Temporal Symmetries in Lossless Linear Networks that Efficiently Transport or Transform Electrical Energy

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Abstract

Pulsed power applications generally involve rapid transport or transformation of electrical energy. They often start from storage in a pure state in which energy is stored in the electric fields in capacitors or the magnetic fields in inductors. This energy may be rapidly moved and reconfigured into another pure state of either electric field energy or magnetic field energy. This work presents formal proofs of theorems that govern the temporal symmetries of voltage and current waveforms in all lossless linear networks used to achieve efficient energy transport and transformation.

I. Introduction

Attempts have been made in recent years to improve the energy transfer efficiency from complex Marx generator networks into external capacitive loads using parametric optimization [1]. This involved the timing of switches, the tuning of internal resonances, and adjustment of parasitic circuit elements. When good but not necessarily perfect results were achieved, current and voltage waveforms showed a tendency toward nearly periodic behavior. This and other symmetries were later found to be necessary conditions for achieving total energy transfer [2]. Control theory and the gradient vector method were applied to the assignment of network resonant frequencies [1], [2], [3], [4], [5]. More recently, spatial and temporal symmetries were utilized to achieve un-dispersed energy and charge transport on lumped-element transmission lines [6]. The final step in solving such problems involves the assignment of network resonant frequencies to specific harmonics of the fundamental mode frequency. Ironically, these inherently time domain problems are transformed into a problem in the frequency domain.

The foundation of these energy transformation optimizations is four periodicity theorems which are presented below. They involve the four possible transformations of pure electric or magnetic field energy into another configuration of either pure electric or magnetic field energy. This paper offers a expanded version of earlier work with improvements and typographical corrections. (Appendix I of [2]).

II. Waveform Symmetries

Consider an arbitrary lossless linear network containing only lossless capacitors C_k and inductors L_k. The distribution of energy in such a network will evolve in time. It will consist of a conserved sum of changing electric and magnetic field energies. If at two distinct times the total energy is contained in either purely electric fields or purely magnetic fields, then certain temporal symmetries in all voltage and current waveforms will result. All measurable quantities will be periodic in time and will exhibit even or odd time-reversal symmetry at specific times. The proof of this conjecture will now be presented.

We restrict our attention to networks in which the voltage across the k'th capacitor $V_k(t)$ and current in the k'th inductor $I_k(t)$ may be described by the coupled first-order differential equations

$$C_k \frac{d}{dt} V_k(t) = \sum_j a_{kj} I_j(t), \text{ and } L_k \frac{d}{dt} I_k(t) = \sum_j b_{kj} V_j(t), \quad (1)$$

where a_{kj} and b_{kj} are dimensionless network-specific constants that will depend upon relative capacitor and inductor values. Some of these equations may be redundant because of variables being interrelated. This occurs when there are capacitive loops or inductors tied to isolated nodes. The coefficients can be found by the repeated application of the Thevenin-Norton theorems. Equations (1) may be further expressed in a manner that incorporates the initial conditions, $V_k(0)$ and $I_k(0)$, as the first-order coupled integral equations

$$V_k(t) = V_k(0) + C_k^{-1} \sum_j a_{kj} \int_0^t dt'\, I_j(t'), \text{ and} \quad (2)$$

$$I_k(t) = I_k(0) + L_k^{-1} \sum_j b_{kj} \int_0^t dt'\, V_j(t'). \quad (3)$$

The voltages $V_k(t)$ and currents $I_k(t)$ may be expressed as the sums of their even components, $V_k^+(t)$ and $I_k^+(t)$, and their odd components, $V_k^-(t)$ and $I_k^-(t)$, according to

$V_k(t)=V_k^+(t)+V_k^-(t)$, and $I_k(t)=I_k^+(t)+I_k^-(t)$, where
$V_k^+(t)=[V_k(t)+V_k(-t)]/2$, $V_k^-(t)=[V_k(t)-V_k(-t)]/2$,
$I_k^+(t)=[I_k(t)+I_k(-t)]/2$, and $I_k^-(t)=[I_k(t)-I_k(-t)]/2$. (4)

Equations (2) and (3) may then be written in terms of the even and odd variables of voltage and current defined in (4).

$$V_k^+(t)+V_k^-(t)=V_k(0)+C_k^{-1}\sum_j a_{kj}\int_0^t dt'\left[I_j^+(t')+I_j^-(t')\right] \quad (5)$$

$$I_k^+(t)+I_k^-(t)=I_k(0)+L_k^{-1}\sum_j b_{kj}\int_0^t dt'\left[V_j^+(t')+V_j^-(t')\right] \quad (6)$$

The integral of an odd function of time can generate only an even function of time and vise versa. Therefore, (5) and (6) can be separated into four separate coupled equation sets in which the even voltages and odd currents are decoupled from the odd voltages and even currents.

$$V_k^+(t)=V_k(0)+C_k^{-1}\sum_j a_{kj}\int_0^t dt'\, I_j^-(t'), \text{ with} \quad (7)$$

$$I_k^-(t)=L_k^{-1}\sum_j b_{kj}\int_0^t dt'\, V_j^+(t'), \text{ and} \quad (8)$$

$$V_k^-(t)=C_k^{-1}\sum_j a_{kj}\int_0^t dt'\, I_j^+(t'), \text{ with} \quad (9)$$

$$I_k^+(t)=I_k(0)+L_k^{-1}\sum_j b_{kj}\int_0^t dt'\, V_j^-(t') \quad (10)$$

A. Case I: No Initial or Final Currents

Combining (9) and (10) and setting all $I_k(0)=0$, the second-order coupled integral equations for the odd voltage components $V_k^-(t)$ become

$$V_k^-(t)=C_k^{-1}\sum_{i,j} a_{kj}b_{ji}L_j^{-1}\int_0^t dt'\int_0^{t'} dt''\, V_i^-(t''). \quad (11)$$

From the definition in (4) we see that all of the odd $V_k^-(0)$'s are zero. With no initial odd voltages there can be no seed to initiate the growth of the $V_k^-(t)$'s. The only possible solution to (11) is therefore $V_k^-(t)=0$. Applying this result in (10) gives $I_k^+(t)=0$. Thus, from (4) it follows that all voltages must be even in time, and all currents must be odd in time about $t = 0$.

$$V_k(t)=V_k^+(t)=V_k(-t), \text{ and } I_k(t)=I_k^-(t)=-I_k(-t). \quad (12)$$

Suppose that in addition to $t=0$ there is another time, say $t=\tau/2$, when all currents $I_k(\tau/2)=0$. Then all voltages $V_k(t)$ and currents $I_k(t)$ must exhibit the same time reversal symmetry as in (12) about the time $t = \tau/2$ as well, i.e.

$$V_k(\tau/2+t)=V_k(\tau/2-t),\ I_k(\tau/2+t)=-I_k(\tau/2-t). \quad (13)$$

Thus, all voltages are even in time, and all currents are odd in time about both $t = 0$ and $t = \tau/2$.

We may now substitute $t \rightarrow t + \tau/2$ in (13) and combine the result with (12) to get

$$V_k(t+\tau)=V_k(-t)=V_k(t),\ I_k(t+\tau)=-I_k(-t)=I_k(t). \quad (14)$$

It is now possible to show by induction that all waveforms are periodic in time with period τ. Suppose it is true that for some integer n we have

$$V_k(t)=V_k(t+n\tau), \text{ and } I_k(t)=I_k(t+n\tau). \quad (15)$$

We may now substitute $t \rightarrow t \pm \tau$ into (15) and combine the result with (14) to see that (15) will be true for $n\pm1$ as well. Therefore, (15) is valid for all integers n, and all current and voltage waveforms will be periodic in time with period τ.

It also follows from (15) that all waveforms must have the same time-inversion symmetries as evidenced in (12) and (13) at all integral multiples of $\tau/2$.

$$V_k(n\tau/2+t)=V_k(n\tau/2-t),$$
$$I_k(n\tau/2+t)=-I_k(n\tau/2-t) \quad (16)$$

B. Case II: No Initial or Final Voltages

Combining (7) and (8) and setting all $V_k(0)=0$, the second-order coupled integral equations for the odd current components $I_k^-(t)$ become

$$I_k^-(t)=L_k^{-1}\sum_{i,j} a_{ji}b_{kj}C_j^{-1}\int_0^t dt'\int_0^{t'} dt''\, I_i^-(t'') \quad (17)$$

From the definition in (4) we see that all of the odd $I_k^-(0)$'s are zero. With no initial odd currents there can be no seed to initiate the growth of the $I_k^-(t)$'s. The only possible solution to (17) is therefore $I_k^-(t)=0$. Applying this result in (7) gives $V_k^+(t)=0$. Thus, from (4) it follows that all voltages must be odd in time, and all currents must be even in time about $t = 0$.

$$V_k(t)=V_k^-(t)=-V_k(-t), \text{ and } I_k(t)=I_k^+(t)=I_k(-t). \quad (18)$$

Suppose that in addition to $t=0$ there is another time, say $t=\tau/2$, when all voltages $V_k(\tau/2)=0$. Then all voltages $V_k(t)$ and currents $I_k(t)$ must exhibit the same time reversal symmetry as in (18) about the time $t=\tau/2$ as well, i.e.

$$V_k(\tau/2+t)=-V_k(\tau/2-t),\ I_k(\tau/2+t)=I_k(\tau/2-t). \quad (19)$$

Thus, all voltages are odd in time, and all currents are even in time about both $t=0$ and $t=\tau/2$.

We may now substitute $t \to t + \tau/2$ in (19) and combine the result with (18) to get

$$V_k(t+\tau) = -V_k(-t) = V_k(t),\ I_k(t+\tau) = I_k(-t) = I_k(t). \qquad (20)$$

It is now possible to show by induction that all waveforms are periodic in time with period τ. Suppose it is true that for some integer n we have

$$V_k(t) = V_k(t+n\tau), \text{ and } I_k(t) = I_k(t+n\tau). \qquad (21)$$

We may now substitute $t \to t \pm \tau$ in (21) and combine the result with (20) to see that (21) will be true for $n\pm1$ as well. Therefore, (21) is valid for all integers n, and all current and voltage waveforms will be periodic in time with period τ.

It also follows from (21) that all waveforms must have the same time-reversal symmetries as evidenced in (18) and (19) at all integral multiples of $\tau/2$.

$$\begin{aligned} &V_k(n\tau/2+t) = -V_k(n\tau/2-t) \\ &I_k(n\tau/2+t) = I_k(n\tau/2-t) \end{aligned} \qquad (22)$$

C. Case III: No Initial Currents or Final Voltages

The initial conditions here are identical with Case I. Therefore, all voltages must be even in time, and all currents must be odd in time about $t = 0$, as was the case in (12).

$$V_k(t) = V_k(-t), \text{ and } I_k(t) = -I_k(-t). \qquad (23)$$

Suppose that at a later time $t=\tau/4$ there is a condition when all voltages $V_k(\tau/4)=0$. This is the same as the initial conditions in *Case II*. Therefore all voltages $V_k(t)$ and currents $I_k(t)$ must exhibit the same time-reversal symmetry about this time $t=\tau/4$ as was the case in (18), i.e.

$$V_k(\tau/4+t) = -V_k(\tau/4-t),\ I_k(\tau/4+t) = I_k(\tau/4-t). \qquad (24)$$

Thus, all voltages are odd in time and all currents are even in time around $t=\tau/4$, which is the reverse of (23).
Substituting $t \to t + \tau/4$ in (24) and applying (23) gives

$$\begin{aligned} &V_k(t+\tau/2) = -V_k(-t) = -V_k(t), \text{ and} \\ &I_k(t+\tau/2) = I_k(-t) = -I_k(t). \end{aligned} \qquad (25)$$

Thus, all voltages and currents undergo a sign reversal at times differing by $\tau/2$. Applying this result to (24) gives

$$\begin{aligned} &V_k(3\tau/4+t) = -V_k(\tau/4+t) = V_k(\tau/4-t) = -V_k(3\tau/4-t), \\ &I_k(3\tau/4+t) = -I_k(\tau/4+t) = -I_k(\tau/4-t) = I_k(3\tau/4-t). \end{aligned} \qquad (26)$$

Thus, all voltages are odd in time and all currents are even in time around $t=3\tau/4$, as they were around $t=\tau/4$ in (24). Now substituting $t \to t + \tau/4$ in (26) gives

$$\begin{aligned} &V_k(t+\tau) = -V_k(t+\tau/2) = V_k(t), \text{ and} \\ &I_k(t+\tau) = -I_k(t+\tau/2) = I_k(t). \end{aligned} \qquad (27)$$

Once again it can be shown by induction that all waveforms are periodic in time with period τ. Suppose it is true that for some integer n we have

$$V_k(t) = V_k(t+n\tau), \text{ and } I_k(t) = I_k(t+n\tau). \qquad (28)$$

We may now substitute $t \to t \pm \tau$ in (28) and combine the result with (27) to see that (28) will be true for $n\pm1$ as well. Therefore, (28) is valid for all integers n, and all current and voltage waveforms will be periodic in time with period τ.

It also follows from (23) through (28) that even or odd time-inversion symmetries occur at four distinct times per period τ.

$$\begin{aligned} &V_k(n\tau/2+t) = V_k(n\tau/2-t), \\ &I_k(n\tau/2+t) = -I_k(n\tau/2-t), \\ &V_k(n\tau/2+\tau/4+t) = -V_k(n\tau/2+\tau/4-t), \\ &I_k(n\tau/2+\tau/4+t) = I_k(n\tau/2+\tau/4-t), \end{aligned} \qquad (29)$$

where n is any integer. Also, voltages and currents undergo sign reversals at odd multiples of $\tau/2$.

$$V_k(n\tau/2+t) = (-1)^n V_k(t),\ I_k(n\tau/2+t) = (-1)^n I_k(t) \qquad (30)$$

D. Case IV: No Initial Voltages or Final Currents

This case is simply the equivalent of Case III time shifted by $\tau/4$. Waveforms will be periodic in time with period τ, as they were in (28). For this case, the final result of (29) may be derived by inspection to be

$$\begin{aligned} &V_k(n\tau/2+t) = -V_k(n\tau/2-t), \\ &I_k(n\tau/2+t) = I_k(n\tau/2-t), \\ &V_k(n\tau/2+\tau/4+t) = V_k(n\tau/2+\tau/4-t), \\ &I_k(n\tau/2+\tau/4+t) = -I_k(n\tau/2+\tau/4-t), \end{aligned} \qquad (31)$$

where n is any integer. Also, voltages and currents undergo sign reversals at even multiples of $\tau/2$.

$$V_k(n\tau/2+t) = -(-1)^n V_k(t),\ I_k(n\tau/2+t) = -(-1)^n I_k(t) \qquad (30)$$

This concludes the derivations for the four types of electrical energy transformation and transport.

III. Conclusions

The main results of the four theorems presented here may be summarized in the following way. About any time when all currents are zero, these same currents will be odd in time, and all voltages will be even in time. About any time when all voltages are zero, these same voltages will be odd in time, and all currents will be even in time. If either all currents or all voltages are zero at two different times, then all current and voltage waveforms will be periodic in time. This periodicity is illustrated in the flow diagram of Fig. 1. The period τ is shown for the four possible paths.

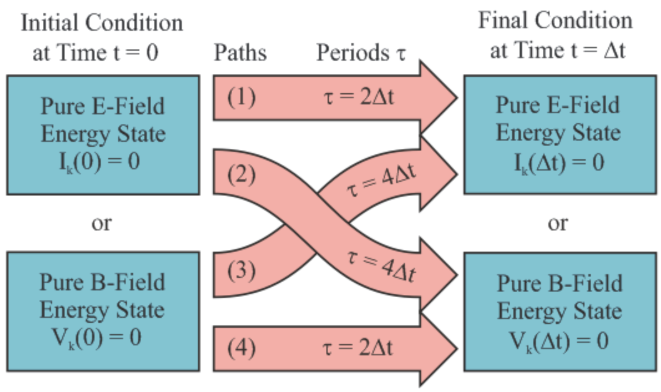

Figure 1. Flow diagram for lossless linear networks that transport or transform electrical energy. All current and voltage waveforms are periodic in time with period τ.

It must be stated that the four theorems presented in this paper apply only to linear systems and to time intervals between switching events. Such switching events are nonlinear processes. The periodicity and time-reversal symmetries apply to conditions that assume the switches remain unchanged outside of that time interval. The theorems presented here provide a starting point for designing real-world systems where compromises and approximations will be prudent. Modes of oscillation that receive minimal excitation may be ignored. Their resonant frequencies may be unconstrained, thus allowing less than perfect yet practical solutions with fewer variables [2]. When losses are introduced or encountered in a network, the starting point will still be the lossless solutions. Further parametric optimization of the ideal network solutions may lessen these losses [6].

IV. References

[1] C. J. Buchenauer, “Optimizing Compact Marx Generator Networks,” in *Proc. 17th IEEE International Pulsed Power Conference*, June 28 – July 1, 2009, pp. 357–362.

[2] C. J. Buchenauer, “Optimizing Compact Marx Generator Networks,” *IEEE Transactions on Plasma Science: Special Issue on Pulsed Power Science and Technology*, vol. 38, no. 10, pp. 2771–2784, Oct. 2010.

[3] L. Zaccarian, S. Galleani, C. T. Abdallah, E. Schamiloglu, and C. J. Buchenauer, “A Control Theory Approach on the Design of a Marx Generator Network,” in *Proc. 17th IEEE International Pulsed Power Conference*, June 28 – July 1, 2009, pp. 363–367.

[4] A. Verzi, “Algoritmi Numerici per la Progettazione de Generatori di Marx,” Thesis, 2009, Dipartimento di Informatica, Sistemi e Produzione, Universita' di Roma, Tor Vergata

[5] S. Galeani, D. Henrion, A. Jacquemarde, and L. Zaccarian, “Design of Marx Generators as a Structured Eigenvalue Assignment,” *Automatica*, 50, pp. 2709-2717, June 2014.

[6] C. J. Buchenauer, “Dispersionless Charge Transfer on Lumped-Element Transmission Lines Synthesized via Resonant Frequency Assignment,” in *Proc. 20th IEEE International Pulsed Power Conference*, May 31 – June 4, 2015, (to be published).