Physics Notes

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Lagrangian Formulation of the Combined-Field Form of the Maxwell Equations

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Abstract

A classical question in electromagnetic theory concerns the construction of a Lagrangian from which the Maxwell equations can be derived. In this paper a Lagrangian is formulated for the (complex) combined Maxwell equation, including both electric and magnetic potentials.

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1. Introduction

One way to formulate the Maxwell equations is via a Lagrangion density, L, which can be found in various tests, e.g., [2]. In this formalism the Lagrangian is a scalar function of the various electromagnetic field components, current-density components, and potential (vector and scalar) components. A certain combination of these (forming L) is differentiated with respect to the coordinates (space and time) and with respect to the spatial derivatives of the potentials, where the potential components and their derivatives (forming the fields) are regarded as separate variables for this purpose. (This is also referenced as the Euler-Lagrange equation.)

One can also write the Maxwell equations in combined-field form, reducing these to a single curl equation extended to the four-vector form of the Maxwell equations, as well as the quaternion form [1]. In this paper we investigate the application of the Lagrangian to the four-vector form of the combined Maxwell equation.

2. Combined-Field Form of the Maxwell Equations

2.1 Three-vector form

The combined Maxwell equation is [3]

$$\left[\nabla\times-\frac{jq}{c}\frac{\partial}{\partial t}\right]\vec{E}_q(\vec{r},t) = jq\ Z\ \vec{J}_q(\vec{r},t) \tag{2.1}$$

where

$$\vec{E}_q(\vec{r},t) = \vec{E}(\vec{r},t) + jq\,Z\,\vec{H}(\vec{r},t) \equiv \text{combined field}$$

$$\vec{J}_q(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{jq}{Z}\vec{J}_m(\vec{r},t) \equiv \text{combined current density}$$

$$Z = \left[\frac{\mu}{\varepsilon}\right]^{1/2} \equiv \text{wave impedance of uniform, isotropic, frequency-independent medium (e.g. free space)}$$

$$\text{c} = [\mu\,\varepsilon]^{-1/2} \equiv \text{wave propagation speed (light speed in free space)} \tag{2.2}$$

$$q = \pm 1 \equiv \text{separation index}$$

Taking the divergence of (2.1) we have

$$\varepsilon\frac{\partial}{\partial t}\nabla\bullet\vec{E}_q(\vec{r},t) \equiv \frac{\partial}{\partial t}\nabla\bullet\vec{D}_q(\vec{r},t) = -\nabla\bullet\vec{J}_q(\vec{r},t)$$

$$\nabla\bullet\vec{J}_q(\vec{r},t) \equiv -\frac{\partial}{\partial t}\rho_q(\vec{r},t)$$

$$\rho_q(\vec{r},t) \equiv \rho(\vec{r},t) + \frac{jq}{Z}\rho_m(\vec{r},t) \equiv \text{combined charge density} \tag{2.3}$$

This is extended to the potentials

$$\begin{aligned} \vec{A}_q(\vec{r},t) &\equiv \vec{A}(\vec{r},t) + jq\, Z\, A_m(\vec{r},t) \\ \Phi_q(\vec{r},t) &\equiv \Phi(\vec{r},t) + jq\, \Phi_q(\vec{r},t) \end{aligned} \tag{2.4}$$

This gives the combined field as

$$\vec{E}_q(\vec{r},t) = -\nabla\Phi_q(\vec{r},t) + \left[jqc\nabla\times - \frac{\partial}{\partial t} \right] \vec{A}_q(\vec{r},t) \tag{2.5}$$

with the integral representation

$$\begin{aligned} \vec{A}_q(\vec{r},t) &= \mu \int_V \frac{\vec{J}_q\left(\vec{r}',t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{4\pi|\vec{r}-\vec{r}'|} dV' \\ \Phi_q(\vec{r},t) &= \frac{1}{\varepsilon} \int_V \frac{\rho_q\left(\vec{r}',t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{4\pi|\vec{r}-\vec{r}'|} dV' \end{aligned} \tag{2.6}$$

These are related by the gauge condition

$$\nabla \bullet \vec{A}_q(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi_q(\vec{r},t) = 0 \tag{2.7}$$

2.2 Four-vector form

The space-time coordinates take the form [3]

$$
\begin{aligned}
\overline{r}_p &\equiv (\vec{r}, T_p) = \left(x, y, z, T_p\right) = \left(r_1, r_2, r_3, r_{4_p}\right) \\
&= \times\ \overline{1}_x + y\overline{1}_x + z\overline{1}_x + T_p\overline{1}_{T_p} \\
T_p &= jpct, p = \pm 1 \\
p &\equiv \text{additional separation index}
\end{aligned}
\tag{2.8}
$$

Here an overbar is used to indicate a 4-vector. The del operator and laplacian become

$$
\begin{aligned}
\Box_p &= \left(\nabla, \frac{\partial}{\partial T_p}\right) \\
\Box_p^2 &= \Box_{-p}^2 = \nabla^2 + \frac{\partial^2}{\partial T_p^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}
\end{aligned}
\tag{2.9}
$$

Then we have

$$
\overline{J}_{p,q}(\overline{r}_p) = \left(\vec{J}_q(\vec{r}, t), jpc\rho_q(\vec{r}, t)\right) \qquad \text{(4-current density)}
$$

$$
\Box \bullet \overline{J}_{p,q}(\overline{r}_p) = 0 = \nabla \bullet \vec{J}_q(\vec{r}, t) + \frac{\partial}{\partial t}\rho_q(\vec{r}, t) \qquad \text{(continuity)} \tag{2.10}
$$

$$
\overline{A}_{p,q}(\overline{r}_p) = \left(\vec{A}_q(\vec{r}, t),\ j\frac{p}{c}\Phi_q(\vec{r}, t)\right) \qquad \text{(4-potential)}
$$

$$
\Box_p \bullet \overline{A}_{p,q}(\overline{r}, t) = \nabla \bullet \vec{A}_q(\vec{r}, t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi_q(\vec{r}, t) = 0 \qquad \text{(Lorentz gauge)}
$$

$$
\begin{aligned}
\Box_p^2 \bullet \overline{A}_{p,q}(\overline{r}_p) &= -\mu\overline{J}_{p,q}(\overline{r}_p) \\
\overline{A}_{p,q}(\overline{r}_p) &= \mu\int_V \frac{\overline{J}_{p,q}(\overline{r}_p'')}{4\pi|\vec{r} - \vec{r}'|}dV' \ ,\quad \overline{r}_p'' = \left(\vec{r}', jp\left[ct - |\vec{r} - \vec{r}'|\right]\right)
\end{aligned}
\tag{2.11}
$$

This leads to the 4-vector form of the combined Maxwell equation as

$$
\Box_p \bullet \overline{\overline{E}}_{p,q}(\overline{r}_p) = jqZ\overline{J}_{p,q}(\overline{r}_p)
$$

$$\overline{\overline{E}}_{p,q}(\overline{r}_p) = \begin{pmatrix} 0 & -E_{z_q} & E_{y_q} & -pqE_{x_q} \\ E_{z_q} & 0 & -E_{x_q} & -pqE_{y_q} \\ -E_{y_q} & E_{x_q} & 0 & -pqE_{z_q} \\ pqE_{x_q} & pqE_{y_q} & pqE_{z_q} & 0 \end{pmatrix} \tag{2.12}$$

$$= -\overline{\overline{E}}^{T}_{p,q}(\overline{r}_p) \qquad \text{(skew symmetric)}$$

We also have

$$\overline{\overline{E}}_{p,q}(\overline{r}_p) \bullet \overline{\overline{E}}_{p,q}(\overline{r}_p) \equiv \overline{\overline{E}}^{2}_{p,q}(\overline{r}_p) = -\vec{E}_q(\vec{r},t) \bullet \vec{E}_q(\vec{r},t) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.13}$$

Where

$$\vec{E}_q(\vec{r},t) \bullet \vec{E}_q(\vec{r},t) = \vec{E}(\vec{r},t) \bullet \vec{E}(\vec{r},t) - Z^2 \vec{H}(\vec{r},t) \bullet \vec{H}(\vec{r},t) + j2qZ\, \vec{E}(\vec{r},t) \bullet \vec{H}(\vec{r},t) \tag{2.14}$$

which is preserved under Lorentz transformation.

Assuming that (2.14) is nonzero (which is not the case for a plane wave), then the field four-dyadic has an inverse as

$$\overline{\overline{E}}^{-1}_{p,q}(\overline{r}_p) = \left[-\vec{E}_q(\vec{r},t) \bullet \vec{E}_q(\vec{r},t) \right]^{-1} \overline{\overline{E}}_{p,q}(\overline{r}_p) \tag{2.15}$$

i.e., except for a scalar multiplier it is its own inverse. It has determinant and trace

$$\det\left(\overline{\overline{E}}_{p,q}(\overline{r}_p)\right) = \left[-\vec{E}_q(\vec{r},t) \bullet \vec{E}_q(\vec{r},t) \right]^2 \ , \quad tr\left(\overline{E}_{p,q}(\overline{r}_p)\right) = 0 \tag{2.16}$$

with eigenvalues

$$\lambda_q\left(\overline{\overline{E}}_{p,q}(\overline{r}_p)\right) = \pm \left[-\vec{E}_q(\vec{r},t) \bullet \vec{E}_q(\vec{r},t) \right]^{1/2} \tag{2.17}$$

with each sign taken twice.

2.3 Dual of a four-dyadic

Define

$$d_{n,m,n',m'} = \begin{cases} 1/2 \text{ for any even permutation of } (1, 2, 3, 4) \\ \text{-}1/2 \text{ for any odd permutation of } (1, 2, 3, 4) \\ 0 \text{ otherwise (any two indices equal)} \end{cases} \tag{2.18}$$

$\left(d_{n,m,n',m'}\right) \equiv$ tetradic or fourth-rank tensor

For an antisymmetric dyadic

$$\overline{\overline{X}} = \begin{pmatrix} 0 & X_{1,2} & X_{1,3} & X_{1,4} \\ -X_{1,2} & 0 & X_{2,3} & X_{2,4} \\ -X_{1,3} & -X_{2,3} & 0 & X_{3,4} \\ -X_{1,4} & -X_{2,4} & -X_{3,4} & 0 \end{pmatrix} = -\overline{\overline{X}}^T \tag{2.19}$$

The dual dyadic is also antisymmetric as

$$\begin{aligned} \text{dual}\left(\overline{\overline{X}}\right) &\equiv \left(\sum_{n'm'} d_{n,m,n',m'} X_{n',m'}\right) \\ &= \begin{pmatrix} 0 & X_{3,4} & -X_{2,4} & X_{2,3} \\ -X_{3,4} & 0 & X_{1,4} & -X_{1,3} \\ X_{2,4} & -X_{1,4} & 0 & X_{1,2} \\ -X_{2,3} & X_{1,3} & -X_{1,2} & 0 \end{pmatrix} \end{aligned} \tag{2.20}$$

with

$$\text{dual}\left(\text{dual}\left(\overline{\overline{X}}\right)\right) = \overline{\overline{X}} \tag{2.21}$$

Applying this to the combined field dyadic gives [3]

$$\text{dual}\left(\overline{\overline{E}}_{p,q}\left(\overline{r}_p\right)\right) = pq\overline{\overline{E}}_{p,q}\left(\overline{r}_p\right) \tag{2.22}$$

which, except for a possible sign change, makes the combined-field four-dyadic self dual.

2.4 Four-dyadic field tensor related to potentials

From [3] we have

$$\overline{\overline{E}}_{p,q}(\overline{r}_p) = -jc\left[q\left[\square_p \overline{A}_{p,q}(\overline{r}_p) - \left[\square_p \overline{A}_{p,q}(\overline{r}_p)\right]^T\right] + p\ \text{dual}\left[\square_p \overline{A}_{p,q}(\overline{r}_p) - \left[\square_p \overline{A}_{p,q}(\overline{r}_p)\right]^T\right]\right] \tag{2.23}$$

3. Classical Lagrangian Formulation

Following [2] we have (in his units)

$$L = -\frac{1}{4}\sum_{\mu,\nu} F_{\mu,\nu}^2 + \frac{1}{c}\overline{J}\bullet\overline{A}$$

$$\sum_{\nu}\frac{\partial}{\partial r_\nu}\frac{\partial L}{\partial\left(\frac{\partial A_\mu}{\partial r_\nu}\right)} = -\sum_{\nu}\frac{\partial}{\partial r_\nu}F_{\nu,\mu} \tag{3.1}$$

$$\frac{\partial L}{\partial A_\mu} = \frac{1}{c}\overline{J}_\mu$$

giving the Maxwell equations

$$\left(\sum_{\nu}\frac{\partial F_{\mu,\nu}}{\partial r_\nu}\right) = \frac{1}{c}\overline{J}$$

$$\left(F_{\mu,\nu}\right) = \begin{pmatrix} 0 & B_z & -B_y & -jE_x \\ -B_z & 0 & B_x & -jE_y \\ B_y & -B_x & 0 & -jE_z \\ jE_x & jE_y & jE_z & 0 \end{pmatrix} \tag{3.2}$$

4. Combined-Field Lagrangian

So now let us try something of the form (in MKS units)

$$L = -\frac{1}{4}\sum_{n,m} E_{n,m;p,q}^2(\overline{r}_p) + \alpha\overline{J}_{p,q}(\overline{r}_p) \bullet \overline{A}_{p,q}(\overline{r}_p) \tag{4.1}$$

$\alpha \equiv$ constant to be determined

4.1 First term ($\partial A_n / \partial r_m$ terms)

The first term involves the derivatives of the potentials with respect to the four coordinates. So we write

$$\begin{aligned} L_1 &= -\frac{1}{4}\sum_{n,m} E_{n,m;p,q}^2\left(\overline{r}_p\right) \\ &= \vec{E}_q(\vec{r},t)\bullet\vec{E}_q(\vec{r},t) \equiv \vec{E}_q^{\,2}(\vec{r},t) \end{aligned} \tag{4.2}$$

from (2.12) (or (2.13)). This also takes the form in (2.14) in terms of $\vec{E}$ and $\vec{H}$.

In terms of the potentials we have

$$
\begin{aligned}
\bar{\bar{E}}_{p,q}\left(\bar{r}_p\right) &= jqc\bar{\bar{X}}^{(1)} + jpc\bar{\bar{X}}^{(2)} \\
\bar{\bar{X}}^{(1)} &= -\Box_p \bar{A}_{p,q}(\vec{r}_p) + \left[\Box_p \bar{A}_{p,q}(\vec{r}_p)\right]^T \\
\bar{\bar{X}}^{(2)} &= \text{dual}\left(\bar{\bar{X}}^{(1)}\right) \\
\text{dual}\left(\bar{\bar{E}}_{p,q}\left(\bar{r}_p\right)\right) &= jqc\bar{\bar{X}}^{(2)} + jqc\bar{\bar{X}}^{(1)} = pq\bar{\bar{E}}_{p,q}\left(\bar{r}_p\right)
\end{aligned}
\tag{4.3}
$$

Where $\bar{\bar{X}}^{(1)}$ and $\bar{\bar{X}}^{(2)}$ are explicitely exhibited in [3]. For L_1 we then have (suppressing the p, q subscripts on the vector-potential components, and space-time coordinates)

$$
\begin{aligned}
L_1 &= -\frac{1}{4}\sum_{n,m} E^2_{n,m;p,q}(\bar{r}_p) \\
&= -\frac{1}{4}\sum_{n,m}\left[jqcX^{(1)}_{n,m} + jpcX^{(2)}_{n,m}\right]^2 \\
&= \frac{c^2}{4}\sum_{n,m} X^{(1)^2}_{n,m} + X^{(2)^2}_{n,m} + 2pqX^{(1)}_{n,m}X^{(2)}_{n,m} \\
&\equiv \frac{c^2}{4}\left[L^{(1)}_1 + L^{(2)}_1 + L^{(3)}_1\right] \\
L^{(1)}_1 &= L^{(2)}_1 = \sum_{n,m} X^{(1)^2}_{n,m} = \sum_{n,m} X^{(2)^2}_{n,m} \\
&= 2\sum_{\substack{n,m \\ m>n}}\left[\frac{\partial A_n}{\partial r_m} - \frac{\partial A_n}{\partial r_n}\right]^2 \quad \text{(zero for } n = m\text{, 6 nonzero terms)} \\
L^{(3)}_1 &= 4pq\left[\left[\frac{\partial A_1}{\partial r_2} - \frac{\partial A_2}{\partial r_1}\right]\left[\frac{\partial A_3}{\partial r_4} - \frac{\partial A_4}{\partial r_3}\right]\right. \\
&\quad -\left[\frac{\partial A_1}{\partial r_3} - \frac{\partial A_3}{\partial r_1}\right]\left[\frac{\partial A_2}{\partial r_4} - \frac{\partial A_4}{\partial r_2}\right] \\
&\quad \left.+\left[\frac{\partial A_2}{\partial r_3} - \frac{\partial A_3}{\partial r_2}\right]\left[\frac{\partial A_1}{\partial r_4} - \frac{\partial A_4}{\partial r_1}\right]\right]
\end{aligned}
\tag{4.4}
$$

Now we form the 4 x 4 dyadic

$$
\left(\frac{\partial L_1^{(1)}}{\partial\left(\partial A_n / \partial r_m\right)}\right) = \left(\frac{\partial L_1^{(2)}}{\partial\left(\partial A_n / \partial r_m\right)}\right) =
$$

$$
2\begin{pmatrix} 0 & \frac{\partial A_1}{\partial r2} - \frac{\partial A_2}{\partial r_1} & \frac{\partial A_1}{\partial r_3} - \frac{\partial A_3}{\partial r_1} & \frac{\partial A_1}{\partial r_4} - \frac{\partial A_4}{\partial r_1} \\ \frac{\partial A_2}{\partial r_1} - \frac{\partial A_1}{\partial r_2} & 0 & \frac{\partial A_2}{\partial r_3} - \frac{\partial A_3}{\partial r_2} & \frac{\partial A_2}{\partial r_4} - \frac{\partial A_4}{\partial r_2} \\ \frac{\partial A_3}{\partial r_1} - \frac{\partial A_1}{\partial r_3} & \frac{\partial A_3}{\partial r_2} - \frac{\partial A_2}{\partial r_3} & 0 & \frac{\partial A_3}{\partial r_4} - \frac{\partial A_4}{\partial r_3} \\ \frac{\partial A_4}{\partial r_1} - \frac{\partial A_1}{\partial r_4} & \frac{\partial A_4}{\partial r_2} - \frac{\partial A_2}{\partial r_4} & \frac{\partial A_4}{\partial r_3} - \frac{\partial A_3}{\partial r_4} & 0 \end{pmatrix} \tag{4.5}
$$

$$
= 2\overline{\overline{X}}^{(1)} = -2\overline{\overline{X}}^{(1)^T}
$$

Similarly we form

$$
\left(\frac{\partial L_1^{(3)}}{\partial\left(\partial A_n / \partial r_m\right)}\right) =
$$

$$
4pq\begin{pmatrix} 0 & \frac{\partial A_3}{\partial r_4} - \frac{\partial A_4}{\partial r_3} & \frac{\partial A_4}{\partial r_2} - \frac{\partial A_2}{\partial r_4} & \frac{\partial A_2}{\partial r_3} - \frac{\partial A_3}{\partial r_2} \\ \frac{\partial A_4}{\partial r_3} - \frac{\partial A_3}{\partial r_4} & 0 & \frac{\partial A_1}{\partial r_4} - \frac{\partial A_4}{\partial r_1} & \frac{\partial A_3}{\partial r_1} - \frac{\partial A_1}{\partial r_3} \\ \frac{\partial A_2}{\partial r_4} - \frac{\partial A_4}{\partial r_2} & \frac{\partial A_4}{\partial r_1} - \frac{\partial A_1}{\partial r_4} & 0 & \frac{\partial A_1}{\partial r_2} - \frac{\partial A_2}{\partial r_1} \\ \frac{\partial A_3}{\partial r_2} - \frac{\partial A_2}{\partial r_3} & \frac{\partial A_1}{\partial r_3} - \frac{\partial A_3}{\partial r_1} & \frac{\partial A_2}{\partial r_1} - \frac{\partial A_1}{\partial r_2} & 0 \end{pmatrix} \tag{4.6}
$$

$$
= 4pq\overline{\overline{X}}^{(2)} = -4pq\overline{\overline{X}}^{(2)^T}
$$

Summing these we have

$$
\begin{aligned} \left(\frac{\partial L_1}{\partial\left(\partial A_n / \partial r_m\right)}\right) &= c^2\left[\overline{\overline{X}}^{(1)} + pq\overline{\overline{X}}^{(2)}\right] \\ &= -jqc\overline{\overline{E}}_{p,q}(\overline{r}_p) \\ &= -c^2\left[q\left[\square_p \overline{A}_{p,q}(\overline{r}_p) - \left[\square_p \overline{A}_{p,q}(\overline{r}_p)\right]^T\right]\right. \\ &\qquad \left. + p\,\mathrm{dual}\left(\square_p \overline{A}_{p,q}(\overline{r}_p) - \left[\square_p \overline{A}_{p,q}(\overline{r}_p)\right]^T\right)\right] \end{aligned} \tag{4.7}
$$

with the last result from (2.23).

Now we have (from (2.12))

$$\square_p \bullet \left(\frac{\partial L_1}{\partial(\partial A_n / \partial r_m)} \right) = -jqc\square_p \bullet \overline{\overline{E}}_{p,q}(\overline{r}_p)$$
$$= cZ\overline{J}_{p,q}(\overline{r}_p) \tag{4.8}$$
$$= \frac{1}{\varepsilon}\overline{J}_{p,q}(\overline{r}_p)$$

4.2 Second term (∂A_n terms)

From (4.1) we have the second term

$$L_2 = \alpha \overline{J}_{p,q}(\overline{r}_p) \bullet \overline{A}_{p,q}(\overline{r}_p) \tag{4.9}$$

Taking the derivatives with respect to the $\overline{A}_{p,q}$ four components gives a gradient as

$$\left(\frac{\partial L_2}{\partial A_n} \right) = \alpha \overline{J}_{p,q}(\overline{r}_p) \tag{4.10}$$

4.3 Combining both terms

Combining the $\partial A_n / \partial r_m$ and ∂A_n derivatives of the Lagrangian to give the required zero variational form, we have

$$\square_p \bullet \left(\frac{\partial L_1}{\partial(\partial A_n / \partial r_m)} \right) + \left(\frac{\partial L_2}{\partial A_n} \right) = \overline{0}$$
$$= \frac{1}{\varepsilon}\overline{J}_{p,q}(\overline{r}_p) + \alpha \overline{J}_{p,q}(\overline{r}_p) \tag{4.11}$$

This requires

$$\alpha = -1/\varepsilon \tag{4.12}$$

To give our Lagrangian

$$
\begin{aligned}
L &= -\frac{1}{4}\sum_{n,m} E^2_{n,m;p,q}(\overline{r}_p) - \frac{1}{\varepsilon}\overline{J}_{p,q}(\overline{r}_p) \bullet \overline{A}_{p,q}(\overline{r}_p) \\
&= \overrightarrow{E}_q(\overrightarrow{r},t) \bullet \overrightarrow{E}_q(\overrightarrow{r},t) - \frac{1}{\varepsilon}\overrightarrow{J}_q(\overrightarrow{r},t) \bullet \overrightarrow{A}_q(\overrightarrow{r},t) + \frac{1}{\varepsilon}\rho_q(\overrightarrow{r},t)\Phi_q(\overrightarrow{r},t)
\end{aligned}
\tag{4.13}
$$

By our previous derivation this Lagrangian is differentiable in the usual Lagrangian way to give the Maxwell equation (2.12) in 4-vector/tensor form.

5. Concluding Remarks

It is thus possible to construct a Lagrangian for the combined-field Maxwell equations, expressed in four-vector/dyadic form. This works for all choices of *q* (field-separation index) and *p* (time-separation index).

References

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