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The Non-Relativistic Stationary State Equation of Motion and a Modified Schrödinger Equation

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The Stationary State equation of motion for charged particles developed in an earlier note is used as the basis for deriving a wave equation by introducing a Hamiltonian that takes into account the imaginary radiation time constant. The result is a slightly modified Schrödinger equation, the solutions for the expected values of the orbital radius, the orbital angular velocity and the ground state energy are in agreement with the earlier results.

1. INTRODUCTION

1.1 The Non-Relativistic Stationary State equation of motion for a point charged particle is given by[1]

$$m\dot{\mathbf{v}} + mi\tau\frac{\dot{v}^2}{v^2}\mathbf{v} = -\nabla.\mathbf{V} \tag{1}$$

The successful application of this equation, in conjunction with the relativistic equation, to obtaining solutions for the energy levels of the hydrogen atom and other results of quantum mechanics suggests that it should be possible to derive an associated Schrödinger-type equation.

2. THE HAMILTONIAN

2.1 The Lagrangian for one-dimensional motion is given by

$$\frac{dp}{dt} = \frac{\partial\mathscr{L}}{\partial x} = -\frac{\partial V}{\partial x} - mi\tau\frac{\dot{v}^2}{v} \tag{2}$$

where V is the potential energy. Integrating this equation wrt x

$$\mathscr{L} = \frac{p^2}{2m} - mi\tau\int\dot{v}^2 dt - V \tag{3}$$

The Hamiltonian is then given by

$$\mathscr{H} = pv - \mathscr{L} = \frac{p^2}{2m} + mi\tau\int\dot{v}^2 dt + V \tag{4}$$

2.2 To check that this is a suitable function, the equation of motion is determined via

$$\dot{p} = -\frac{\partial\mathscr{H}}{\partial x} = -\frac{\partial V}{\partial x} - mi\tau\frac{\partial}{\partial x}\int\dot{v}^2 dt = -\frac{\partial V}{\partial x} - mi\tau\frac{\dot{v}^2}{v} \tag{5}$$

i.e.

$$m\dot{v} + mi\tau\frac{\dot{v}^2}{v} = -\frac{\partial V}{\partial x} \tag{6}$$

2.3 Writing the Hamiltonian in the form

$$\mathscr{H} = \frac{p^2}{2m} + \frac{i\tau}{m}\int\dot{p}^2 dt + V \tag{7}$$

the momentum is replaced by

$$p \to -i\hbar\frac{\partial}{\partial x} \qquad \dot{p}^2 \to -\hbar^2\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial x^2} \tag{8}$$

The operational form of the Hamiltonian becomes

$$\mathscr{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{i\tau\hbar^2}{m}\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2} + V \tag{9}$$

3 THE WAVE EQUATION

3.1 Letting E be the total energy of the particle, and noting

$$\mathscr{H}\psi = E\psi \tag{10}$$

the wave equation becomes

$$\frac{\partial^2\psi}{\partial x^2} + 2i\tau\frac{\partial}{\partial t}\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E - V)\psi = 0 \tag{11}$$

or, generalising to three dimensions

$$\nabla^2\psi + 2i\tau\frac{\partial}{\partial t}\nabla^2\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0 \tag{12}$$

3.2 Expressing the time dependency as

$$\psi = \psi_0 e^{\frac{iE}{\hbar}t} \tag{13}$$

the modified Schrödinger equation is

$$\left[1 - \frac{2E\tau}{\hbar}\right]\nabla^2\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0 \tag{14}$$

Observe that setting τ to zero reduces this to Schrödinger's equation and that dividing by the coefficient of the first term, the equation is identical in form.

4 SUITABILITY OF EQUATION

4.1 As a check on the suitability of this equation, it must predict that the current in a one-dimensional beam of electrons is independent of x. This requires that

$$\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x} \tag{15}$$

is independent of x. Multiplying the modified Schrödinger equation by ψ^*, the conjugate equation by ψ and subtracting

$$\left(1 - \frac{2E\tau}{\hbar}\right)\left[\psi\frac{\partial^2\psi^*}{\partial^2 x} - \psi^*\frac{\partial^2\psi}{\partial x^2}\right] = 0 \tag{16}$$

or

$$\frac{\partial}{\partial x}\left[\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right] = 0 \tag{17}$$

i.e., the current is independent of x.

5. SOLUTION FOR THE GROUND STATE

5. 1 Writing this equation in the standard form for the hydrogen atom

$$\nabla^2\psi + \left(-a^2 + \frac{P}{r}\right)\psi = 0 \tag{18}$$

where

$$\frac{2mE}{\hbar^2\left(1 - \frac{2E\tau}{\hbar}\right)} = -a^2 \qquad P = \frac{-2me^2}{\hbar^2} \tag{19}$$

The solution of this equation then gives for the ground state

$$P = 2a \tag{20}$$

which implies

$$E\left(1 - \frac{2E\tau}{\hbar}\right) = -\frac{me^4}{2\hbar^2} \tag{21}$$

5.2 Solving this quadratic and expanding to terms of order α^3

$$E = -\frac{me^4}{2\hbar^2}\left(1 + \frac{2\alpha^3}{3}\right) \tag{22}$$

Introducing this expression for E into P

$$a = \frac{me^2}{\hbar^2}\frac{1}{\left(1 + \frac{2\alpha^3}{3}\right)} = \frac{1}{a_0\left(1 + \frac{2\alpha^3}{3}\right)} \tag{23}$$

5.3 With the usual probability interpretation of ψ followed by integration over space, the mean radius of the orbit is

$$r_0 = a_0\left(1 + \frac{2\alpha^3}{3}\right) \tag{24}$$

5.4 The orbital angular momentum is then given by

$$r_0^2\omega = \hbar \tag{25}$$

or

$$\omega = \omega_0\left(1-\frac{4\alpha^3}{3}\right) \tag{26}$$

These results are in agreement with those presented in [2]

6. THE GENERAL SOLUTION FOR THE HYDROGEN ATOM

6.1 More general solutions are obtained following the standard procedure. To be specific we again consider the hydrogen atom and write the equation in terms of spherical polar coordinates obtaining

$$\frac{1}{r}\frac{\partial^2 r\psi}{\partial r^2}+\frac{1}{r^2}\Omega\psi+\left(-a^2+\frac{P}{r}\right)\psi = 0 \tag{27}$$

where

$$\Omega\psi = \frac{1}{\mathbf{sin}\theta}\frac{\partial}{\partial\theta}\left(\mathbf{sin}\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\mathbf{sin}^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{28}$$

The equation having been reduced to the standard form for the hydrogen atom, we now make the trial solution

$$r\psi = R(r)Y(\theta,\phi)e^{\frac{iEt}{\hbar}} \tag{29}$$

yielding the two equations

$$\frac{d^2R}{dr^2}+\left(-a^2+\frac{B}{r}-\frac{\ell(\ell+1)}{r^2}\right)R = 0 \tag{30}$$

$$\Omega Y + \ell(\ell+1)Y = 0$$

6.2 This leads to a solution in terms of r^ℓ times Laguerre polynomials. The above solution was obtained by setting $\ell = 0$ for the ground state. For ℓ being any integer, the solution for the energy becomes

$$E_n \approx -\frac{me^4}{2\hbar^2 n^2}\left(1+\frac{2}{3}\alpha^3\right) \tag{31}$$

The transition frequency from $n_2 \rightarrow n_1$ is then

$$\upsilon = \frac{me^4}{4\pi\hbar^3}\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)\left(1+\frac{2}{3}\alpha^3\right) \tag{31}$$

The magnitude of the correction for a transition from state 2 to state 1is

$$\frac{\delta\nu}{\nu} = \frac{2\alpha^3}{3} \approx 2.59\times 10^{-7} \tag{32}$$

References

[1] An Investigation into the Motion of a Classical Charged Particle, I.L.Gallon, Physics Note No 15, University of New Mexico, Albuquerque

[2] A Note on the Stationary State Model of the Hydrogen Atom, Ian L. Gallon, Physics Notes Note 16, December 2005, University of New Mexico, Albuquerque.