Sensor and Simulation Notes

Note 418

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Intermediate Field of an Impulse-Radiating Antenna

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Abstract

In considering the far field of an impulse-radiating antenna there is a problem in that infinite frequencies are included significantly in the impulse, making it difficult to properly define the far field in such an ideal case. This paper adopts a different approach by taking an asymptotic expansion parallel to the aperture normal (z axis) with the aperture step illuminated simultaneously over the entire aperture. By this means we obtain a detailed description of the early-time field which we call the intermediate field. This has a convenient scaling relationship involving narrowing it in time as z increases.

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SSN 417

Gif, January 13, 1998

Dr.C.Baum

Dear Dr.Baum,

Please, find here enclosed a new version of our manuscript entiteld:

"Spherical Near-Field Facility for Microwave Coupling Assessments in the 100 MHz - 6 GHz Frequency range" by D.Sérafin et al.

The first page has been remodeled, the pagination has been centered according to your recommandations. Furthermore, I made my best to improve the quality of the bad figures or in providing the originals.

The fax number you have used is wrong. Please note the right one below.

Best regards,

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1. Introduction

Much has been done to characterize an impulse radiating antenna (IRA), such as summarized in [14, 17, 18, 20]. These include TEM-fed paraboloidal reflectors and TEM-fed lenses. Both kinds are characterized as aperture antennas with fields being calculated by integration over the assumed TEM fields on an aperture plane, at least for the impulsive part of the radiated far field (with assumed step excitation). (The prepulse associated with the TEM feed of a reflector IRA is not discussed here.) As discussed in [3] which began this odyssey, the impulsive part of the waveform (on boresight) is not a true impulse. It has a peak which is just the aperture field, but its width decreases like r^{-1}, so that its time integral decreases like r^{-1}, appropriate to a far field.

Extending the model based on an aperture integral one can look in more detail at the behavior for large distances from the aperture, so as to obtain a higher-order approximation and obtain the shape of the approximate delta function, called $\delta_a(t)$ in [5] where

$$\int_{-\infty}^{\infty} \delta_a(t)dt = 1 \tag{1.1}$$

This, of course, is a normalized form of the actual waveform and the various vector components need to be considered for positions off symmetry planes. As we shall see, we can define a vector waveform function which has the time variable (width for each amplitude) scale like z^{-1} where the z axis is normal to the aperture.

Some development in this regard has already been made [16, 19, 21]. For a given retarded time there is a circle on the aperture plane, the radius of which expands as time increases. The field is represented as an integral of the aperture field on this circle. For a uniform, single-polarization aperture field closed-form expressions for the field are developed. Other approaches to this problem are contained in [12, 13].

For the special case of a uniform aperture field on a circular aperture, the fields can be evaluated exactly on the z axis perpendicular to the aperture plane and centered in the circular aperture [6, 9–11, 15]. While in most cases, the aperture integration is performed in the frequency domain, with the resulting field transformed to the time domain, in [6] the integration is performed directly in the time domain, simplifying the derivation. Furthermore, it is also shown in [6] that this result applies to any TEM wave (uniform phase) on the aperture provided that one use the field at the aperture center ($\vec{r} = \vec{0}$) in the expressions. This result will be of significant help in the present paper.

Our starting point is the general expressions for the fields from a specified tangential electric field on the aperture S_a [2, 3, 6] as

$$E_{\substack{x\\y}}(\vec{r},t) = \frac{1}{2\pi}\int_{S_a}\frac{z}{R^3}\left[\frac{R}{c}\frac{\partial}{\partial t} + 1\right]E'_{\substack{x\\y}}\left(x',y';t-\frac{R}{c}\right)dS'$$

$$E_z(\vec{r},t) = \frac{1}{2\pi}\int_{S_a}\frac{x-x'}{R^3}\left[\frac{R}{c}\frac{\partial}{\partial t} + 1\right]E'_x\left(x',y';t-\frac{R}{c}\right)dS'$$

$$+ \frac{1}{2\pi}\int_{S_a}\frac{y-y'}{R^3}\left[\frac{R}{c}\frac{\partial}{\partial t} + 1\right]E'_y\left(x',y';t-\frac{R}{c}\right)dS'$$

$$Z_0H_x(\vec{r},t) = -\frac{1}{2\pi}\int_{S_a}\frac{(x-x')(y-y')}{R^4}\left[\frac{R}{c}\frac{\partial}{\partial t} + 3 + \frac{3c}{R}\,I_t\right]E'_x\left(x',y';t-\frac{R}{c}\right)dS'$$

$$+ \frac{1}{2\pi}\int_{S_a}\frac{1}{R^2}\left\{2+\frac{2c}{R}I_t - \frac{(y-y')^2+z^2}{R^2}\left[\frac{R}{c}\frac{\partial}{\partial t} + 3 + \frac{3c}{R}\,I_t\right]\right\}E'_y\left(x',y';t-\frac{R}{c}\right)dS'$$

$$Z_0H_y(\vec{r},t) = \frac{1}{2\pi}\int_{S_a}\frac{(x-x')(y-y')}{R^4}\left[\frac{R}{c}\frac{\partial}{\partial t} + 3 + \frac{3c}{R}\,I_t\right]E'_y\left(x',y';t-\frac{R}{c}\right)dS'$$

$$- \frac{1}{2\pi}\int_{S_a}\frac{1}{R^2}\left\{2+\frac{2c}{R}I_t - \frac{(x-x')^2+z^2}{R^2}\left[\frac{R}{c}\frac{\partial}{\partial t} + 3 + \frac{3c}{R}\,I_t\right]\right\}E'_x\left(x',y';t-\frac{R}{c}\right)dS'$$

$$Z_0H_z(\vec{r},t) = -\frac{1}{2\pi}\int_{S_a}\frac{(y-y')\,z}{R^4}\left[\frac{R}{c}\frac{\partial}{\partial t} + 3 + \frac{3c}{R}\,I_t\right]E'_x\left(x',y';t-\frac{R}{c}\right)dS'$$

$$+ \frac{1}{2\pi}\int_{S_a}\frac{(x-x')\,z}{R^4}\left[\frac{R}{c}\frac{\partial}{\partial t} + 3 + \frac{3c}{R}\,I_t\right]E'_y\left(x',y';t-\frac{R}{c}\right)dS'$$

$$Z_0 = \left[\frac{\mu_0}{\varepsilon_0}\right] \equiv \text{wave impedance of free space}$$

$$c = [\mu_0\,\varepsilon_0]^{-\frac{1}{2}} \equiv \text{speed of light}$$

$$R = |\vec{r}-\vec{r}\,'| = \left[[x-x']^2 + [y-y']^2 + z^2\right]^{\frac{1}{2}}$$

$$I_t \equiv \text{time integral operator} \tag{1.2}$$

$$I_t\,\Xi(t) \equiv \int_{-\infty}^{t}\Xi(t')\,dt'$$

$$\vec{E}'_t(x',y';t) = E'_x(x',y';t)\ \vec{1}_x + E'_y(x',y';t)\ \vec{1}_y$$

$$\equiv \text{tangential electric field on aperture } S_a$$

Referring to fig. 1.1, note that a prime designates a parameter on the aperture S_a which lies on the $z = 0$ plane. Parameters at the observation point $\vec{r}$ are unprimed.

Our coordinates are as usual

$$\vec{r} = (x,y,z) = x\ \vec{1}_x = y\ \vec{1}_y = z\ \vec{1}_z = r\ \vec{1}_r \qquad (1.3)$$
$$r = |\vec{r}|$$

Cylindrical coordinates (Ψ, ϕ) are formed via

$$x = \Psi \cos(\phi)\ , \quad y = \Psi \sin(\phi) \qquad (1.4)$$

with primes to place these on the aperture plane.

Referring to fig. 1.1, the aperture boundary C_a encloses the aperture S_a consisting of a set of points $(x',y') \in S_a$. If one extends the points $(x',y') \in C_a$ as the points (x, y) with $x = x'$ and $y = y'$ for all positive z, there is formed a cylinder with generators parallel to the z axis. The same extension of $(x',y') \in S_a$ to equal (x, y) gives the interior of this cylinder which we call

$$V_a \equiv \text{aperture extension} \qquad (1.5)$$

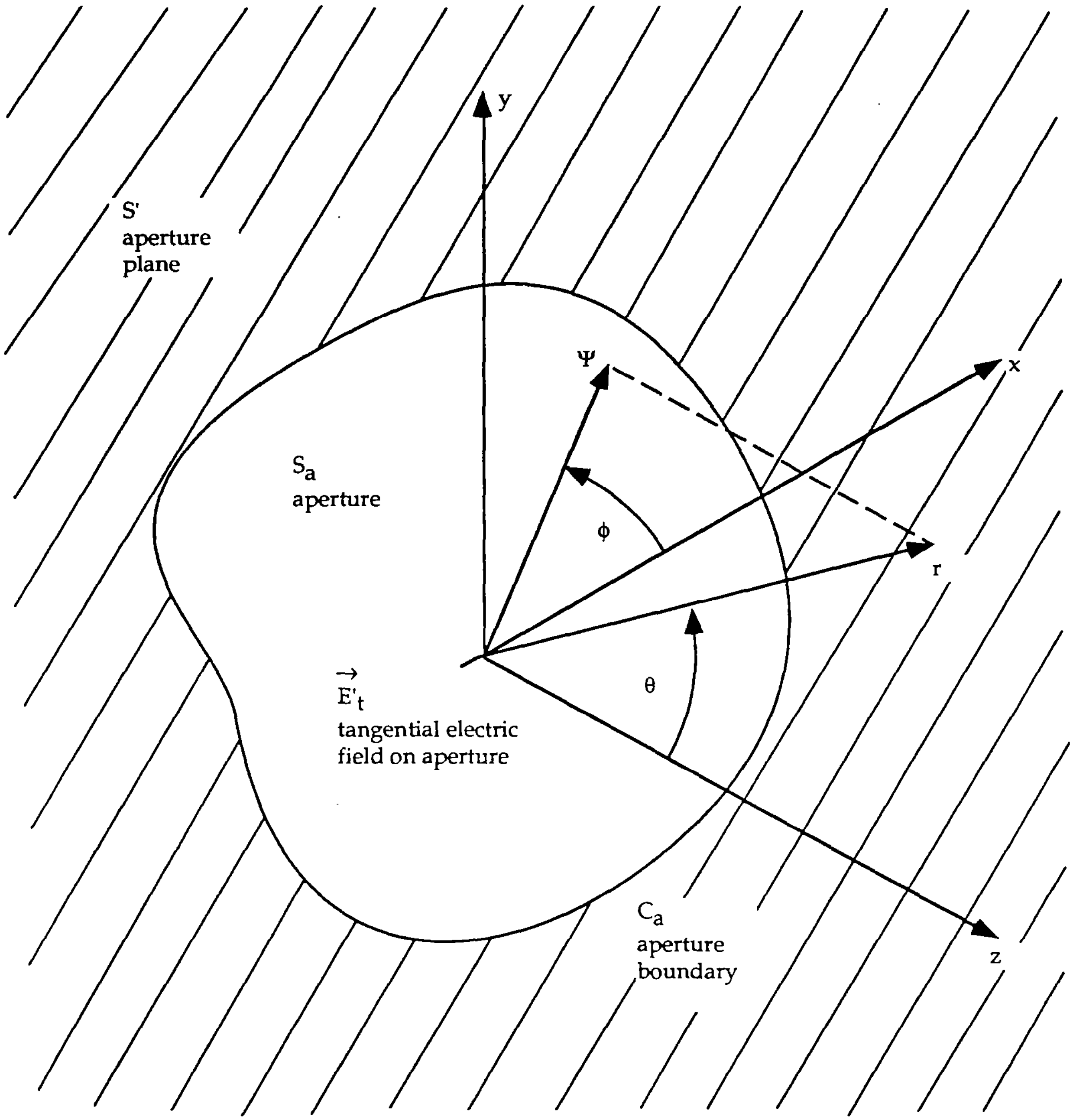

Fig. 1.1 Aperture Antenna with Specified Targential Electric Field

2. Formulation of the Intermediate Field

For the intermediate-field formulation of the impulsive part of the waveform (becoming an impulse in the limit of large r on the beam axis) let us first consider only the R^{-1} terms in (1.2). This defines what is sometimes called the "radiating near field" [23]. Furthermore, let us consider the asymptotic form for $z \to \infty$ with fixed x and y. First we have

$$\begin{aligned} R &= z\left[1 + \left[\frac{x-x'}{z}\right]^2 + \left[\frac{y-y'}{z}\right]^2\right]^{\frac{1}{2}} \\ &= z\left[1 + \frac{1}{2z^2}\left[[x-x']^2 + [y-y']^2\right] + O\left(z^{-4}\right)\right] \\ &= z + O\left(z^{-1}\right) \text{ as } z \to \infty \end{aligned} \tag{2.1}$$

Note that x' and y' are bounded by the assumed finite dimensions of the aperture. Now in (1.2) retain only the leading z^{-1} terms in the coefficients of the aperture fields giving

$$\begin{aligned} \vec{E}(\vec{r},t) &= \frac{1}{2\pi cz}\frac{\partial}{\partial t}\int_{S_a}\vec{E}'\left(x',y';t-\frac{R}{c}\right)dS' + O\left(z^{-2}\right) \text{ as } z \to \infty \\ Z_0\,\vec{H}(\vec{r},t) &= \vec{1}_z \times \vec{E}(\vec{r},t) + O\left(z^{-2}\right) \text{ as } z \to \infty \\ \vec{E}(\vec{r},t) &= -Z_0\,\vec{1}_z \times \vec{H}(\vec{r},t) + O\left(z^{-2}\right) \text{ as } z \to \infty \end{aligned} \tag{2.2}$$

Here the dependence on R is retained in the retarded time in the temporal argument for the aperture field. This is equivalent to retaining the exact phase of each elementary radiator on the aperture. Note that in this approximation the fields are TEM with respect to the z axis with errors in the fields of order z^{-2}. This is like the usual far field (taken in the z direction) except for the dependence on R in the aperture field.

Defining retarded time

$$t_r \equiv t - \frac{z}{c} \tag{2.3}$$

and keeping only up to quadratic terms in (2.1) for the expansion of R we have

$$\vec{E}(\vec{r},t) \simeq \vec{E}_i(\vec{r},t) = \frac{1}{2\pi cz}\frac{\partial}{\partial t}\int_{S_a}\vec{E}'\left(x',y';t_r - \frac{1}{2cz}\left[[x'-x]^2 + [y'-y]^2\right]\right)dS' \tag{2.4}$$

It is this that we call the intermediate field, a far field with quadratic time (or phase) correction on the aperture. The approximation here will give a small error as to when an event is seen at the observer (as compared to an error in amplitude).

Now let

$$\vec{E}'(x',y';t) = E_0\,\vec{e}_0(x',y')\,f(t)$$

$f(t) \equiv$ waveform function

$\vec{e}_0 \equiv$ normalized electric-field distribution (tangential) on aperture (2.5)

$E_0 \equiv$ normalizing electric-field magnitude (V/m)

so that there is a uniform waveform (or phase) for the entire aperture, giving focusing in the z direction (at ∞). (One can focus in a different direction, say $\vec{1}_0$, but then one should do the expansion in that direction instead of $\vec{1}_z$ [3].) Our canonical aperture excitation function is a step function [3, 5]. Let us then write

$$\vec{E}_i(\vec{r},t) = E_0\left[\frac{df(t)}{dt}\right] \circ \vec{e}_i(\vec{r},t)$$

$\circ \equiv$ convolution with respect to time

(2.6)

$$\begin{aligned}\vec{e}_i(\vec{r},t_r) &= \frac{1}{2\pi cz}\frac{\partial}{\partial t}\int_{S_a}\vec{e}_0(x',y')\,u\left(t_r - \frac{1}{2cz}\left[[x'-x]^2 + [y'-y]^2\right]\right)dS' \\ &= \frac{1}{2\pi cz}\int_{S_a}\vec{e}_0(x',y')\,\delta\left(t_r - \frac{1}{2cz}\left[[x'-x]^2 + [y'-y]^2\right]\right)dS'\end{aligned}$$

Define

$$\begin{aligned}\xi &\equiv 2czt_r \text{ (units } m^2) \\ &\equiv \text{time parameter or normalized retarded time}\end{aligned}$$

$$C_\xi \equiv \left\{(x',y')\,\middle|\,[x'-x]^2 + [y'-y]^2 = \xi\ ,\ \ \xi > 0\right\}$$

S_ξ ≡ circular disk in aperture plane S' of radius $\xi^{1/2}(>0)$ centered on (x, y) with boundary C_ξ

S_a ≡ aperture (disk) in S' with specified tangential electric field $\vec{E'}_t$ and normalized form $\vec{e}_0$, excluding any feed-arm projections (zero field here)

C_a ≡ aperture (S_a) boundary including any deviation around feed-arm projections (even if of zero thickness)

S_b ≡ $S_\xi \cap S_a$ ≡ portion of S_a contributing to surface integrals

C_b ≡ boundary of S_b (noting deviation around feed-arm projections)

$C_c \equiv C_\xi \cap S_a$ (a proper subset of C_b when C_ξ lies partly or entirely outside of S_a) (2.7)

$dS' = dx'\,dy'$ ≡ unoriented or scalar surface element

$d\ell'$ ≡ unoriented or scalar line element

$d\vec{\ell}'$ ≡ oriented or vector line element

We then have the representations

$$\begin{aligned}
\vec{e}_i(\vec{r},t_r) &= \frac{1}{\pi}\frac{\partial}{\partial\xi}\int_{S_a}\vec{e}_0(x',y')\,u\left(\xi-[x'-x]^2-[y'-y]^2\right)dS' \\
&= \frac{1}{\pi}\frac{\partial}{\partial\xi}\int_{S_b}\vec{e}_0(x',y')\,dS' \\
&= \frac{1}{2\pi}\frac{\partial}{\partial\xi}\int_{S_a}\vec{e}(x',y')\,u\left(\xi^{1/2}-\left[[x'-x]^2-[y'-y]^2\right]^{1/2}\right)dS' \text{ (for } \xi\geq 0) \\
&= \frac{1}{2\pi}\,\xi^{-\frac{1}{2}}\int_{S_a}\vec{e}_0(x',y')\,\delta\left(\xi^{\frac{1}{2}}-\left[[x'-x]^2-[y'-y]^2\right]^{\frac{1}{2}}\right)dS' \\
&= \frac{1}{2\pi}\,\xi^{-\frac{1}{2}}\int_{C_c}\vec{e}_0(x',y')\,d\ell' \\
&= \vec{e}_i(x,y;\xi)
\end{aligned} \tag{2.8}$$

where, in converting to a contour integral over C_c, one needs to be careful so that the integral of the delta function (one dimensional) is over a path normal to the circle of radius $\sqrt{\xi}$ with this radial coordinate as the integration variable. Furthermore, one needs to be careful near singularities of the field (e.g., near edges of feed arm projections), but such singularities are quite integrable. Our normalized intermediate field is then a function of x, y, and ξ. Space z and time t_r are combined as a product. So for a given (x, y) this normalized field need be computed only as a function of ξ and this temporal waveform applies for all z (within the limits of our approximations). This scaling also applies to the frequency spectrum of the intermediate field as discussed in Appendix A.

As we go to larger and larger z the waveform becomes shorter and shorter in retarded time. If we take the complete time integral the circle of radius $\xi^{1/2}$ on the aperture plane expands to include all of S_a giving

$$\int_{-\infty}^{\infty} \vec{e}_i(x,y;\xi)\,dt_r = \frac{1}{2cz}\int_{-\infty}^{\infty} \vec{e}_i(x,y;\xi)\,d\xi = \frac{1}{2\pi cz}\int_{S_a} \vec{e}_0(x',y')\,dS' \tag{2.9}$$

This is a rather simple result and it falls off as z^{-1} as a far field should. Of course, here we are only concerned with the impulsive part of the waveform, not prepulse (for reflector IRA) or postpulse. Furthermore, this integral (the area of the intermediate-field waveform) is independent of x and y, and hence of whether or not $\vec{r}$ is contained in the aperture extension V_a (of course limited to $\Psi << z$). Note also that the vector orientation of this integral is the same as the average of $\vec{e}_0$ over S_a. For typical cases of one or more symmetry planes this direction is found by inspection (e.g., $\vec{1}_y$ for $\vec{e}_0$ *symmetric* with respect to $x' = 0$ [5, 8, 24]).

As a simple example one can consider the observer on a symmetry plane which we take as the (x, z) plane. With *symmetric* fields on the aperture the x component of the field is zero giving

$$\begin{aligned} \vec{e}_i(0,y;\xi) &= \vec{e}_{i_y}(0,y;\xi)\,\vec{1}_y \\ \int_{-\infty}^{\infty} e_{i_y}(0,y;\xi)\,dt &= \frac{1}{2\pi cz}\int_{S_a} e_{0_y}(x',y')\,dS' \end{aligned} \tag{2.10}$$

This scalarization allows us to find the approximate delta function

$$\begin{aligned} \delta_a(t_r) &= 2\pi cz\left[\int_{S_a} e_{0_y}(x',y')\,dS'\right]^{-1} e_{i_y}(0,y;\xi) \\ \int_{-\infty}^{\infty} \delta_a(t_r)dt &= 1 \end{aligned} \tag{2.11}$$

as discussed in [3]. The present development extends this to include its vector properties and scaling with z and t_r.

3. Initial Step in Intermediate Field Within Aperture Extension

If the observer is within the aperture extension V_a, the first contribution to the intermediate field comes from $(x',y')=(x,y)$. As it turns out, this $\vec{e}_i(x,y;0_+)$ is easy to evaluate. Our first approach follows the development in [6]. In that case the TEM property on the aperture, which can be given by

$$\vec{e}_0(x',y') = \nabla' u(x',y') \quad , \quad \nabla'^2 u(x',y')=0 \tag{3.1}$$

allows one to expand $\vec{e}_0$ around the origin $(\Psi'=0)$ in terms of non-negative integer powers of Ψ' (say $\Psi'^{m-1}, m\geq 1$) times $\cos(m\,\phi')$ and $\sin(m\,\phi')$. Then integrating over a circular aperture only the $m = 1$ term contributes for an observer on the z axis, and the field at the observer is initially the same as the field on S_a at $\Psi'=0$.

Extending this result, consider some small $\xi>0$. The contour C_r is a circle of radius $\xi^{1/2}$ lying completely within S_a for ξ sufficiently small, i.e., $\xi=0_+$. The integration is only over the circular disk contained inside C_r. The aperture field is TEM there, and the observer initially sees $(t_r=0_+)$ that aperture field. This result applies not only on the z axis, but inside the aperture extension V_a (at least a small distance away from the cylindrical boundary), since the field is TEM at all (x',y') within S_a. (Equivalently the z axis can be shifted to any position within V_a.) So we therefore conclude

$$\vec{e}_i(x,y;0_+) = \vec{e}_0(x,y) \tag{3.2}$$

a remarkably simple result. Note then that the early-time $\vec{e}_i$ need not have the same orientation as its complete time integral as can be seen from the symmetric aperture in (2.10) and (2.11). There the complete time integral is parallel to $\vec{1}_y$, but for $x \neq 0$, a symmetric $\vec{e}_i$ with respect to the symmetry plane can have an orientation other than $\vec{1}_y$. Furthermore, note that this result does not depend on the intermediate-field approximation; it applies right up to the aperture.

Referring to fig. 3.1, we can see that as ξ increases the radius $\xi^{1/2}$ of C_ξ increases until a non-TEM domain is reached. This could be the aperture boundary C_a or some discontinuity at conductors forming boundaries for the assumed planar TEM mode on S_a. Defining

$$\begin{aligned} \xi^{(1)}(x,y) &\equiv [\text{radius of largest circular disk wholly } \textit{within} \text{ the TEM region}]^2 \\ &= 2czt_r^{(1)} \end{aligned} \tag{3.3}$$

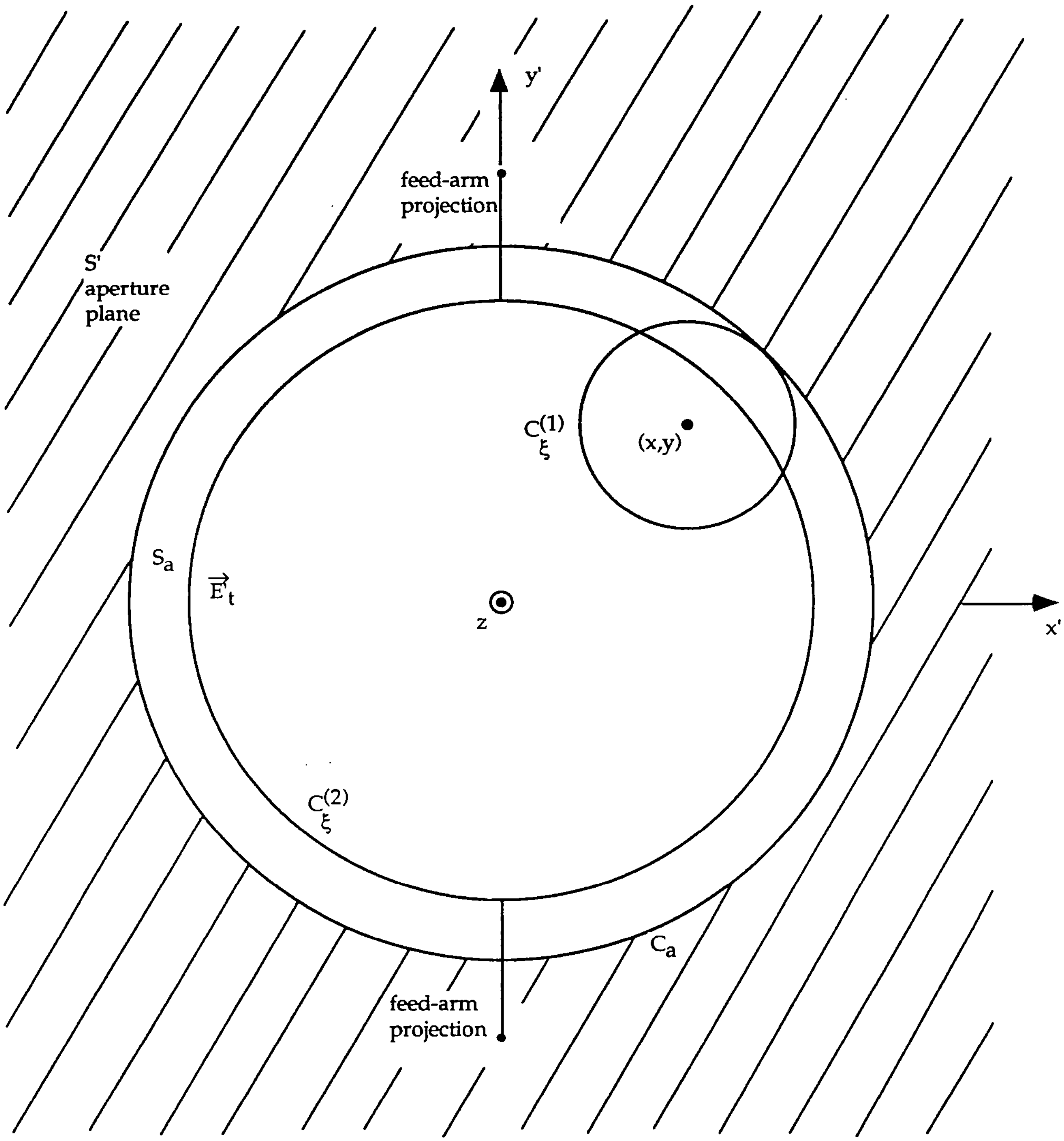

Fig. 3.1 Example Aperture: Circular with Projection of Two Coplanar Feed Arms

we have the result

$$\vec{e}_i(x,y;\xi) = \vec{e}_0(x,y)\ u(\xi)\ \text{for}\ \xi < \xi^{(1)}(x,y) \tag{3.4}$$

So we have a step function valid out to some readily calculable ξ (or equivalent retarded time). Note in fig. 3.1 we have shown two contours: $C_\xi^{(1)}$ centered on (x, y) and $C_\xi^{(2)}$ centered on the origin. In the first case $\xi^{(1)}$ is determined by the circular C_a; in the second case it is determined by the distance to the conductors introducing a singularity in the TEM mode. If one wishes $\xi^{(1)}$ can be more accurately calculated based on R as in (2.2), thereby avoiding the intermediate-field approximations for this early-time window.

Note that for observers outside the aperture extension (3.3) can be extended to

$$\begin{aligned}\xi^{(1)}(x,y) &\equiv [\text{radius of largest circular disk wholly } \textit{outside} \text{ the TEM region (aperture)}]^2 \\ &= 2czt_r^{(1)}\end{aligned} \tag{3.5}$$

In this case we have

$$\vec{e}_i(x,y;\xi) = \vec{0}\ \ \text{for}\ \ \xi < \xi^{(1)}(x,y) \tag{3.6}$$

so that $\xi^{(1)}$ now gives the first retarded time that the observer can detect any signal, the tangential electric field on the aperture outside S_a being assumed zero.

Going a step further we can define

$$\begin{aligned}\xi^{(2)}(x,y) &\equiv [\text{radius of smallest circular disk wholly } \textit{containing} \text{ the TEM region (aperture)}]^2 \\ &= 2czt_r^{(2)}\end{aligned} \tag{3.7}$$

In this case (2.8) gives

$$\vec{e}_i(x,y;\xi) = \vec{0}\ \ \text{for}\ \xi > \xi^{(2)}(x,y) \tag{3.8}$$

Between $\xi^{(1)}$ and $\xi^{(2)}$ the waveform is more complicated and polarization may even rotate. One constraint that we can obtain concerns the time integral of this portion. Combining (2.9) with (3.2) gives

$$
\begin{aligned}
\int_{t_r^{(1)}}^{t_r^{(2)}} \vec{e}_i(x,y;\xi)\,dt_r &= \frac{1}{2cz}\int_{\xi^{(1)}}^{\xi^{(2)}} \vec{e}_i(x,y;\xi)\,d\xi \\
&= \frac{1}{2\pi cz}\int_{S_a} \vec{e}_0(x',y')\,dS' - \vec{e}_0(x,y)\,t_r^{(1)} \\
&= \frac{1}{2cz}\left[\frac{1}{\pi}\int_{S_a} \vec{e}_0(x',y')\,dS' - \begin{Bmatrix} \vec{e}_0(x,y)\,\xi^{(1)}(x,y) \text{ for } (x,y) \in S_a \\ \vec{0} \text{ for } (x,y) \notin S_a \end{Bmatrix}\right]
\end{aligned}
\tag{3.9}
$$

This last result can also be interpreted by recognizing that $\vec{e}_0 = \vec{0}$ outside S_a.

A second approach to the early-time result in (3.4) involves a gedankenexperiment. Again, referring to fig. 3.1, consider first a boundary value problem in which the entire aperture plane S' ($z = 0$ plane) has specified on it the TEM tangential electric field given by (2.5) with $f(t)$ as the unit step at $t = 0$. Furthermore, let the fed-arm projections be extended as conductors (perfect) parallel to the z axis toward $z = +\infty$. Noting that the TEM mode $\vec{e}_0(x,y)$ is exactly the transverse field for such a mode on a TEM transmission line matching the requisite boundary condition on the above transmission-line conductors, then

$$
\vec{E}(x,y;t) = E_0\,\vec{e}_0(x,y)\,u\left(t-\frac{z}{c}\right) \tag{3.10}
$$

satisfies the Maxwell equations and boundary conditions (tangential $\vec{E}$ on S', zero tangential $\vec{E}$ on the conductors, and radiation condition ($+z$ propagation)) and is thereby the unique solution for positive z.

Next, change the field on S' to be zero outside S_a, but unchanged on S_a. Then by causality, this change cannot be observed inside the aperture extension V_a until some time after the wave in (3.10) first arrives at the observer. This retarded time is $t_r^{(1)}$ except that the non-TEM part of the aperture (feed-arm projection) is not considered to limit the expansion of C_c, now to the aperture boundary.

Finally, remove the transmission-line conductors from the $z > 0$ half space. There being no currents on these to support the wave in (3.10), now an observer in the aperture extension can observe the fact of their absence after a retarded time $t_r^{(1)}$ with the feed-arm projection included as part of the aperture boundary (boundary of non-TEMness).

Since this gedankenexperiment does not rely on the asymptotic expansion of R for large z as in (2.1), then one can use exact values of R to determine a more accurate value of $t_r^{(1)}$. Thereby one can extend the early-time result in (3.4) right back to the aperture plane, i.e., z need not be considered large.

4. Complex Potentials and Fields on Aperture

Having constrained the aperture fields to be TEM (as on a cylindrical transmission line) we can derive the aperture tangential electric field from a potential function as in (3.1). This allows us to use the complex-variable formulation in [5]. Summarizing we have

$$
\begin{aligned}
\zeta &= x' + jy' \equiv \text{complex coordinate (on aperture)} \\
\zeta_0 &= x + jy \equiv \text{complex observer coordinate} \\
w(\zeta) &= u(\zeta) + jv(\zeta) \equiv \text{complex potential} \\
\vec{e}_0(x',y') &= \nabla' u(x',y') \ , \quad \vec{h}_0(x',y') = \nabla' v(x',y') \\
e_0(\zeta) &= e_{0_x}(x',y') - je_{0_y}(x',y'), \ h_0(\zeta) = h_{0_x}(x',y') - jh_{0_y}(x',y') \\
e_{0_x}(x',y') &= h_{0_y}(x',y') \ , \quad e_{0_y}(x',y') = -h_{0_x}(x',y') \\
e_0(\zeta) &= jh_0(\zeta) = \frac{dw(\zeta)}{d\zeta} \equiv \text{complex aperture field (normalized)} \qquad (4.1) \\
E(\zeta) &= jZ_0 H(\zeta) = -\frac{V}{\Delta u}\frac{dw(\zeta)}{d\zeta} = -\frac{V}{\Delta u} e_0(\zeta) \\
&= E_x(x',y') - j\,E_y(x',y') \\
f_g &= \frac{\Delta u \ \text{(between appropriate conductors)}}{\Delta v \ \text{(around appropriate conductor(s))}} \\
&\equiv \text{geometric impedance factor for TEM transmission line feeding aperture}
\end{aligned}
$$

Here V is the voltage across the aperture (via the feed arms in the case of a reflector IRA); this can be a function of time to match (2.5) as desired.

In complex form (2.8) becomes

$$
\begin{aligned}
e_i(\zeta_0,\xi) &= e_{i_x}(x,y;\xi) - j\,e_{i_y}(x,y;\xi) \\
&= \frac{1}{\pi}\int_{S_b} e_0(\zeta)\,dS' \\
&= \frac{1}{2\pi}\xi^{-\frac{1}{2}}\int_{C_c} e_0(\zeta)\,d\ell' \qquad (4.2) \\
&= \frac{1}{2\pi}\xi^{-\frac{1}{2}}\int_{C_c} e_0(\zeta)\,|d\zeta|
\end{aligned}
$$

Consider first the early-time behavior. For $\xi < 0$ the integrals for $\vec{e}_i$ give zero. For the time interval (the first time interval)

$$0 < \xi < \xi^{(1)}(x,y) \tag{4.3}$$

S_b is a circular disk and its boundary C_ξ is a circle consistent with (3.3). One way to approach this problem is from Cauchy's theorem

$$e_0(\zeta) = \frac{1}{2\pi j} \oint_C \frac{e_0(\zeta')}{\zeta' - \zeta} d\zeta' \ , \quad \zeta \text{ inside } C \tag{4.4}$$

which requires that $e_0(\zeta)$ be analytic within and on C. Let us then choose C as a circle centered on ζ and change variables as

$$\begin{aligned} \zeta' - \zeta &= \zeta'' = |\zeta''| e^{j\arg(\zeta'')} \\ d\zeta' &= je^{j\arg(\zeta'')} d\ell'' \text{ on } C \\ d\ell'' &= \text{line element (real) on } C \\ |\zeta''| &= \text{radius of circle} \end{aligned} \tag{4.5}$$

giving

$$\begin{aligned} e_0(\zeta) &= \frac{1}{2\pi|\zeta''|} \oint_C e_0(\zeta')\, d\ell'' \\ &= \text{average value of } e_0(\zeta) \text{ on circular contour } C \end{aligned} \tag{4.6}$$

Applying this general result to our circle of radius $\xi^{\frac{1}{2}}$ centered on ζ_0 gives

$$\begin{aligned} e_0(\zeta_0) &= \frac{1}{2\pi} \xi^{-\frac{1}{2}} \oint_{C_\xi} e_0(\zeta)|d\zeta| \\ d\ell' &= |d\zeta| \\ \vec{e}_0(x,y) &= \frac{1}{2\pi} \xi^{-\frac{1}{2}} \oint_{C_\xi} \vec{e}_0(x',y')d\ell' \quad \text{(real vector form)} \end{aligned} \tag{4.7}$$

Applying this result to (4.2) gives

$$e_i(\zeta_0, \xi) = e_0(\zeta_0) \quad \text{for} \quad 0 < \xi < \xi^{(1)}(x,y) \tag{4.8}$$

This is the same result as (3.4) stated in complex form and derived by a different technique (Cauchy's theorem). For observers outside the aperture extension this gives the same result as (3.6) noting that $e_0(\zeta_0)$ is zero for ζ_0 outside S_a.

For the time interval (the third time interval)

$$\xi^{(2)}(x,y) < \xi \tag{4.9}$$

the circular contour C_ξ contains S_a completely inside (as per (3.5)). The contour C_c in (4.2) and (2.7) is then the null set and

$$e_i(\zeta_0,\xi) = 0 \quad \text{for} \quad \xi^{(2)}(x,y) < \xi \tag{4.10}$$

consistent with (3.6). This leaves the second time interval

$$\xi^{(1)}(x,y) < \xi < \xi^{(2)}(x,y) \tag{4.11}$$

to consider.

For the second time interval we need to apply the contour integral to a circular arc C_c centered on ζ_0 in S_a. Consider then a circular arc of radius ξ which we can parameterize as

$$\begin{aligned} \zeta - \zeta_0 &= \xi^{\frac{1}{2}} e^{j\psi} \ , \quad \psi_1 \le \psi \le \psi_2 \ , \quad \psi \text{ real} \\ d\zeta &= j\xi^{\frac{1}{2}} e^{j\psi}\, d\psi \\ d\ell' &= |d\zeta| = \xi^{\frac{1}{2}}\, d\psi \end{aligned} \tag{4.12}$$

Noting the positive sense of ψ, the range of ψ ($-\pi$ to π etc.) can be chosen at our convenience. Then (4.2) becomes

$$e_i(\zeta_0,\xi) = \frac{1}{2\pi} \int_{\psi_1}^{\psi_2} e_0\left(\xi\, e^{j\psi} + \zeta_0\right) d\psi \tag{4.13}$$

If C_c consists of more than one such arc (say due to interruption by any feed arm projection(s)), then other arc portions (e.g., ψ_3 to ψ_4) need to be included in (4.13).

A special case has ζ_0 on the aperture boundary. Then we have

$$\begin{aligned} \xi^{(1)}(x,y) &= 0_+ \\ e_i(\zeta_0, 0_+) &= \frac{1}{2\pi}[\psi_2 - \psi_1] e_0(\zeta_{0\_}) \\ \zeta_{0\_} &\notin S_a \text{ (just inside from } \zeta_0) \end{aligned} \tag{4.14}$$

The 0_+ value of $\xi^{(1)}$ indicates the limit as $\xi \to 0$ from positive values. Note that there is assumed a zero field just outside the boundary, so an infinitesimally thin feed-arm projection does not apply here, but one could include two such terms in that case. Assuming that the boundary is smooth (locally straight) at ζ_0, then we have the simple result

$$\begin{aligned} \psi_2 - \psi_1 &= \pi \\ e_i(\zeta_0, 0_+) &= \frac{1}{2} e_0(\zeta_{0\_}) \end{aligned} \tag{4.15}$$

5. Symmetrical Two-Wire Transmission Line as Feed-Arm Projection with Circular Aperture

For a concrete example for which one can compute $e_i(\zeta_0, \xi)$ let us consider the canonical example [5] of the aperture fields as given by those of a symmetrical two-wire transmission line as illustrated in fig. 5.1. The positions of the equivalent line charges are made coincident with the edge of the circle of radius a (at $\zeta = \pm ja$), while the wires of radius b are centered at $\zeta = \pm ja_c$. Using previous results [1, 5] we have

$$
\begin{aligned}
a_c^2 &= a^2 + b^2 \\
f_g &= \frac{\Delta u}{\Delta v} = \frac{1}{\pi} \operatorname{arccosh}\left(\frac{a_c}{b}\right) = \frac{1}{\pi} \operatorname{arcsinh}\left(\frac{a}{b}\right) \\
&= \frac{1}{\pi} \ell n\left(\frac{a_c}{b} + \left(\left(\frac{a_c}{b}\right)^2 - 1\right)^{\frac{1}{2}}\right) = \frac{1}{\pi} \ell n\left(\frac{a}{b} + \left(\left(\frac{a}{b}\right)^2 + 1\right)^{\frac{1}{2}}\right) \\
&= \frac{1}{\pi} \ell n\left(\frac{2a}{b}\right) + O\left(\left(\frac{b}{a}\right)^2\right) \text{ as } \frac{b}{a} \to 0
\end{aligned}
\tag{5.1}
$$

For the complex potential and field we use

$$
\begin{aligned}
w(\zeta) &= w_0 \, \ell n\left(\frac{\zeta + ja}{\zeta - ja}\right) \\
e_0(\zeta) &= \frac{dw(\zeta)}{d\zeta} = w_0\left[[\zeta + ja]^{-1} - [\zeta - ja]^{-1}\right] = -\frac{j2w_0 a}{\zeta^2 + a^2}
\end{aligned}
\tag{5.2}
$$

For convenience we choose

$$
\begin{aligned}
\vec{e}_0(0,0) &= \vec{1}_y \\
e_0(0) &= -j \\
w_0 &= \frac{a}{2}
\end{aligned}
\tag{5.3}
$$

so that E_0 in (2.5) represents the field amplitude at the aperture center. With this we have the potential difference between the conductors as

$$
\begin{aligned}
\Delta u &= a \operatorname{arcsinh}\left(\frac{a}{b}\right) = \pi a f_g \\
\Delta v &= a\pi \quad \text{(circling one conductor)}
\end{aligned}
\tag{5.4}
$$

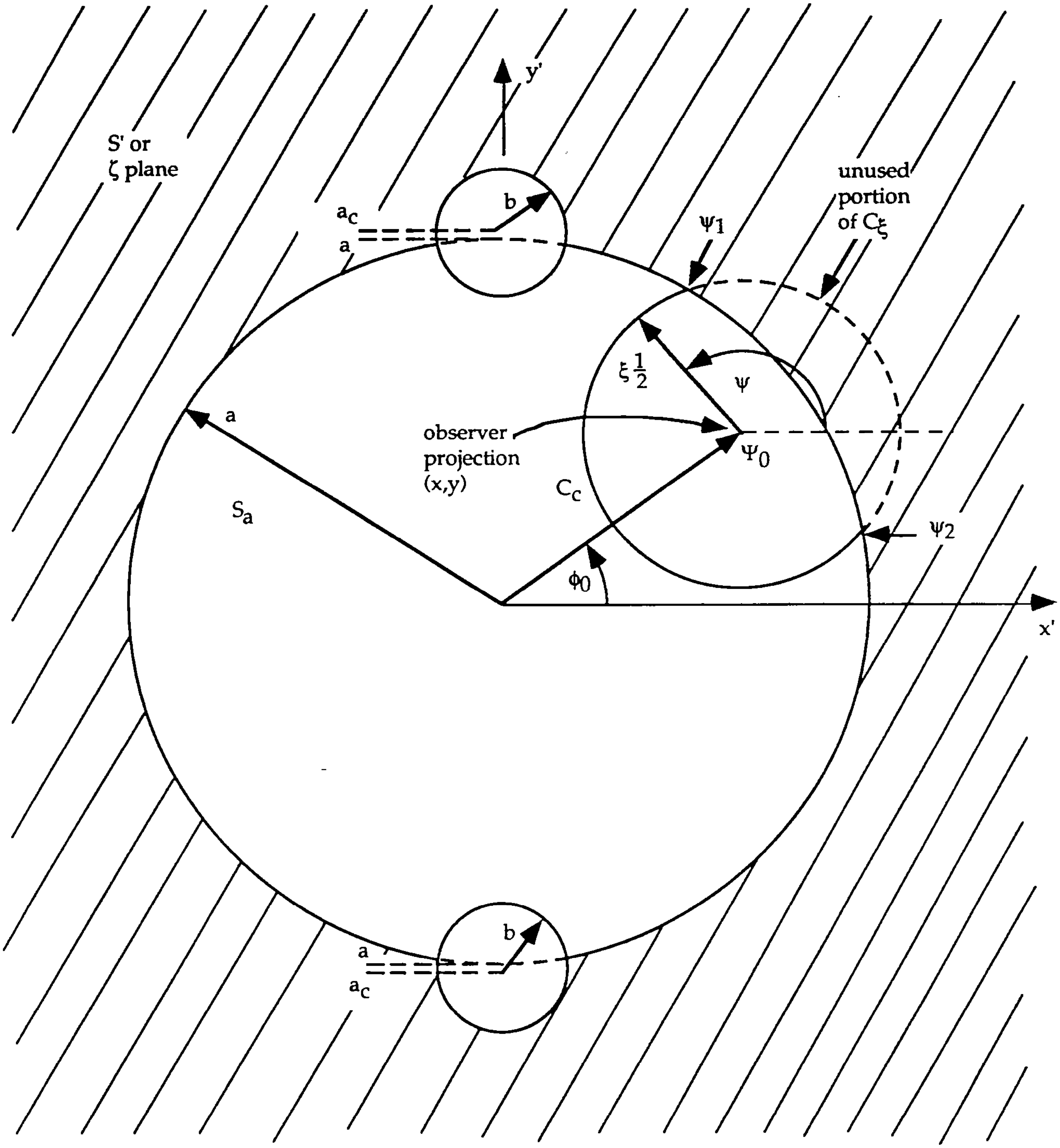

Fig. 5.1 Symmetrical Two-Wire Transmission Line Feeding Circular Aperture

We can then identify Δu with the voltage V between the conductors giving

$$E_0 = -\frac{v}{\Delta u} = -\frac{V}{\pi\, a\, f_g} \tag{5.5}$$

where the upper conductor has convention positive and the bottom has negative. Note, however, the sign reversal in the case of a reflector IRA due to the negative reflection at the paraboloidal reflector.

In fig. 5.1, the observer location (projected on the aperture plane) is

$$\zeta_0 = x + jy = \Psi_0\, e^{j\phi_0} \tag{5.6}$$

As previously determined the intermediate field is initially (the first time interval) the aperture field, i.e.,

$$\begin{aligned} e_i(\zeta_0,\xi) &= e_0(\zeta_0) = -\frac{ja^2}{\zeta_0^2 + a^2} \\ &\quad \text{for } 0<\xi<\xi^{(1)}(x,y)\ ,\ \Psi_0 < a \\ \xi^{(1)}(x,y) &= [a-\Psi_0]^2 \end{aligned} \tag{5.7}$$

where b has been assumed negligible compared to a for certain ϕ_0 (e.g., $\pi/2, 3\pi/2$) so that the feed projections do not significantly lessen $\xi^{(1)}$. On the two symmetry planes ($x = 0$ and $y = 0$) the field is polarized in the y direction. The field is *symmetric* with respect to $x = 0$ and *antisymmetric* with respect to $y = 0$ [8, 24]. The individual field components for the first time interval are

$$\begin{aligned} e_{i_x} &= \frac{1}{2}\left[\frac{\frac{x}{a}}{\left[\frac{x}{a}\right]^2 + \left[1+\frac{y}{a}\right]^2} - \frac{\frac{x}{a}}{\left[\frac{x}{a}\right]^2 + \left[1-\frac{y}{a}\right]^2}\right] \\ e_{i_y} &= \frac{1}{2}\left[\frac{1+\frac{y}{a}}{\left[\frac{x}{a}\right]^2 + \left[1+\frac{y}{a}\right]^2} + \frac{1-\frac{y}{a}}{\left[\frac{x}{a}\right]^2 + \left[1-\frac{y}{a}\right]^2}\right] \end{aligned} \tag{5.8}$$

Note that for $\Psi_0 > a$ the observer is outside the aperture extension, and the field is initially zero, i.e.,

$$e_i(\zeta_0,\xi) = 0 \ \text{ for } 0<\xi<\xi^{(1)}(x,y)\ ,\ \Psi_0 > a \tag{5.9}$$

consistent with the extended definition of $\xi^{(1)}$ in (3.5).

In the third time interval we have

$$\begin{aligned} e_i(\zeta_0,\xi) &= 0 \quad \text{for} \quad \xi^{(2)}(x,y) < \xi \\ \xi^{(2)}(x,y) &= [a + \Psi_0]^2 \end{aligned} \tag{5.10}$$

As in (2.9) the complete time integral is

$$\int_{-\infty}^{\infty} e_i(\zeta_0,\xi)dt_r = \frac{1}{2cz}\int_{-\infty}^{\infty} e_i(\zeta_0,\xi)d\xi = \frac{1}{2\pi cz}\int_{S_a} e_0(\xi)dS' \tag{5.11}$$

which varies as z^{-1}, but is independent of ζ_0 (or x and y). As in (3.9) the time integral over the second time interval is

$$\begin{aligned} \int_{t_r^{(1)}}^{t_r^{(2)}} e_i(\zeta_0,\xi)dt_r &= \frac{1}{2cz}\int_{\xi^{(1)}}^{\xi^{(2)}} e_i(\zeta_0,\xi)d\xi \\ &= \frac{1}{2cz}\left[\frac{1}{\pi}\int_{S_a} e_0(\zeta)dS' - e_0(\zeta_0)\,\xi^{(1)}(x,y)\right] \end{aligned} \tag{5.12}$$

For the special case of an observer on the z axis we have no second time interval. The waveform is explicitly

$$\begin{aligned} e_i(0,\xi) &= e_0(0)\ u(\xi)\ u\left(a^2 - \xi\right) \\ \zeta_0 &= 0\ ,\quad \xi^{(1)} = \xi^{(2)} = a^2 \end{aligned} \tag{5.13}$$

The complete time integral is

$$\int_0^{t_r^{(1)}} e_i(0,\xi)\,dt_r = \frac{1}{2cz}\int_0^{a^2} e_i(0,\xi)\,d\xi = \frac{a^2}{2cz}e_0(0) \tag{5.14}$$

This is consistent with the results of [6] for the leading part (impulsive part) of the on-axis fields from a circular aperture. This applies to any TEM distribution on a circular aperture where there is no significant penetration of the aperture by feed-arm projections. Thus the results apply to four (or any number) of thin feed arms driving the aperture provided one calculates the tangential electric field at the aperture center.

Now we come to the second time interval for general ζ_0. As illustrated in fig. 5.1, we have the circular arc C_c of radius $\xi^{1/2}$ centered on ζ_0 on which to integrate as in (4.13). The end points of the contour are given by

$$
\begin{aligned}
&\left|\Psi_0\, e^{j\phi_0} + \xi^{1/2}\, e^{j\psi}\right| = a \\
&\Psi_0^2 + \xi + 2\,\Psi_0\,\xi^{1/2}\cos(\psi - \phi_0) = a^2 \\
&\psi_{\substack{1\\2}} - \phi_0 = \pm \arccos\left(\frac{a^2 - \Psi_0^2 - \xi}{2\Psi_0\,\xi^{1/2}}\right)
\end{aligned}
\tag{5.15}
$$

For $\Psi_0 < a(\zeta_0 \in S_a)$ the cosine goes from +1 to −1 as ξ increases. For $\Psi_0 > a(\zeta_0 \notin S_a)$, the cosine goes from −1 to some maximum value and then back to −1 as ξ increases.

For our aperture field as in (5.2) the contour integral for the second time interval as in (4.13) becomes

$$
e_i(\zeta_0,\xi) = \frac{a}{4\pi}\left[\int_{\psi_1}^{\psi_2}\left[\xi^{1/2}\, e^{j\psi} + \zeta_0 + ja\right]^{-1} d\psi \;-\; \int_{\psi_1}^{\psi_2}\left[\xi^{1/2}\, e^{j\psi} + \zeta_0 - ja\right]^{-1} d\psi\right]
\tag{5.16}
$$

The general form of these integrals is given by the indefinite integral [22]

$$
\int \frac{dv}{\alpha_1 + \alpha_2 e^{\alpha_3 v}} = \frac{1}{\alpha_1\alpha_3}\left[\alpha_3\, v - \ell n\left(\alpha_1 + \alpha_2\, e^{\alpha_3 v}\right)\right]
\tag{5.17}
$$

which can be verified by differentiation. Noting that we have a logarithm of a complex argument we will need to observe the location of any branch cut(s) in evaluating the integrals. A good place for such a cut is where it does not cross C_c, but goes from ζ_0 outward in the $+\Psi$ direction, crossing C_ξ in a place that is not part of C_c. Note also that as ψ_1 and ψ_2 sweep around C_a and go by the wires, the end points of C_c pass the wires slightly into the aperture (where $e_0(\zeta)$ is analytic).

Applying (5.17) to (5.16) we have

$$
\begin{aligned}
e_i(\zeta_0,\xi) &= \frac{a}{4\pi}\left[-\frac{j}{\zeta_0+ja}\left[j[\psi_2-\psi_1]-\ell n\left(\frac{\zeta_0+ja+\xi^{1/2}e^{j\psi_2}}{\zeta_0-ja+\xi^{1/2}e^{j\psi_1}}\right)\right]\right.\\
&\qquad\left.+\frac{j}{\zeta_0-ja}\left[j[\psi_2-\psi_1]-\ell n\left(\frac{\zeta_0-ja+\xi^{1/2}e^{j\psi_2}}{\zeta_0-ja+\xi^{1/2}e^{j\psi_1}}\right)\right]\right]\\
&= -\frac{j}{2\pi}\,\frac{a^2}{\zeta_0^2+a^2}\,[\psi_2-\psi_1]\\
&\quad+\frac{j}{4\pi}\,\frac{\zeta_0 a}{\zeta_0^2+a^2}\,\ell n\left(\frac{\zeta_0+ja+\xi^{1/2}\,e^{j\psi_2}}{\zeta_0+ja+\xi^{1/2}\,e^{j\psi_1}}\;\frac{\zeta_0-ja+\xi^{1/2}\,e^{j\psi_1}}{\zeta_0-ja+\xi^{1/2}\,e^{j\psi_2}}\right)\\
&\quad+\frac{1}{4\pi}\,\frac{a^2}{\zeta_0^2+a^2}\,\ell n\left(\frac{\zeta_0+ja+\xi^{1/2}\,e^{j\psi_2}}{\zeta_0+ja+\xi^{1/2}\,e^{j\psi_1}}\;\frac{\zeta_0-ja+\xi^{1/2}\,e^{j\psi_2}}{\zeta_0-ja+\xi^{1/2}\,e^{j\psi_1}}\right)\\
&= -\frac{j}{2\pi}\,\frac{a^2}{\zeta_0^2+a^2}\,[\psi_2-\psi_1]\\
&\quad+\frac{j}{4\pi}\,\frac{\zeta_0 a}{\zeta_0^2+a^2}\,\ell n\left(\frac{\zeta_0^2+a^2+\xi\,e^{j[\psi_1+\psi_2]}+\zeta_0\xi^{1/2}\left[e^{j\psi_1}+e^{j\psi_2}\right]+ja\xi^{1/2}\left[e^{j\psi_1}-e^{j\psi_2}\right]}{\zeta_0^2+a^2+\xi\,e^{j[\psi_1+\psi_2]}+\zeta_0\xi^{1/2}\left[e^{j\psi_1}+e^{j\psi_2}\right]+ja\xi^{1/2}\left[e^{j\psi_2}-e^{j\psi_1}\right]}\right)\\
&\quad+\frac{1}{4\pi}\,\frac{a^2}{\zeta_0^2+a^2}\,\ell n\left(\frac{\zeta_0^2+a^2+\xi\,e^{j2\psi_2}+2\zeta_0\xi^{1/2}e^{j\psi_2}}{\zeta_0^2+a^2+\xi\,e^{j2\psi_1}+2\zeta_0\xi^{1/2}e^{j\psi_1}}\right)
\end{aligned}
\tag{5.18}
$$

In this formula ψ_1 and ψ_2 are obtained from (5.15). Note that (for simplicity)

$$
\psi_2-\psi_1 = 2\arccos\left(\frac{a^2-\Psi_0^2-\xi}{2\Psi_0\xi^{1/2}}\right) \tag{5.19}
$$

with $\psi_1-\psi_2$ progressing from 2π to 0 for increasing ξ with $|\zeta_0|<a$ (inside the aperture), but from 0 to some maximum value and then back to zero for $|\zeta_0|>a$ (outside the aperture).

To better understand this result, let us consider some special cases. At the beginning of the second time interval for ζ_0 within the aperture we have

$$
\begin{aligned}
\psi_2-\psi_1 &= 2\pi \quad \text{(full circular contour)}\\
e_i(\zeta_0,\xi) &= -j\frac{a^2}{\zeta_0^2+a^2} = e_0(\zeta_0)
\end{aligned}
\tag{5.20}
$$

Note that the logarithmic terms are zero. As we should expect the waveform begins the second time interval at the same value as at the end of the first time interval. For ζ_0 outside the aperture we have

$$\begin{aligned} \psi_2 - \psi_1 &= 0 \quad \text{(contour just touching the aperture)} \\ e_i(\zeta_0,\xi) &= 0 \end{aligned} \tag{5.21}$$

At the end of the second time interval for ζ_0 either inside or outside the aperture we have

$$\begin{aligned} \psi_2 - \psi_1 &= 0 \\ e_i(\zeta_0,\xi) &= 0 \end{aligned} \tag{5.22}$$

again the logarithmic terms being zero. We can also recall from (4.15) that on the circular boundary

$$\begin{aligned} \psi_2 - \psi_1 &= \pi \quad , \quad \xi = 0_+ \\ e_i(\zeta_0, 0_+) &= -\frac{j}{2}\frac{a^2}{\zeta_0^2 + a^2} = \frac{1}{2} e_0(\zeta_{0_-}) \end{aligned} \tag{5.23}$$

which is consistent with (5.18)

On the $y = 0$ plane (H plane) the symmetry simplifies the results for the second time interval as (branch cut at $\psi = 0, 2\pi$)

$$\begin{aligned} \phi_0 &= 0 \text{ for } x > 0 \quad , \quad 0 \le \psi_1 \le \pi \quad , \quad 2\pi \ge \psi_2 \ge \pi \\ \psi_1 &= 2\pi - \psi_2 = \arccos\left(\frac{a^2 - x^2 - \xi}{2x\,\xi^{1/2}}\right) \\ e_{i_y}(x,\xi) &= j\, e_i(x,\xi) \\ &= \left[1 + \left[\frac{x}{a}\right]^2\right]^{-1} \left[1 - \frac{\psi_1}{\pi}\right] \\ &- \frac{1}{4\pi}\,\frac{x}{a}\left[1 + \left[\frac{x}{a}\right]^2\right]^{-1} \ell n \left(\frac{x^2 + a^2 + \xi + 2\times\xi^{1/2}\cos(\psi_1) - 2a\xi^{1/2}\sin(\psi_1)}{x^2 + a^2 + \xi + 2\times\xi^{1/2}\cos(\psi_1) + 2a\xi^{1/2}\sin(\psi_1)}\right) \\ &- \frac{1}{2\pi}\left[1 + \left[\frac{x}{a}\right]^2\right]^{-1} \arg\left(x^2 + a^2 + \xi\, e^{-j2\psi_1} + 2x\,\xi^{1/2} e^{-j\psi_1}\right) \end{aligned} \tag{5.24}$$

For negative x one can change ϕ_0 to π or simply recognize from the symmetry

$$e_{i_y}(-x,\xi) = e_{i_y}(x,\xi) \tag{5.25}$$

On the $x = 0$ plane (E plane) the symmetry simplifies the results for the second time interval as (branch cut at $\psi = \pi/2,\ 5\pi/2$)

$$
\begin{aligned}
&\phi_0 = \frac{\pi}{2} \text{ for } y>0 \ , \quad \frac{\pi}{2} \le \psi_1 \le \frac{3\pi}{2} \ , \quad \frac{5\pi}{2} \ge \psi_2 \ge \frac{3\pi}{2} \\
&\psi_1 = 3\pi - \psi_2 = \frac{\pi}{2} + \arccos\left(\frac{a^2 - y^2 - \xi}{2y\xi^{1/2}}\right) = \arcsin\left(\frac{a^2 - y^2 - \xi}{2y\xi^{1/2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
e_{i_y}(jy,\xi) &= je_i(jy,\xi) \\
&= \left[1 - \left[\frac{y}{a}\right]^2\right]^{-1} \left[\frac{3}{2} - \frac{\psi_1}{\pi}\right] \\
&+ \frac{1}{2\pi}\frac{y}{a}\left[1 - \left[\frac{y}{a}\right]^2\right]^{-1} \arg\left(-y^2 + a^2 - \xi - 2y\xi^{1/2}\sin(\psi_1) + ja\xi^{1/2}\cos(\psi_1)\right) \\
&- \frac{1}{2\pi}\left[1 - \left[\frac{y}{a}\right]^2\right]^{-1} \arg\left(-y^2 + a^2 + \xi e^{-j2\psi_1} + j2y\xi^{1/2}e^{-j\psi_1}\right)
\end{aligned}
\tag{5.26}
$$

For negative y one can change ϕ_0 to $-\pi/2$ or simply recognize from the symmetry

$$
e_{i_y}(-jy,\xi) = e_{i_y}(jy,\xi) \tag{5.27}
$$

Other choices of ϕ_0 are also of interest. In particular $\pi/4$ plus integer multiples of $\pi/2$ are of interest for the four-wire feed-arm configuration. As discussed in [4], if two thin feed-arm pairs (including the case of coplanar-plate pairs) are oriented on a circular aperture with their planes at right angles to each other, the two pairs do not interact by symmetry. The fields of each may be added to give the total resulting field. With the fields given as in (5.18) we can construct those for the symmetric four-wire configuration from the two-wire results (superscript 2) via

$$
e_i^{(4)}(\zeta_0,\xi) = \frac{1}{\sqrt{2}}\left[e^{-j\frac{\pi}{4}} e_i^{(2)}\left(e^{j\frac{\pi}{4}}\zeta_0,\xi\right) + e^{j\frac{\pi}{4}} e_i^{(2)}\left(e^{-j\frac{\pi}{4}}\zeta_0,\xi\right)\right] \tag{5.28}
$$

The first term rotates coordinates 45° in a positive sense, accounting for the feed-arm pair at $\pm e^{j\frac{\pi}{4}}a$, and –45° for the other pair. The factor of $1/\sqrt{2}$ normalizes the field to be of unit amplitude in the y direction at the origin. One needs to allow for this in calculating the antenna response for a given voltage applied to the feed arms [7].

For both two-wire and four-wire configurations we need only calculate the fields in the first quadrant ($0 \leq \phi_0 \leq \pi/2$) due to symmetry. The fields are *symmetric* [8, 24] with respect to the $x = 0$ plane (R_x symmetry) giving an extension to the fourth quadrant via

$$\begin{aligned} e_{i_x}(-x + jy, \xi) &= -e_{i_x}(x + jy, \xi) \quad , \quad e_{i_y}(-x + jy, \xi) = e_{i_y}(x + jy, \xi) \\ e_i(-x + jy, \xi) &= -e_i^*(x + jy, \xi) \\ e_i\left(-\zeta^*, \xi\right) &= -e_i^*(\zeta, \xi) \end{aligned} \tag{5.29}$$

They are *antisymmetric* with respect to the $y = 0$ plane (R_y symmetry) giving an extension to the second quadrant via

$$\begin{aligned} e_{i_x}(x - jy, \xi) &= -e_{i_x}(x + jy, \xi) \quad , \quad e_{i_y}(x - jy, \xi) = e_{i_y}(x + jy, \xi) \\ e_i(x - jy, \xi) &= -e_i^*(x + jy, \xi) \\ e_i\left(\zeta^*, \xi\right) &= -e_i^*(\zeta, \xi) \end{aligned} \tag{5.30}$$

Applying both of the above ($C_{2a} = R_x \otimes R_y$ symmetry) gives an extension to the third quadrant via

$$e_i(-\zeta, \xi) = e_i(\zeta, \xi) \tag{5.31}$$

This can also be considered as two-dimensional inversion symmetry.

6. Jumps in Waveform as Contour Sweeps by Feed-Wire Projections

There are some peculiarities in the formulae for the intermediate field as the contour end points sweep past the wire projections where they touch S_a. For the observer in the first quadrant of the ζ_0 plane this happens first for ψ_1 when $\zeta \to ja$. From (5.15) and (5.18) this occurs when

$$\zeta_0 - ja + \xi^{1/2} e^{j\psi_1} \to 0 \tag{6.1}$$

Referring to fig. 5.1 we can see that the argument of the logarithm (denominator in first form in (5.18), noting the form in (6.1)) changes from 0 to π, noting that the contour passes *under* the line charge at $\zeta = ja$. This gives a jump in the intermediate field as

$$\Delta_i e_i(\zeta_0, \xi) = \frac{1}{4} \frac{a}{\zeta_0 - ja} = \frac{1}{4} \left[\frac{\zeta_0}{a} - j\right]^{-1} \tag{6.2}$$

near the origin this is $\simeq j/4$ which is a decrease noting the minimum sign with e_{i_y} in the complex field. Here Δ_1 is the change as value after minus value before. There is a logarithmic singularity from the log magnitude as well, but this is integrable without discontinuity in the integral.

Similarly when ψ_2 passes $\zeta = -ja$ we have

$$\zeta_0 + ja + \xi^{1/2} e^{j\psi_2} \to 0 \tag{6.3}$$

Now the contour passes above $\zeta = -ja$ with the argument of the logarithm (numerator in first form in (5.18), noting the form of (6.3)) changing from 0 to π. This gives a jump in the intermediate field as

$$\Delta_2 e_i(\zeta_0, \xi) = -\frac{1}{4} \frac{a}{\zeta_0 + ja} = \frac{1}{4}\left[-\frac{\zeta_0}{a} - j\right]^{-1} \tag{6.4}$$

Again near the origin this is $\simeq j/4$ which is also a decrease in e_{i_y}. There is again an integrable logarithmic singularity.

Curiously enough adding the two changes gives

$$
\begin{aligned}
\Delta_1\, e_i(\zeta_0,\xi) + \Delta_2\, e_i(\zeta_0,\xi) &= \frac{a}{4}\left[[\zeta_0 - ja]^{-1} - [\zeta_0 + ja]^{-1}\right] \\
&= \frac{j}{2}\,\frac{a^2}{\zeta_0^2 + a^2} = -\frac{1}{2}\,e_0(\zeta_0) \quad \text{for } \zeta_0 \text{ in aperture projection}
\end{aligned} \tag{6.5}
$$

which, in the aperture projection, accounts for one half of the initial fields at the observer (first time interval).

7. Concluding Remarks

So here we have some interesting ways to evaluate the intermediate field. The contour integrals for thin-wire TEM aperture distributions can be expressed in closed form. Other types of TEM fields are also of interest (e.g., for coplanar plates). So there is much that can be done to exploit the present results. Furthermore, the aperture need not be a circular disk (e.g., a semi-circular disk, as in a half IRA).

Here we have assumed that the TEM fields on the aperture are propagating parallel to the aperture normal. As discussed in [3] this need not be the case. One can assume that the aperture fields take the form of a TEM plane wave propagating in some direction, say $\vec{1}_0$, into the positive half space $(z > 0)$. Then by taking the asymptotic expansion in the $\vec{1}_0$ direction (instead of z), one might expect to obtain similar results.

Note that the results here apply to only one part of the waveform, the "impulsive" part. For a reflector IRA one needs to include the prepulse associated with the TEM field. Then there is also what follows the impulse (the postpulse).

Appendix A. Laplace/Fourier Transform of the Intermediate Field

In Section 2, we have found that the intermediate field is expressible as

$$\vec{e}_i(x,y;\xi) \equiv \text{intermediate field}$$
$$\xi = 2czt_r \tag{A.1}$$
$$t - \frac{z}{c} \equiv \text{retarded time (referred to the } z \text{ direction)}$$

The two-sided Laplace transform is expressed as

$$s = \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency}$$
$$\sim \equiv \text{Laplace transform (two-sided)}$$
$$\begin{aligned}\tilde{\vec{e}}_i(x,y;\Sigma) &= \int_{-\infty}^{\infty} \vec{e}_i(x,y;\xi)\, e^{-st_r} dt_r \\ &= \frac{1}{2cz} \int_{-\infty}^{\infty} \vec{e}_i(x,y;\xi)\, e^{-\Sigma\xi} d\xi\end{aligned} \tag{A.2}$$
$$\Sigma = \sigma + j\nu = \frac{s}{2cz} \equiv \text{normalized complex frequency (units m}^{-2}\text{)}$$

Here we have used the retarded time as the time variable to avoid a large phase shift (phase wrapping) for large z.

The important thing here is that the scaling in time domain represented by the parameter ξ goes over to a similar scaling in frequency domain via the parameter Σ. As we can see, if we double z the waveform has the same amplitude but is squeezed into half the time. In frequency domain this corresponds to a doubling of the frequency, but dividing the value of the spectrum by 2 (from the z^{-1} coefficient). So what one better calculates is

$$\tilde{\vec{e}}_i^{(0)}(x,y;\Sigma) = 2cz\,\tilde{\vec{e}}_i(x,y;\Sigma) = \int_{-\infty}^{\infty} \vec{e}(x,y;\Sigma)\, e^{-\Sigma\xi} d\xi \tag{A.3}$$

This then applies to any z for a given (x, y) and can be calculated once and scaled as above.

For observers in the aperture extension we have derived several different ways (Sections 3 and 4) the result

$$\vec{e}_i(x,y;0_+) = \vec{e}_0(x,y) \tag{A.4}$$

Noting that this is initially a step function we have

$$\begin{aligned} \tilde{\vec{e}}_i^{(0)}(x,y;\Sigma) &= \frac{1}{\Sigma} = \frac{2cz}{s} \quad \text{as } s \to \infty \text{ in RHP} \\ \tilde{\vec{e}}_i(x,y;\Sigma) &= \frac{1}{s} \quad \text{as } s \to \infty \text{ in RHP} \end{aligned} \tag{A.5}$$

We can let s go to the $j\omega$ axis (on the way to $\pm j\infty$) provided there are no further discontinuities (steps) later in time in the waveform. Lower order discontinuities are allowed since they give contributions that go to zero faster than $1/s$.

These results can also be applied to the complex form of the intermediate field as in Section 4. However, the Laplace transform also gives a complex spectrum which can be confused with the field-component combination in the complex field. This can be sorted out by considering s and s^* together (or $j\omega$ and $-j\omega$) and taking sum and difference to give the transforms of the two field components.

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