System Design and Assessment Notes

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SURVIVABLE COMMUNICATION NETWORKS WITH NON-DIRECTED AND DIRECTED GRAPHS

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ABSTRACT

Ad-hoc networks such as Mobil Ad-hoc Networks (MANETs) are comprised of mobile and typically random nodes that have enough power to originate, receive, and relay packets by multi-hop transport. The nodes are randomly positioned and their statistical properties (e.g. average concentration) are space and time dependent. It is necessary to demonstrate their resilience against enemy attack by showing that connectedness is maintained at a sufficient throughput. In this study we examine the issue of connectedness for MANETS comprised of both non-directed and directed graphs. A new formula for the percolation threshold for a network with arbitrary fractions of non-directed and directed links is derived.

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1. INTRODUCTION

The term "survivable communications" was created in the Cold War where it was necessary to ensure that extremely low data rate Emergency Action Messages (EAMs) could reach the critical command posts around the world in a timely way. **Figure 1** shows a generic survivability model used for strategic communications in the Cold War. The network is comprised of communication links (solid black lines) and nodes (black dots). The critical concern was to achieve the most basic communications between the command posts.

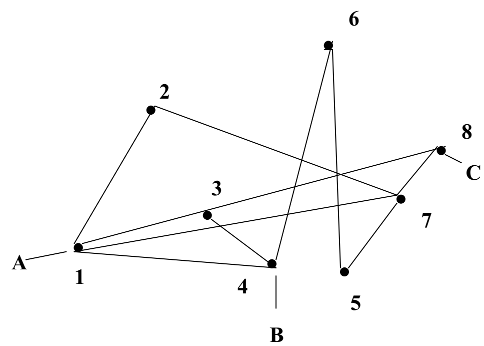

Figure 1 Survivable communications model

The nodes in **Figure 1** are labeled 1 to 8 and the command posts are labeled A, B, and C. There are five communication paths between A and C. These paths have some links in common.

path 1: 1-2-7-8
path 2: 1-3-8
path 3: 1-7-8
path 4: 1-4-6-5-7-8
path 5: 1-3-4-6-5-7-8

We assume all the links survive so that survivability is determined only by the survivability of the nodes. Now let A_i be the event that node "i" survives, and E_μ be the event that path μ exists: $E_\mu = A_\alpha A_\beta ... A_\lambda$, where $A_\alpha A_\beta ... A_\lambda$ is the intersection of the events that comprise path μ. The probability that path μ exists is $P[E_u] = P[A_\alpha A_\beta ... A_\lambda]$. For statistically independent events, $P[A_\alpha A_\beta ... A_\lambda] = P[A_\alpha]P[A_\beta].....P[A_\lambda]$. Using basic probability theory the general formula for computing the probability of successful communication between any two command posts for L possible paths connecting them is **[1]**

$$P[\bigcup_{\mu=1}^{L} E_\mu] = \sum_{\mu=1}^{L} P[E_\mu] - \sum_{\mu=1}^{L-1} \sum_{\eta=\mu+1}^{L} P[E_\mu E_\eta]........ + ...(-1)^{L+1} P[E_1 E_2 ....E_L] \tag{1.1}$$

The foregoing formulation was used successfully to evaluate communication survivability of the NATO networks. This approach was practical because the number of paths connecting the relatively small number of command posts was small.

MANETs are extremely different from Cold War networks. It is computationally impractical to use equation (1) to compute the probability of connectedness for MANETs because of their large number of nodes. By considering a MANET to be modeled as a random graph Kohlberg proposed **[2-6]** that survivability of MANETs should be measured by its ability to support a network-spanning cluster—being able to communicate over most of the physical size of the network with only a fraction of the nodes surviving. In contrast to the initial random graph model of Erdos and Renyi **[7]** in which all nodes can be connected to one another, for MANETs a typical node is only connected to a few other nodes. Destroying a fraction of the nodes reduces the information rate, and if bad enough can cause the network to break into isolated clusters.

Today's ad-hoc networks may have hundreds to thousands of nodes and each node can have many links to other nodes. It is typically more important for a node to communicate to another node than to touch base with a central node. **Figure 2** depicts a hypothetical ad-hoc network for nearest neighbor communications and **Figure 3** shows a network spanning cluster of that network after attack. It is necessary to define the mathematical structure of MANETs under these conditions and the mathematical tools that are necessary to determine their survivability. Starting from this modest basis that each node had a probability of surviving, *p*, and there are *k* links connected to it one can compute the probability that the network will form a "network spanning cluster". This cluster of nodes allows end-to-end communication at a minimum rate and thereby establishes a baseline level of survivability.

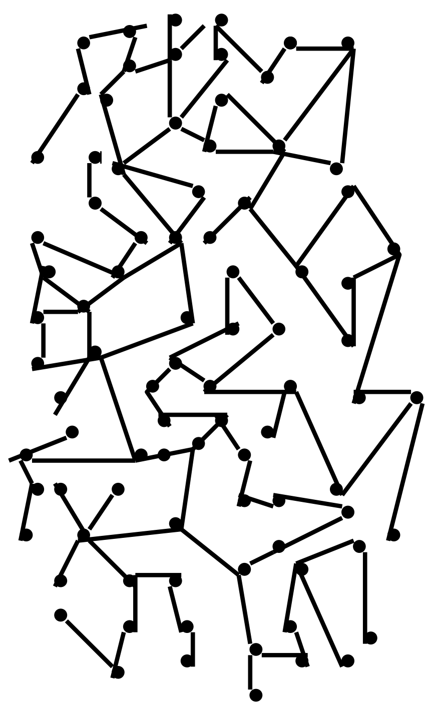

Figure 2 Hypothetical MANET

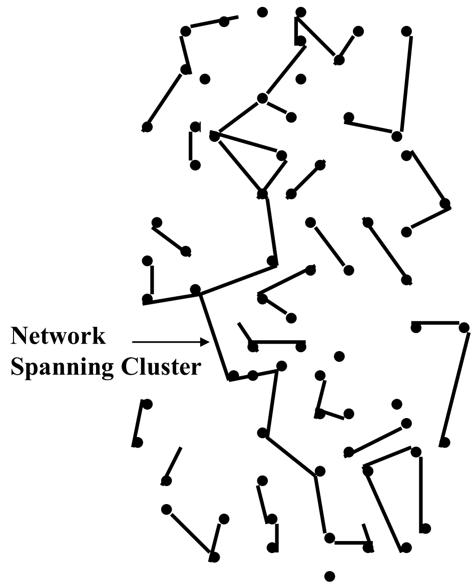

Figure 3 Network spanning cluster after attack

What may be of great interest here is not the issue of information rate, but rather the ability of the network to communicate at any information rate over a significant extent—"a connected cluster of sites that spans the entire network even for an arbitrarily large fraction of sites, f , that are randomly removed". For this case the fraction of nodes that survive is $p_s = 1 - f$ which must be equal to or greater than p_c , the percolation threshold of the network—the critical probability that a giant cluster forms. When $p_s \geq p_c$ a giant cluster appears that spans the entire network, but below p_c the network is composed of isolated clusters. We propose that a measure of survivability is p_c , the percolation threshold. The lower the value of p_c the larger the fraction of nodes that have to be removed before the network collapses.

2. Review and Summary of Survivability using Random Graphs and Percolation Theory

Let's examine the consequences of removing node 7 from **Figure 1**. The network now looks like **Figure 4**, which as observed changes dramatically from **Figure 1**.

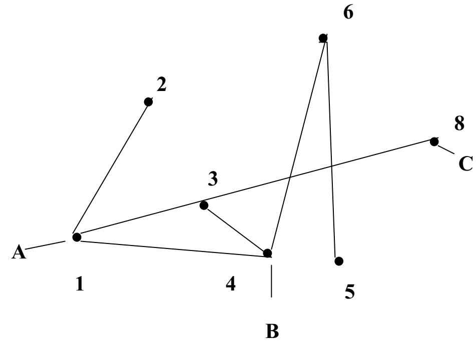

Figure 4 Removal of node 7

There's now only one path connecting A and C, and the bulk of the nodes are connected in a useless sub-network. This situation can be remedied by increasing the interconnections between the nodes. For a network consisting of N nodes each node can be connected to $N-1$ other nodes and the total number of links, N_L is

$$N_L = \frac{1}{2}N(N-1) \tag{2.1}$$

Equation (2.1) applies to a fully connected network. The total number of connections to the nodes is $2N_L$ since each link connects two nodes. The removal of a single node in a fully connected network is easily tolerated.

For a given bandwidth the maximum information rate decreases as the number of nodes increases [**8**], producing a tradeoff between survivability and communications capability that can be addressed from a random graph and percolation viewpoint **[9]**. Now imagine a very large network that has a top and a bottom, and allow communications to occur with any other node in the system. For a desired information rate: What is the minimum connectedness required that ensures end-to-end communication? Alternatively, what is the dependence of the percolation threshold on the Signal-to-Noise/Interference Ratio (SINR)?

Random graph theory starts from a fully connected network; it demonstrates the conditions that must exist between the number of nodes and the number of links in order to generate a network spanning cluster, and provides estimates of the cluster size distribution. In particular, it shows that if a network spanning cluster is not achieved the remaining nodes are minimally connected to one another **[9]**. Now assume that for every link there is an independent probability, p, for that connection to actually exist. The smaller the value of p the fewer links are available to support network communication. In general, p can be a function of N. The average number of links, n, is then

$$n = pN_L \tag{2.2}$$

The number of distinguishable different possible networks is then

$$C = \frac{N_L!}{n!(N_L - n)!} \tag{2.3}$$

Albert and Barabasi examine three classes of random networks in which N is very large **[9]**:

a) $$p < \frac{1}{N} \equiv \tilde{p}_c \tag{2.4a}$$

Network spanning cluster will *not* form. Nodes group in tree structures. The largest tree has $\ln N$ nodes.

b) $$p = \frac{1}{N} = \tilde{p}_c \tag{2.4b}$$

Network spanning cluster appears that includes $N^{2/3}$ nodes.

$$\text{c)} \quad p > \frac{1}{N} = \tilde{p}_c \tag{2.4c}$$

This is the network spanning regime, our region of interest, where the network spanning cluster grows rapidly.

Now consider a communication system that has N nodes and n links before interference and/or noise is imposed on it. Such a network has a value

$$p = \frac{n}{N_L} = \frac{2n}{N(N-1)} \tag{2.5}$$

By comparing equation (2.5) with equation (2.4c) we determine whether the network is totally connected and can support a network spanning cluster. For example, if $p < \tilde{p}_c$, it means that there aren't even enough links to connect all the nodes together in series. Communication systems of interest are in the domain: $p > \tilde{p}_c$. In this regime the degree (number of edges connected to a node) distribution for large N with a random selection of supporting links is often given by the Poisson distribution

$$P(k) = \exp(-pN)\frac{(pN)^k}{k!} \tag{2.6}$$

The average number of edges is

$$\langle k \rangle \cong \sum_0^\infty kP(k) = pN\,, \tag{2.7}$$

and

$$\langle k^2 \rangle = \sum_{k=0}^{k=k_m} k^2 P(k) = \langle k \rangle^2 + \langle k \rangle \tag{2.8}$$

While random graph theory does provide much insight into the structure of the network in the sub-network spanning cluster regime it does not provide quantitative information to determine the thresholds of network spanning clusters as a function of p or N in the $p > \tilde{p}_c$ regime. Despite its limitations random graph theory can provide an estimate of a network's tolerance to noise and interference. If

$$Np = \frac{2n}{(N-1)} > \tilde{p}_c \tag{2.9}$$

a network spanning cluster will likely exist in the limit of large N. Suppose now that interference reduces n to n' but we again have the condition

$$Np' = \frac{2n'}{(N-1)} > \tilde{p}_c \tag{2.10}$$

Equation (2.10) shows that our network is tolerant to a fractional decrease in available links *h*, given by

$$h = \frac{n - n'}{n} \tag{2.11}$$

It is instructive to now show how random graph theory can be applied in network communication conditions. For a fixed bandwidth, *W*, assume that the capacity per node, *T*, diminishes as the number of nodes in the network increases. A generic model is

$$T = CWN^{-\alpha}, \tag{2.12}$$

where $\alpha > 0$ and *C* is a constant. Suppose we require

$$CWN^{-\alpha} \geq T_0 \tag{2.13}$$

We then have

$$N \leq \left(\frac{CW}{T_0} \right)^{1/\alpha} \equiv N_0 \tag{2.14}$$

As long as

$$pN_0 > 1 \tag{2.15}$$

and N_0 is large enough the network will function as desired and be capable of forming a network spanning cluster. On the other hand, if

$$pN_0 < 1 \tag{2.16}$$

a network spanning cluster cannot be formed. We can also address the question of connectedness by examining its limiting behavior as a function of the ratio: W / T_0. Improvements in assessing the effect of information flow on connectedness can be better handled by percolation theory, which has been applied to flows generally found in the physical world of fluids, particle transport, electrical conductivity, etc. in a random media.

The starting point for percolation theory is the normalized degree distribution function, $P(k)$, the number of edges connected to a node. We have

$$1 = \sum_{k=2}^{k=k_{\max}} P(k) \tag{2.17}$$

where $k_{\max}$ is the maximum number of edges considered. Cohen et al **[10]** have shown that a network spanning cluster will be formed with a fraction, p_c, of the nodes given by

$$p_c = \frac{1}{K-1} \tag{2.18}$$

$$K = \frac{\langle k^2 \rangle}{\langle k \rangle} \tag{2.19}$$

$$\langle k \rangle = \sum_{k=0}^{k=k_{max}} kP(k) \tag{2.20}$$

$$\langle k^2 \rangle = \sum_{k=0}^{k=k_{max}} k^2 P(k) \tag{2.21}$$

Whereas random graph theory provides rough estimates for predicting the conditions of a network spanning cluster, percolation theory provides a more precise criterion. Potrykus and Kohlberg **[4]** applied the percolation theory approach to a wireless ad-hoc network in a square area *A* of side *D*. Their inter-node range for wireless communication is *R,* spatial density of nodes is n_0, and total number of nodes $N = An_0 = D^2 n_0$ is large. The dimensions *A* and *D* are very large so that boundary effects are not significant, but that the ratio, n_0, is finite and $R<<D$. Assuming independent Bernoulli trials of successfully dropping nodes within a selected area of size πR^2 with probability ($\pi R^2 / D^2$) centered on any node generates the Poisson distribution of equation (2.6) for $P(k)$. We then get

$$K = \frac{\langle k^2 \rangle}{\langle k \rangle} = \langle k \rangle + 1 \tag{2.22}$$

$$p_c = \frac{1}{K-1} = \frac{1}{\langle k \rangle + 1 - 1} = \frac{1}{\langle k \rangle} \tag{2.23}$$

Using their basic physical model Potrykus and Kohlberg showed

$$\langle k \rangle = \beta R^2 n_0 = \beta \left(\frac{R}{D} \right)^2 N \tag{2.24}$$

$$p_c = \frac{1}{\beta R^2 n_0} = \frac{D^2}{\beta R^2 N} \tag{2.25}$$

where β is a dimensionless constant of order unity. Since the largest value of p_c is unity, equation (2.25) only makes sense when

$$N > \frac{D^2}{\beta R^2} \tag{2.26}$$

The Gupta-Kumar theory showed that if large information rates per node are required then ($W/N^{1/2}$) must be greater than a certain amount. For a fixed bandwidth N can't be greater than a certain amount, and hence if hundreds to thousands of communication and sensor nodes are involved the network must be comprised of linked subnets. Analogous to the discussion of equations (2.12) to (2.16) assume that parameter of interest is the information rate per node which we now write as

$$T = CWN^{-1/2} \tag{3.27}$$

If T_0 is the required information rate per node and C is a constant, we get

$$CWN^{-1/2} \geq T_0 \tag{2.28}$$

$$N \leq \left(\frac{CW}{T_0}\right)^{1/2} \equiv N_0 \tag{2.29}$$

Inserting equation (2.29) into equation (2.26) gives the condition for concurrently supporting a network spanning cluster.

$$N_0 > N > \frac{D^2}{\beta R^2} \tag{2.30}$$

In contrast to the random graph model in which only limits could be evaluated, the percolation theory result of equation (2.25) provides an algebraic relationship between the percolation threshold and the network parameters for all values of p_c. We also note that as the data transmission rate requirement T_0 goes to zero, the percolation threshold goes to zero, consistent with the state of maximum connectedness.

3. Non-Directed and Directed Graphs

The new result, derivation of the percolation threshold for combined directed and non-directed graphs is rendered in Section 3.3. In order to render the derivation user-friendly the author has provided sufficient discussion of the probabilistic theoretical basis in Sections 3.1 and 3.2.

3.1 Non-Directed Graph

Consider a random graph with $P(k)$ being the fraction of nodes having k links connected to it. The normalizing condition is

$$\sum_{k=2}^{k=M} P(k) = 1 \tag{3.1}$$

The lower limit, $k = 2$, is a requirement that for a node to contribute to the network connectivity it must as a minimum have 2 links, and M is the maximum number of links that can be connected to it. Following the notation of reference [**10**] we let (i)and (j) be two distinct nodes in the system. We define

$P(k_i) =$ Probability that node (i) has connectivity k_i

$P(k_i \mid i \leftrightarrow j) =$ Conditional probability that node (i) has connectivity k_i , given that it is connected to node (j)

$P(k_i, i \leftrightarrow j) =$ Joint probability that node (i) has connectivity k_i , and is connected to node (j)

$P(i \leftrightarrow j \mid k_i) =$ Conditional probability that node (i) is connected to node (j) given that it has connectivity k_i

$P(i \leftrightarrow j) =$ a-priori probability that any two nodes are connected to each other—essentially a network ensemble average

For a randomly connected large network of size N we have

$$P(i \leftrightarrow j) = \frac{\langle k \rangle}{(N-1)} \rightarrow \frac{\langle k \rangle}{N}; \quad \langle \; \rangle \text{ indicates network ensemble average} \tag{3.2}$$

$$P(i \leftrightarrow j \mid k_i) = \frac{k_i}{(N-1)} \rightarrow \frac{k_i}{N} \tag{3.3}$$

As proposed by Cohen et al, a network spanning cluster is created when node (i) is connected to node (j) and at least one other node. This is given by [**10**]

$$\langle k_i \mid i \leftrightarrow j \rangle = \sum_{k_i} k_i P(k_i \mid i \leftrightarrow j) \geq 2 \tag{3.4}$$

The foregoing equation is equivalent to the classic result of Molloy and Reed [**11**]. Their proof for the existence of a network spanning cluster is

$$Q = \sum_{k \geq 1} k(k-2)P(k) \geq 0 \tag{3.5}$$

From Bayes' theorem we have

$$P(i \leftrightarrow j \mid k_i)P(k_i) = P(k_i \mid i \leftrightarrow j)P(i \leftrightarrow j) \tag{3.6}$$

Using equations (3.2) and (3.3) we have

$$k_i P(k_i \mid i \leftrightarrow j) = k_i \frac{P(i \leftrightarrow j \mid k_i)P(k_i)}{P(i \leftrightarrow j)} = k_i \frac{(k_i / N)P(k_i)}{\langle k \rangle / N} = \frac{k_i^2}{\langle k \rangle} P(k_i) \tag{3.7}$$

Inserting equation (3.7) into equation (3.3) gives

$$\sum_{k_i} k_i^2 P(k_i) \geq 2\langle k \rangle = 2\sum_{k_i} k_i P(k_i) \tag{3.8}$$

$$\sum_{k_i} (k_i^2 - 2k_i) P(k_i) \geq 0 \tag{3.9}$$

As observed, equation (3.9) is the same as equation (3.3) and the proof is complete. Equation (3.8) can also be written as

$$K = \frac{\langle k^2 \rangle}{\langle k \rangle} \geq 2 \tag{3.10}$$

for the existence of a networking spanning cluster.

Cohen et al show [**10**] that if a fraction of nodes, f, uniformly distributed over the network, is removed from a network with initial link distribution $P(k_0)$, the new probability distribution, $\hat{P}(\hat{k})$, is given by

$$\hat{P}(\hat{k}) = \sum_{k_0=\hat{k}}^{\infty} P(k_0) \frac{k_0!}{(k_0 - \hat{k})!\hat{k}!} (1-f)^{\hat{k}} f^{k_0 - \hat{k}} \tag{3.11}$$

and it is easy to show that

$$\langle \hat{k} \rangle = \langle k_0 \rangle (1-f) \tag{3.12}$$

$$\langle \hat{k}^2 \rangle = \langle k_0^2 \rangle (1-f)^2 + \langle k_0 \rangle f(1-f) \tag{3.13}$$

The new condition for a network spanning cluster is

$$\hat{K} = \frac{\langle \hat{k}^2 \rangle}{\langle \hat{k} \rangle} \geq 2 \tag{3.14}$$

$$\frac{\langle k_0^2 \rangle (1-f) + \langle k_0 \rangle f}{\langle k_0 \rangle} \geq 2 \tag{3.15a}$$

$$K_0 (1-f) + f \geq 2 \tag{3.15b}$$

$$K_0 = \frac{\langle k_0^2 \rangle}{\langle k_0 \rangle} \tag{3.16}$$

$$K_0 \geq \frac{2-f}{1-f} \tag{3.17a}$$

$$\frac{K_0 - 2}{K_0 - 1} \geq f \tag{3.17b}$$

The quantity $1-f = p_s$ is the fraction of nodes that survive; it satisfies the equation

$$p_s = 1 - f \geq \frac{1}{K_0 - 1} \equiv p_c \tag{3.18}$$

where p_c is defined as the percolation threshold.

For the case where all the nodes have exactly the same number of links we have

$$K_0 = \langle k_0 \rangle \tag{3.19}$$

$$p_s \geq \frac{1}{\langle k_0 \rangle - 1} \tag{3.20}$$

3.2 Directed Graphs

Following the methodology of Schwartz et al [**12**], node b will be a member of a giant cluster when a link from node a reaches it, and it has at least one outgoing link. Every node is now characterized not only by the degree k, but by the double index (j) and (k) which are the in and out degrees respectively. The requirement for the creation of a network spanning cluster is

$$\langle k_b \mid a \to b \rangle = \sum_{j_b} \sum_{k_b} k_b P(j_b, k_b \mid a \to b) = \geq 1 \tag{3.21}$$

We define the following:

$P(j_b, k_b \mid a \to b) =$ Conditional probability that node a has a link leading to b

$P(j_b, k_b, a \to b) = P(j_b, k_b \mid a \to b) P(a \to b) =$ Joint probability that node a has a link leading to b

$$P(a \to b) = \frac{\langle k \rangle}{(N-1)} \to \frac{\langle k \rangle}{N} = \text{Ensemble average for random networks} \tag{3.22}$$

$$P(a \to b \mid j_b, k_b) = \frac{j_b}{(N-1)} \to \frac{j_b}{N} = \text{ Conditional average for random networks} \tag{3.23}$$

From Bayes' theorem we have

$$P(j_b, k_b \mid a \to b) = \frac{P(a \to b \mid j_b, k_b) P(j_b, k_b)}{P(a \to b)} = \frac{j_b}{\langle k \rangle} P(j_b, k_b) \tag{3.24}$$

Inserting equation (3.24) into equation (3.21) gives

$$\langle k_b \mid a \to b \rangle = \frac{1}{\langle k \rangle} \sum_{j_b} \sum_{k_b} k_b j_b P(j_b, k_b \mid = \frac{\langle jk \rangle}{\langle k \rangle} \geq 1 \tag{3.25}$$

Where we have used the normalization condition

$$\sum_{j_b} \sum_{k_b} P(j_b, k_b) = 1 \tag{3.26}$$

Schwartz et al show that if a fraction of nodes, f, uniformly distributed over the network, is removed from a network with initial link distribution $P(j_0, k_0)$, the new probability distribution, $\hat{P}(\hat{j}, \hat{k})$, is given by

$$\hat{P}(\hat{j}, \hat{k}) = \sum_{j_0 = \hat{j}} \sum_{k_0 = \hat{k}} P(j_0, k_0) \frac{j_0!}{(j_0 - \hat{j})! \hat{j}!} (1-f)^{\hat{j}} f^{j_0 - \hat{j}} \frac{k_0!}{(k_0 - \hat{k})! \hat{k}!} (1-f)^{\hat{k}} f^{k_0 - \hat{k}} \tag{3.27}$$

$$\langle \hat{k} \rangle = \sum_{\hat{j}} \sum_{\hat{k}} \hat{k} P(\hat{j}, \hat{k}) \tag{3.28}$$

$$\langle \hat{j}\hat{k} \rangle = \sum_{\hat{j}} \sum_{\hat{k}} \hat{j}\hat{k} P(\hat{j}, \hat{k}) \tag{3.29}$$

$$\hat{K} = \frac{\langle \hat{j}\hat{k} \rangle}{\langle \hat{k} \rangle} \tag{3.30}$$

In the case where the j and k are separable we have

$$\langle \hat{k} \rangle = \langle k_0 \rangle (1-f) \tag{3.31}$$

$$\langle \hat{j}\hat{k} \rangle = \sum_{\hat{j}} \sum_{\hat{k}} \hat{j}\hat{k} P(\hat{j}, \hat{k}) = \langle \hat{j} \rangle \langle \hat{k} \rangle = \langle j_0 \rangle \langle k_0 \rangle (1-f)^2 \tag{3.32}$$

$$1 - f \geq \frac{1}{\langle j_0 \rangle} \tag{3.33}$$

3.3 Combined Graphs

It is anticipated that real military networks will consist of combinations of non-directed and directed graphs. Fortunately, it is possible to handle this case by decomposing every edge of a non-directed graph into two opposite directed edges [**13**]. This mathematical feature renders all graphs as directed graphs provided that the aforementioned mathematical feature is properly accounted. Consider a node with *k* non-directed edges. According to the aforementioned mathematical principle we have *k* outgoing edges and exactly $j = k$ incoming edges. For a collection of non-directed nodes the joint probability density function is then

$$P_{ND}(j,k) = \delta_{jk} P_{ND}(k) \tag{3.34}$$

By using equation (3.33) we incorporate non-directed graphs and directed graphs in the same framework.

Consider a system where there are N_D nodes that have directed edges and N_{ND} nodes that have directed edges. The total number of nodes is

$$N = N_D + N_{ND} \tag{3.35}$$

Now let

$n_{ND}(k) =$ Number of nodes with *k* non-directed edges

We then have

$$\sum_k n_{ND}(k) = N_{ND} \tag{3.36}$$

$$P_{ND}(k) = \frac{n_{ND}(k)}{N_{ND}} \tag{3.37}$$

$$n_{ND}(j,k) \equiv \delta_{jk} n_{ND}(k) \tag{3.38}$$

$$P_{ND}(j,k) = \frac{n_{ND}(j,k)}{N_{ND}} = \frac{\delta_{jk} n_{ND}(k)}{N_{ND}} = \delta_{jk} P_{ND}(k) \tag{3.39}$$

$$\sum_j \sum_k n_{ND}(j,k) = N_{ND} \tag{3.40}$$

Following the same procedure for the directed case we define

$n_D(j,k) =$ Number of nodes with *j* and *k* directed edges

We then get

$$\sum_k n_D(j,k) = N_D \tag{3.41}$$

$$P_D(j,k) = \frac{n_D(j,k)}{N_D} \tag{3.42}$$

The total number of nodes with *j,k* edges is

$$n_T(j,k) = n_D(j,k) + n_{ND}(j,k) \qquad (3.43)$$

and the combined normalized probability density function is

$$P_T(j,k) = \frac{n_D(j,k) + n_{ND}(j,k)}{N} = \frac{N_D P_D(j,k) + N_{ND} P_{ND}(j,k)}{N} = W_D P_D(j,k) + W_{ND} P_{ND}(j,k) \qquad (3.44)$$

$$W_D = \frac{N_D}{N} \quad \text{(a)}, \quad W_D = \frac{N_{ND}}{N} \quad \text{(b)}, \quad W_D + W_{ND} = 1 \quad \text{(c)} \qquad (3.45)$$

The condition for the emergence of a network spanning cluster for directed graphs is initially obtained from the equation [**14**]

$$\sum_j \sum_k (2jk - j - k) P_T(j,k) = 0 \qquad (3.46)$$

Since the net average of edges entering a node is zero we have

$$\sum_j \sum_k (j - k) P_T(j,k) = 0 \qquad (3.47)$$

Combining equations (3.46) and (3.47) we get.

$$\sum_j \sum_k k(j-1) P_T(j,k) = 0 \qquad (3.48)$$

The same equation has previously been derived when only a directed graph network was considered [**15**].

Equation (3.48) can be written as

$$\langle jk \rangle - \langle k \rangle = 0 \qquad (3.49)$$

$$\langle jk \rangle = \sum_j \sum_k jk P_T(j,k) = W_D \sum_j \sum_k jk P_D(j,k) + W_{ND} \sum_j \sum_k jk P_{ND}(j,k) \qquad (3.50)$$

$$\langle k \rangle = \sum_j \sum_k k P_T(j,k) = W_D \sum_j \sum_k k P_D(j,k) + W_{ND} \sum_j \sum_k k P_{ND}(j,k) \qquad (3.51)$$

From equation (3.39) we get

$$\sum_j \sum_k jk P_{ND}(j,k) = \sum_j \sum_k jk \delta_{jk} P_{ND}(k) = \langle k^2 \rangle_{ND} \qquad (3.52)$$

$$\sum_j \sum_k k P_{ND}(j,k) = \sum_j \sum_k k \delta_{jk} P_{ND}(k) = \langle k \rangle_{ND} \qquad (3.53)$$

Similarly,

$$\sum_j \sum_k jk P_D(j,k) = \langle jk \rangle_D \qquad (3.54)$$

$$\sum_j \sum_k k P_D(j,k) = \langle k \rangle_D \qquad (3.55)$$

Using equations (3.50) to (3.55) in equation (3.49) we have

$$\langle jk \rangle = W_D \langle jk \rangle_D + W_{ND} \langle k^2 \rangle_{ND}$$

$$\langle k \rangle = W_D \langle k \rangle_D + W_{ND} \langle k \rangle_{ND}$$

$$\langle jk \rangle - \langle k \rangle = W_D \langle jk \rangle_D + W_{ND} \langle k^2 \rangle_{ND} - W_D \langle k \rangle_D - W_{ND} \langle k \rangle_{ND} = 0 \qquad (3.56)$$

If the combined network is now put under attack as in Sections 3.1 and 3.2 we use

$$\langle k \rangle_{ND} = \langle k_0 \rangle (1-f) \qquad (3.57)$$

$$\langle k^2 \rangle_{ND} = \langle k_0^2 \rangle (1-f)^2 + \langle k_0 \rangle f(1-f) = \langle k_0^2 \rangle (1-f)^2 + \langle k_0 \rangle (1-f) - \langle k_0 \rangle (1-f)^2 \qquad (3.58)$$

$$\langle k \rangle_D = \langle k_0 \rangle (1-f) \qquad (3.59)$$

$$\langle jk \rangle_D = \langle j_0 \rangle \langle k_0 \rangle (1-f)^2 \qquad (3.60)$$

As before, we let $p_s = 1 - f = p_c$ be the fraction of nodes that survive. Inserting equations (3.57) to (3.60) into equation (3.56) gives

$$W_D \langle j_0 \rangle \langle k_0 \rangle p_s^2 + W_{ND} \left(\langle k_0^2 \rangle p_s^2 + \langle k_0 \rangle p_s - \langle k_0 \rangle p_s^2 \right) - W_D \langle k_0 \rangle p_s - W_{ND} \langle k_0 \rangle p_s = 0 \qquad (3.61)$$

Recalling $W_D + W_{ND} = 1$, the solution of (3.61) is

$$p_s = p_c = \frac{\langle k_0 \rangle}{W_D \langle j_0 \rangle \langle k_0 \rangle + W_{ND} \left(\langle k_0^2 \rangle - \langle k_0 \rangle \right)} = \frac{1}{W_D \langle j_0 \rangle + W_{ND} (K_0 - 1)} \qquad (3.62)$$

$$K_0 = \frac{\langle k_0^2 \rangle}{\langle k_0 \rangle} \qquad (3.63)$$

Equation (3.62) provides exact agreement with limiting cases.

4. Way Forward

In this paper the theoretical principles random graph theory and percolation theory are used to evaluate the resilience of MANETs with non-directed and directed graphs. The proposed metric for survivability is the network's ability to remain radio-frequency connected with a suitable surviving fraction of nodes. A new formula is derived for the percolation threshold that includes arbitrary amounts of directed and non-directed links.

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