Physics Notes
Note 15

October 2005

An Investigation into the

Motion of a Classical Charged Particle

Ian L. Gallon

41, St. Katherine's Avenue, Bridport, DT6 3DE, UK
ilandpm@gallon4151.fsnet.co.uk

The equation of motion of a classical charged particle is obtained by equating the rate of change of momentum acquired, plus the rate of change of momentum lost by radiation, to the applied force. This leads to a non-linear equation with solutions that are consistent with the conservation of energy and the maintenance of causality, two essential requirements which are conspicuously absent in both the classical equations of motion derived according to Abraham- Lorentz and later by Dirac. The equation is first derived non-relativistically and later is expressed in the four-vector notation of special relativity. The addition of one additional assumption based on the observation of stationary states and the imposition of certain symmetry conditions leads to solutions consistent with quantum mechanics. In particular, consistent models of the electron, the positron and the photon are produced. The energy levels of the hydrogen atom and the zero point energy for the harmonic oscillator are also generated. It is shown that the Dirac relativistic quantum mechanical equation for the electron leads to a similar model of the electron. A consequence of incorporating the new model of the electron into the model of the hydrogen atom together with the imposition of certain symmetry conditions leads to an implicit formula for the fine structure constant. It is not suggested or intended that the above models in any way replace the standard approach of quantum mechanics, but merely show that classical physics is not as far removed from quantum mechanics as usually appears.

I. INTRODUCTION

1.1 This investigation is concerned with the equation of motion of a classical charged particle in which the radiation due to the acceleration is properly accounted for. Previous attempts[1] appear to have lost their way by attempting to produce a linear equation, resulting in a third degree differential equation. This not only means that three initial or two initial and a final condition are required, but that we must have the particle moving before the force is applied, or we must know the future behaviour of the particle. In addition we have solutions that suggest that particles in zero field accelerate to infinity, and even solutions that have the initial appearance of reasonable behaviour are found to violate the conservation of energy and the conservation of momentum.

1.2 Classical physics is a set of laws based on observation combined with the principles of causality, ie the cause precedes, or is at least concurrent with the effect, the conservation of energy, mass-energy in the case of relativistic physics, the conservation of momentum, and the conservation of angular momentum. In addition relativity gives a limiting velocity, that of em radiation in free space. Any derivation of equations attempting to account for observations or predict as yet unobserved effects must be consistent with these principles, and when velocities are encountered that approach the velocity of light, the equations must be formulated in the four vector notation of special relativity, and the resulting equations must be consistent with this limiting velocity. Any violation of these principles means that the equation is wrong. It is argued that the pre-acceleration[2] appearing in the solutions of earlier proposed equations of motion occurred over very small time scales, typically of order 10^{-24} seconds and indicated the breakdown of classical physics. Classical physics knows nothing of quantum limitations, so why should it be inconsistent? It is true that it may give predictions that are in error compared to observation, and that accurate predictions are made by quantum mechanics, but this is irrelevant. The only breakdown that is demonstrated by the erroneous solutions is in the logic of the derivation of the equation.

1.3 A lengthy discussion of the Abraham-Lorentz equation and an attempt to justify the solutions is provided by J. D. Jackson[3], and philosophical implications are additionally discussed by J. L. Jimenez and I. Campus[4]. In Diracs[5] contribution it is suggested that the pre-acceleration can be accounted for by the finite size of the electron, the equation of motion being concerned with the centre of mass. However this leads to the prediction that radiation travels through the electron at superluminal velocities! There are further problems that arise if an attempt is made to remove the difficulties by considering a finite distribution of charge from the beginning. Apart from the almost prohibitive complexity of the equations that have now to take into account internal stresses, some essentially unobservable charge distribution has to be assumed, and assumptions have to be made about the elastic nature or otherwise of the finite electron. A heroic effort in this direction was made by A Yaghjian[6] . To obtain a solution further assumptions have to be made. While this process undoubtedly leads to the solution for a macroscopic charged particle with prescribed real properties and of sufficient size to allow the continuum approximation to apply, it can only be applied to an electron by selecting charge and mass distributions and specifying elastic properties.

1.4 The non-relativistic Abraham -Lorentz equation can be written[1][3]

$$m(\dot{\mathbf{v}} - \tau\ddot{\mathbf{v}}) = \mathbf{f} \tag{1.1}$$

where

$$\tau = \frac{q^2}{6\pi\varepsilon_0 c^3 m} \tag{1.2}$$

and τ = 6.2664218.10^{-24} seconds. Notice that the second term can be positive or negative, implying that the radiation of energy can, for a given positive acceleration of the charge augment or detract from the applied force. The radiated energy actually depends on the square of the acceleration, and so we should expect that the radiation will always detract from the applied force. The rate of change of acceleration in general is not parallel to the velocity and it will be shown that the deceleration effect of the radiation is parallel.

1.5 We can see how the N-R A-L equation was obtained by writing

$$m\dot{\mathbf{v}} = \mathbf{f} + \mathbf{f}_{\mathbf{rad}} \tag{1.3}$$

$$\int \mathbf{f}_{\mathbf{rad}}.\mathbf{v}dt = -\int m\tau\dot{\mathbf{v}}.\dot{\mathbf{v}}dt \tag{1.4}$$

$\mathbf{f}_{rad}$ being the radiation reaction on the charge. Integrating the rhs side by parts over the time during which the charge is accelerated, we find

$$\int \mathbf{f}_{\mathbf{rad}}.\mathbf{v}dt = \int m\tau\ddot{\mathbf{v}}.\mathbf{v}dt \tag{1.5}$$

The argument is that this suggests that

$$\mathbf{f}_{\mathbf{rad}} = m\tau\ddot{\mathbf{v}} \tag{1.6}$$

1.6 We can proceed differently by multiplying the integrand by unity in the form giving

$$\mathbf{v}.\mathbf{v}/\mathbf{v}^2 \tag{1.7}$$

$$\mathbf{f}_{\mathbf{rad}}.\mathbf{v} = m\tau\frac{\dot{v}^2}{v^2}\mathbf{v}.\mathbf{v} \tag{1.8}$$

The equation of motion can then be written

$$\dot{\mathbf{v}} + \tau\frac{\dot{v}^2}{v^2}\mathbf{v} = \mathbf{f}/m \tag{1.9}$$

which is the equation to be developed in the next section by a different route. This equation was in fact developed by Planck[7] and discarded as being too complex.

1.7 All later studies appear to start with the Abraham-Lorentz equation, putting a great deal of effort into explaining away the clearly erroneous solutions. Professor Erber[1] proved that the A_L equation only applied to quasi periodical motion, and suspected that '*we may be missing something dynamically important in averaging'*.

1.8 Apart from the demonstration of the failure of the solutions of the A-L equation, it is easy to demonstrate the mathematical error in the derivation. As shown above, the derivation transforms a definite integral by integration by parts, and then equates integrands, which in general is not allowed. If we imagine a force that leads to a velocity given by

$$\mathbf{v} = 2\mathbf{v}_0\lambda t(1-\lambda t) \tag{1.10}$$

and carry out the required differentiations, the two integrands become respectively proportional to

$$\left.\begin{array}{l} 4v_0^2\lambda^2[1-2\lambda t]^2 \\ -8v_0^2\lambda^3 t[1-\lambda t] \end{array}\right\} \tag{1.11}$$

1.9 The equation to be developed in this study will assume a point particle, and we immediately have a problem in that relativity theory will not allow this. In particular if the density of a massive particle exceeds a critical value, its radius falls below the Schwarzschild radius and forms a black hole. If this occurs, a black hole will evaporate via the Hawking radiation. The mass of the electron is such that it would evaporate extremely rapidly and so we are faced with the necessity of considering a finite electron with a radius greater than the Schwarzschild radius. Fortunately the Schwarzschild radius for an electron is of order 10^{-57} m and is some 42 orders of magnitude less than the classical electron radius. The electron is well approximated as a point particle provided electrons approaching to distances of order 10^{-15}m are not considered.

1.10 The effective radius of the electron is discussed in a later section where support for the electron radius to be 1.5 times the classical radius is obtained independently from the mass/energy consideration.

2 NON-RELATIVISTIC MOTION

2.1 The Equation of Motion

2.1.1 Consider a charged particle released in a force field. The force causes the particle to move with a resultant change of momentum. In the absence of radiation the equation is

$$\frac{d\mathbf{p}}{dt} = \mathbf{f} \tag{2.1}$$

To find how to modify this equation to take into account the fact that accelerating charges radiate, write for the kinetic energy of a particle

$$E_{kin} = \frac{1}{2}m\mathbf{v.v} = \frac{1}{2}\frac{\mathbf{p.p}}{m} \tag{2.2}$$

Differentiating

$$\frac{dE_{kin}}{dt} = \frac{1}{m}\mathbf{p.}\frac{d\mathbf{p}}{dt}\mathbf{cos}\,\theta \tag{2.3}$$

However, if the change in momentum is entirely due to a loss of kinetic energy, and not to some force that does no work,

$$\cos\theta = 1 \tag{2.4}$$

In other words it is only the component of the change in momentum that is parallel to the momentum that can cause a loss of kinetic energy. A loss of kinetic energy implies a change of momentum parallel to the momentum.

2.1.2 Denoting the loss of kinetic energy due to radiation as E_{rad}, write

$$\frac{dE_{rad}}{dt} = \frac{1}{m}\mathbf{p}\frac{d\mathbf{p}}{dt} \tag{2.5}$$

and accordingly the magnitude of the momentum not acquired by the charge is given by

$$\frac{dp_{rad}}{dt} = \frac{m}{p}\frac{dp_{rad}}{dt} \qquad (2.6)$$

This loss of momentum is in the direction of the momentum, and thus

$$\frac{d\mathbf{p}_{rad}}{dt} = \frac{m}{p}\frac{dE_{rad}}{dt}\frac{\mathbf{p}}{p} \qquad (2.7)$$

Summing the rate at which the charge is acquiring momentum and the rate of loss of momentum due to radiation, and equating this to the applied force, the equation of motion becomes

$$\frac{d\mathbf{p}}{dt} + \frac{m}{p^2}\frac{dE_{rad}}{dt}\mathbf{p} = \mathbf{f} \qquad (2.8)$$

The rate of loss of energy from an accelerating charged particle is

$$\frac{dE_{rad}}{dt} = m\tau\dot{v}^2 \qquad (2.9)$$

Substituting this expression in the momentum equation and expressing the result in terms of velocity, the non-relativistic equation of motion for a radiating charged particle becomes

$$\dot{\mathbf{v}} + \tau\frac{\dot{v}^2}{v^2}\mathbf{v} = \frac{\mathbf{f}}{m} \qquad (2.10)$$

2.1.3 Note that setting $\tau = 0$, ie no radiation occurring, the equation reduces to the Newtonian form. Inclusion of the radiation term leads to non-linear term as expected, and the equation remains of second order, ie first order in velocity. If the scalar product of this equation with velocity is formed,

$$\frac{dE_{kin}}{dt} + \frac{dE_{rad}}{dt} = \frac{dW}{dt} \qquad (2.11)$$

where dW/dt is the rate at which work is being done by the force.

2.1.4 Consider the equation of motion with zero applied force, and make the assumption that the acceleration is non-zero,

$$\dot{\mathbf{v}} + \tau\frac{\dot{v}^2}{v^2}\mathbf{v} = 0 \qquad \mathbf{(2.12)}$$

This implies that the acceleration is parallel to the velocity, and so the equation reduces to the scalar equation

$$1 + \tau\frac{\dot{v}}{v} = 0 \qquad (2.13)$$

which has a solution, which is the only solution

$$v = v_0 e^{-\frac{t}{\lambda}} \qquad (2.14)$$

This states that any velocity will decrease rapidly to zero in the absence of an applied force. This non-physical result implies that the initial assumptions are false. Specifically, if the applied force is zero, the acceleration cannot be non-zero!

2.1.5 It has now been demonstrated that this equation preserves causality and conserves energy, two of the fundamental requirements for a classical equation of motion.

2.2 Rectilinear Motion

2.2.1 For linear motion the three vectors in the equation of motion become co-linear, and the equation reduces to the scalar equation

$$\dot{v} + \tau \frac{\dot{v}^2}{v} = \frac{f}{m} \tag{2.15}$$

Treating this equation as a quadratic in the acceleration

$$\dot{v} = -\frac{v}{2\tau}\left[1 \pm \sqrt{1 + \frac{4f\tau}{mv}}\right] \tag{2.16}$$

Inspection shows that the negative sign must be taken, a positive sign giving negative acceleration for positive forces. The equation to be solved then becomes

$$\dot{v} = \frac{v}{2\tau}\left[\sqrt{1 + \frac{4\tau f}{mv}} - 1\right] \tag{2.17}$$

2.2.2 **N**ow consider a constant force. Rearranging the above and integrating

$$\int_0^v \frac{dv}{v\left[\sqrt{1 + \frac{4\tau f}{mv}} - 1\right]} = \frac{t}{2\tau} \tag{2.18}$$

Observing that

$$\dot{v} = v\frac{dv}{dx} \tag{2.19}$$

$$\int \frac{dv}{\left[\sqrt{1 + \frac{4\tau f}{mv}} - 1\right]} = \frac{x}{2\tau} \tag{2.20}$$

The integrands in these two equations are reduced to rational algebraic functions by the substitution

$$w = \sqrt{1 + \frac{4\tau f}{mv}} \tag{2.21}$$

yielding

$$\int \frac{wdw}{(w+1)(w-1)^2} = \frac{-t}{4\tau} \tag{2.22}$$

$$\int \frac{wdw}{(w+1)^2(w-1)^3} = -\frac{mx}{16\tau^2 f} \tag{2.23}$$

Carrying out the indicated integrations and reverting to the original variables

$$\frac{t}{\tau} = \frac{2}{\left[\sqrt{1 + \frac{v_0}{v}} - 1\right]} + \ln\left\{\frac{\sqrt{1 + \frac{v_0}{v}} + 1}{\sqrt{1 + \frac{v_0}{v}} - 1}\right\} \tag{2.24}$$

$$\frac{mx}{\tau^2 f} = \frac{2}{\sqrt{1+\frac{v_0}{v}}} + \frac{2}{\left[\sqrt{1+\frac{v_0}{v}}-1\right]^2} - \ln\left\{\frac{\sqrt{1+\frac{v_0}{v}}+1}{\sqrt{1+\frac{v_0}{v}}-1}\right\} \tag{2.25}$$

where

$$\frac{4f\tau}{m} = v_0 \tag{2.26}$$

Introducing the dimension-less variable

$$\mathbf{T} = \frac{\mathbf{t}}{\tau} \tag{2.27}$$

the graph obtained for the velocity is presented in Fig1.

2.3 Discontinuous Motion

2.3.1 A problem arises when a retarding force acting on the charge is considered. Writing

$$f \to -f \tag{2.28}$$

$$\dot{v} = \frac{v}{2\tau}\left\{\sqrt{1-\frac{4f\tau}{mv}}-1\right\} \tag{2.29}$$

For

$$v < \frac{4\tau f}{m} \tag{2.30}$$

this expression becomes complex, and so for a real solution, there is a minimum positive velocity given by

$$v_m = \frac{4f\tau}{m} \tag{2.31}$$

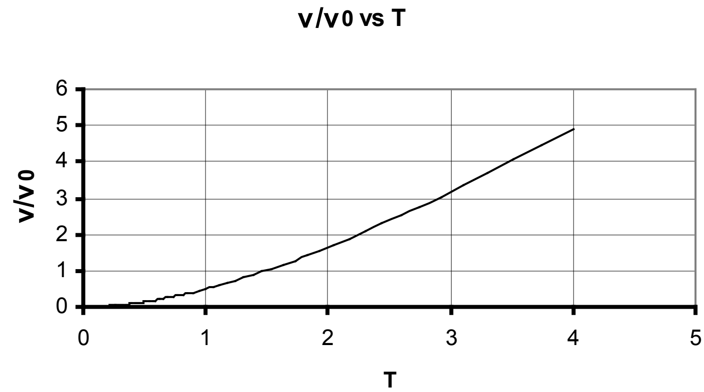


Fig1

2.3.2 To sketch the phase plane of the motion it is to be observed that

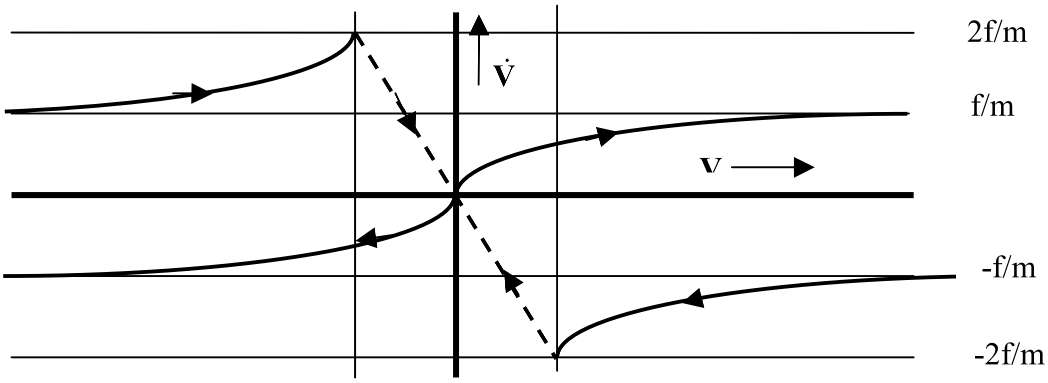
$$\begin{array}{ll} \dot{v}_{v\to 0} \to 0 & \dot{v}_{v\to\infty} \to -\frac{f}{m} \\ \dot{v}_{v\to\infty} \to \frac{f}{m} & \dot{v}_{v\to v_m} \to -\frac{2f}{m} \\ \left.\frac{d\dot{v}}{dv}\right._{v\to 0} \to -\infty & \left.\frac{d\dot{v}}{dv}\right._{v\to v_m} \to -\infty \end{array} \tag{2.31}$$

In addition for a positive force

$$\dot{v}_{v \to -v_m} = \frac{2f}{m} \tag{2.32}$$

2.3.3 The discontinuous nature of the solution as zero velocity is approached in either direction is to be noted. Such behaviour is typical of the solutions to non-linear equations, and may be interpreted as the radiation reaction becoming impulsive. To see this, write

$$\dot{v} + A\delta(t - a) = \frac{f}{m} \tag{2.33}$$



Sketch of Phase Plane Motion of Electron Under a Constant Force
Fig 2

where a is the time to v_m. Integrating from a-ε to a and proceeding to the limit

$$\left\{ \frac{1}{2} v_m + \frac{1}{2} A \right\} u(t - a) = 0 \tag{2.34}$$

$$A = -\frac{4\tau f}{m} \tag{2.35}$$

The radiated energy is then

$$E_{rad} = m\tau \int \dot{v}^2 dt = \frac{8f^2\tau^2}{m} \tag{2.36}$$

The kinetic energy lost by the electron is

$$E_{kin} = \frac{1}{2} m v_m^2 = \frac{8f^2\tau^2}{m} \tag{2.37}$$

2.4 Total Radiation During Linear Acceleration

2.4.1 The standard approach to calculating the energy radiated by an accelerating electron is to calculate the motion ignoring the radiation and to follow this by integrating the radiation loss over the acceleration history. If this process is carried out for linear motion, ignoring radiation,

$$v = \frac{ft}{m} \tag{2.38}$$

where a constant force has been assumed and the initial velocity was taken to be zero. The kinetic energy of the electron is then

$$E_{kin} = \frac{1}{2} m v^2 = \frac{m}{2}\left(\frac{ft}{m}\right)^2 \qquad \textbf{(2.39)}$$

The radiation rate is

$$\frac{dE_{rad}}{dt} = m\tau\dot{v}^2 = m\tau\left(\frac{f}{m}\right)^2 \qquad \textbf{(2.40)}$$

Integrating

$$E_{rad} = m\tau\left(\frac{f}{m}\right)^2 t \qquad (2.41)$$

The fractional loss of energy is then

$$\left(\frac{E_{rad}}{E_{kin}}\right)_{st} = \frac{2\tau}{t} \qquad (2.42)$$

To compare this result with deductions based on the new equation of motion, it is necessary to obtain an approximate explicit solution for the velocity. With the assumption

$$\frac{v_0}{v} << 1 \qquad (2.43)$$

the time taken to attain a given velocity is

$$t = \frac{1 + \frac{f\tau}{mv} \ell n\left[1 + \frac{mv}{f\tau}\right]}{\frac{f}{mv}} \qquad (2.44)$$

Solving for v

$$v = \frac{\frac{ft}{m}}{1 + \frac{f\tau}{mv} \ell n\left[1 + \frac{mv}{f\tau}\right]} \qquad (2.45)$$

The first iterative solution is obtained by noting that

$$v \approx \frac{ft}{m} \qquad (2.46)$$

yielding

$$v \approx \frac{\frac{ft}{m}}{1 + \frac{\tau}{t} \ell n\left(1 + \frac{t}{\tau}\right)} \qquad (2.47)$$

Substitution into the kinetic energy and making use of the binomial theorem

$$E_{kin} = \frac{m}{2}\left(\frac{ft}{m}\right)^2 \left(1 - \frac{2\tau}{t} \ell n\left(1 + \frac{t}{\tau}\right)\right) \qquad (2.48)$$

The logarithmic term represents the radiated energy. Comparing the two calculations

$$\frac{E_{rad}}{E_{radst}} = \ell n\left(1 + \frac{t}{\tau}\right) \qquad (2.49)$$

For a rectangular pulse lasting 50 ns, the ratio is 36.6!

3 THE RELATIVISTIC EQUATION OF MOTION

3.1 The relativistic equation of motion is now derived in the covariant 4-vector notation of relativity. Ensuring that the terms of the equation are 4-vectors guarantees that the equation is invariant under a Lorentz transformation. All 4-vectors are denoted by bold face capitals, and 3-vectors by bold face lower case as before. Introducing T as the proper time, the equation of motion for a non-radiating particle is

$$\frac{\mathbf{dP}}{\mathrm{dT}} = \mathbf{F} \tag{3.1}$$

where **P** is the 4-momentum and **F** the 4-force, and this may be written displaying the space-like and time-like components

$$\frac{\mathrm{d}}{\mathrm{dT}}(\mathbf{p}, \mathrm{imc}) = \left(1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{-1/2} \left(\mathbf{f}, \frac{\mathrm{i}}{\mathrm{c}} \mathbf{f.v}\right) \tag{3.2}$$

where **p is** the 3-momentum

$$\mathbf{p} = \mathrm{m}\mathbf{v} = \frac{\mathrm{m}_0 \mathbf{v}}{\sqrt{1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}}} \tag{3.3}$$

and it is to be noted that the time-like component of the 4-force is the rate at which work is being done by the force on the particle. To modify the equation of motion a 4-vector is introduced that represents the momentum not acquired by the particle by virtue of the particle radiating,

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{dT}} + \frac{\mathrm{d}\mathbf{P}_{\mathrm{rad}}}{\mathrm{dT}} = \mathbf{F} \tag{3.4}$$

The relativistic equation for the rate of loss of energy by an accelerating particle is

$$\dot{\mathrm{E}}_{\mathrm{rad}} = \frac{\mathrm{m}_0 \tau}{\left(1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^2} \left\{ \dot{\mathbf{v}}.\dot{\mathbf{v}} + \frac{(\dot{\mathbf{v}}.\mathbf{v})^2}{\mathrm{c}^2 \left(1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{3/2}} \right\} \tag{3.5}$$

3.2 To relate this loss to the components of $\mathbf{p}_{\mathrm{rad}}$ the energy is written as

$$\mathrm{E} = \frac{\mathrm{m}_0 \mathrm{c}^2}{\sqrt{1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}}} \tag{3.6}$$

Differentiating with respect to t

$$\dot{\mathrm{E}} = \frac{\mathrm{m}_0 \mathrm{v}\dot{\mathrm{v}}}{\left(1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{3/2}} \tag{3.7}$$

The rate of energy loss by radiation may then be written

$$\dot{\mathrm{E}}_{\mathrm{rad}} = \frac{\mathrm{m}_0 \mathrm{v}\dot{\mathrm{v}}_{\mathrm{rad}}}{\left(1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{3/2}} \tag{3.8}$$

where $\dot{\mathrm{v}}_{\mathrm{rad}}$ is the rate of change of velocity due to the radiation of energy. This radiation is acting to retard the particle and so the rate of change of momentum due to the radiation emission is in the direction of the momentum. Accordingly an expression must be obtained for the rate of change of momentum due to the radiation under the condition that the direction of the particle does not change, that is

$$\dot{\mathbf{p}}_{\mathrm{rad}} = \frac{\partial}{\partial t}[\mathbf{p}]_{\left(\frac{\mathbf{v}}{v}\right)} \tag{3.9}$$

where the notation has been borrowed from thermodynamics. To obtain this derivative the momentum is written in the form

$$\mathbf{p} = \frac{m_0 v}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}\frac{\mathbf{v}}{v} \tag{3.10}$$

making the unit vector in the direction of the momentum explicit. Carrying out the differentiation

$$\dot{\mathbf{p}}_{\mathrm{rad}} = \frac{m_0 \dot{v}_{\mathrm{rad}}}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}.\frac{\mathbf{v}}{v} \tag{3.11}$$

This becomes, on introducing the radiation rate

$$\dot{\mathbf{p}}_{\mathrm{rad}} = \frac{\dot{E}_{\mathrm{rad}}}{v^2}\mathbf{v} \tag{3.12}$$

Noting that

$$\frac{dt}{dT} = \left(1-\frac{v^2}{c^2}\right)^{-1/2} \tag{3.13}$$

the rate of change of mass is

$$\dot{m} = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\frac{v\dot{v}}{c^2} \tag{3.14}$$

and similarly

$$\dot{m}_{\mathrm{rad}} = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\frac{v\dot{v}_{\mathrm{rad}}}{c^2} = \frac{\dot{E}_{\mathrm{rad}}}{c^2} \tag{3.15}$$

3.3 With these results the equation of motion becomes

$$\left\{\frac{d}{dt}\frac{m_0\mathbf{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}, ic\frac{d}{dt}\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}\right\} + \left\{\dot{E}_{\mathrm{rad}}\frac{\mathbf{v}}{v^2}, i\frac{\dot{E}_{\mathrm{rad}}}{c}\right\} = \left\{\mathbf{f}, \frac{i}{c}\mathbf{f.v}\right\} \tag{3.16}$$

Separating this equation into its space-like and time-like components

$$\frac{d}{dt}\frac{m_0\mathbf{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + \dot{E}_{\mathrm{rad}}\frac{\mathbf{v}}{v^2} = \mathbf{f} \tag{3.17}$$

$$\frac{d}{dt}\frac{m_0c^2}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}+\dot{E}_{rad}=\mathbf{f.v} \tag{3.18}$$

The time-like component is just a statement of the conservation of energy. Substituting for $\dot{E}_{rad}$ in the space-like component, the three-vector equation of motion becomes

$$\frac{d}{dt}\frac{m_0\mathbf{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}+\frac{m_0\tau}{\left(1-\frac{v^2}{c^2}\right)^{2}}\left\{\dot{\mathbf{v}}.\dot{\mathbf{v}}+\frac{(\mathbf{v}.\dot{\mathbf{v}})^2}{c^2\left(1-\frac{v^2}{c^2}\right)}\right\}\frac{\mathbf{v}}{v^2}=\mathbf{f} \tag{3.19}$$

It is to be observed that this is the result that can be obtained directly from the non-relativistic equation by replacing both the momentum and the radiation loss by their relativistic forms.

3.4 One dimensional motion can be studied as in the non-relativistic case, similar results being obtained concerning the discrepancy in radiated energy. Motion in a magnetic field is as in the standard approach with the addition of a low amplitude modulation at a frequency $\omega^2\tau$. The study of oscillatory motion leads to the interesting observation that strictly harmonic oscillation is not possible, a certain drift velocity being required to remove the need for infinite forces. In addition it can be shown that it is only possible to give finite energy to an electron, the radiated energy increasing at a more rapid rate than the increase in kinetic energy. Intrinsically interesting as these effects may be, their development is delayed, being considered in Section 15. Rather we wish to pursue the consequences of imposing a further observation on electron motion, that of the existence of stationary states.

4. THE STATIONARY STATE HYPOTHESIS

4.1 The Hypothesis

A number of results were obtained for various forces acting on an electron, and as is to be expected the radiation term leads to decay of the motion. It can be shown that only a finite amount of energy can be imparted to an electron either impulsively or by sinusoidal fields. The unexpected result of discontinuous motion in the case of electron-electron bremsstrahlung suggested that this equation might be closer to describing the quantum world than previous formulations of the electron equation of motion. The problem that then presents itself is what minimum assumption needs to be added to classical physics to obtain results consistent with quantum mechanics. The simplest hypothesis is that for stationary states *the radiation time constant becomes imaginary*.

4.2 A Heuristic Solution

A heuristic justification for this hypothesis is that changing the coefficient of a dissipatory term from real to imaginary will turn the solutions that decay into solutions that oscillate. Making the analogy with an LC circuit connected to an antenna which presents a radiation resistance R, the solution is a decaying oscillation with parameter

$$\lambda = \frac{\left\{-1 \pm i\sqrt{1 - \frac{R^2C}{2L}}\right\}}{2RC} \qquad (4.1)$$

If the radiation resistance is increased to infinity, leaving the inductance and capacitance the same, the parameter becomes

$$\lambda = \frac{-i}{\sqrt{LC}} = -\frac{i}{\tau} \qquad (4.2)$$

The increase of a real parameter to infinity is more appealing than the switching of a parameter from real to imaginary! 'Elementary' inductance and capacitance can be identified by adopting the approach of Hallen[8], writing

$$L' = \frac{Z_0^2}{mc^2}\frac{q^2}{4\pi} \qquad C' = \frac{q^2}{4\pi mc^2} \qquad (4.3)$$

We then have

$$\tau = \frac{2}{3}\sqrt{L'C'} \qquad (4.4)$$

This analogy will be pursued in a later section with interesting results**.**

4.3 The following sections are concerned with applying the above hypothesis to an assumed point electron in zero external field. In so doing a model of the electron emerges that can not only be visualised, but is also consistent with the known properties of the electron. In addition a limiting process shows that the model is consistent with neutrinos and photons. A reinterpretation of Dirac's relativistic quantum mechanical equation for the electron gives rise to essentially the same electron model.

5 THE STATIONARY STATE EQUATION

5.1 Rectilinear Motion

We consider first rectilinear motion. The non-relativistic equation for linear motion is

$$\dot{v} + \tau\frac{\dot{v}^2}{v} = \frac{f}{m} \qquad (5.1)$$

Making the replacement

$$\tau \to i\tau \qquad (5.2)$$

the equation becomes, on setting f to zero and cancelling out the acceleration now assumed not to be zero

$$\frac{\dot{v}}{v} = -\frac{i}{\tau} \qquad (5.3)$$

and this has solution

$$v = v_0\, e^{\frac{-it}{\tau}} \qquad (5.4)$$

We assumed linear motion and have obtained a complex solution. The imaginary part of the solution is not to be discarded- it means that motion also occurs in a direction orthogonal to the one chosen, that is we have two-dimensional motion and in particular we have motion in a circle. We have arrived at a model of a spinning electron!

5.2 Two Dimensional Motion

To confirm that this really is the case we now consider two-dimensional motion, and we write

$$z = x + iy \tag{5.5}$$

The equation for stationary states becomes

$$-\ddot{z} + \tau \frac{[-\ddot{z}.-\ddot{z}^*]}{i\dot{z}.-i\dot{z}^*}(-i\dot{z}) = 0 \tag{5.6}$$

Reducing to its simplest terms

$$\frac{\ddot{z}^*}{\dot{z}^*} = -\frac{1}{i\tau} \tag{5.7}$$

Taking the complex conjugate

$$\frac{\ddot{z}}{\dot{z}} = -\frac{i}{\tau} \tag{5.8}$$

Integrating, the solution becomes

$$\dot{z} = \dot{z}_0 e^{\frac{-it}{\tau}} \tag{5.9}$$

Integrating again

$$z = i z_0 e^{-\frac{it}{\tau}} \tag{5.10}$$

The motion is circular with radius $v_0\tau$ and constant speed v_0, and we observe that this motion is consistent with the 'zitterbewegung'[5] of quantum mechanics. If this spinning charge is to represent the observed electron, the associated parameters must agree, and so we attempt to match the angular momentum

$$m_0 v_0^2 \tau = \frac{\hbar}{2} \tag{5.11}$$

Assuming m_0 is the observed rest mass of the electron, we obtain for the velocity

$$v_0 = \sqrt{\frac{\hbar}{2m_0\tau}} \sim 3.10^9\,\mathrm{m/s} \tag{5.12}$$

and so we see that a relativistic treatment is necessary.

6 THE RELATIVISTIC SPINNING ELECTRON

6.1 The Relativistic Equation

The relativistic equation of motion is, for zero applied force,

$$\frac{d}{dt}\frac{m\mathbf{v}}{[1-\frac{v^2}{c^2}]^{1/2}} + \frac{m\tau}{[1-\frac{v^2}{c^2}]^2}\{\dot{v}^2 + \frac{(\mathbf{v}.\dot{\mathbf{v}})^2}{c^2[1-\frac{v^2}{c^2}]}\}\frac{\mathbf{v}}{v^2} = 0 \tag{6.1}$$

We have previously considered the electron to have a rest mass m_0, but on this model the electron is taken to be moving at speeds close to c, and it will be the spinning electron that will have an apparent rest mass m_0. The postulated particle will have an intrinsic mass m_i, assumed to be non-electromagnetic, such that the relativistic speed gives the observed rest mass. We observe that the radiation time constant depends on m, and so we write

$$m_i \tau_i = m_0 \tau_0 \tag{6.2}$$

Heitler[9]discusses the need to consider mechanical mass for an electron '*we must---attribute to the electron a mechanical inert mass*-----'. Classically internal stresses have to be taken into account for the electron to have the right relativistic transformation properties. Quantum mechanics shows that the internal stresses vanish for an electron at rest. Detailed analysis of the relativistic motion of a finite charged sphere is carried out in [6]

6.2 We now make the assumption of circular motion at constant speed as indicated in the non-relativistic solution. We then have

$$\frac{d}{dt} v^2 = 0 \tag{6.3}$$

$$v.\dot{v} = 0 \tag{6.4}$$

and the equation reduces to

$$\dot{v} + \frac{\tau \dot{v}^2}{[1 - \frac{v^2}{c^2}]^{3/2}} \frac{v}{v^2} = 0 \tag{6.5}$$

Making the replacements

$$r \rightarrow z \qquad \tau \rightarrow i\tau \tag{6.6}$$

the equation becomes

$$-\ddot{z} + \frac{\tau_i}{[1 - \frac{v^2}{c^2}]^{3/2}} \frac{\ddot{z}\,\ddot{z}^*}{i\dot{z}^*} = 0 \tag{6.7}$$

Simplifying

$$\frac{\ddot{z}^*}{\dot{z}^*} = \frac{i}{\tau_i}[1 - \frac{v^2}{c^2}]^{3/2} \tag{6.8}$$

Taking the complex conjugate

$$\frac{\ddot{z}}{\dot{z}} = \frac{-i}{\tau_i}[1 - \frac{v^2}{c^2}]^{3/2} \tag{6.9}$$

and the solution is

$$\dot{z} = v_0 \exp\{-i[1 - \frac{v^2}{c^2}]^{3/2} \frac{t}{\tau}\} \tag{6.10}$$

Integrating again

$$z = \frac{iv_0 \tau_i}{[1 - \frac{v^2}{c^2}]^{3/2}} \exp\{-i[1 - \frac{v^2}{c^2}]^{3/2} \frac{t}{\tau}\} \tag{6.11}$$

and the radius of the circle is

$$r = \frac{v \tau_i}{[1 - \frac{v^2}{c^2}]^{3/2}} \tag{6.12}$$

7. THE ELECTRON SPIN

7.1 The Electron Spin Velocity

The angular momentum is

$$\Omega = m_0 v \times r = \frac{m_0 v^2 \tau_i}{[1 - \frac{v^2}{c^2}]^{3/2}} \hat{n} = \frac{m_0 v^2 \tau}{[1 - \frac{v^2}{c^2}]^2} \hat{n} \tag{7.1}$$

Equating the magnitude to the known value of the electron spin

$$m_0 \tau \frac{v^2}{\left[1 - \frac{v^2}{c^2}\right]^2} = \frac{\hbar}{2} \tag{7.2}$$

or

$$\frac{v}{[1 - \frac{v^2}{c^2}]} = \pm \sqrt{\frac{\hbar}{2m_0 \tau}} \tag{7.3}$$

Introducing α, the fine structure constant

$$\alpha = \frac{3}{2} \frac{m_0 c^2 \tau}{\hbar} \tag{7.4}$$

we have

$$\frac{v}{[1 - \frac{v^2}{c^2}]} = \pm c \sqrt{\frac{3}{4\alpha}} \tag{7.5}$$

Rearranging we obtain a quadratic for v/c

$$\frac{v^2}{c^2} \sqrt{\frac{3}{4\alpha}} + \frac{v}{c} + \sqrt{\frac{3}{4\alpha}} = 0 \tag{7.6}$$

Taking the upper sign

$$\frac{v}{c} = \frac{-1 \pm \sqrt{1 + \frac{3}{\alpha}}}{\sqrt{\frac{3}{\alpha}}} \tag{7.7}$$

We must take the positive sign for v <c, with the result that

$$\frac{v}{c} = \frac{\sqrt{1 + \frac{3}{\alpha}} - 1}{\sqrt{\frac{3}{\alpha}}} = 0.9518956034 \tag{7.8}$$

7.2 The Electron Intrinsic Mass

The intrinsic mass of the electron is then given by

$$m_i = m_0 \sqrt{\left(1 - \frac{v^2}{c^2}\right)}. = 0.306422519 m_0 \tag{7.9}$$

7.3 Angular Velocity

The angular velocity of the charge is

$$\omega = \frac{\left[1-\dfrac{v^2}{c^2}\right]^{3/2}}{\tau_i} = \frac{\left[1-\dfrac{v^2}{c^2}\right]^{2}}{\tau} \qquad (7.10)$$

From the solution for the spin velocity

$$[1 - \frac{v^2}{c^2}]^2 = [\frac{v}{c}]^2.\frac{4\alpha}{3} \qquad (7.11)$$

and so the angular velocity on the present theory is

$$\omega = \left[\frac{v}{c}\right]^2 \frac{4\alpha}{3\tau} \qquad (7.12)$$

The quantum mechanical result for the 'zitterbewegung' gives the frequency

$$\omega = \frac{2E}{\hbar} = \frac{2m_0c^2}{\hbar} = \frac{4\alpha}{3\tau} \qquad (7.13)$$

Comparison of the two results shows that the present model agrees with the Dirac equation on setting the intrinsic mass to zero, ie v=c.
The spin radius of the circle is

$$r_s = \frac{v\tau}{[1 - \dfrac{v^2}{c^2}]^2} \qquad (7.14)$$

This reduces to

$$r_s = \frac{c^2\tau}{4v}\frac{3}{4\alpha} \qquad (7.15)$$

Introducing the Compton wavelength via the relation

$$ƛ_c = \frac{3c\tau}{2\alpha} \qquad (7.16)$$

the expression becomes

$$2r_s = \left(\frac{c}{v}\right)ƛ = 1.05053537ƛ \qquad (7.17)$$

i.e. the spin circle diameter is just over a Compton wavelength.

8. THE QUANTUM MECHANICAL ‘ZITTERBEWEGUNG’

8.1 Solution to Dirac’s Equation

It can be shown[10] that Dirac’s relativistic equation for the electron, using the Heisenberg representation, gives

$$\dot{x} = \frac{i}{\hbar}(Hx - xH) = \frac{i}{\hbar}c\alpha_x(p_x x - xp_x) = c\alpha_x \qquad \textbf{(8.1)}$$

and in general

$$\mathbf{v} = c\boldsymbol{\alpha} \qquad \text{or} \quad v = c \qquad (8.2)$$

For an electron with no classical momentum it can be shown that

$$\dot{x} = \frac{i}{2} c\hbar \left(\dot{\alpha}_x\right)_{t=0} e^{-\frac{2iH_0}{\hbar}t} H_0^{-1} \tag{8.3}$$

and we have

$$H^2 H^{-1} = H \ I = E^2 H^{-1} \tag{8.4}$$

$$H^{-1} = H / E^2 \tag{8.5}$$

A measurement of $\dot{x}$ then gives

$$\dot{x} = \frac{ic\hbar}{2E_0} \left(\dot{\alpha}_x\right)_{t=0} e^{-\frac{2iE_0}{\hbar}t} \tag{8.6}$$

where E_0 is the rest mass energy. In explanation of the 'Zitterbewegung' the imaginary part is ignored[10], and the rapid oscillation of the real part is taken and it is shown that the velocity of this motion is c. However the imaginary part should not be ignored. Obtaining a complex solution to a 1-dimensional problem indicates that the solution is 2-dimensional. Accordingly we write

$$\begin{aligned} x &\Rightarrow Z = re^{i\omega t} \\ \dot{x} &\Rightarrow \dot{Z} = i\omega re^{i\omega t} + \dot{r}e^{i\omega t} \end{aligned} \tag{8.7}$$

Z being used to avoid confusion with the z-axis. Rotating the x-axis to correspond to the instantaneous position of r

$$\dot{Z} = \frac{ic\hbar}{2E_0} \left(\dot{\alpha}_r\right)_{t=0} e^{-\frac{2iE_0}{\hbar}t} \tag{8.8}$$

Equating the two forms for $\dot{Z}$

$$\dot{r} = 0 \tag{8.9}$$

$$\omega = -\frac{2E_0}{\hbar} \tag{8.10}$$

$$r = \left| \frac{ic\hbar^2}{4E_0^2} \left(\dot{\alpha}_r\right)_{t=0} \right| \tag{8.11}$$

Observing that

$$\alpha_x = \frac{i\hbar}{2} \frac{\left(\dot{\alpha}_x\right)_{t=0}}{E_0} e^{-\frac{2iE_0}{\hbar}t} \tag{8.12}$$

we obtain

$$r = \frac{c\hbar}{2E_0} \alpha_r \tag{8.13}$$

a measurement of r then yielding

$$r = \frac{c\hbar}{2E_0} = \frac{\lambda\!\!\!^{-}}{2} \tag{8.14}$$

and so

$$v = r\omega = \frac{c\hbar}{2E_0} \frac{2E_0}{\hbar} = c \tag{8.15}$$

This shows that the proper interpretation of 'Zitterbewegung' is that a 'stationary' electron is in fact moving around a circle of diameter $\lambda\!\!\!^{-}$ at a velocity of c. Notice that E_0 is the rest mass energy, and that this is entirely associated with the spin.

8.2 The Angular Momentum

The angular momentum is

$$\boldsymbol{\Omega} = m\mathbf{v}\times\mathbf{r} = \frac{E_0}{c^2}.c\frac{c\hbar}{2E_0}\hat{\mathbf{n}} = \frac{\hbar}{2}\hat{\mathbf{n}} \tag{8.17}$$

8.3 Zero Intrinsic Mass

The intrinsic mass model that has been developed is consistent with this result if $m_i=0$ while still retaining a charge, the radius and the frequency depending directly on c/v. A difficulty is that the limiting processes indicate that the charge vanishes with mass, as is shown in §10 and electromagnetic fields are not known to transport charge.

9. THE ELECTRON MAGNETIC MOMENT

9.1 The magnetic moment is the turning force per unit magnetic flux density,

$$\boldsymbol{\mu} = q\mathbf{v}\times\mathbf{r} = \mathbf{q}\frac{\mathbf{v}^2\tau}{[1-\frac{v^2}{c^2}]^2}\hat{\mathbf{n}} = \frac{q\hbar}{2m_0}\hat{\mathbf{n}} = \boldsymbol{\mu}_\mathbf{B} \tag{9.1}$$

that is the magnetic moment is the Bohr Magneton.

10. THE NEUTRINO AND THE PHOTON

10.1 The equation of motion is not restricted to electrons and we are at liberty to consider other parameters. We write for an arbitrary spin half particle

$$m_i\tau_i\frac{v^2}{[1-\frac{v^2}{c^2}]^2} = \frac{\hbar}{2} \tag{10.1}$$

If we consider the possibility of $m_i \to 0$ we must expect $v \to c$ in such a way that

$$\underset{\substack{m_i\to 0\\ v\to c}}{\mathrm{Lt}}\frac{m_i}{\left[1-\frac{v^2}{c^2}\right]^{\frac{1}{2}}} = \frac{E}{c^2} \tag{10.2}$$

where E is the energy of the particle. Inserting this result

$$\underset{\substack{m_i\to 0\\ v\to c}}{\mathrm{Lt}}\frac{E}{c^2}.\frac{\tau_i c^2}{\left[1-\frac{v^2}{c^2}\right]^{\frac{3}{2}}} = \frac{\hbar}{2} \tag{10.3}$$

For this limit to exist, we must have $\tau_i \to 0$ as $v \to c$ in such a way that

$$\underset{\substack{m_i\to 0\\ v\to c}}{\mathrm{Lt}}\frac{\tau_i}{\left[1-\frac{v^2}{c^2}\right]^{\frac{3}{2}}} = \frac{\hbar}{2E} \tag{10.4}$$

Note that as $\tau_i = 0$, we have $q = 0$ and so we observe that charge and mass are tied together, suggesting that they are aspects of the same reality. In particular this result strongly suggests that the electron must have intrinsic mass. The radius of rotation is given by

$$2r = 2 \operatorname*{Lt}_{\substack{\tau_i \to 0 \\ v \to c}} \frac{c\tau_i}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{3}{2}}} = \frac{\hbar c}{E} = \lambda\!\!\!^{-}_c \tag{10.5}$$

The spin diameter is then seen to be the Compton wavelength of a massless neutrino.

10.2 Considering a particle with spin 1, i.e. a photon, and writing

$$E = h\nu \tag{10.6}$$

we have

$$2r = \frac{c}{\nu} = \lambda \tag{10.7}$$

The stationary state equation is then consistent with the existence of photons.

10.3 Summary and Discussion

The relativistic equation of motion supplemented by the stationary state hypothesis has resulted in a visualizable model of a spinning electron, that is a point charge rotating around a centre with a velocity close to c, the motion being such that the observed mass is given by the relativistic increase in an intrinsic mass $m_i \sim m_0/3$. The rotation diameter is just greater than the Compton wavelength of the electron and the angular frequency is slightly less than the quantum mechanical result for 'Zitterbewegung', this being attributable to a non-zero intrinsic mass on this model. Further analysis of the 'Zitterbewegung' shows that the circular motion solution is consistent with Dirac's theory of the electron. The magnetic moment of the electron on this model is one Bohr magneton. A mathematical limiting process leads to a model of the neutrino. It appears to follow from the equations that mass and charge vanish together, suggesting that they are different aspects of the same reality. However the principle results, the radius of rotation and the circular frequency, are directly dependent on the ratio v/c. Setting v=c is equivalent to setting $m_i=0$, and the results agree with quantum mechanics.

10.4 The results appear to be deterministic, but this is illusory as we cannot know the initial conditions. The results simply tell us *how* the electron moves, not what its speed or position is at any particular time. The results of any measurement of position will be uncertain by $\Delta x=r$, while the uncertainty in momentum will be $\Delta p=m_0 v$, and from the appropriate equations

$$\Delta x.\Delta p = \frac{\hbar}{2} \tag{10.8}$$

and the result is seen to be consistent with the uncertainty principle.

10.5 The essential point that has been made is that classical physics supplemented by the stationary state hypothesis is surprisingly consistent with quantum mechanics, at least insofar as the electron is concerned.

11 THE HARMONIC OSCILLATOR

11.1 The Stationary State Equation

The $\tau \rightarrow i\tau$ hypothesis is now applied to the equation of motion for an electron subject to a returning force. The non-relativistic 1-d equation of motion is

$$\ddot{x} + \tau \frac{\ddot{x}^2}{\dot{x}} = -\omega_0 x^2 \tag{10.9}$$

Making the replacements

$$\tau \rightarrow i\tau \qquad x \rightarrow z \tag{10.10}$$

the equation becomes

$$\ddot{z} + i\tau \frac{\dddot{z} z^*}{\dot{z}} = -\omega_0^2 z \tag{10.11}$$

The trial solution

$$z = r\ e^{i\omega t} \tag{10.12}$$

leads to the cubic equation

$$\omega^2 [1 + \omega\tau] = \omega_0^2 \tag{10.13}$$

An approximate solution is

$$\omega \sim \pm \omega_0 \tag{10.14}$$

leading to a second approximation

$$\omega \sim \pm \omega_0 [\ 1 \pm \frac{\omega_0 \tau}{2}] \tag{10.15}$$

The effect of the radiation term is to split the fundamental resonance into two close frequencies.

11.2 Quantisation of the Energy Levels

The energy associated with the frequency ω is

$$\mathcal{E} = \frac{1}{2} m_0\, \omega^2 r^2 \tag{11.1}$$

So far we have ignored the spin of the electron, but if we now take this partially into account we see that for different orbits in the complex plane to be distinct it is necessary for there to be a minimum radius of $2r_s$, the radius increasing in odd integral multiples of this initial radius. That is we have

$$\mathcal{E}_n = \frac{1}{2} m_0\, \omega^2 [2n+1]^2 (2r_s)^2 = 8\, m_0 \omega^2 r_s^2 \left[n + \frac{1}{2} \right]^2 \tag{11.2}$$

We also observe that symmetry on the x-axis requires an odd number of revolutions in a half period

$$\left.\begin{aligned} \frac{\pi}{\omega} &= (2n+1) \frac{2\pi}{\omega_e} \\ \omega &= \frac{\omega_e}{2(2n+1)} \end{aligned}\right\} \tag{11.3}$$

Replacing one of the terms in ω^2, the energy becomes

$$\mathcal{E}_n = 8m_0\left(\frac{\left(\frac{3}{\alpha}\right)^{\frac{3}{2}} c\tau}{4\left(\sqrt{\left[\frac{3}{\alpha}-1\right]}\ -\ 1\right)}\right)^2 \frac{\omega_e}{4}\left[n+\frac{1}{2}\right]\omega \qquad (11.4)$$

Substituting for ω_e

$$\omega_e = \left[\frac{v}{c}\right]^2 \frac{4\alpha}{3\tau} \qquad (11.5)$$

where

$$\frac{v}{c} = \frac{\sqrt{1+\frac{3}{\alpha}}\ -1}{\sqrt{\frac{3}{\alpha}}} \qquad (11.6)$$

and replacing the fine structure constant by

$$\alpha = \frac{3}{2}\frac{m_0 c^2 \tau}{\hbar} \qquad (11.7)$$

we find that

$$\mathcal{E}_n = (n + \tfrac{1}{2})\hbar\omega = (n + \tfrac{1}{2})h\nu \qquad (11.8)$$

and we have the quantum mechanical result for the energy levels of the harmonic oscillator, including the zero point energy.

12. THE HYDROGEN ATOM

12.1 The Stationary State hypothesis $\tau \rightarrow i\tau$ having been used successfully to obtain the quantum properties of the electron and the quantisation of the harmonic oscillator it is now applied to the hydrogen atom. The motion of the point charge on the assumed model presented above will be highly complex, and a relativistic treatment is strictly necessary. Here we make the simplifying assumption that the motion of the centre of mass may be determined by assuming a point charge with mass m_0, that is, initially we are going to ignore the spin motion of the intrinsic mass. The velocity of electrons within atoms is known to be low enough for realistic calculations with non-relativistic mathematics. The complexity and high degree of non-linearity of the relativistic equation when applied to the present electron model will contain many features and fine detail of the potential motions that will be lost in the simplified approach, but despite this we will find that the simplified approach with the addition of an obvious and reasonable assumption will result in a model of the hydrogen atom in close agreement in many aspects with the quantum mechanical model.

12.2 The Stationary State Equation for the Hydrogen Atom

The non-relativistic vector equation of motion is

$$.\ \dot{\mathbf{v}} + \tau\frac{\dot{\mathbf{v}}^2}{v^2}\ \mathbf{v} = \frac{f}{m} = k\nabla\frac{1}{r} \qquad (12.1)$$

where for the hydrogen atom we have put

$$\mathbf{f} = -\frac{q^2}{4\pi\varepsilon_0 r^3}\mathbf{r} = -m_0\frac{k}{r^3}\mathbf{r} \qquad (12.2)$$

Making the replacement

$$\tau \rightarrow i\tau \qquad (12.3)$$

the equation becomes

$$\dot{\mathbf{v}} + i\tau\frac{\dot{\mathbf{v}}^2}{v^2}\mathbf{v} = -\frac{k}{r^3}\mathbf{r} \qquad (12.4)$$

We now specialise to 2-d motion and make the replacements

$$\left.\begin{array}{l} \mathbf{r} \rightarrow z \quad \mathbf{r}^2 \rightarrow zz^* \quad \dot{\mathbf{r}} = v \rightarrow \dot{z} \\ \mathbf{v}^2 \rightarrow \dot{z}\dot{z}^* \quad \dot{\mathbf{v}} \rightarrow \ddot{z} \quad \dot{\mathbf{v}}^2 \rightarrow \ddot{z}\ddot{z}^* \end{array}\right\} \qquad (12.5)$$

with the result

$$\ddot{z} + i\tau\frac{\ddot{z}\ddot{z}^*}{\dot{z}\dot{z}^*}\dot{z} = \frac{-kz}{[zz^*]^{3/2}} \qquad (12.6)$$

We now look for a circular orbit and set

$$z = \mathbf{r}_0\, e^{i\theta} \qquad (12.7)$$

and the equation becomes,

$$(i\ddot{\theta} - \dot{\theta}^2) - \tau\frac{(\ddot{\theta}^2 + \dot{\theta}^4)}{\dot{\theta}} = -\frac{k}{r_0^3} \qquad (12.8)$$

We then have immediately

$$\ddot{\theta} = 0 \qquad \dot{\theta} = \omega \qquad (12.9)$$

$$\tau\omega^3 + \omega^2 - \frac{k}{r_0^3} = 0 \qquad (12.10)$$

Writing in dimensionless form by setting $\rho = \tau\omega$

$$\rho^3 + \rho^2 - \frac{\tau^2 k}{r_0^3} = 0 \qquad (12.11)$$

The discriminant is <0 and so the equation has three real roots. Noting that the constant is <<1 and setting r_0 equal to the s×Bohr radius,a_0, we have an approximate solution,

$$\rho \sim \mathbf{-1\ or} \pm\sqrt{\frac{\boldsymbol{\tau}^2\mathbf{k}}{\mathbf{a}_0^3}} = \pm\frac{\mathbf{2}}{\mathbf{3}}\,\mathbf{s}^{-3/2}\boldsymbol{\alpha}^3 \qquad (12.12)$$

where α is the fine structure constant, s is an arbitrary constant and in addition we have assumed an infinite mass for the nucleus. Discarding the ρ=-1 solution, this then gives

$$r \sim a_0 \exp\{\pm i\frac{2s^{-3/2}\alpha^3}{3\tau}t\} \qquad (12.13)$$

and the orbital speed is

$$v \sim \frac{2}{3}\frac{\alpha^3}{\tau}s^{-3/2}a_0 = s^{-1/2}\alpha c \qquad (12.14)$$

Writing

$$\omega_0^2 = \frac{k}{a_0^3} \qquad (12.15)$$

a further approximation for the angular frequency may be obtained by iteration

$$\omega_{orbit} \sim \omega_0 \left(1-\frac{\omega_0 \tau}{2}\right) \qquad (12.16)$$

12.3 Quantised Orbits

12.3.1 The approximation ignores the non-zero spin diameter, and, for a true steady state, the spin period should be commensurate with the orbit period; that is there should be an integral number of spin revolutions in the time for one orbit. Imposing this condition

$$p = \frac{\omega_e}{\omega_{orbit}} = \left\{\frac{v_e}{c}\right\}^2 .\frac{4\alpha}{3\tau}.\frac{s^{3/2}a_0}{\alpha c} = \frac{2s^{3/2}}{\alpha^2}\left\{\frac{v_e}{c}\right\}^2 \qquad (12.17)$$

where we have used the electron parameters. Substituting for v_e it reduces to a function of α

$$p = 2\frac{s^{3/2}}{3\alpha}\left[\sqrt{1+\frac{3}{\alpha}}-1\right]^2 = 34031.2605s^{3/2} \qquad (12.18)$$

For s=1 the numerical coefficient must be integer. This calculation has ignored the effect of the orbit velocity on the electron spin velocity and a correction can be obtained by combining the two velocities relativistically. The relativistic correction to this estimate is very small because of the low orbital velocity, but only a small correction is required. To obtain the necessary correction we find the spin velocity in the orbital frame of reference. The relativistic sum of two velocities is given by[11]

$$\mathbf{w} = \mathbf{v_1} \oplus \mathbf{v_2} = \frac{\mathbf{\Phi_1.v_2} + \beta_1\mathbf{v_1}}{\beta_1\left(1+\frac{\mathbf{v_1.v_2}}{\mathbf{c^2}}\right)} \qquad (12.19)$$

where

$$\mathbf{\Phi} = \mathbf{I} + \frac{(\beta-1)}{\mathbf{v^2}}\mathbf{vv} \qquad (12.20)$$

and

$$\mathbf{v}_1 = \gamma c\hat{\theta} \qquad v_2 = \alpha c\,\mathbf{sin}\,\theta\hat{\theta} \qquad (12.21)$$

This correction will vary as the electron spins, and so an average is taken. The average is

$$\bar{\gamma} = \frac{\gamma}{2\pi}\int_0^{2\pi}\frac{\left(1-\frac{\alpha}{\gamma}\mathbf{sin}\theta\right)}{(1-\alpha\gamma\mathbf{sin}\theta)}d\theta \qquad (12.22)$$

the integrand being the relativistic sum of the spin velocity with the component of the orbital velocity in the direction of the spin velocity and we have put $\gamma = v_e / c$. The result is

$$\bar{\gamma} = \gamma\left[1-\frac{\alpha^2}{2}(1-\gamma^2)\left(1+\frac{3\alpha^2\gamma^2}{2}\right)\right] \qquad (12.23)$$

The conservation of angular velocity requires that the mean orbit radius be reduced by the same factor as the decrease in velocity with the result that the corrected value of p is given by

$$p = 2\frac{s^{3/2}}{3\alpha}\left[\sqrt{1+\frac{3}{\alpha}}-1\right]^2 \times\left[1-\frac{\alpha^2}{2}(1-\gamma^2)\left(1+\frac{3\alpha^2\gamma^2}{2}\right)\right]^3 = 34031.000812s^{3/2} \qquad (12.26)$$

where the accepted value of α **(7.297352568.10^{-3})** has been used. The quoted standard error is 24×10^{-11}.

12.3.2 Accordingly it seems reasonable to accept the coefficient to be an integer, and we have $s^{-3/2}$ p = 34031, which happens to be a prime number. It follows that s must be a perfect square otherwise p would not be integral and moreover it must be an integer as fractional values would require 34031 to have factors. Accordingly we put

$$s = n^2 \tag{12.27}$$

The orbital angular velocity is found to be

$$\omega_{orb} = n^{-3} \frac{2\alpha^3}{3\tau} \tag{12.28}$$

and the orbital velocity is

$$v_{orb} = \frac{\alpha c}{n} \tag{12.29}$$

The total energy is then

$$E_{tot} = -\frac{\alpha^2 m_0 c^2}{2n^2} \tag{12.30}$$

which is the quantum mechanical result.

12.4 Determination of the Fine Structure Constant

The above success suggests that the formula for p is to be regarded as giving an implicit formula for α. To remove the decimal excess requires that α be increased by 3.5 standard errors to $7.297352653 \times 10^{-3}$. The relative error is~ 1.2×10^{-8}.

12.5 Comparison with the Dirac Equation Result

Assuming the mass of the electron is all electromagnetic, the electrons spin angular velocity is

$$\omega_e = \frac{4\alpha}{3\tau} \tag{12.32}$$

and the orbital velocity is as before

$$\omega_0 = \frac{2\alpha^3}{3\tau} \tag{12.33}$$

The ratio yields n=37557.731. Even with the relativistic correction this is not close to an integer, never mind a prime. The accurate calculation of **α** with the assumption of an intrinsic mass suggests that an intrinsic mass should be incorporated into the quantum mechanical approach.

12.6 Orbital Angular Momentum

Using these results to calculate the angular momentum due to motion in the lowest energy orbit, we have

$$\Omega_{orbit} = mv\, r_{0B} \sim m_0 \alpha c a_0 = \hbar \tag{12.34}$$

this being the result for the component measured normal to the electron orbit, Ω_z, a_0 being the radius of the Bohr orbit.

12.7 The Zeeman Effect

Imposition of a weak magnetic field causes the energy levels to split into a number of discrete levels separated by $\boldsymbol{\mu_B}.\mathbf{B}$, the scalar product of the Bohr magneton and the magnetic flux density. The 2-D model demonstrates this splitting of the levels into two in the presence of a weak magnetic field. The equation of motion for a stationary state with an applied magnetic field becomes

$$v + i\tau\frac{\dot{v}^2}{v^2}\,v = -\frac{k}{r^3}\,r + \frac{q}{m_0}v\times B \tag{12.35}$$

For 2-D motion we may write

$$v\times B \rightarrow i\,\dot{z}B \tag{12.36}$$

With this addition the equation becomes

$$(\,i\ddot{\theta} - \dot{\theta}^2) - \tau\frac{(\ddot{\theta}^2 + \dot{\theta}^4)}{\dot{\theta}} = -\frac{k}{a_0^3} - \frac{q}{m_0}B\dot{\theta} \tag{12.37}$$

We again have

$$\dot{\theta} = \omega \tag{12.38}$$

and in addition

$$\tau\omega + \omega^2 - \frac{qB}{m_0}\omega = \frac{k}{r_0^3}\,r \tag{12.39}$$

$$\rho^3 + \rho^2 - p\rho = \rho_0^2 \tag{12.40}$$

where

$$p = \frac{qB\tau}{m_0} \tag{12.41}$$

We may again make the approximation

$$\rho^3 \sim 0 \tag{12.42}$$

and the equation becomes

$$\rho^2 - p\rho - \rho_0^2 \sim 0 \tag{12.43}$$

Solving this equation

$$\rho = \frac{p \pm \sqrt{p^2 + 4\rho_0^2}}{2} \tag{12.44}$$

For weak fields we may assume

$$p \gg 2\rho_0 \tag{12.45}$$

and we have

$$\rho = \pm\,\rho_0 + \frac{p}{2} \tag{12.46}$$

Taking the modulus and replacing p and γ_0 ,

$$\omega = \omega_0 \pm \frac{qB}{2m_0} \tag{12.47}$$

This is the correct quantum mechanical result for the two new levels introduced by the magnetic field.

12.8 Perturbed Orbits

We can consider more general orbits by allowing the radial coordinate to be a function of time. We again set
Differentiating

$$\dot{z} = [\dot{r} + i r \dot{\theta}] e^{i\theta} \qquad (12.49)$$

$$\ddot{z} = \{[\ddot{r} - r\dot{\theta}^2] + i[2\dot{r}\dot{\theta} + r\ddot{\theta}]\} e^{i\theta} \qquad (12.50)$$

Substituting into the stationary state equation of motion and separating real and imaginary parts we obtain the two equations

$$(\ddot{r} - r\dot{\theta}^2) + \tau \frac{[(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta})^2]}{\dot{r}^2 + r^2\dot{\theta}^2} r\dot{\theta} = -\frac{k}{r^2} \qquad (12.51)$$

$$-[2\dot{r}\dot{\theta} + r\ddot{\theta}] + \tau \frac{[(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta})^2]}{\dot{r}^2 + r^2\dot{\theta}^2} \dot{r} = 0 \qquad (12.52)$$

As a check on these equations we put

$$\ddot{r} = 0 = \dot{r} \qquad (12.53)$$

obtaining

$$\dot{\theta}^2 - \tau \frac{(\dot{\theta}^4 + \ddot{\theta}^2)}{\dot{\theta}} = \frac{k}{r^3} \qquad (12.54)$$

$$r\ddot{\theta} = 0 \qquad (12.55)$$

The second of these gives

$$\dot{\theta} = \omega \qquad (12.56)$$

and this is seen to be compatible with the previous results. If we now make the trial solution $\dot{\theta} = \omega$ and r remain time dependent, these equations reduce to

$$(\ddot{r} - r\omega^2)(\dot{r}^2 + r^2\omega^2) + \tau \left[\left(\ddot{r} - r\omega^2\right) + \left(2\dot{r}\omega\right)^2\right] r\omega\omega = -\frac{k^2}{r^2}\left(\dot{r}^2 + r^2\omega^2\right) \qquad (12.57)$$

$$2\omega\dot{r}\left(\dot{r}^2 + r^2\omega^2\right) + \tau\left[\left(\ddot{r} - r\omega^2\right)^2 + \left(2\dot{r}\omega\right)^2\right] = 0 \qquad (12.58)$$

Eliminating τ, ensuring that the solution is compatible with both equations, the first of the pair of equations becomes

$$\ddot{r} - r\omega^2 = -\frac{k}{r^2} \qquad (12.59)$$

For a perturbation of the circular orbit we put

$$r = \left[\frac{k}{\omega^2}\right]^{1/3}(1+\delta) = r_0(1+\delta) \qquad (12.60)$$

and the equation becomes

$$\ddot{\delta} - \omega^2(1+\delta) = \frac{-k}{r_0^3}\frac{1}{(1+\delta)^2} \sim \frac{-k}{r_0^3}(1-2\delta) \qquad (12.61)$$

and this reduces to

$$\ddot{\delta} = 3\omega^2\delta \qquad (12.62)$$

This indicates that the motion for small disturbances is unstable. However we note that there is a singular solution $\delta = \dot{\delta} = \ddot{\delta} = 0$, and this is a point of equilibrium in the phase plane and is the stationary state and further, this solution exists for square integral multiples of a_0. *For non-zero values of δ the trajectory is outside the stationary state, and so the appropriate equation of motion is*

$$\dot{v} + \tau \frac{\dot{v}^2}{v^2}\, v = \frac{f}{m} = -k\frac{r}{r^3} \tag{12.63}$$

Introducing polar coordinates as before, the equations become

$$(\ddot{r} - r\dot{\theta}^2)(\dot{r}^2 + r^2\dot{\theta}^2) + \tau[(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta} + r\ddot{\theta})^2]\dot{r} = -\frac{k}{r^2}(\dot{r}^2 + r^2\dot{\theta}^2) \tag{12.64}$$

$$(2\dot{r}\dot{\theta} + r\ddot{\theta})(\dot{r}^2 + r^2\dot{\theta}^2) + \tau\ \ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta} + r\ddot{\theta})^2]r\dot{\theta} = 0 \tag{12.65}$$

Eliminating τ we obtain

$$(\ddot{r} - r\dot{\theta}^2) - \frac{2\dot{r}\dot{\theta} + r\ddot{\theta}}{r\dot{\theta}}\dot{r} = -\frac{k}{r^2} \tag{12.66}$$

We now look for decaying solutions of the form

$$r = r_A \exp\{-\theta\} \tag{12.67}$$

Substituting this trial solution, there results

$$\dot{\theta}^2 = \frac{k}{2r^3} \tag{12.68}$$

or

$$\dot{\theta} = \sqrt{\frac{k}{2r_A^3}}\exp\left\{\frac{3\theta}{2}\right\} \tag{12.69}$$

This has a solution

$$e^{-\theta} = \left\{1 - \frac{3}{2}\sqrt{\frac{k}{2r_A^3}}\ t\right\}^{2/3} \tag{12.70}$$

giving

$$r = r_A\left\{1 - \frac{3}{2}\sqrt{\frac{k}{2r_A^3}}\ t\right\}^{2/3} \tag{12.71}$$

We may now consider a small positive radial displacement from a_0 writing

$$r = a_0(1 + \delta_0)\left[1 - \frac{3}{2}\sqrt{\frac{k}{2a_0^3(1 + \delta_0)^3}}\ t\right]^{2/3} \tag{12.72}$$

Setting the radius to r_0 and solving for t_r, the time to return to the stationary state.

$$t_r = \frac{2\sqrt{2}}{3\omega_0}\left[[1 + \delta_0]^{3/2} - 1\right] \sim \sqrt{2}\frac{\delta_0}{\omega_0} \tag{12.73}$$

For a negative displacement, consider an infinitesimal displacement from the state with n=2. We may then take $r_A=4a_0$, and determine the time to reach the ground state. Setting $r=a_0$ we obtain

$$1 = 4\left[1 - \frac{3}{16\sqrt{2}}\omega_B t_{rg}\right]^{2/3} \tag{12.74}$$

Solving for the time to return to the ground state,

$$t_{rg} = \frac{14\sqrt{2}}{3\omega_B} \sim 1.05T_B \tag{12.75}$$

where T_B is the period and ω_B the angular frequency of the Bohr orbit.

12.9 Transition Radiation

The rate of energy emission during the transition from n=2 to n=1 is given by

$$\dot{\mathcal{E}} = m_0\tau\left[(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta})^2\right] \tag{12.76}$$

Substituting from the previous solution and integrating

$$\mathcal{E} = \frac{m_0\tau a_0^2\omega_B^4}{64^2}\int_0^{t_{rg}}\left[\frac{20+4u^{2/3}+u^{4/3}}{u^{8/3}}\right]dt \tag{12.77}$$

where

$$u = \left[1 - \frac{3}{16\sqrt{2}}\omega t\right] \tag{12.78}$$

Carrying out the integration,

$$\mathbf{E} = 47.494 m_0\tau a_0^2\omega_B^3 = \frac{\alpha^5}{15.83} m_0 c^2 \tag{12.79}$$

Comparing this with the difference in energy between the two levels, we have

$$\frac{\mathcal{E}}{\Delta E} = 0.0168\alpha^3 \tag{12.80}$$

This low value of the fraction of energy that is radiated in the transition from the n=2 to n=1 state implies that the majority of the radiation occurs on entering the ground state.

12.10 Discussion

The stationary state hypothesis has now resulted in a model of the hydrogen atom that has many of the properties in agreement with quantum mechanical results, in particular the contribution of the orbit angular momentum, the magnetic moment of the orbit, and the quantised nature of the allowed energy levels. A significant omission so far is the lack of a prediction of degenerate states. It is expected that there are sets of elliptical orbits having the same energy and so resulting in degeneracy. The model indicates that a sudden transition is required on entering the ground state as the radiated energy during the motion to the ground state is only a small fraction of the energy to be lost. The nature of the forces involved in this transition could in principle be determined by matching the force to that required to produce the measured line widths. The Zeeman effect is accounted for in that the two extra levels are correctly predicted. However, the most important result is the formula for the fine structure constant that can, in principle, be developed to any required accuracy. The continued success with this approach suggests that Dirac's equation should be modified to incorporate the intrinsic mass of the electron.

13 THE POSITRON

13.1 Intrinsic Mass Derivation

The derivation of the intrinsic mass for the electron failed to find all the solutions. The observed rest mass of an electron is given by

$$m_e = \frac{m_i}{\left[1 - \frac{v^2}{c^2}\right]^{1/2}} \tag{13.1}$$

and so

$$\frac{v^2}{c^2} = 1 - \left[\frac{m_i}{m_e}\right]^2 \tag{13.2}$$

Writing

$$\frac{m_i}{m_e} = k \tag{13.3}$$

and substituting in the spin equation

$$\frac{3}{4\alpha}k^4 + k^2 - 1 = 0 \tag{13.4}$$

The solutions are

$$k = \pm\left[\frac{2\alpha}{3}\left\{\sqrt{\frac{3}{\alpha}+1} - 1\right\}\right]^{1/2} \quad \text{and} \quad \pm i\left[\frac{2\alpha}{3}\left\{\sqrt{\frac{3}{\alpha}+1} + 1\right\}\right]^{1/2} \tag{13.5}$$

13.2 The Real Negative Solution

Assuming the positive intrinsic mass is associated with the electron, we now associate the negative mass with the positron. To maintain the positivity of the observed mass all that is required is to write

$$m_0 = \frac{\pm m_i}{\pm\sqrt{1 - \frac{v^2}{c^2}}} \tag{13.6}$$

for both the electron and the positron. As a consequence the intrinsic mass is not directly observable. In a collision between an electron and a positron the total energy released is still twice the relativistic energy, the intrinsic mass energy cancelling along with the charge.

13.3 Imaginary Mass

To accept the concept of imaginary mass it is necessary to provide an interpretation, and a possible way, proposed by Dr Carl Baum[12], is to consider the complex entity

$$\mathfrak{M} = -\sqrt{G}m_0 + \frac{iq}{\sqrt{4\pi\varepsilon_0}} \tag{13.7}$$

Imaginary charges are equivalent to masses, and conversely imaginary masses can now be interpreted as charges

$$q_m = im_0\sqrt{4\pi\varepsilon_0 G} \qquad m_q = -\frac{iq}{\sqrt{4\pi\varepsilon_0 G}} \tag{13.8}$$

We can then write

$$\mathcal{F} = \frac{1}{r^2}\mathfrak{M}^2 = \frac{1}{r^2}\left\{-\sqrt{G}m_0 + \frac{iq}{\sqrt{4\pi\varepsilon_0}}\right\}^2 = \frac{G}{r^2}\{m_0 + m_q\}^2 \qquad (13.9)$$

This suggests the possibility of a particle with the above transformed mass and charge,

$$m_q = 1.8594\times10^{-9}\,\mathrm{kg} = 1.043\times10^{18}\,\mathrm{GeV} \approx 10^{-4}M_P \qquad (13.10)$$

$$q_m = 1.01\times10^{-40}\,\mathrm{C} = 6.33\times10^{-22}\,q \qquad (13.11)$$

which could be called the G-particle. Note that the fractional charge is not in conflict with observation as it is well below the limits of detection, and that the mass of the particle is enormous. These particles would behave somewhat like super heavy neutrons and would be very difficult to detect. Should a G and a $\overline{G}$ annihilate each other the resulting 18 GeV-plus gamma rays would be contenders for the gamma burst photons that have been detected.

14. THE ELECTRON ANALOGUE

14.1 In §4.2 the steady state hypothesis was supported by the analogy with an LC circuit. With the model of an electron moving in a circle, the analogy is now extended to consider this model as such a circuit. The exact inductance of a single turn loop[8]

is given by

$$L = \mu_0(a-b)\left(\mathbf{ln}\left(\frac{8a}{b}\right) - \frac{7}{4}\right) \qquad \mathbf{(14.1)}$$

Setting a to the spin radius, $\alpha a_0 / 2\gamma$ and b to the electron radius, this becomes

$$L = \mu_0 \frac{\alpha a_0(1-2\alpha\gamma)}{2\gamma}\left[\mathbf{ln}\left(\frac{4}{\alpha\gamma}\right) - \frac{7}{4}\right] \qquad (14.2)$$

while the capacitance of the electron may be taken as

$$C = 4\pi\varepsilon_0\alpha^2 a_0 \qquad (14.3)$$

We then have

$$\omega = \frac{c}{a_0\sqrt{2\pi\frac{\alpha^3(1-2\alpha\gamma)}{\gamma}\left[\mathbf{ln}\left(\frac{4}{\alpha\gamma}\right) - \frac{7}{4}\right]}} \qquad (14.4)$$

Evaluating this expression as ω_{an} and comparing with the qm and ss values

$$\omega_{an}=1.659815670\times10^{21} \quad \omega_0=1.552475633\times10^{21} \quad \omega_{ss}=1.406706306\times10^{21} \qquad (14.5)$$

The electron radius has been taken as the classical value. It might be expected that the actual radius would lie between r_e and $\sqrt{8/3}r_e$. Changing the radius to 1.166083275r_e reduces the analogue frequency to the quantum mechanical value, while changing the radius to 1.5508893r_e reduces it to the steady state solution. A factor of 3/2 can be introduced by considering the charge to be uniformly distributed throughout the electron or considering the electron to be a shell of charge with radius 1.5 times r_e. The result is $\omega_{ss} = 1.4092457\times10^{21}$. The fact the analogy results in an estimate as close as it does is quite remarkable.

.

15 NON-RELATIVISTIC MECHANICS

15.1 General Approach

It is difficult to extract solutions for the motion of electrons under prescribed forces due to the non-linearity of the equation of motion, so we invert the problem and determine the forces for a prescribed motion. This will frequently give rise to the requirement for infinite forces or infinite strength impulses. The motion is then modified when possible to remove the infinities resulting in a number of analytic solutions.

15.2 Impulsive Motion

15.2.1 The non-relativistic equation of linear motion is

$$\dot{v} + \tau \frac{\dot{v}^2}{v} = \frac{f}{m} \tag{15.1}$$

We impose an impulsive acceleration

$$\dot{v} = v_0 \delta_p(t) \tag{15.2}$$

Substituting into the equation of motion

$$v_0 \delta_p(t) + \tau v_0 \delta_p^2(t) = \frac{f}{m} \tag{15.3}$$

From Appendix A this can be written

$$v_0 \delta_p(t) + \tau v_0 S \delta_p^*(t) = \frac{f}{m} \tag{15.4}$$

and the equation of motion becomes

$$\dot{v} + \tau \frac{\dot{v}^2}{v} = v_0 \delta_p(t) + \tau v_0 S \delta_p^*(t) \tag{15.5}$$

Integrating over a, the equation reduces to

$$\int_0^a \frac{\dot{v}^2}{v} dt = v_0 S \tag{15.6}$$

The energy radiated in the interval 0→a is $m\tau v_0^2 S$ and as the two proto-delta functions tend to delta functions, the radiated energy tends to infinity while the kinetic energy of the electron remains at $\frac{1}{2} m v_0^2$.

15.2.2 To estimate the impulse required to attain a given velocity we note that

$$S \approx \frac{1}{a} \qquad \text{and} \qquad \delta_p \approx \delta_p^{\cdot} \tag{15.7}$$

Then

$$I \approx fa \approx m v_0 \left(1 + \frac{\tau}{a}\right) \tag{15.8}$$

and the effect is exceedingly small for any practical case.

15.3 'Harmonic' Motion

15.3.1 We now consider a particle moving under a restoring force proportional to the displacement, obtaining the equation for rectilinear motion

$$\dot{v} + \tau\frac{\dot{v}^2}{v} = -\omega^2 x \tag{15.9}$$

Noting that the equation is homogeneous, a trial solution is

$$x = x_0 e^{\lambda t} \tag{15.10}$$

and there results

$$\lambda^2 + \tau\lambda^3 + \omega^2 = 0 \tag{15.11}$$

This has one real root and two complex roots, which can be expressed approximately as

$$\lambda_1 = -\frac{\left(1+\omega^2\tau^2\right)}{\tau} \quad \lambda_{2,3} = -\omega^2\tau \pm i\omega \tag{15.12}$$

by expanding to second order in $\omega\tau$. The real root is of no interest as the motion is damped out in times of order 10^{-24} seconds. The complex roots represent 2-D motion, specifically positive or negative decaying spirals

$$\mathbf{r} = \mathbf{r}_0 e^{-\omega^2\tau t} \tag{15.13}$$

15.3.2 It is clear on physical grounds that 1-D oscillatory solutions must exist, but as demonstrated in 2.3 there will be a discontinuity as zero velocity is approached. Reformulating the problem as an integral allows such discontinuities and leads to an implicit solution. Regarding the equation as a quadratic in acceleration, we obtain

$$\dot{v} = -\frac{v}{2\tau}\left\{1 \pm \sqrt{1 - \frac{4\omega^2\tau x}{v}}\right\} \tag{15.14}$$

We can then write

$$v = v_0 \exp\left\{-\frac{1}{2\tau}\int_0^t \left[1 - \left(1 \pm \frac{4\omega^2\tau x}{v}\right)^{1/2}\right] dt\right\} \tag{15.15}$$

This solution must correspond to the radiationless case as $\tau \rightarrow 0$, and this requires that we adopt the negative sign. If we now consider an electron passing through x=0 with a velocity v_0 at t=0, the radiation rate will be very small and an approximate solution will be

$$v = v_0 \cos\omega t \tag{15.16}$$

$$x = \frac{v_0}{\omega}\sin\omega t \tag{15.17}$$

Substituting these into the implicit solution, there results

$$v = v_0 \exp\left\{-\frac{1}{2\tau}\int_0^t \left[1 - \left(1 - 4\omega\tau\tan\omega t\right)^{1/2}\right]\right\} \tag{15.18}$$

Expanding to second order in τ this reduces to

$$v = v_0 \exp\left\{-\int_0^t \tan(\omega t)\,d(\omega t) - \int_0^t \tan^2(\omega t)\,d(\omega t)\right\} \tag{15.19}$$

Carrying out the integrations and simplifying

$$v \approx v_0 \cos(\omega t)\exp\{-\omega\tau[\tan(\omega\tau) - \omega\tau]\} \tag{15.20}$$

and we observe that this solution drops very rapidly to zero as ωt approaches $\pi/2$. The limit of validity of the solution is somewhat less than $\pi/2$, the radical in the implicit solution implying that the maximum value of ωt is

$$\omega t_{\mathbf{max}} \approx \frac{\pi}{2} - 4\omega\tau \tag{15.21}$$

At this time the velocity jumps to zero and then increases in the negative direction. This will result in a delta-function pulse of energy

$$E_{pulse} = 8\omega^2\tau^2 m v_0^2 \tag{15.22}$$

The following motion will be initially the same as acceleration under a constant force.

15.4 Forced Oscillatory Motion

15.4.1 To study forced oscillation we determine the force required to maintain sinusoidal motion by writing

$$\left.\begin{aligned} v &= v_0 \sin \omega t \\ \dot{v} &= \omega v_0 \cos \omega t \end{aligned}\right\} \tag{15.23}$$

The required force is then

$$f = m v_0 \cos \omega t \left[1 + \omega\tau \cot \omega t\right] \tag{15.24}$$

A periodically infinite force is needed to maintain the motion, and it is readily seen that the impulse integral diverges. As a consequence electrons cannot oscillate precisely with simple harmonic motion.

15.4.6 Oscillatory motion can be obtained while avoiding the need for infinite forces by choosing a motion that has the velocity vanishing with the acceleration. Such a form is

$$v = v_0 \sin^2 \omega t \tag{15.24}$$

yielding

$$\dot{v} = \omega v_0 \sin 2\omega t \tag{15.25}$$

Substituting into the equation of motion, there results

$$f = 2m\omega v_0 \cos \omega t \left[\sin \omega t + 2\omega t \cos \omega t\right] \tag{15.26}$$

Elementary trigonometry reduces this to

$$f = m\omega v_0 \left[1 + \omega^2\tau^2\right]^{1/2} \left[\sin(2\omega t + \phi) + \omega\tau\right] \tag{15.27}$$

$$\phi = \tan^{-1} \omega\tau \tag{15.28}$$

and the effect of the radiation term is seen to be a phase difference between the oscillations and the driving force, together with the need for a constant force.

15.4.2 The energy radiated in one oscillation is given by

$$E_{rad} = m\omega^2\tau v_0^2 \int_0^{\pi/\omega} \sin^2 2\omega t dt = \pi\omega\tau.\frac{1}{2} m v_0^2 \tag{15.29}$$

The velocity is always positive, and so the electron will have a mean drift velocity of

$$\overline{v} = \frac{\omega}{\pi} v_0 \int_0^{\pi/\omega} \sin^2 \omega t dt = \frac{v_0}{2} \tag{15.30}$$

15.4.3 The solution to the equation of motion can be simplified slightly by making the following changes.

$$2\omega \to \omega \tag{15.31}$$

$$\omega t \to \frac{\omega}{2}t - \tan^{-1}\left(\frac{\omega}{2}\tau\right) \tag{15.32}$$

$$f_0 = \frac{1}{2}m\omega v_0\left[1 + \frac{(\omega\tau)^2}{4}\right]^{1/2} \tag{15.33}$$

The equation then becomes

$$\dot{v} + \tau\frac{\dot{v}^2}{v} = \frac{f_0}{m}\left[\sin\omega t + \frac{\omega\tau}{2\left\{1 + \frac{\omega^2\tau^2}{4}\right\}^{1/2}}\right] \tag{15.34}$$

and the solution becomes

$$v = \frac{f_0}{m}\frac{2}{\omega}\frac{1}{\left[1 + \frac{\omega^2\tau^2}{4}\right]^{1/2}}\left[1 - \cos\left(\omega\left[t - \frac{\tau}{2}\right]\right)\right] \tag{15.35}$$

15.4.4 The velocity is always positive and so the electron will have a mean drift velocity given by

$$\overline{v} = \frac{1}{\pi}v_0\int_{\tau/2}^{\pi+\tau/2}\left[1 - \cos\omega\left(t - \frac{\tau}{2}\right)\right]d\omega t = v_0 \tag{15.36}$$

15.4.5 The radiated energy is

$$m\tau\int \dot{v}^2 dt = 4\pi\omega\tau m v_0^2 \tag{15.37}$$

The constant force is the force required to maintain the sinusoidal oscillation. The net distance moved in one cycle is

$$x = \int_{\tau/2}^{\pi+\tau/2} v dt = 4\pi\frac{f_0}{\omega^2 m}\frac{1}{\left[1 + \frac{\omega^2\tau^2}{4}\right]^{1/2}} \tag{15.38}$$

The work done by the constant force is then

$$W = xf_c = 4\pi\omega\tau m v_0^2 \tag{15.39}$$

15.4.6 The implication of this result is that if a sinusoidal force is applied, a drift velocity develops equal to the maximum velocity to be expected with no radiation. This conjecture needs to be verified by a numerical solution to the equation of motion with a sinusoidal force.

15.5 Motion in a Magnetic Field

15.5.1 The force on an electron in a uniform magnetic field is

$$f = q\mathbf{v} \times \mathbf{B} \tag{15.40}$$

The equation of motion then becomes

$$\dot{v} + \tau\frac{\dot{v}^2}{v^2}v = \frac{q}{m}\mathbf{v} \times \mathbf{B} \tag{15.41}$$

Consider an electron injected into a field with its velocity vector perpendicular to the field. Expressing the equation of motion in Cartesian coordinates

$$\dot{v}_x\hat{i}+\dot{v}_y\hat{j}+\tau\frac{\dot{v}^2}{v^2}\left[v_x\hat{i}+v_y\hat{j}\right]=\frac{q}{m}\left[v_x\hat{i}+v_y\hat{j}\right]\times B\hat{k} \tag{15.42}$$

Separating the two component equations

$$\dot{v}_x+\tau\frac{\dot{v}^2}{v^2}v_x=\frac{qB}{m}v_y \qquad \dot{v}_y+\tau\frac{\dot{v}^2}{v^2}v_y=-\frac{qB}{m}v_x \tag{15.43}$$

Approximate solutions ignoring radiation are

$$\dot{v}_x=\frac{qB}{m}v_y \qquad \dot{v}_y=-\frac{qB}{m}v_x \tag{15.44}$$

Introducing these approximations into the radiation terms, the equations become

$$\dot{v}_x+\tau\left[\frac{qB}{m}\right]^2 v_x=\frac{qB}{m}v_y \tag{15.45}$$

$$\dot{v}_y+\tau\left[\frac{qB}{m}\right]^2 v_y=-\frac{qB}{m}v_x \tag{15.45}$$

15.5.2 Eliminating the y-components

$$\ddot{v}_x+2\tau\left(\frac{qB}{m}\right)^2\dot{v}_x+\left(\frac{qB}{m}\right)^2\left\{1+\left(\frac{\tau qB}{m}\right)^2\right\}v_x=0 \tag{15.46}$$

the initial motion is along the x-axis and it can be shown that the solution and a similar equation is obtained for the y-components. This is the equation for damped harmonic motion, and the combined motion is that of a decaying spiral. Without loss of generality we can assumecan be expressed as

$$v_x=v_{\mathbf{0}}e^{-\alpha\tau t}\left\{\mathbf{cos}(\omega t)+\frac{\alpha\tau}{\omega}\mathbf{sin}(\omega t)\right\} \tag{15.47}$$

$$v_y=v_{\mathbf{0}}e^{-\alpha\tau}\,\mathbf{sin}(\omega t) \tag{15.48}$$

which gives

$$v\approx v_0e^{-\omega^2\tau t}\left\{1+\frac{\omega\tau}{2}\mathbf{sin}\,2\omega t\right\} \tag{15.49}$$

The standard approach gives just the exponential decay without the modulation.

15.5.3 The radiation rate is given by

$$\dot{E}=m\tau\dot{v}^2=\omega^2\tau v_0^2e^{-2\omega^2\tau t}\{1-\omega\tau\,\mathbf{sin}\,\omega t\} \tag{15.59}$$

Integrating from zero to infinity, we obtain

$$E=\frac{1}{2}mv^2\left[1-\frac{(2\omega\tau)^2}{2\left[1+(2\omega\tau)^2\right]}\right] \tag{15.60}$$

the term in parentheses indicating the error made in the approximate solution.

15.6 Motion under a General Force

15.6.1 Non-relativistic linear motion was shown to be solutions of

$$\dot{v}=\frac{v}{2\tau}\left\{\pm\sqrt{1+\frac{4f\tau}{mv}}-1\right\} \tag{15.61}$$

where f was unspecified and so is applicable to arbitrary forces and we still have the lower velocity limit given by

$$v < -\frac{4f\tau}{m} = v_m \qquad \mathbf{(15.62)}$$

the velocity then jumping to zero, while energy equal to this loss of kinetic energy is radiated. The relativistic equation for linear motion is

$$\frac{\dot{v}}{\left[1-\frac{v^2}{c^2}\right]^{3/2}} + \tau\frac{\dot{v}^2}{v\left[1-\frac{v^2}{c^2}\right]^3} = \frac{f}{m} \qquad (15.63)$$

and this leads to

$$\dot{v} = \frac{v}{2\tau}\left(1-\frac{v^2}{c^2}\right)^{3/2}\left\{\sqrt{1+\frac{4f\tau}{mv}}-1\right\} \qquad \mathbf{(15.64)}$$

and the same minimum velocity is obtained as for the non-relativistic case**.**

15.6.2 The maximum value of the minimum positive velocity is c. Solving for the force

$$f = -\frac{mc}{4\tau} \qquad (15.65)$$

If this force is due to the presence of an electron, we can write

$$f = -\frac{q^2}{4\pi\varepsilon_0 r^2} \qquad (15.66)$$

Equating the two forces and solving for r we obtain the distance of closest approach of two electrons

$$r_m = \sqrt{\frac{8}{3}}\frac{q^2}{4\pi\varepsilon_0 m_0 c^2} = \sqrt{\frac{8}{3}}.r_e \qquad \mathbf{(15.67)}$$

where r_e is the classical electron radius and r_m turns out to be the equivalent radius for the Thomson cross section. This is not to be taken as necessarily indicative of any structure to the electron. At this distance the radiation loss due to any virtual displacement is equal to the gain in potential energy.

15.6.3 We may consider a spatial variation of the force by making the substitution

$$\dot{v} = v\frac{dv}{dr} \qquad \mathbf{(15.68)}$$

and the equation for linear motion becomes

$$v\frac{dv}{dr} + \tau v\left[\frac{dv}{dr}\right]^2 = \frac{f}{m_0} \qquad \mathbf{(15.69)}$$

We first obtain an approximate solution by ignoring the radiation term

$$v_0\frac{dv_0}{dx} = \frac{f}{m_0} \qquad \mathbf{(15.70)}$$

Integrating from x=0,v=0 to x,v

$$v_0^2 = 2\int_0^x \frac{fdx}{m_0} \qquad \mathbf{(15.71)}$$

The approximate equation is then

$$v\frac{dv}{dx}\left[1+\frac{\tau}{v_0}\frac{f}{m_0}\right] = \frac{f}{m_0} \qquad \mathbf{(15.72)}$$

or

$$\frac{v^2}{2} = \int_0^x \frac{\dfrac{f}{m_0}}{1+\tau\dfrac{f}{v_0 m_0}}dx \tag{15.73}$$

As an example, consider

$$\frac{f}{m_0} = kx \tag{15.74}$$

Then

$$v \sim \frac{\sqrt{k}.x}{\sqrt{1+\tau\sqrt{k}}} \tag{15.75}$$

16 CONCLUSIONS

16.1 The derivation of an equation of motion for a charged particle in the point particle approximation, which conserves energy and causality, has resulted in two major predictions, that of the finite transfer of energy to electrons by infinite amplitude pulses, and the occurrence of a drift velocity when the driving force is sinusoidal. It should be possible to test these predictions experimentally.

16.2 The application of the steady state hypothesis resulting in a model of the spinning electron, the determination of the zero point energy for a harmonic oscillator, the energy levels of the hydrogen atom, the determination of the fine structure constant, and the derivation of the Thomson cross-section for the electron indicates that the hypothesis has some degree of validity and shows how close classical physics is to quantum mechanics. This is further emphasised by the remarkably close agreement of the analogue electrical circuit of the rotating electron with the postulate of a uniform density of charge, leading to an electron radius one and a half times the classical radius.

16.3 There are obvious possible extensions to the theory to account for the existence of degenerate states and the decay of excited states, and these are being actively pursued. The degree of success also leads to the hope that a deeper connection between the stationary state hypothesis and quantum mechanics will be found.

17 APPENDIX

A1. Proto-step function

A proto-step function may be defined as a continuous function satisfying the following

$$u_p(0) = 0$$

$$u_p(a) = 1 \qquad x \geq a$$

$$u_p^{(n)}(0) = 0 \qquad \forall n, x \leq 0$$

$$u_p^{(n)}(a) = 1 \qquad \forall n$$

A2. Proto-delta function

A proto delta-function is then defined by

$$\delta_p(t) = u_p'(t)$$

It follows that

$$\delta_p^{(n)}(0) = \delta_p^{(n)}(a) = 0$$

$$\int_0^a \delta_p(t)dt = 1$$

A3. Delta-function Squared

Consider the square of the proto-delta function

$$\left(\delta_p^2(0)\right)^{(n)} = \left(\delta_p^2(a)\right)^{(n)} = 0$$

$$\int \delta_p^2(t)dt = S$$

Accordingly the squared proto-delta function is equivalent to a new proto-delta function of strength S

$$\delta_p^2(t) \equiv S\delta_p^*(t)$$

From the definition of u_p

$$\delta_p\left(\frac{a}{2}\right) > \frac{1}{a}$$

We then have

$$\int_0^a \delta_p^2(t)dt > \int_0^a \frac{1}{a^2}dt = \frac{1}{a}$$

and so

$$S\int_0^a \delta_p^*(t)dt > \frac{1}{a} \to \infty$$

We may then write formally

$$\delta^2(t) \equiv \infty\delta(t)$$

17 REFERENCES

[1] The Classical Theories of Radiation Reaction, Thomas Erber, Fortschritte der Physik 9,343-392,(1961)

[2][3] Classical Electrodynamics, J. D. Jackson, John Wiley & Son

[4] A Critical Examination of the Abraham- Lorentz Equation for a Radiating Charged Particle, J.L. Jimenez and I. Campos, Am Journal of Physics 55(11)Nov 1987.

[5] Classical Theory of Radiating Electrons, P.A.M. Dirac, Proceedings of the Royal Society, 1938.

[6] Relativistic Dynamics of a Charged sphere: Updating the Lorentz-Abraham Model, RL-TR-92-46, A Yaghjian, Rome Laboratory, Air Force Systems Command, Griffiss Air Force Base, NY 13441-5700.

[7] Private communication, Thomas Erber.

[8] Electromagnetic Theory, Eric Hallén, Chapman Hall 1962

[9] The Quantum Theory of Radiation, W Heitler, Oxford Clarendon Press, 1954

[10] Quantum Particle Dynamics, J. McConnell, North Holland Publishing Company,

[11] Advanced Vector Analysis, C.E. Weatherburn, G. Bell and Sons Ltd., 1966

[12] Private Communication, Professor Carl Baum, University of New Mexico