}_2(0,s) \\
&= \tilde{Y}_{1,1}(0,s)\,\tilde{V}_1(0,s) - \tilde{Y}_{1,2}(0,s)\left[Y_1 + \tilde{Y}_{2,2}(0,s)\right]^{-1}\tilde{Y}_{2,1}(s)\,\tilde{V}_1(0,s) \\
\tilde{Y}_{in}(s) &= \tilde{Y}_{1,1}(0,s) - \left[Y_1 + \tilde{Y}_{2,2}(0,s)\right]^{-1}\tilde{Y}_{1,2}(0,s)\,\tilde{Y}_{2,1}(0,s) \\
&= \left[Y_1 + \tilde{Y}_{2,2}(0,s)\right]^{-1}\left[Y_1\,\tilde{Y}_{1,1}(0,s) + \tilde{Y}_{1,1}(0,s)\,\tilde{Y}_{2,2}(0,s) - \tilde{Y}_{1,2}(0,s)\,\tilde{Y}_{2,1}(0,s)\right] \\
&= \left[Y_1 + \tilde{Y}_{2,2}(0,s)\right]^{-1}\left[Y_1\,\tilde{Y}_{1,1}(0,s) + \det\left(\left(\tilde{Y}_{n,m}(0,s)\right)\right)\right]
\end{aligned} \tag{7.6}$$

Carrying this further we have

$$\begin{aligned}
\det\left(\left(\tilde{Y}_{n,m}(0,s)\right)\right) &= \det\left(Y_1\left(F_{n,m}(0)\right)^{-\frac{1}{2}} \cdot \left(\tilde{\zeta}_{n,m}(0,s)\right)^{-1}\left(F_{n,m}(0)\right)^{-\frac{1}{2}}\right) \\
&= Y_1^2\,F_{n,m}^{-1}(0)\,\det\left(\left(\tilde{\zeta}_{n,m}(0,s)\right)^{-1}\right) \\
&= Y_1^2 F_{n,m}^{-1}(0)\,\tilde{\zeta}_1^{-1}(0,s)\,\tilde{\zeta}_2^{-1}(0,s) \\
\frac{\tilde{Y}(s)}{Y_1} &= \frac{\left[1 + F_{1,1}(0)\right]^{-1}\left[F_{1,1}^{-1}(0)\tilde{\zeta}_1^{-1}(0,s) + \tilde{\zeta}_2^{-1}(0,s)\right] + F_{n,m}^{-1}(0)\tilde{\zeta}_1^{-1}(0,s)\,\tilde{\zeta}_2^{-1}(0,s)}{1 + \left[1 + F_{1,1}(0)\right]^{-1}\left[F_{1,1}(0)\tilde{\zeta}_1^{-1}(0,s) + \tilde{\zeta}_2^{-1}(0,s)\right]} \\
&= \frac{1 + F_{1,1}^{-1}(0) + \tilde{\zeta}_1(0,s) + F_{1,1}^{-1}(0)\tilde{\zeta}_2(0,s)}{\left[1 + F_{1,1}(0)\right]\tilde{\zeta}_1(0,s)\,\tilde{\zeta}_2(0,s) + \tilde{\zeta}_1(0,s) + F_{1,1}(0)\tilde{\zeta}_2(0,s)}
\end{aligned} \tag{7.7}$$

8. Transmission of Wave from Wave Launcher

From Sections 3 and 5, the waves in the launcher section are described by

$$\begin{pmatrix} (\tilde{V}_n(z,s)) \\ Z_1(\tilde{I}_n(z,s)) \end{pmatrix} = \left((\tilde{\Phi}_{n,m}(z,0;s))_{u,v}\right) \odot \begin{pmatrix} (\tilde{V}_n(0,s)) \\ Z_1(\tilde{I}_n(0,s)) \end{pmatrix}$$

$$\begin{pmatrix} (\tilde{v}_n(z,s)) \\ (\tilde{i}_n(z,s)) \end{pmatrix} = \left((\tilde{\Xi}_{n,m}(z,0;s))_{u,v}\right) \odot \begin{pmatrix} (\tilde{v}_n(0,s)) \\ (\tilde{i}_n(0,s)) \end{pmatrix}$$

$$\left((\tilde{\Phi}_{n,m}(z,0;s))_{u,v}\right) = \begin{pmatrix} (F_{n,m}(0))^{\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{-\frac{1}{2}} \end{pmatrix} \odot \left((\tilde{\Xi}_{n,m}(z,0;s))_{u,v}\right) \odot \begin{pmatrix} (F_{n,m}(0))^{-\frac{1}{2}} & (0_{n,m}) \\ (0_{n,m}) & (F_{n,m}(0))^{\frac{1}{2}} \end{pmatrix} \tag{8.1}$$

$$\left((\tilde{\Xi}_{n,m}(z,0;s))_{u,v}\right) = \sum_{\beta=1}^{2}\left((\Xi_{n,m}(z,0;s))_{u,v}\right)_{\beta}$$

$$\left((\tilde{\Xi}_{n,m}(z,0;s))_{u,v}\right)_{\beta} = \left[(x_n)_{\beta}(x_n)_{\beta}\right] \otimes \left(\tilde{\psi}_{u,v}(z,0;s)\right)_{\beta}$$

Our interest lies with $z = \ell_{-}$, noting as before the possible singularity in the marizant at $z = \ell$

What we need is a transfer function defined by

$$\tilde{T}(s) \equiv e^{\tilde{\gamma}(s)\ell}\,\frac{\tilde{V}_1(\ell,s)}{\tilde{V}(s)} = e^{\tilde{\gamma}(s)\ell}\,\frac{\tilde{V}_2(\ell,s)}{\tilde{V}(s)} \tag{8.2}$$

where the factor $e^{\tilde{\gamma}(s)\ell}$ removes the delay of the wave propagating through the wave launcher. Then $\tilde{T}(s)\,\tilde{V}(s)$ represents the wave in retarded time

$$t_r \equiv t - \frac{z}{c} \tag{8.3}$$

launched in the forward direction. Note the equality of the voltages at $z = \ell$. This gives various ways to calculate $\tilde{T}(s)$ via

$$\tilde{V}_1(\ell,s) = \tilde{V}_2(\ell,s) = Z_1\left[\tilde{I}_1(\ell,s) + \tilde{I}_2(\ell,s)\right] \tag{8.4}$$

In terms of our normalized variables we have

$$\begin{pmatrix}(\tilde{v}_n(z,s))\\(\tilde{i}_n(z,s))\end{pmatrix} = \begin{pmatrix}(F_{n,m}(0))^{\frac{1}{2}} & (0_{n,m})\\(0_{n,m}) & (F_{n,m}(0))^{-\frac{1}{2}}\end{pmatrix} \odot \begin{pmatrix}(\tilde{V}_n(z,s))\\Z_1(\tilde{I}_n(z,s))\end{pmatrix}$$

$$\tilde{v}_1(z,s) = F_{1,1}^{\frac{1}{2}}(0)\,\tilde{V}_1(z,s) \quad , \quad \tilde{v}_2(z,s) = \tilde{V}_2(z,s) \tag{8.5}$$

$$\tilde{T}(s) = e^{\tilde{\gamma}(s)\ell}\, F_{1,1}^{\frac{1}{2}}(0)\frac{\tilde{v}_1(\ell,s)}{\tilde{V}(\ell,s)} = e^{\tilde{\gamma}(s)\ell}\,\frac{\tilde{v}_2(\ell,s)}{\tilde{V}(\ell,s)}$$

Solving for $\tilde{v}_2(\ell_-,s)$ we have

$$\begin{aligned}(\tilde{v}_n(\ell_-,s)) &= \left(\tilde{\Xi}_{n,m}(\ell_-,0;s)\right)_{1,1}\cdot(\tilde{v}_n(0,s)) + \left(\tilde{\Xi}_{n,m}(\ell_-,0;s)\right)_{1,2}\cdot(\tilde{i}_n(0,s))\\ &= \left[\left(\tilde{\Xi}_{n,m}(\ell_-,0;s)\right)_{1,1} + \left(\tilde{\Xi}_{n,m}(\ell_-,0;s)\right)_{1,2}\cdot\left(\zeta_{n,m}(0,s)\right)^{-1}\right]\cdot(\tilde{v}_n(0,s))\\ (\tilde{v}_2(\ell,s)) &= \begin{pmatrix}0\\1\end{pmatrix}\cdot\left[\left(\tilde{\Xi}_{n,m}(\ell_-,0;s)\right)_{1,1} + \left(\tilde{\Xi}_{n,m}(\ell_-,0;s)\right)_{1,2}\cdot\left(\zeta_{n,m}(0,s)\right)^{-1}\right]\cdot(\tilde{v}_n(0,s))\end{aligned} \tag{8.6}$$

giving

$$\tilde{T}(s) = e^{s\frac{\ell}{c}}\begin{pmatrix}0\\1\end{pmatrix}\cdot\left[\left(\tilde{\Xi}_{n,m}(\ell_-,0;s)\right)_{1,1} + \left(\tilde{\Xi}_{n,m}(\ell_-,0;s)\right)_{1,2}\cdot\left(\zeta_{n,m}(0,s)\right)^{-1}\right]\cdot\begin{pmatrix}F_{1,1}^{-\frac{1}{2}}(0)\\-\left[Y_1+\tilde{Y}_{2,2}(0,s)\right]^{-1}\tilde{Y}_{2,1}(0,s)\end{pmatrix} \tag{8.7}$$

This means that we need to consider $\tilde{\psi}_{1,1;\beta}$ and $\tilde{\psi}_{1,2;\beta}$.

As in Section 6 let us now consider the limit as $\ell_- - \ell$. For $\beta = 1$, we have

$$\begin{aligned}&\tilde{\psi}_{1,1;1}(\ell,0;s)\\ &\quad = \tilde{\gamma}(s)\left[\ell+\ell_1\right]\left[I_0(\tilde{\gamma}(s)\ell_1)K_1(\tilde{\gamma}(s)[\ell+\ell_1]) + K_0(\tilde{\gamma}(s)\ell_1)I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]\\ &\tilde{\psi}_{1,2;1}(\ell,0;s)\\ &\quad = \tilde{\gamma}(s)\left[\ell+\ell_1\right]\left[I_1(\tilde{\gamma}(s)\ell_1)K_1(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)\ell_1)I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]\\ &\ell + \ell_1 = \ell\left[1+F_{1,1}(0)\right]\end{aligned} \tag{8.8}$$

For $\beta = 2$ we have (with limit as $\Gamma \to 0$ or $\ell_- \to \ell$)

$$\tilde{\psi}_{1,1;2}(\ell,0;s) = I_0(\tilde{\gamma}(s)\ell)$$
$$\tilde{\psi}_{1,2;2}(\ell,0;s) = -I_1(\tilde{\gamma}(s)\ell) \tag{8.9}$$

Writing out (8.7) in greater detail we now have

$$\begin{aligned}
\tilde{T}(s) &= e^{\tilde{\gamma}(s)\ell}\begin{pmatrix}0\\1\end{pmatrix}\cdot\Big[(x_n)_1(x_n)_1\,\tilde{\psi}_{1,1;1}(\ell,0;s) + (x_n)_2(x_n)_2\,\tilde{\psi}_{1,1;2}(\ell,0;s) \\
&\quad + (x_n)_1(x_n)_1\,\tilde{\psi}_{1,2;1}(\ell,0;s)\tilde{\zeta}_1^{-1}(0,s) + (x_n)_2(x_n)_2\,\tilde{\psi}_{1,2;2}(\ell,0;s)\tilde{\zeta}_2^{-1}(0,s)\Big] \\
&\quad \cdot\begin{pmatrix} F_{1,1}^{-\frac{1}{2}}(0) \\ -\left[Y_1+\tilde{Y}_{2,2}(0,s)\right]^{-1}\tilde{Y}_{2,1}(0,s)\end{pmatrix} \\
&= e^{\tilde{\gamma}(s)\ell}\begin{pmatrix}0\\1\end{pmatrix}\cdot\Big[(x_n)_1(x_n)_1\left[\tilde{\psi}_{1,1;1}(\ell,0;s) + \tilde{\psi}_{1,2;1}(\ell,0;s)\,\tilde{\zeta}_1^{-1}(0,s)\right] \\
&\quad + (x_n)_2(x_n)_2\left[\tilde{\psi}_{1,1;2}(\ell,0;s) + \tilde{\psi}_{1,2;2}(\ell,0;s)\,\tilde{\zeta}_2^{-1}(0,s)\right]\Big] \\
&\quad \cdot\begin{pmatrix} F_{1,1}^{-\frac{1}{2}}(0) \\ -\left[Y_1+\tilde{Y}_{2,2}(0,s)\right]^{-1}\tilde{Y}_{2,1}(0,s)\end{pmatrix} \\
&= e^{\tilde{\gamma}(s)\ell}\begin{pmatrix}0\\1\end{pmatrix}\cdot\left[(x_n)_1(x_n)_1\right]\cdot\begin{pmatrix} F_{1,1}^{-\frac{1}{2}}(0) \\ -\left[Y_1+\tilde{Y}_{2,2}(0,s)\right]^{-1}\tilde{Y}_{2,1}(0,s)\end{pmatrix} \\
&\quad \left[\tilde{\psi}_{1,1;1}(\ell,0;s) + \tilde{\psi}_{1,2;1}(\ell,0;s)\,\tilde{\zeta}_1^{-1}(0,s)\right]
\end{aligned} \tag{8.10}$$

where conveniently (8.9) and (6.15) make the terms involving $(x_n)_2$ vanish.

Next we compress

$$\begin{aligned}
&\tilde{\psi}_{1,1;1}(\ell,0;s) + \tilde{\psi}_{1,2;1}(\ell,0;s)\,\tilde{\zeta}_1^{-1}(0,s) \\
&= N_1\tilde{\gamma}(s)\left[\ell+\ell_1\right]\Big[\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]K_1(\tilde{\gamma}(s)\ell_1) + \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]I_1(\tilde{\gamma}(s)\ell_1)\Big]^{-1} \\
&\quad N_1 \equiv \left[K_1(\tilde{\gamma}(s)[\ell+\ell_1])\,I_0(\tilde{\gamma}(s)\ell_1) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\,K_0(\tilde{\gamma}(s)\ell_1)\right] \\
&\qquad \Big[\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]K_1(\tilde{\gamma}(s)\ell_1) + \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]I_1(\tilde{\gamma}(s)\ell_1)\Big] \\
&\quad + \left[K_1(\tilde{\gamma}(s)[\ell+\ell_1])\,I_1(\tilde{\gamma}(s)\ell_1) - I_1(\tilde{\gamma}(s)[\ell+\ell_1])\,K_1(\tilde{\gamma}(s)\ell_1)\right] \\
&\qquad \Big[\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]K_0(\tilde{\gamma}(s)\ell_1) - \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]I_0(\tilde{\gamma}(s)\ell_1)\Big]
\end{aligned}$$

$$= \left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]K_1(\tilde{\gamma}(s)[\ell+\ell_1])\left[I_0(\tilde{\gamma}(s)\ell_1)K_1(\tilde{\gamma}(s)\ell_1) + I_1(\tilde{\gamma}(s)\ell_1)\,K_0(\tilde{\gamma}(s)\ell_1)\right]$$
$$+ \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]I_1(\tilde{\gamma}(s)[\ell+\ell_1])\left[I_0(\tilde{\gamma}(s)\ell_1)K_1(\tilde{\gamma}(s)\ell_1) + I_1(\tilde{\gamma}(s)\ell_1)\,K_0(\tilde{\gamma}(s)\ell_1)\right]$$
$$= \left[I_0(\tilde{\gamma}(s)[\ell+\ell_1])\,K_1(\tilde{\gamma}(s)[\ell+\ell_1]) + K_0(\tilde{\gamma}(s)[\ell+\ell_1])\,I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]$$
$$\left[I_0(\tilde{\gamma}(s)\ell_1)\,K_1(\tilde{\gamma}(s)\ell_1) + I_1(\tilde{\gamma}(s)\ell_1)\,K_0(\tilde{\gamma}(s)\ell_1)\right]$$
$$= \frac{1}{\tilde{\gamma}(s)[\ell+\ell_1]}\,\frac{1}{\tilde{\gamma}(s)\ell_1} \tag{8.11}$$

where the Wronskian (5.11) is applied twice.

This gives

$$\tilde{\psi}_{1,1;1}(\ell,0;s) + \tilde{\psi}_{1,2;1}(\ell,0;s)\,\tilde{\zeta}^{-1}(0,s)$$
$$= \frac{1}{\tilde{\gamma}(s)\ell_1}\left[\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]K_1(\tilde{\gamma}(s)\ell_1) + \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]I_1(\tilde{\gamma}(s)\ell_1)\right] \tag{8.12}$$

Also, we have

$$\begin{pmatrix}0\\1\end{pmatrix} \cdot \left[(x_n)_1\;(x_n)_1\right] \cdot \begin{pmatrix} F_{1,1}^{-\frac{1}{2}}(0) \\ -\left[Y_1 + \tilde{Y}_{2,2}(0,s)\right]^{-1}\tilde{Y}_{2,1}(0,s)\end{pmatrix}$$
$$= \left[1+F_{1,1}(0)\right]^{-1}\left[1 - F_{1,1}(0)\left[Y_1 + \tilde{Y}_{2,2}(0,s)\right]^{-1}\tilde{Y}_{2,1}(0,s)\right]$$
$$= \left[1+F_{1,1}(0)\right]^{-1}\left[1 - F_{1,1}(0)\frac{\tilde{\zeta}_1^{-1}(0,s) - \tilde{\zeta}_2^{-1}(0,s)}{1 - F_{1,1}(0) + F_{1,1}(0)\,\tilde{\zeta}_1^{-1}(0,s) + \tilde{\zeta}_1^{-1}(0,s)}\right] \tag{8.13}$$
$$= \left[1+F_{1,1}(0)\right]^{-1}\frac{\left[1+F_{1,1}(0)\right]\left[\tilde{\zeta}_2^{-1}(0,s)+1\right]}{F_{1,1}(0)\left[\tilde{\zeta}_1^{-1}(0,s)+1\right] + \tilde{\zeta}_2^{-1}(0,s)+1}$$
$$= \left[F_{1,1}(0)\frac{\tilde{\zeta}_1^{-1}(0,s)+1}{\tilde{\zeta}_2^{-1}(0,s)+1} + 1\right]$$

thereby having all the terms needed in (8.10) for $\tilde{T}(s)$.

9. Low-Frequency performance

In order to better understand the character of the foregoing solutions for the wave-launcher response, it is instructive to consider special cases. For low frequencies the expansion of the Bessel functions for small arguments as in (6.12) is appropriate. Consider the eigenvalues of $(\tilde{\zeta}_{n,m}(0,s))$ which appear in various formulas. For low frequencies these are

$$
\begin{aligned}
&\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] K_1(\tilde{\gamma}(s)\ell_1) + \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] I_1(\tilde{\gamma}(s)\ell_1) \\
&= \left[1 + O(\tilde{\gamma}(s)\ell)\right]\left[\left[\tilde{\gamma}(s)\ell_1\right]^{-1} + O(\tilde{\gamma}(s)\ell\ \ell n(\tilde{\gamma}(s)\ell))\right] \\
&\qquad -\left[\left[\tilde{\gamma}(s)[\ell+\ell_1]\right]^{-1} + O(\ell n(\tilde{\gamma}(s)\ell))\right]\left[\frac{\tilde{\gamma}(s)\ell_1}{2} + O\left(\left[\tilde{\gamma}(s)\ell\right]^3\right)\right] \\
&= \frac{1}{\tilde{\gamma}(s)\ell_1} + O(1) \quad \text{as} \quad s \to 0 \\
&\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\gamma[\ell+\ell_1])\right] K_0(\tilde{\gamma}(s)\ell_1) - \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] I_0(\tilde{\gamma}(s)\ell_1) \\
&= \left[1 + O(\tilde{\gamma}[\ell+\ell_1])\right]\left[-\ell n\left(\frac{\tilde{\gamma}(s)[\ell+\ell_1]}{2}\right) + \Gamma_0\right]\left[1 + O\left(\left[\tilde{\gamma}(s)\ell\right]^2\right)\right] \\
&\qquad +\left[\left[\tilde{\gamma}(s)[\ell+\ell_1]\right]^{-1} + O(\ell n(\tilde{\gamma}(s)\ell))\right]\left[1 + O\left(\left[\tilde{\gamma}(s)\ell\right]^2\right)\right] \\
&= \frac{1}{\tilde{\gamma}(s)[\ell+\ell_1]} + O(\ell n(\tilde{\gamma}(s)\ell)) \quad \text{as} \quad s \to 0 \\
&\tilde{\zeta}_1^{-1}(0,s) = \frac{\ell_1}{\ell+\ell_1} + O(\tilde{\gamma}(s)\ell\ \ell n(\tilde{\gamma}(s)\ell)) \\
&\qquad = \frac{F_{1,1}(0)}{1+F_{1,1}(0)} + O(\tilde{\gamma}(s)\ell\ \ell n(\tilde{\gamma}(s)\ell)) \quad \text{as} \quad s \to 0 \\
&\tilde{\zeta}_2^{-1}(0,s) = \frac{I_0(\tilde{\gamma}(s)\ell)}{I_1(\tilde{\gamma}(s)\ell)} = \left[1 + O\left(\left[\tilde{\gamma}(s)\ell\right]^2\right)\right]\left[\frac{\tilde{\gamma}(s)\ell}{2} + O\left(\left[\tilde{\gamma}(s)\ell\right]^3\right)\right]^{-1} \\
&\qquad = \frac{2}{\tilde{\gamma}(s)\ell} + O(\tilde{\gamma}(s)\ell) \quad \text{as} \quad s \to 0
\end{aligned}
\tag{9.1}
$$

The low frequency admittance from (7.7) is then

$$
\begin{aligned}
\frac{\tilde{Y}_{in}(s)}{Y_1} &= \frac{1 + F_{1,1}^{-1}(0)\, 1 + \tilde{F}_{1,1}^{-1}(0) + O(\tilde{\gamma}(s)\ell\ \ell n(\tilde{\gamma}(s)\ell))}{1 + F_{1,1}^{-1}(0) + O(\tilde{\gamma}(s)\ell\ \ell n(\tilde{\gamma}(s)\ell))} \\
&= 2 + O(\tilde{\gamma}(s)\ell\ \ell n(\tilde{\gamma}(s)\ell))
\end{aligned}
\tag{9.2}
$$

which is readily interpretable as the admitance Y_1 for the forward propagating wave $(z > \ell)$ in parallel with the admittance Y_1 for the backward propagating wave $(z < 0)$.

The low-frequency transfer function from (8.10) is then

$$
\begin{aligned}
\tilde{T}(s) &= \left[1 + O(\tilde{\gamma}(s)\ell)\right] \left[F_{1,1}(0) \frac{\frac{\ell}{\ell+\ell_1} + 1 + O\left(\tilde{\gamma}(s)\ell \, \ell n(\tilde{\gamma}(s)\ell)\right)}{\frac{2}{\tilde{\gamma}(s)\ell} + O(\tilde{\gamma}(s)\ell)} + 1 \right]^{-1} \\
&\quad \frac{1}{\tilde{\gamma}(s)\ell_1} \left[\frac{1}{\tilde{\gamma}(s)\ell_1} + O(1) \right]^{-1} \\
&= \left[1 + O(\tilde{\gamma}(s)\ell)\right] \left[1 + O\left(\left[\tilde{\gamma}(s)\ell\right]^2 \ell n(\tilde{\gamma}(s)\ell)\right)\right]^{-1} \left[1 + O(\tilde{\gamma}(s)\ell)\right]^{-1} \\
&= 1 + O(\tilde{\gamma}(s)\ell) \quad \text{as} \quad s \to 0
\end{aligned}
\tag{9.3}
$$

This is also as we would expect. There is also a transfer function to the backward propagating wave $(z < 0)$ which goes to 1 in this limit, consistent with the admittance in (9.2).

10. High-Frequency Performance

For high frequencies the expansion of the Bessel functions for large arguments (right half plane) [16] is

$$
\begin{aligned}
I_n(\Gamma) &= [2\pi\Gamma]^{-\frac{1}{2}} e^{\Gamma}\left[1 + O\left(\Gamma^{-1}\right)\right] \\
K_n(\Gamma) &= \left[\frac{\pi}{2\Gamma}\right]^{\frac{1}{2}} e^{-\Gamma}\left[1 + O\left(\Gamma^{-1}\right)\right] \text{ as } \Gamma \to \infty \text{ (RHP)}
\end{aligned}
\tag{10.1}
$$

Again consider the eigenvalues of $(\tilde{\zeta}_{n,m}(0,s))$. For high frequencies these are

$$
\begin{aligned}
&\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] K_1(\tilde{\gamma}(s)\ell_1) + \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] I_1(\tilde{\gamma}(s)\ell_1) \\
&\quad = \frac{e^{\tilde{\gamma}(s)\ell}}{\tilde{\gamma}(s)} \left[[\ell+\ell_1]\ell_1\right]^{-\frac{1}{2}}\left[1 + O\left(\left[\tilde{\gamma}(s)\ell\right]^{-1}\right)\right] \text{ as } s \to \infty \\
&\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] K_0(\tilde{\gamma}(s)\ell_1) - \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] I_0(\tilde{\gamma}(s)\ell_1) \\
&\quad = \frac{e^{\tilde{\gamma}(s)\ell}}{\tilde{\gamma}(s)} \left[[\ell+\ell_1]\ell_1\right]^{-\frac{1}{2}}\left[1 + O\left(\left[\tilde{\gamma}(s)\ell\right]^{-1}\right)\right] \text{ as } s \to \infty \\
&\tilde{\zeta}_1^{-1}(0,s) = 1 + O\left(\left[\tilde{\gamma}(s)\ell\right]^{-1}\right) \text{ as } s \to \infty \\
&\tilde{\zeta}_2^{-1}(0,s) = \frac{I_0(\tilde{\gamma}(s)\ell)}{I_1(\tilde{\gamma}(s)\ell)} = 1 + O\left(\left[\tilde{\gamma}(s)\ell\right]^{-1}\right) \text{ as } s \to \infty
\end{aligned}
\tag{10.2}
$$

The high-frequency admittance from (7.7) is then

$$
\begin{aligned}
\frac{\tilde{Y}_{in}(s)}{Y_1} &= \frac{1 + F_{1,1}^{-1}(0) + 1 + F_{1,1}^{-1}(0) + O(\tilde{\gamma}(s)\ell)}{\left[1 + F_{1,1}^{-1}(0)\right] + 1 + F_{1,1}(0) + O(\tilde{\gamma}(s))} \\
&= F_{1,1}^{-1}(0) + O(\tilde{\gamma}(s)\ell) \text{ as } s \to \infty
\end{aligned}
\tag{10.3}
$$

This corresponds to the impedance looking into terminal 1 without regard to the other terminals (causality), which is just the 1,1 element in $(f_{g_{n,m}})$ multiplied by Z_1.

The high-frequency transfer function from (8.10) is then

$$\tilde{T}(s) = e^{s\frac{\ell}{c}}\left[F_{1,1}(0)\frac{2+O\left(\left[\tilde{\gamma}(s)\ell\right]^{-1}\right)}{2+O\left(\left[\tilde{\gamma}(s)\ell\right]^{-1}\right)} + 1\right]^{-1}\left[\left[\frac{\ell+\ell_1}{\ell_1}\right]^{\frac{1}{2}} e^{-\tilde{\gamma}(s)\ell}\left[1+O\left(\left[\tilde{\gamma}(s)\ell\right]^{-1}\right)\right]\right]$$

$$= \left[1+F_{1,1}(0)\right]^{-\frac{1}{2}} F_{1,1}^{-\frac{1}{2}}(0)\left[1+O\left(\left[\tilde{\gamma}(s)\right]^{-1}\right)\right] \tag{10.4}$$

$$= \left[1+F_{1,1}(0)\right]^{-\frac{1}{2}} F_{1,1}^{-\frac{1}{2}}(0)\left[1+O\left(\left[\tilde{\gamma}(s)\right]^{-1}\right)\right] \quad \text{as } s \to \infty$$

One can interpret this is terms of power. From (10.3) the pwer from the source is proporational to $F_{1,1}^{-1}(0)$, while that radiated in the forward direction is proportional to $[1+F_{1,1}(0)]^{-1}\ F_{1,1}^{-1}(0)$. The difference, proportional to $[1+F_{1,1}(0)]^{-1}$, accounts for power in other waves (back and later reflections). A particular case of interest has the high-frequency transfer function set to unity, for which we have

$$F_{1,1}(0) = \frac{\sqrt{5}-1}{2} \simeq 0.618 \quad (\text{for } \tilde{T}(s)=1 \text{ as } s\to\infty) \tag{10.5}$$

Another interesting case concerns the early-time step-like response to a step excitation. After the initial step the next term is a ramp (constant times t_r). By appropriate choice of $F_{1,1}(0)$, one can try to set this term to zero to make a close approximation to a step function at early time. In terms of the transfer function $\tilde{T}(s)$, one can take the asymptotic expansion to one more term in the Bessel functions in (10.1). The first term is as in (10.4) and one can set the second term to zero by the choice

$$F_{1,1}(0) = \left[\frac{3}{2}\right]^{\frac{1}{2}} \simeq 1.22 \tag{10.6}$$

This choise is seen later in the numerical calculations.

The reader should note that these asymptotic expansions take a different form if one sets

$$\tilde{\gamma}(s) = j\frac{\omega}{c} \equiv jk \tag{10.7}$$

In this case, one has (for $k > 0$) [16]

$$\ell_3 > 0$$

$$I_0(jk\ell_3) = J_0(k\ell_3) \;, \quad I_1 = (jk\ell_3) = j\,J_1(k\ell_3)$$

$$K_0(jk\ell_3) = -j\frac{\pi}{2}\,H_0^{(2)}(k\ell_3) \tag{10.8}$$

$$K_1(jk\ell_3) = -\frac{\pi}{2}\,H_1^{(2)}(k\ell_3)$$

For the high-frequency adymptotic forms the K_n functions (representing waves traveling in a single direction (single exponential)) take the same form on the $j\omega$ axis as in (10.1). However, the I_n or J_n functions (representing waves traveling in two directions (two exponentials)) take the form

$$J_n(K\ell_3) = \left[\frac{2}{\pi K\ell_3}\right]^{\frac{1}{2}}\left[\cos\left(k\ell_3 - \frac{n\pi}{2}\right) + O\left([k\ell_3]^{-1}\right)\right] \tag{10.9}$$

This changes the character of the asymptotic expansions in (10.2) to bring in multiple high-frequency reflections.

Evaluating the eigenvalues of $(\tilde{\zeta}_{n,m}(0,j\omega))$ for high frequencies these are

$$\left[I_0(jk[\ell+\ell_1]) + I_1(jk[\ell+\ell_1])\right]K_1(jk[\ell_1]) + \left[K_0(jk[\ell+\ell_1]) - K_1(jk[\ell+\ell_1])\right]I_1(jk\ell_1)$$

$$= \frac{1}{k}\left[\ell_1[\ell+\ell_1]\right]^{-\frac{1}{2}}\Bigg[\left[\cos\left(k[\ell+\ell_1]-\frac{\pi}{4}\right) + j\cos\left(k[\ell+\ell_1]-\frac{3\pi}{4}\right) + O\left((k[\ell+\ell_1])^{-1}\right)\right]$$

$$e^{-jk\ell_1 - j\frac{\pi}{4}}\left[1+O\left((k\ell_1)^{-1}\right)\right]$$

$$+\left[e^{-jk[\ell+\ell_1]-j\frac{\pi}{4}}\,e^{-jk\ell_1 - j\frac{\pi}{4}} + O\left((k[\ell+\ell_1])^{-1}\right)\right]\left[j\cos\left(k\ell_1 - \frac{3\pi}{4}\right) + O\left((k\ell_1)^{-1}\right)\right]\Bigg]$$

$$= \frac{1}{k}\left[\ell_1[\ell+\ell_1]\right]^{-\frac{1}{2}}\,e^{jk[\ell+\ell_1]-j\frac{\pi}{4}}\,e^{-jk\ell_1 - j\frac{\pi}{4}} + O\left((k\ell)^{-1}\right)$$

$$= -\frac{j}{k}\left[\ell_1[\ell+\ell_1]\right]^{-\frac{1}{2}}\,e^{jk\ell}\left[1 + O\left((k\ell)^{-1}\right)\right] \text{ as } k\ell \to \infty$$

$$\left[I_0(jk[\ell+\ell_1]) + I_1(jk[\ell+\ell_1])\right]K_0(jk\ell_1) - \left[K_0(jk[\ell+\ell_1]) - K_1(jk[\ell+\ell_1])\right]I_0(jk\ell_1)$$

$$= \frac{1}{k}\left[\ell_1[\ell+\ell_1]\right]^{-\frac{1}{2}}\Bigg[\left[\cos\left(k[\ell+\ell_1]-\frac{\pi}{4}\right) + j\cos\left(k[\ell+\ell_1]-\frac{3\pi}{4}\right) + O\left((k[\ell+\ell_1])^{-1}\right)\right]$$

$$e^{-jk\ell_1 - j\frac{\pi}{4}}\left[1+O\left((k\ell_1)^{-1}\right)\right]$$

$$- \left[e^{-jk[\ell+\ell_1] - j\frac{\pi}{4}} - e^{-jk[\ell+\ell_1] - j\frac{\pi}{4}} + O\left(\left(k[\ell+\ell_1] \right)^{-1} \right) \right] \left[j\cos\left(k\ell_1 - \frac{\pi}{4} \right) + O\left((k\ell_1)^{-1} \right) \right] \Bigg]$$

$$= \frac{1}{k} \left[\ell_1 [\ell + \ell_1] \right]^{-\frac{1}{2}} e^{jk[\ell+\ell_1] - j\frac{\pi}{4}} e^{-jk\ell_1 - j\frac{\pi}{4}} + O\left((k\ell)^{-1} \right)$$

$$= -\frac{j}{k} \left[\ell_1 [\ell + \ell_1] \right]^{-\frac{1}{2}} e^{jk\ell} \left[1 + O\left((k\ell)^{-1} \right) \right] \quad \text{as} \quad k\ell \to \infty$$

$$\tilde{\zeta}_1^{-1}(0,s) = 1 + O\left((k\ell)^{-1} \right) \quad \text{as} \quad k\ell \to \infty$$

$$\begin{aligned}
\tilde{\zeta}_2^{-1}(0,s) &= \frac{I_0(jk\ell)}{I_1(jk\ell)} = j \frac{\cos\left(k\ell - \frac{\pi}{4} \right) + O\left((k\ell)^{-1} \right)}{\cos\left(k\ell - \frac{3\pi}{4} \right) + O\left((k\ell)^{-1} \right)} \\
&= j \frac{\cos\left(k\ell - \frac{\pi}{4} \right) + O\left((k\ell)^{-1} \right)}{\sin\left(k\ell - \frac{3\pi}{4} \right) + O\left((k\ell)^{-1} \right)} \quad \text{as} \quad k\ell \to \infty \\
&\to j \cot\left(k\ell - \frac{\pi}{4} \right) \quad \text{as} \quad k\ell \to \infty
\end{aligned} \tag{10.10}$$

The high-frequency admittance is then

$$\begin{aligned}
\frac{\tilde{Y}_{in}(j\omega)}{Y_1} &= \\
&\frac{\left[2 + F_{1,1}^{-1}(0) + O\left((k\ell)^{-1} \right) \right] \left[j\cos\left(k\ell - \frac{\pi}{4} \right) + O\left((k\ell)^{-1} \right) \right] + F_{1,1}^{-1}(0) \left[\sin\left(k\ell - \frac{\pi}{4} \right) + O\left((k\ell)^{-1} \right) \right]}{\left[1 + 2F_{1,1}(0) + O\left((k\ell)^{-1} \right) \right] \left[\sin\left(k\ell - \frac{\pi}{4} \right) + O\left((k\ell)^{-1} \right) \right] + \left[1 + O\left((k\ell)^{-1} \right) \right] \left[j\cos\left(k\ell - \frac{\pi}{4} \right) + O\left((k\ell)^{-1} \right) \right]} \\
&= \frac{1}{F_{1,1}(0)} \frac{j\left[2F_{1,1}(0) + 1 \right] \cos\left(k\ell - \frac{\pi}{4} \right) + \sin\left(k\ell - \frac{\pi}{4} \right)}{\left[2F_{1,1}(0) + 1 \right] \sin\left(k\ell - \frac{\pi}{4} \right) + j\cos\left(k\ell - \frac{\pi}{4} \right)} + O\left((k\ell)^{-1} \right) \quad \text{as} \quad k\ell \to \infty
\end{aligned} \tag{10.11}$$

The high-frequency transfer function is then

$$\tilde{T}(j\omega) = e^{jk\ell}\left[F_{1,1}(0)\frac{2 + O\left((k\ell)^{-1}\right)}{j\dfrac{\cos\left(k\ell - \frac{\pi}{4}\right) + O\left((k\ell)^{-1}\right)}{\sin\left(k\ell - \frac{\pi}{4}\right) + O\left((k\ell)^{-1}\right)} + 1} + 1 \right]^{-1}$$

$$\frac{1}{jk\ell_1}\left[\frac{-j}{k}\left[\ell_1[\ell + \ell_1]\right]^{-\frac{1}{2}} e^{jk\ell}\left[1 + O(k\ell)^{-1}\right]\right]^{-1}$$

$$= \left[1 + F_{1,1}^{-1}(0)\right]^{\frac{1}{2}}\left[F_{1,1}(0)\frac{-j\,2\sin\left(k\ell - \frac{\pi}{4}\right) + O\left((k\ell)^{-1}\right)}{e^{-jk\ell + j\frac{\pi}{4}} + O\left((k\ell)^{-1}\right)} + 1\right]^{-1}$$

$$= \left[1 + F_{1,1}^{-1}(0)\right]^{\frac{1}{2}}\left[F_{1,1}(0)\left[-j\,2\sin\left(k\ell - \frac{\pi}{4}\right)e^{jk\ell - j\frac{\pi}{4}} + O\left((k\ell)^{-1}\right)\right] + 1\right]^{-1}$$

$$= \left[1 + F_{1,1}^{-1}(0)\right]^{\frac{1}{2}}\left[F_{1,1}(0)\left[1 - e^{j2k\ell - j\frac{\pi}{2}} + O\left((k\ell)^{-1}\right)\right] + 1\right]^{-1} \tag{10.12}$$

$$= \left[1 + F_{1,1}^{-1}(0)\right]^{\frac{1}{2}}\left[1 + F_{1,1}(0)\left[1 + j\,e^{j2k\ell}\right]\right]^{-1} + O\left((k\ell)^{-1}\right) \quad \text{as} \quad k\ell \to \infty$$

which oscillates between the bounds

$$\left[1 + F_{1,1}^{-1}(0)\right]^{\frac{1}{2}}\left[1 + 2\,F_{1,1}(0)\right]^{-1} \le \left|\tilde{T}(j\omega)\right| \le \left[1 + F_{1,1}^{-1}(0)\right]^{\frac{1}{2}} \tag{10.13}$$

This shows some benefit for small $F_{1,1}(0)$, at least for high frequencies.

11. Time-Domain Considerations

In converting to time domain it is instructive to consider an excitation of the form

$$\tilde{V}(s) = \frac{V_0}{s} \quad , \quad V(t) = V_0 u(t) \tag{11.1}$$

Then one can consider aspects of the temporal response such as amplitude of the initial rise, rate of rise, transition (smoothness) from early-time to late-time response, etc., as appropriate to the particular problem at hand. Defining a response V_r as a voltage launched to the right $(z > \ell)$ of the array we have

$$\begin{aligned} \tilde{V}_r(s) &= \tilde{T}(s)\tilde{V}(s) = \frac{V_0}{s}\tilde{T}(s) \\ V_r(t_r) &= V_0 \int_{0_-}^{t_r} T(t')dt' \end{aligned} \tag{11.2}$$

noting that we have formulated the response in retarded time.

From Sections 9 and 10 we already have some results that can be interpreted in time domain. From (9.3) we have

$$V_r(\infty) = \lim_{t_r \to \infty} V_r(t_r) = V_0 \tag{11.3}$$

From (10.4) we have

$$V_r(0_+) \lim_{t_r \to 0_+} V_r(t_r) = \left[1 + F_{1,1}(0)\right]^{-\frac{1}{2}} F_{1,1}^{-\frac{1}{2}}(0)\; V_0 \tag{11.4}$$

which gives the special case in (10.5) as

$$V_r(0_+) = V_0 = V_r(\infty) \quad \text{for} \quad F_{1,1}(0) = \frac{\sqrt{5} - 1}{2} \tag{11.5}$$

In such a case one is concerned in how small is the deviation of the intermediate time response from V_0. One can also use the low-frequency admittance in (9.2) to find

$$I_1(\infty) = \lim_{t\to\infty} I(t) = 2\, Y_1\, V_0 \tag{11.6}$$

and the high-frequency admittance in (10.3) to find

$$I_1(0_+) = \lim_{t\to 0_+} I(t) = F_{1,1}^{-1}(0)\, Y_1\, V_0 \tag{11.7}$$

Using the full expression (8.10) for the transfer function, together with its constituents in (8.12) and (8.13), one can of course calculate $\tilde{V}_r(j\omega)$ and numerically inverse Fourier transform to find $V_r(t_r)$ with $F_{1,1}(0)$ treated as a parameter. (One can similarly find $I_1(t)$.) These expressions, however, are still rather complicated, and one would like to further simpify them for analytic understanding.

One approach to this simplification involves separating out the various reflections back and forth on the wave-launcher. The round-trip time for such reflections is $2\ell/c$, so one may look at time windows of this characteristic width. In particular let us look at the first such window. Then we need to segregate various terms in Laplace domain by their arrival in time domain. For this purpose we need to refine the asymptotic evaluation as in Section 10 in terms of exponential order [10 (Appendix A)] symbolized by

$$\tilde{H}(s) = O_e\left(\tilde{h}(s)\right) \text{ as } s \to s_0 \text{ (typically } \pm\infty) \tag{11.8}$$

and interpreted in terms of the usual order symbol as

$$\tilde{H}(s) = O\left(e^{\tilde{h}(s)+\chi|s|}\right) = e^{\chi|s|} O\left(e^{\tilde{h}(s)}\right) \text{ as } s \to s_0 \text{ for all } \chi > 0 \tag{11.9}$$

Besides the important exponential term we allow any function which grows less rapidly toward ∞ than any exponential function (e.g., polynomial, logarithm, etc., in s). While (11.9) gives an upper bound formagnitude, one can similarly define $0_{e\_}$ to give a lower bound.

Looking at the exponential behavior (in the RHP) of the modified Bessel functions in (10.1) we can segregate the various terms. For the eigenvalues we have terms

$$
\begin{aligned}
&\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] K_1(\tilde{\gamma}(s)\ell_1) + \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] I_1(\tilde{\gamma}(s)\ell_1) \\
&\qquad = \left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] K_1(\tilde{\gamma}(s)\ell_1)\left[1 + O_e(-2\tilde{\gamma}(s)\ell)\right] \\
&\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] K_0(\tilde{\gamma}(s)\ell_1) - \left[K_0(\tilde{\gamma}(s)[\ell+\ell_1]) - K_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] I_0(\tilde{\gamma}(s)\ell_1) \\
&\qquad = \left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right] K_0(\tilde{\gamma}(s)\ell_1)\left[1 + O_e(-2\tilde{\gamma}(s)\ell)\right] \qquad (11.10) \\
&\tilde{\zeta}_1^{-1}(0,s) = \frac{K_0(\tilde{\gamma}(s)\ell_1)}{K_1(\tilde{\gamma}(s)\ell_1)}\left[1 + O_e(-2\tilde{\gamma}(s)\ell)\right] \\
&\tilde{\zeta}_2^{-1}(0,s) = \frac{I_0(\tilde{\gamma}(s)\ell_1)}{I_1(\tilde{\gamma}(s)\ell_1)}
\end{aligned}
$$

The neglected terms correspond to arrival at a time later by $2\ell/c$ due to the exponential kernel of the Laplace tranform, the above being exact when applied to times before this.

From this we find

$$
\begin{aligned}
&\left[F_{1,1}(0)\frac{\tilde{\zeta}_1^{-1}(0,s) + 1}{\tilde{\zeta}_2^{-1}(0,s) + 1} + 1\right]^{-1} \\
&= \left[F_{1,1}(0)\frac{\dfrac{K_0(\tilde{\gamma}(s)\ell_1)}{K_1(\tilde{\gamma}(s)\ell_1)} + 1}{\dfrac{I_0(\tilde{\gamma}(s)\ell_1)}{I_1(\tilde{\gamma}(s)\ell_1)} + 1}\right]^{-1}\left[1 + O_e(-2\tilde{\gamma}(s)\ell)\right] \qquad (11.11) \\
&\tilde{\psi}_{1,1;1}(\ell,0;s) + \tilde{\psi}_{1,2;1}(\ell,0;s)\,\tilde{\zeta}^{-1}(0,s) \\
&= \frac{1}{\tilde{\gamma}(s)\ell_1}\left[\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]K_1(\tilde{\gamma}(s)\ell_1)\right]^{-1}\left[1 + O_e(-2\tilde{\gamma}(s)\ell)\right]
\end{aligned}
$$

This in turn gives the response as

$$
\begin{aligned}
&\tilde{V}_r(s) = \frac{V_0}{s}\tilde{T}(s) \\
&= \frac{1}{s}\left[F_{1,1}(0)\frac{\dfrac{K_0(\tilde{\gamma}(s)\ell_1)}{K_1(\tilde{\gamma}(s)\ell_1)} + 1}{\dfrac{I_0(\tilde{\gamma}(s)\ell_1)}{I_1(\tilde{\gamma}(s)\ell_1)} + 1}\right]^{-1} \\
&\frac{e^{\tilde{\gamma}(s)\ell}}{\tilde{\gamma}(s)\ell_1}\left[\left[I_0(\tilde{\gamma}(s)[\ell+\ell_1]) + I_1(\tilde{\gamma}(s)[\ell+\ell_1])\right]K_1(\tilde{\gamma}(s)\ell_1)\right]^{-1}\left[1 + O_e(-2\tilde{\gamma}(s)\ell)\right] \qquad (11.12)
\end{aligned}
$$

Note the grouping of the exponential with terms in the denominator of the same exponential order. The above product of two terms then corresponds to the convolution of two time-domain functions, both of which begin at zero retarded time. The above result can also be rearranged as

$$\begin{aligned}\tilde{V}_r(s) &= \frac{e^{\tilde{\gamma}(s)\ell}}{s}\,\frac{I_0(\tilde{\gamma}(s)\ell)+I_1(\tilde{\gamma}(s)\ell)}{I_0(\tilde{\gamma}(s)[\ell+\ell_1])+I_0(\tilde{\gamma}(s)[\ell+\ell_1])} \\ &\quad \frac{1}{\tilde{\gamma}(s)\ell_1}\Big[I_0(\tilde{\gamma}(s)\ell)+K_1(\tilde{\gamma}(s)\ell_1) \;+\; F_{1,1}(0)I_1(\tilde{\gamma}(s)\ell)\,K_0(\tilde{\gamma}(s)\ell_1) \\ &\qquad +[1+F_{1,1}(0)]\,I_1(\tilde{\gamma}(s)\ell)\,K_1(\tilde{\gamma}(s)\ell)_1\Big]^{-1} \\ &\quad \left[1+O_e(-2\tilde{\gamma}(s)\ell)\right]\end{aligned} \tag{11.13}$$

A special case which simplifies this formula has

$$\begin{aligned}&F_{1,1}(0) \equiv 1 \;,\; \ell_1 = \ell \\ &V_r(s) = \frac{e^{\tilde{\gamma}(s)\ell}}{s}\,\frac{I_0(\tilde{\gamma}(s)\ell)\;+\;I_1(\tilde{\gamma}(s)\ell)}{I_0(2\tilde{\gamma}(s)\ell)\;+\;I_1(2\tilde{\gamma}(s)\ell)}\left[1+2\tilde{\gamma}(s)\ell\;I_1(\tilde{\gamma}(s)\ell)\,K_1(\tilde{\gamma}(s)\ell)\right]^{-1} \\ &\qquad \left[1+O_e(-2\tilde{\gamma}(s)\ell)\right]\end{aligned} \tag{11.14}$$

where the Wronskian (5.11) has been used.

12. Numerical Evaluation of Important Relations

In this section, several of the important quantities derived in the preceding sections are computed numerically and plotted. All computations presented herein were carried out using Matlab (ver. 5.1) on a Pentium-based PC platform. The frequency-domain relation for the normalized input admittance $\tilde{Y}(s)/Y_1$ given in (7.7) is evaluated on the imaginary axis $s = j\omega$ and is plotted in fig. 12.1. This is the Fourier transform and the magnitude and phase of $\tilde{Y}(s)/Y_1$ are plotted as a function of normalized frequency f/f_0, where f_0 is the frequency such that $\lambda_0 = \ell; f_0 = c/\ell$. As discussed in Section 10, the asymptotic form of (7.7) evaluated on the $j\omega$ axis does not approach $F_{1,1}^{-1}$ as $\omega \to \infty$ (10.3) due to multiple reflections, and the expression for $\tilde{Y}(s)/Y_1$ at high frequencies is given by (10.11). Note however that the normalized input admitance does indeed asymptote to 2 at low frequencies as predicted in (9.2). Similarly, the expression given in (8.10) for the forward transmission from the array is plotted in Fig. 12.2. Once again, the low-frequency limit predicted in (9.3) is obeyed, but the high-frequency expression is given in (10.12) rather than the limit given in (10.4).

To demonstrate more fully the asymptotic limits derived in (10.3) and (10.4), these expressions are evaluated just *off* of the $j\omega$ axis. In figs. 12.3 and 12.4. The axis of evaluation is inclined by 1° with respect to the imaginary axis, i.e.,

$$\lambda = \left[\frac{\omega}{c}\right] e^{j\theta}; \theta = 89°. \qquad (12.1)$$

The input admittance (fig. 12.3) and transfer function (fig. 12.4) are each evaluated at several values of $F_{1,1}(0)$ to illustrate the validity of (10.3) and (10.4).

Perhaps of more importance is the evaluation of the temporal response of the array of wavelaunchers. To determine the temporal response, (10.4) is evaluated on the $j\omega$ axis and then is multiplied by a Gaussian low-pass-filter. This filtering is necessary because, as predicted by (10.12) and demonstrated in fig. 12.1, the high-frequency transfer function is non-zero. When using the discrete Fourier transform, frequencies above the Nyquist limit must be filtered out in order to avoid aliasing. The result of the inversion just described is the impulse response of the system, which is then integrated numerically to yield the step response. The relationship between the temporal sampling interval, frequency sampling interval, and the cutoff frequency of the low-pass-filter must be chosen adequately to prevent aliasing in *both* time- and frequency domains while still achieving the theoretical high-frequency response predicted in (10.4). The step-responses for various values of $F_{1,1}(0)$ are plotted in Fig. 12.5 as a function of normalized time t/t_0; $t_0 = \ell/c$, where t_0 is the one-way transit time of the wavelauncher. Note the characteristic ringing that occurs at multiples of $2\ell / c$ that is expected. Also note that while the prompt response can be tailored by suitable choice of $F_{1,1}(0)$, the late-time response is very similar for all curves.

As a point of interest, the forward and backward radiation from the array is compared for $F_{1,1}(0) = (\sqrt{5}-1)/2$ in fig. 12.6. The expression for the backward radiation is given by $\tilde{V}_2(0,s)$ in (7.3). Because the wavelaunchers are nothing more than narrow flare angle TEM horns, we expect them to be very directional at high-frequencies. This is manifested in fig. 2.6 in the prompt response, which is given by (10.4) in the forward direction but is exactly 0 in the backward direction. However, at late times (low-frequency), we expect equal radiation in both directions, as demonstrated in fig. 12.6.

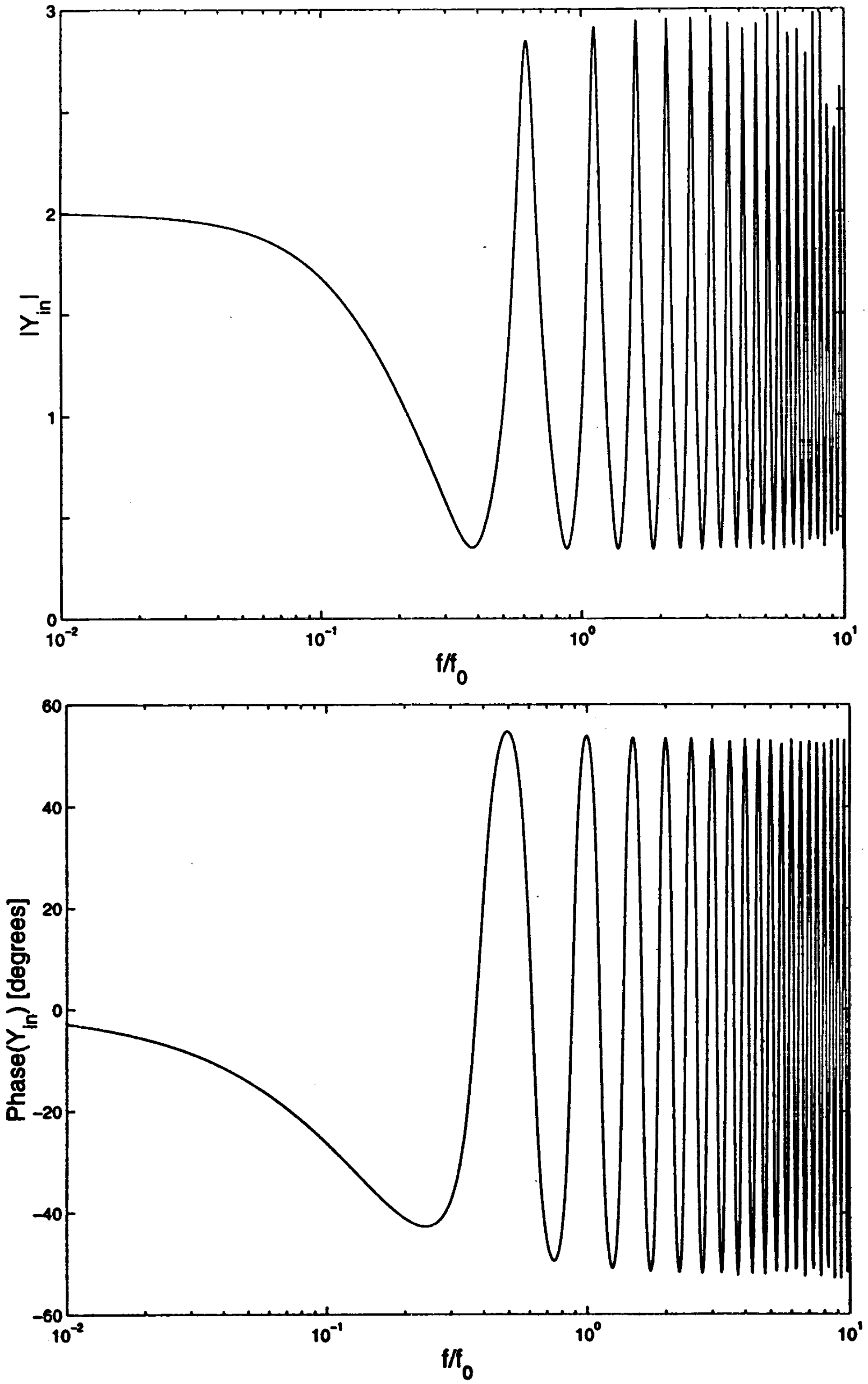

Fig. 12.1. Input Admittance as a Function of Normalized Frequency, $F_{1,1}(0) = 0.618$.

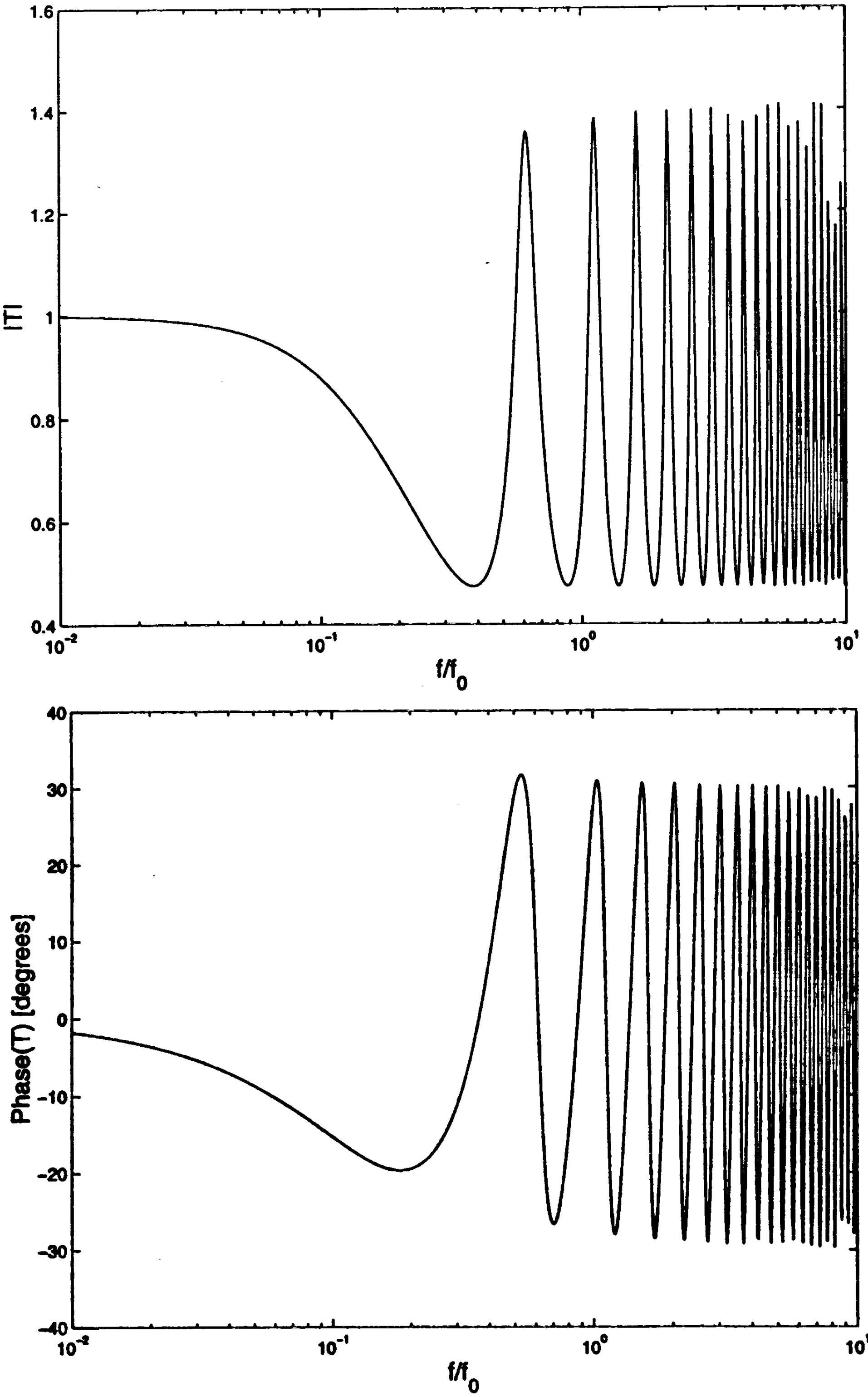

Fig. 12.2. Voltage Transfer Function vs. Normalized Frequency, $F_{1,1}(0) = 0.618$.

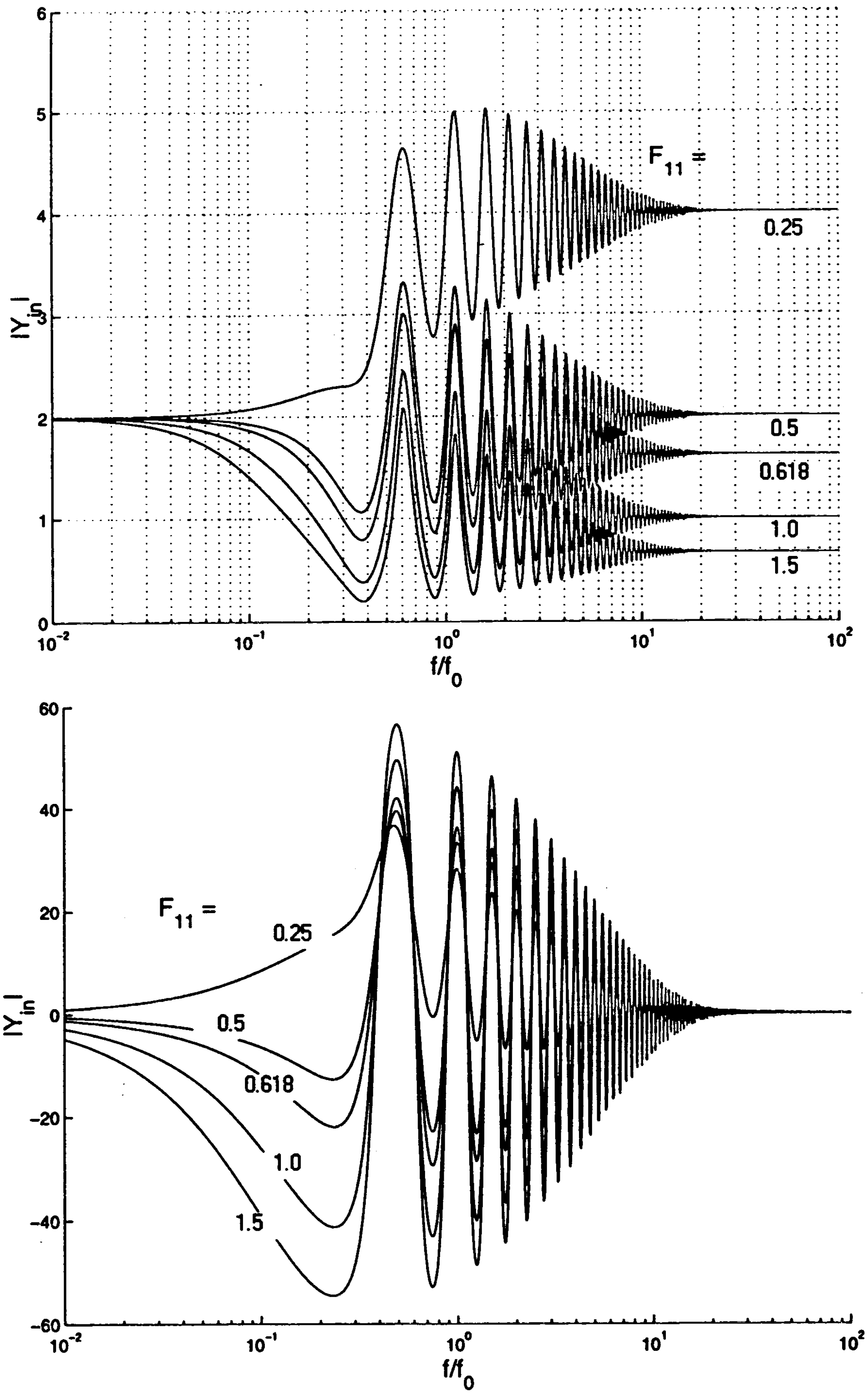

Fig. 12.3. Input Admittance Evaluated off the $j\omega$ Axis, $F_{1,1}(0)$ Indicated on Graph.

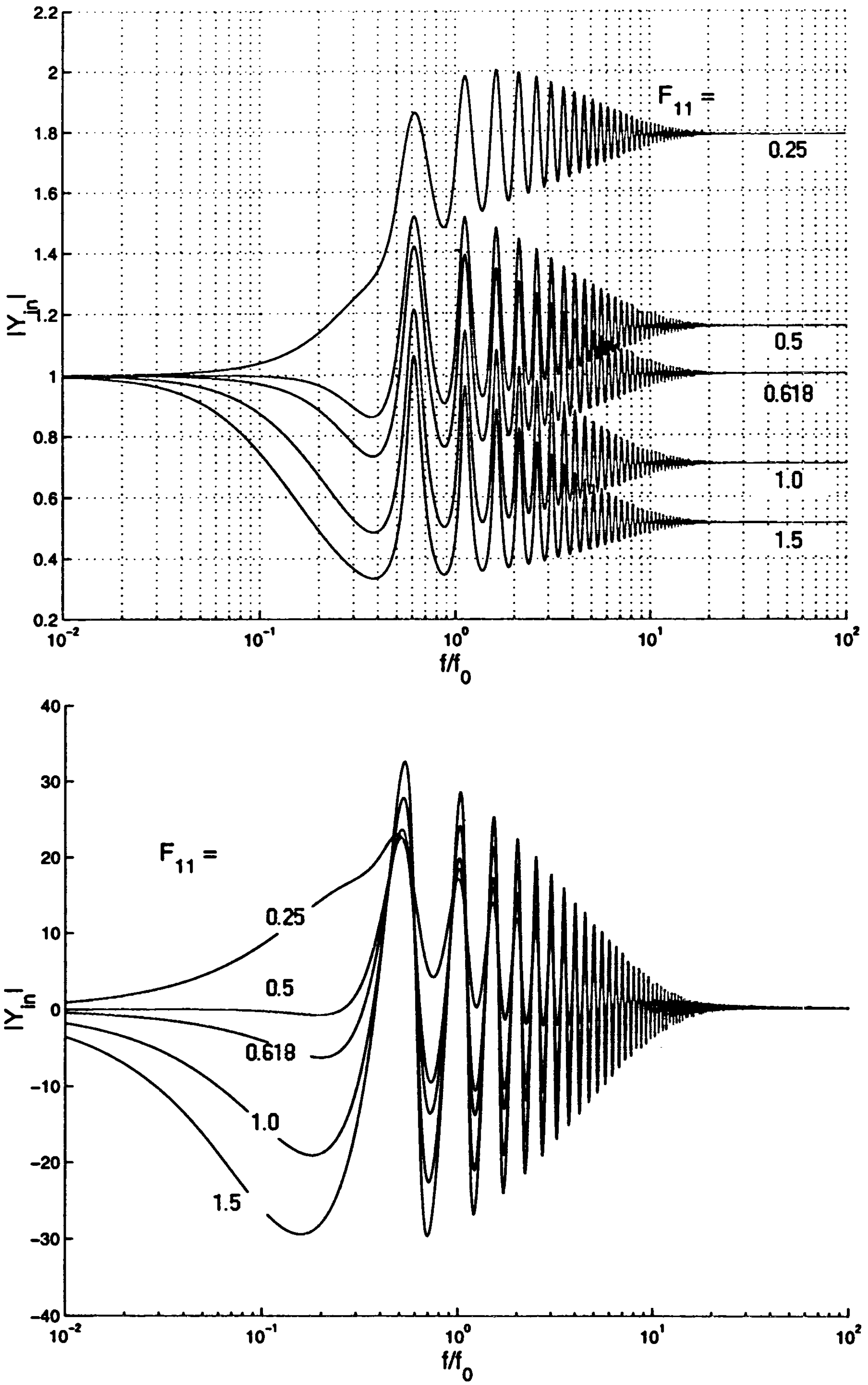

Fig. 12.4. Voltage Transfer Function Evaluated off the $j\omega$ Axis, $F_{1,1}(0)$ Indicated on Graph.

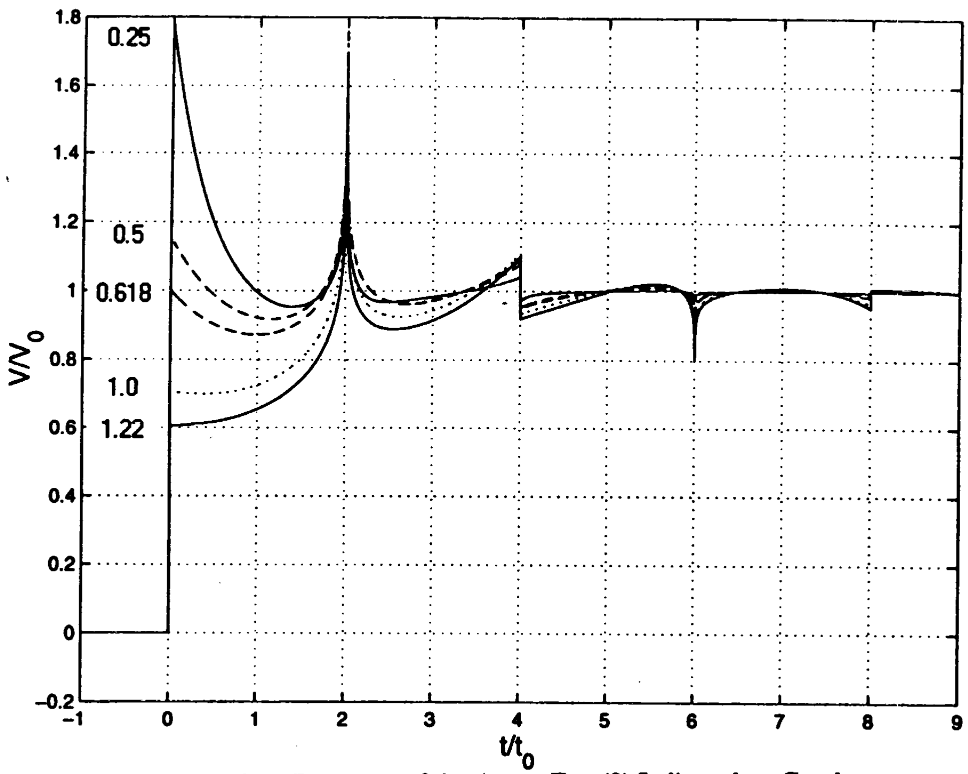

Fig. 12.5. Step Response of the Array, $F_{1,1}(0)$ Indicated on Graph..

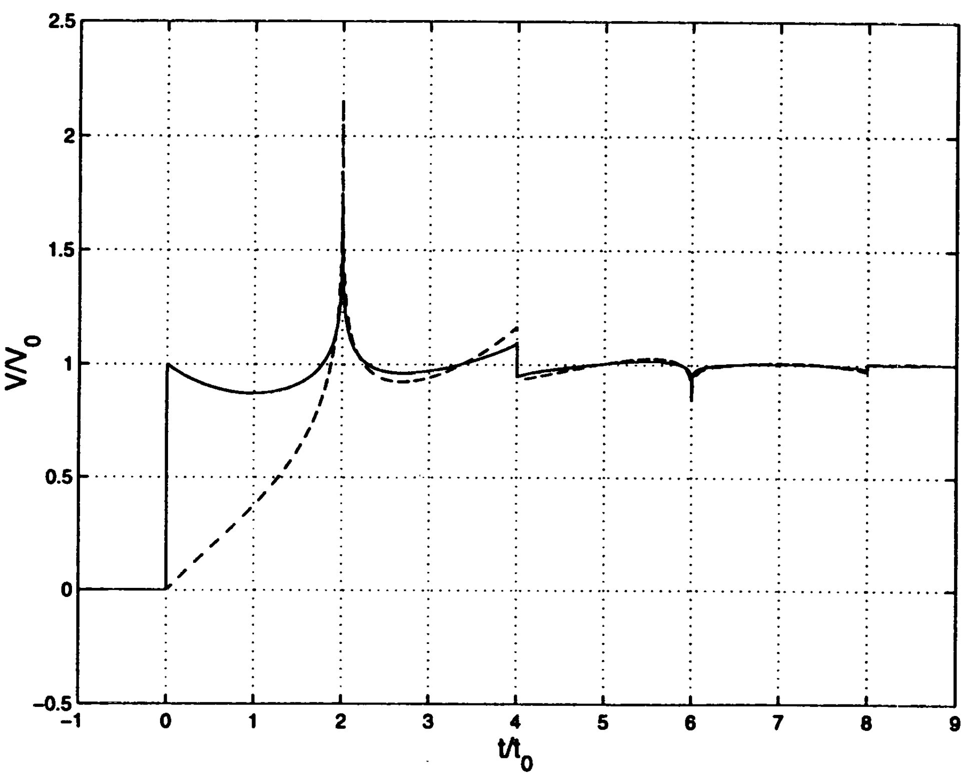

Fig. 12.6. Forward (Solid) and backward (Dotted) Step Response, $F_{1,1}(0) = 0.618$.

13. Concluding Remarks

So now we have some interesting results from the transmission-line model for a certain kind of wave-launcher unit cell described by (4.22). The plate spacing $2b'(z)$ increases proportional to z, so the horn plates are flat as in fig. 1.2. However, the width $2a'(z)$ is not a simple function of z. The curve describing the plate edges is more complicated, but can be found using the procedure developed in [2].

Of course, the transmission-line approximation has its limitations. Some of the undesirable oscillations may be more damped in the exact solution (full Maxwell equations) due to the presence of higher order modes. In any event the present results give to effect of $F_{1,1}(0)$ in an analytic way which can be used as a starting point for more exact calculations and/or measurements. In this case note that ℓ / a (or ℓ / b) is also a parameter to be varied. In the present paper this scales out simply, but in an exact calculation this not the case (e.g., angles change).

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