Physics Notes

Note 3

10 October 1991

Vector and Scalar Potentials Away from Sources, and Gauge Invariance in Quantum Electrodynamics

Carl E. Baum
Phillips Laboratory

Abstract

A fundamental aspect of the formulation of quantum electrodynamics (QED) is the imposition of gauge invariance. The potentials are not unique but the various acceptable forms all give the same results for measurement. This has implications for the influence of the potentials and their measurability under quasi static conditions, in which case it is possible to have potentials without electric or magnetic fields at positions away from sources. Two different antennas are discussed which emphasize the Lorenz vector potential on the one hand and the Lorenz scalar potential on the other. These have the same electromagnetic fields (those of an electric dipole) to illustrate this property of gauge invariance.

I. Introduction

As is well known in classical electromagnetics the fields described by the Maxwell equations can be derived from a vector and scalar potential. However, there are various forms that are possible,a ll giving the same fields. This is referred to as gauge invariance. In making measurements at some point it is the fields and perhaps current and charge densities that one considers. Potentials are quantities inferred (within the ambiguity of gauge invariance) by integration of the fields along appropriate paths.

In quantum electrodynamics (QED) the potentials assume a more important role in the formulation, being related to a phase shift in the wave function. This is still an integral effect over the path of interest. This manifests itself in the phase shift of an electron around a closed path enclosing a magnetic field, even though there are no fields (approximately) on the path itself (static conditions). As can be shown the result of such an experiment is gauge invariant, allowing the use of various choices of the vector potential (all giving the same result).

Generalizing the question somewhat one can explore the degree to which one can measure vector and scalar potentials, including the implications of gauge invariance. Assuming that there are sources (current and/or charge) in some region of space away from the observer, in what sense can the potentials be distinguished? In particular one can compare static and dynamic conditions. Under static conditions it is possible to have zero fields in the vicinity of the observer (away from the source region), while having non-zero potentials. Two different antennas, one emphasizing the vector potential and the other the scalar potential (Lorenz gauge), are discussed which have the same fields away from the source region.

II. Electromagnetic Fields and Potentials

In standard form we have the Maxwell equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{2.1}$$

and constitutive relations (for linear media)

$$\vec{D} = \vec{\varepsilon} \cdot \vec{E} \tag{2.2}$$

$$\vec{B} = \vec{\mu} \cdot \vec{H}$$

$\vec{\varepsilon} \equiv$ permittivity $= \varepsilon_o \vec{1}$ for free space

$\vec{\mu} \equiv$ permeability $= \mu_o \vec{1}$ for free space

$Z_o = \left[\frac{\mu_o}{\varepsilon_o}\right] \equiv$ wave impedance of free space

$c = [\mu_o \varepsilon_o]^{-\frac{1}{2}} \equiv$ the speed of light

The divergence equations are not independent but can be derived from (2.1) under zero initial conditions

$$\nabla \cdot (\nabla \times \quad) \equiv 0$$

$$\nabla \cdot \vec{B} = 0 \quad \text{(no magnetic charge)} \tag{2.3}$$

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t} \nabla \cdot D \equiv -\frac{\partial \rho}{\partial t} \quad \text{(equation of continuity)}$$

The fields can be derived from the well known scalar and vector potentials in free space [6,7,8,17,18,20,21] as

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \Phi \tag{2.4}$$

$$\vec{H} = \frac{1}{\mu_o} \nabla \times \vec{A}$$

As is well known these potentials are not unique since one can add a potential χ as

$$\vec{A}' \equiv \vec{A} - \nabla\chi$$
$$\Phi' \equiv \Phi + \frac{\partial\chi}{\partial t} \tag{2.5}$$

giving the same result in (2.4). Different choices of χ correspond to different gauge conditions.

The most common form taken uses the Lorenz potentials which satisfy the Lorenz gauge

$$\nabla \cdot \vec{A} + c^{-2}\frac{\partial\Phi}{\partial t} = 0 \tag{2.6}$$

These are taken as retarded potentials (outgoing waves for zero initial conditions) with explicit forms

$$\vec{A}(\vec{r},t) = \mu_o \int_V \frac{\vec{J}(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c})}{4\pi|\vec{r}-\vec{r}\,'|}\,dV'$$
$$\tag{2.7}$$
$$\Phi(\vec{r},t) = \frac{1}{\varepsilon_o}\int_V \frac{\rho(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c})}{4\pi|\vec{r}-\vec{r}\,'|}\,dV'$$

This form has both potentials propagating away from the source at speed c and is relativestically invariant, and for this reason often preferred.

A related potential is the Hertz potential

$$\vec{\Pi}(\vec{r},t) = c^2\int_{-\infty}^{t}\vec{A}(\vec{r},t')dt' \tag{2.8}$$

for which we have

$$\vec{E} = -c^{-2}\frac{\partial^2}{\partial t^2}\vec{\Pi} + \nabla(\nabla\cdot\vec{\Pi})$$
$$\vec{H} = \varepsilon_o\nabla\times\left[\frac{\partial}{\partial t}\vec{\Pi}\right] \tag{2.9}$$

However, this is basically the same as $\vec{A}$, since in complex-frequency domain

$$\tilde{\vec{\Pi}}(\vec{r},s) = \frac{c^2}{s}\tilde{\vec{A}}(\vec{r},s) \tag{2.10}$$

$\sim \equiv$ Laplace transform (two sided)

$s \equiv \Omega + j\omega \equiv$ Laplace transform variable or complex frequency

Note that the retarded potentials can be expressed as a single vector potential via (2.6) as

$$\Phi(\vec{r},t) = -c^2\int_{-\infty}^{t} \nabla\cdot\vec{A}(\vec{r},t')dt' \tag{2.11}$$

with zero initial conditions. Thus we can define what we might call the electric gauge condition for which

$$\begin{aligned}
&\Phi_e(\vec{r},t) = 0 \\
&\vec{A}_e(\vec{r},t) = \vec{A}(\vec{r},t) + \int_{-\infty}^{t} \nabla\Phi(\vec{r},t')dt' \\
&\qquad = \vec{A}(\vec{r},t) - c^2\int_{-\infty}^{t}\int_{-\infty}^{t'} \nabla(\nabla\cdot\vec{A}(\vec{r},t''))dt''dt' \\
&\vec{E} = -\frac{\partial\vec{A}_e}{\partial t} \\
&\vec{H} = \frac{1}{\mu_o}\nabla\times\vec{A}_e
\end{aligned} \tag{2.12}$$

Note that this is also a retarded potential propagating outward at speed c.

Another convenient choice is the Coulomb gauge for which we have [17,20,21]

$$\begin{aligned}
&\nabla\cdot\vec{A}_c = 0 \\
&\Phi_c(\vec{r},t) = \frac{1}{\varepsilon_o}\int_V \frac{\rho(\vec{r}\,',t)}{4\pi|\vec{r}-\vec{r}\,'|}dV' \\
&\vec{A}_c(\vec{r},t) = \vec{A}(\vec{r},t) + \int_{-\infty}^{t}\left[\nabla\Phi(\vec{r},t') - \nabla\Phi_c(\vec{r},t')\right]dt' \\
&\vec{E} = \frac{\partial}{\partial t}\vec{A}_c - \nabla\Phi_c
\end{aligned} \tag{2.13}$$

$$\vec{H} = \frac{1}{\mu_o} \nabla \times \vec{A}_c$$

However, note that Φ_c (and hence $\vec{A}_c$) now "propagates" with infinite speed, but that the fields still propagate with the speed of light c.

III. Potentials and Quantum Mechanics

In formulating quantum electrodynamics (QED) it has been found convenient to introduce the electromagnetic interaction with charged particles via the potentials instead of the fields. Consider a particle of charge q travelling on some path P from $\vec{r}_1$ to $\vec{r}_2$. Then the magnetic change in phase of the wave function is just [12]

$$\phi_h = \frac{q}{\hbar}\int_P \vec{A}(\vec{r},t)\cdot d\vec{l} \tag{3.1}$$

$$\hbar \equiv \frac{h}{2\pi} \quad , \quad h \equiv \text{Planck's constant}$$

and the electric change in phase is just

$$\begin{aligned}\phi_e &= \frac{-q}{\hbar}\int_{-\infty}^{t}\int_P \nabla\Phi(\vec{r},t')\cdot d\vec{l}\,dt' \\ &= \frac{-q}{\hbar}\int_{-\infty}^{t}[\Phi(\vec{r}_2,t') - \Phi(\vec{r}_1,t')]dt'\end{aligned} \tag{3.2}$$

where the total change in phase is just

$$\phi = \phi_h + \phi_e \tag{3.3}$$

Combining these expressions we have

$$\phi = \frac{q}{\hbar}\int_P \left[\vec{A}(\vec{r},t) - \int_{-\infty}^{t}\nabla\Phi(\vec{r},t')dt'\right]\cdot d\vec{l} \tag{3.4}$$

This is interpreted in the sense of changing a quantum wave function ψ in the form

$$\psi \to e^{j\phi}\psi \tag{3.5}$$

Note that if the path is closed we have

$$\vec{r}_2 = \vec{r}_1$$

$$\begin{aligned}\phi &= \frac{q}{\hbar}\oint_P \vec{A}(\vec{r},t)\cdot d\vec{l} = \frac{q}{\hbar}\int_S \left[\nabla\times\vec{A}(\vec{r},t)\right]\cdot d\vec{S} \\ &= \frac{q}{\hbar}\int_S \vec{B}(\vec{r},t)\cdot d\vec{S}\end{aligned} \tag{3.6}$$

with P the boundary of S and the unit normal $\vec{1}_S$ taken in the usual right-handed sense. Note that this is independent of the scalar potential and of the gauge chosen since $\vec{A}$ enters via the curl. This phase shift is the basis of the shifting of the diffraction pattern of electrons from a common source going through two slits with a confined magnetic field between the slits (Aharanov and Bohm experiment).

Electron motion is more generally formulated in a form of the Schrödinger equation including the spin in the presence of external fields known as the Pauli equation. This equation is gauge invariant in the sense that a transformation as in (2.5) also changes the quantum wave function ψ as

$$\psi' = e^{-\frac{q}{\hbar}\chi}\psi \tag{3.7}$$

leaving the Pauli equation unchanged [11]. Noting that $\psi\,\psi^*$ or $|\psi|^2$ is the physical observable, this phase change is not important. In particular one can choose

$$\chi = -\int_{-\infty}^{t}\Phi(\vec{r},t')dt' \tag{3.8}$$

and we have the electric gauge in (2.12) in which only the vector potential $\vec{A}_e$ appears. This can be readily computed from the usual Lorenz vector and scalar potentials from (2.7) and (2.12). Other choices, such as the Coulomb gauge, can also be used where convenient.

Another derivation of gauge invariance concerns the time-independent Schrödinger equation [13]. Here it is shown that a zero-curl vector potential can be absorbed into the scalar potential with no change in the observables, i.e. only a phase change in ψ of the form (3.7). This is consistent with the previous discussion in that it is only the divergence-free part (non-zero curl) of the vector potential that is associated with the measurable effect of an electron path enclosing a magnetic flux. Note that $\nabla\times\vec{A}$ is gauge invariant, and this is the part producing the measurable effect.

In formulating QED a least action principle involving a Lagrangian is often used [10,14,15]. This involves the potentials in various forms. Not only is relativistic invariance (Lorenz potentials) desired, but also gauge invariance. At least in the current state of QED, gauge invariance is included as a fundamental part [9,22].

IV. Implications of Zero Electric and Magnetic Fields Away from Sources

Consider a volume (simply connected) V_s with surface S_s of finite dimensions containing the sources as indicated in figure 4.1. The observer at $\vec{r}$ is assumed away from the sources, i.e.

$$\vec{r} \not\ni V_s \bigcup S_s \tag{4.1}$$

Suppose that $\vec{E}$ and $\vec{H}$ are zero for all times at the observer. Then from (2.12) we have

$$\begin{aligned} &\vec{A}_e(\vec{r},t) = \vec{A}_e(\vec{r},-\infty) = \text{ constant vector} \\ &\nabla \times \vec{A}_e(\vec{r},t) = \nabla \times \vec{A}_e(\vec{r},-\infty) = \vec{0} \end{aligned} \tag{4.2}$$

as the only solution, a zero-curl constant vector for all time. Setting $\vec{J}$ and ρ initial conditions to zero we have

$$\vec{A}_e(\vec{r},t) = \vec{0} \text{ for all } t \tag{4.3}$$

as the only solution. In terms of the Lorenz form of the potentials this gives

$$\vec{A}(\vec{r},t) + \int_{-\infty}^{t} \nabla\Phi(\vec{r},t')dt' \quad = \vec{0} \tag{4.4}$$

Since the initial conditions on $\vec{J}$ and ρ are zero then

$$\vec{A}(\vec{r},-\infty) = 0 \quad , \quad \Phi(\vec{r},-\infty) = 0$$

and constant potentials in this form are also excluded.

This can also be considered in complex-frequency form for which we find

$$\begin{aligned} &\tilde{\vec{A}}_e(\vec{r},s) = \vec{0} \text{ for all } s \\ &\tilde{\vec{A}}(\vec{r},s) + \frac{1}{s}\nabla\tilde{\Phi}(\vec{r},s) = 0 \quad \text{for all } s \end{aligned} \tag{4.5}$$

with all appropriate limit for $s \to 0$.

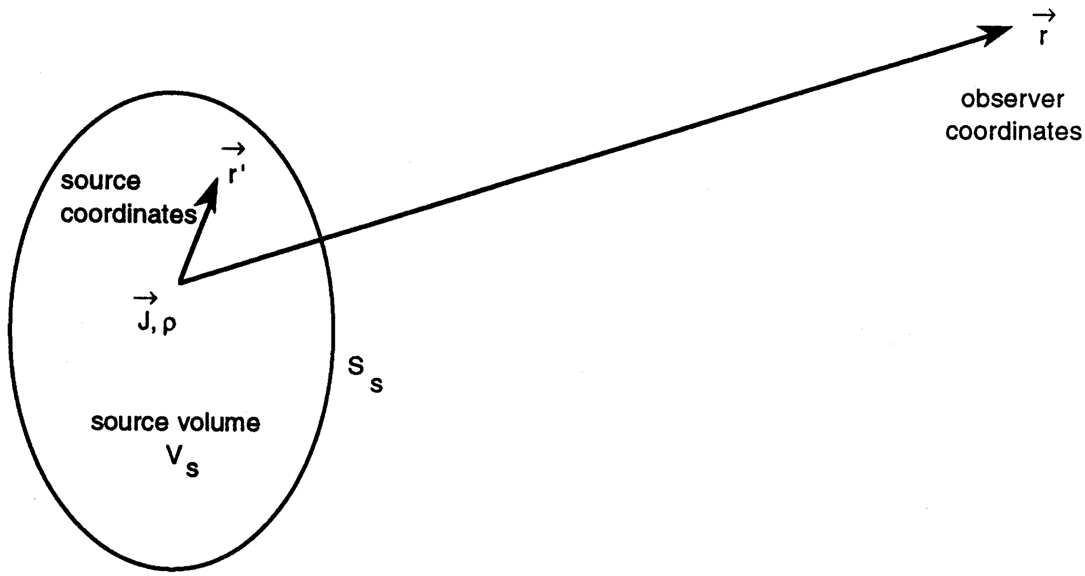

Fig. 4.1. Sources Confined to a Simply-Connected Volume of Finite Linear Dimensions

V. Change from One Static Potential to a Second Static Potential

Let us now consider an initial set of static potentials (subscript 1) followed by a second set (subscript 2) with sufficient time between to allow static conditions to be achieved (at least approximately). Then without loss of generality consider zero initial conditions

$$\vec{A}_{e_1}(\vec{r}) = \vec{0} \quad , \quad \vec{A}_1(\vec{r}) = \vec{0} \quad , \qquad \Phi_1(\vec{r}) = 0 \tag{5.1}$$

Actually, for these potentials, this can be taken in a retarded time sense, i.e., zero for

$$t < t_1 - \frac{r}{c} \tag{5.2}$$

since they are formulated in retarded time via (2.7). Note that the finite linear dimensions of V_s allow t_1 to be adjusted for propagation through V_s.

Next, turn on the sources $\vec{J}$ and ρ and have them reach constant values $\vec{J}_2$ and ρ_2 after some time t_2 less transit times across V_s. Then, in a retarded time sense for

$$t > t_2 - \frac{r}{c} \tag{5.3}$$

we have $\vec{A}_2$ and Φ_2 constant with

$$\vec{A}_{e_2}(\vec{r}) = \vec{A}_2(\vec{r}) \quad + \quad \int_{t_1}^{t_2} \nabla\Phi(\vec{r},t')dt' \tag{5.4}$$

From (2.7) we have constant $\vec{A}_2$ and Φ_2. However, zero electric field implies

$$\vec{E}_2(\vec{r}) = -\nabla\Phi(\vec{r}) = 0 \tag{5.5}$$

allowing only a uniform Φ_2 (independent of $\vec{r}$). This is inconsistent with (2.7) unless

$$\Phi_2 = 0 \tag{5.6}$$

which in turn implies that $\rho_2(\vec{r})$ is constrained to a distribution with no exterior potential. A set of charges inside a constant potential surface (zero potential) such as a conducting cavity with requisite resulting surface charge density on the interior of the surface is an example of such a charge distribution. Furthermore, we have after t_2

$$\nabla \times \vec{A}_{e_2}(\vec{r}) = \vec{0} = \nabla \times \vec{A}_2(\vec{r}) \tag{5.7}$$

since

$$\vec{H}_2(\vec{r}) = \vec{0} \tag{5.8}$$

Then, for both Lorenz and electric gauges (and others such as Coulomb as well) the change from initial (1) to final (2) conditions is characterized by (5.4). Referring to (2.4), we can see that this change is characterized by the electric impulse

$$\int_{t_1}^{t_2} \vec{E}(\vec{r},t')dt' = -\vec{A}_{e_2}(\vec{r})$$

$$= -\vec{A}_2(\vec{r}) - \int_{t_1}^{t_2} \nabla\Phi(\vec{r},t')dt' \tag{5.9}$$

subject to (5.5) and (5.7) as conditions on the potentials. So, while the fields are zero before t_1 and after t_2, they are not in general zero in between these two times. The electric impulse is characterized by $\vec{A}_{e_2}$ (i.e. the change in $\vec{A}_e$ from initial to final conditions). As indicated by (5.4) this can be expressed in various gauges, the electric impulse being gauge invariant.

VI. Potentials from an Electric Dipole

Now, consider how one might realize the conditions set forth in Section V. Consider first an elementary z-directed electric dipole at the coordinate origin in the source region as

$$\vec{p}(t) = p(t)\vec{1}_z \tag{6.1}$$

This can be thought of as elementary charges $\pm Q(t)$ placed at $z = \pm d/2$ with the limit taken as $d \to 0$ with the product

$$p(t) = Q(t)d \tag{6.2}$$

constant.

The usual cylindrical and spherical coordinates are related to the cartesian coordinates via

$$x = \Psi\cos(\phi) \quad , \quad y = \Psi\sin(\phi) \tag{6.3}$$

$$z = r\cos(\theta) \quad , \quad \Psi = r\sin(\theta)$$

The relevant dyadics are

$$\vec{1} \equiv \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z \equiv \text{ identity} \tag{6.4}$$

$$\vec{1}_t \equiv \vec{1} - \vec{1}_r\vec{1}_r = \vec{1}_\theta\vec{1}_\theta + \vec{1}_\phi\vec{1}_\phi \equiv \text{ transverse identity}$$

The Lorenz potentials for this case are from [3]

$$\tilde{\vec{A}}(\vec{r},s) = e^{-\gamma r}\frac{\mu_o}{4\pi r}s\tilde{\vec{p}}(s) \quad , \quad \gamma \equiv \frac{s}{c} \tag{6.5}$$

$$\tilde{\Phi}(\vec{r},s) = e^{-\gamma r}\left\{\frac{s}{4\pi c r} + \frac{1}{4\pi r^2}\right\}\vec{1}_r \cdot \tilde{\vec{p}}(s)$$

and the fields are

$$\tilde{\vec{E}}(\vec{r},s) = e^{-\gamma r}\left\{\frac{-\mu_o}{4\pi r}s^2\vec{1}_t + \frac{Z_o}{4\pi r^2}s\left[3\vec{1}_r\vec{1}_r - \vec{1}\right] + \frac{1}{4\pi\varepsilon_o r^3}\left[3\vec{1}_r\vec{1}_r - \vec{1}\right]\right\}\cdot\tilde{\vec{p}}(s) \tag{6.6}$$

$$\tilde{\vec{H}}(\vec{r},s) = e^{-\gamma r}\left\{\frac{-1}{4\pi c r}s^2 - \frac{1}{4\pi c r^2}s\right\}\vec{1}_r \times \tilde{\vec{p}}(s)$$

these being related by (2.4). Including the electric gauge as in (2.12) we have

$$\tilde{\vec{E}}(\vec{r},s) = -s\tilde{\vec{A}}(\vec{r},s) - \nabla\tilde{\Phi}(\vec{r},s) = -s\tilde{\vec{A}}_e(\vec{r},s)$$

$$\tilde{\vec{H}}(\vec{r},s) = \frac{1}{\mu_o}\nabla\times\tilde{\vec{A}}(\vec{r},s) = \frac{1}{\mu_o}\nabla\times\tilde{\vec{A}}(\vec{r},s) \tag{6.7}$$

So the electric-gauge vector potential is readily expressed in terms of the electric field as in (6.6).

In changing from one static potential to a second, Section V gives this in terms of the electric impulse

$$\int_{t_1}^{t_2} \vec{E}(\vec{r},t')dt' = -\vec{A}_2(r) - \int_{t_1}^{t_2} \nabla\Phi(\vec{r},t')dt'$$

$$= -\vec{A}_{e_2}(\vec{r}) \tag{6.8}$$

where zero initial conditions are assumed, and hence zero initial electric-dipole moment. From (6.6) and (6.7) we have

$$-\vec{A}_{e_2}(\vec{r}) = \int_{t_1}^{t_2} \vec{E}(\vec{r},t')dt'$$

$$= \frac{1}{4\pi\varepsilon_o r^3}\left[3\vec{1}_r\vec{1}_r - \vec{\vec{1}}\right]\cdot\int_{t_1}^{t_2}\vec{p}(t')dt' \tag{6.9}$$

with now

$$\lim_{t\to\infty} \vec{p}(t) = \vec{0} \tag{6.10}$$

or zero final condition for the electric-dipole moment, with non-zero complete time integral.

In terms of the Lorenz potentials, since there is zero current and charge at late time, we have

$$\vec{A}_2(\vec{r}) = \vec{0} \quad , \quad \Phi_2(\vec{r}) = 0$$

$$\int_{t_1}^{t_2} \nabla\Phi(\vec{r},t')dt' = \tilde{\vec{A}}_{e_2}(\vec{r})$$

$$= -\frac{1}{4\pi\varepsilon_o r^3}\left[3\vec{1}_r\vec{1}_r - \vec{\vec{1}}\right]\cdot\int_{t_1}^{t_2}\vec{p}(t')dt' \tag{6.11)]}$$

So both Lorenz potentials are zero for late time (as well as initially), but are, of course, non-zero for intermediate times. The electric-gauge vector potential is, however, non-zero at late-times and is the negative of the electric impulse.

VII. Lorenz Vector Potential Without Scalar Potential: Equivalent Electric Dipole

Now consider the case of no scalar potential (Lorenz gauge) for all time. This implies outside V_s

$$\Phi(\vec{r},t) = 0$$

$$\vec{A}(\vec{r},t) = \vec{A}_e(\vec{r},t)$$

$$\nabla \cdot \vec{A}(\vec{r},t) = 0 = \nabla \cdot \vec{A}_e(\vec{r},t) \tag{7.1}$$

$$\int_{t_1}^{t_2} \vec{E}(\vec{r},t)dt' = \vec{A}_{e_2}(\vec{r}) = \vec{A}_2(\vec{r})$$

For simplicity, one can take

$$\rho(\vec{r},t) = 0 \tag{7.2}$$

consistent with the above, although as discussed before, there are special cases of non-zero ρ which produce no fields outside V_s.

There are various forms that $\vec{J}$ can take with

$$\nabla \cdot \vec{J}(\vec{r},t) = 0 \tag{7.3}$$

One way to synthesize such divergenceless current distributions is to think of the various ways to make closed conducting loops. In a D.C. (low-frequency) sense there is a static current distribution with no (macroscopic) charge density. Then take this same current distribution to non-zero frequencies, either in an approximate sense if the loop is electrically small, or in a more exact sense by distributing current sources around the loop, all producing the same current (both magnitude and phase), thereby suppressing more and more exactly (in the limit of a large number of current sources) any buildup of charge. As the integrals in (2.7) make quite clear, a divergenceless current distribution can produce a vector potential with no scalar potential and hence fields (including radiated or $\frac{e^{-\gamma r}}{r}$ fields) via (2.4). One can also say this directly with the dyadic Green's function of free space [16].

An important class of such loops can be referred to as field containing inductors [4]. In this type of structure, the loop is constructed as a solenoid which is closed on itself to form a toroid-like structure. Such is a doubly connected surface and higher order connectedness is also possible. At DC, the structure is designed to produce <u>no external magnetic field</u>. One can synthesize such geometries by considering a closed, perfectly conducting, multiply connected surface with magnetic field inside, but not through the surface. Solving for the resulting surface current density via the inside tangential magnetic field, one then constructs this current density (at least approximately) via appropriate spacing of wires on the selected surface with one or more sources to drive the resulting loop(s). Note that there are no exterior fields at zero frequency.

As indicated in figure 7.1, consider the simplest case of such a "field-containing" inductor, a body of revolution as a toroid. The cross section of the toroid need not be circular. *It* lies on S_T and contains the volume V_T. With surface current density $\vec{J}_s(\vec{r}_s,t)$ on S_T as indicated we have for zero frequency

$$\tilde{\vec{B}}(\vec{r},0) = \mu_o \tilde{\vec{H}}(\vec{r},0) = \begin{cases} \mu_o \frac{I}{\Psi} \text{ for } \vec{r} \in V_T \\ 0 \text{ for } \vec{r} \notin V_T \cup S_T \end{cases} \tag{7.4}$$

$$\tilde{\vec{J}}_s(\vec{r}_s,0) = \vec{1}_T(\vec{r}_s) \times \tilde{\vec{H}}(\vec{r}_{s\_},0)$$

$\vec{r}_{s\_}$ on inside of S_T

$\vec{1}_T(\vec{r}_s) \equiv$ outward pointing normal to S_T

Now constrain

$$\tilde{\vec{J}}_s(\vec{r}_s,s) = \tilde{f}(s)\tilde{\vec{J}}_s(\vec{r}_s,0) \tag{7.5}$$

where $\tilde{f}(s)$ is some frequency function to be chosen for convenience. This assures a divergenceless current distribution

$$\nabla_s \cdot \tilde{\vec{J}}_s(\vec{r}_s,s) = 0 = -s\tilde{\rho}_s(\vec{r}_s,s) \tag{7.6}$$

Appendix A considers the response of such a toroidal antenna in reception and transmission. In reception, we have the open-circuit voltage from (A.14) for the electrically-small case as

$$\tilde{V}_{oc}(s) = \tilde{\vec{h}}_V(s) \cdot \tilde{\vec{E}}^{(inc)}(s)$$

$$\tilde{\vec{h}}_V(s) = \Xi s^2 \vec{1}_z \tag{7.7}$$

$$\Xi = \frac{\Psi_2^2 - \Psi_1^2}{4} \frac{wN}{c^2}$$

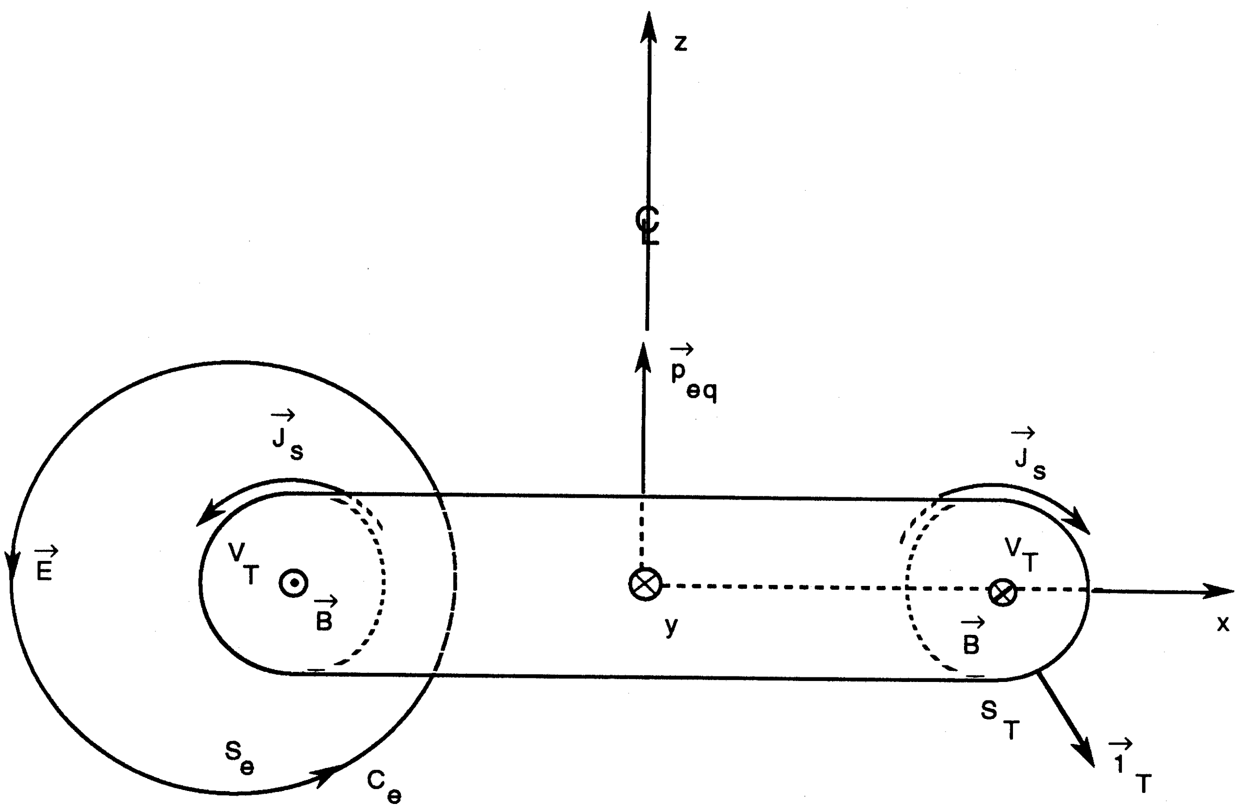

Fig. 7.1. Ideal Toroidal Divergenceless Current Distribution

where the last parameter represents the specific geometry of the antenna as derived in (A.7) (with N turns and dimensions Ψ_2 (outer radius), Ψ_1 (inner radius), and w (height)) for a rectangular cross section. Other cross sections can be similarly calculated.

In transmission, this type of antenna is described in the electrically-small regime by an equivalent electric-dipole moment

$$\tilde{\vec{p}}_{eq}(s) = \frac{1}{s}\tilde{\vec{h}}_V(s)\tilde{I}(s) = \Xi s \vec{1}_z \tilde{I}(s) \tag{7.8}$$

(I being the current driving the antenna port) which produces fields

$$\begin{aligned}\tilde{\vec{E}}(\vec{r},s) &= e^{-\gamma r}\left\{-\frac{\mu_o}{4\pi r}s^2\vec{1}_r + \frac{Z_o}{4\pi r^2}s\left[3\vec{1}_r\vec{1}_r - \vec{1}\right] + \frac{1}{4\pi\varepsilon_o r^3}\left[3\vec{1}_r\vec{1}_r - \vec{1}\right]\right\}\cdot\tilde{\vec{p}}_{eq}(s) \\ \tilde{\vec{H}}(\vec{r},s) &= e^{-\gamma r}\left\{-\frac{1}{4\pi c r}s^2 - \frac{1}{4\pi r^2}\right\}\vec{1}_r \times \tilde{\vec{p}}_{eq}(s)\end{aligned} \tag{7.9}$$

provided r is large compared to antenna dimensions.

An interpretation of this equivalent electric dipole is indicated in figure 7.1. Consider a contour C_e, say on a plane of constant ϕ, enclosing a surface S_e on this plane. Define

$$S_T' \equiv S_e \bigcap V_T \tag{7.10}$$

Note that C_e cannot be shrunk to zero without passing through V_T since S_T is a multiply-connected surface. From the integral form of the first of (2.1) we have

$$\oint_{C_e} \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_{S_e} \vec{B}\cdot\vec{1}_{S_e}\, ds \tag{7.11}$$

where on a plane of constant ϕ

$$\vec{1}_{S_e} = \vec{1}_\phi \tag{7.12}$$

By design, $\vec{B}$ is relatively large in U_T, but note that $\vec{E}$ can only be zero everywhere outside V_T if there is no time variation, i.e. only for a static situation. This is consistent with (6.8) due to the factor of s included.

Having the fields from this equivalent dipole, we are in a position to calculate the potentials from the electric impulse for static initial and final conditions as defined in Section V. Again we have

$$-\vec{A}_{e_2}(\vec{r}) = \int_{t_1}^{t_2} \vec{E}(\vec{r},t')dt'$$

$$= \frac{1}{4\pi\varepsilon_o r^3}\left[3\vec{1}_r\vec{1}_r - \vec{1}\right]\cdot\int_{t_1}^{t_2} \vec{p}_{eq}(t')dt' \tag{7.13}$$

So we require zero initial and final conditions for $\vec{p}_{eq}$ just as for $\vec{p}$ in Section VI. From (7.8) this means we have for the current driving the toroidal antenna

$$I_1 = 0 \quad \text{(initial condition)}$$

$$I_2 = \frac{1}{\Xi}\vec{1}_2\cdot\int_{t_1}^{t_2} \vec{p}_{eq}(t')dt' \tag{7.14}$$

(final condition)

In another form, we have

$$\int_{t_1}^{t_2} \vec{p}_{eq}(t')dt' = \Xi\vec{1}_z I_2$$

$$-\vec{A}_{e_2}(\vec{r}) = \frac{1}{4\pi\varepsilon_o r^3}\left[3\vec{1}_r\vec{1}_r - \vec{1}\right]\cdot\vec{1}_z\Xi I_2 \tag{7.15}$$

Now from (7.1) we have for the Lorenz potentials

$$\Phi_2(\vec{r}) = \Phi_1(\vec{r}) = 0$$

$$\vec{A}_1(\vec{r}) = \vec{0} \tag{7.16}$$

$$\vec{A}_2(\vec{r}) = \vec{A}_{e_2}(\vec{r}) = -\frac{1}{4\pi\varepsilon_o r^3}\left[3\vec{1}_r\vec{1}_r - \vec{1}\right]\cdot\vec{1}_z\Xi I_2 = -\frac{1}{4\pi\varepsilon_o r^3}\left[3\vec{1}_r\vec{1}_r - \vec{1}\right]\cdot\int_{t_1}^{t_2} \vec{p}_{eq}(t')dt'$$

VIII. Comparison of Potentials for Electric Dipole and Toroidal-Antenna Equivalent Electric Dipole

Comparing the two cases in Sections VI and VII, let the two cases be the same in initial and final senses, i.e. set

$$\int_{t_1}^{t_2} \vec{p}(t')dt' = \int_{t_1}^{t_2} \vec{p}_{eq}(t')dt' \tag{8.1}$$

Then we have the same electric impulse in both cases. This gives the same electric-gauge vector potential $\vec{A}_{e_2}$. However, the Lorenz gauge potentials are quite different. For the electric dipole in Section VI both $\vec{A}_2$ and Φ_2 are zero. For the toroidal-antenna equivalent electric dipole in Section VII, while Φ_2 is zero, $\vec{A}_2$ is non-zero. How then are these two cases different? Within the gauge condition (2.6) they are the same.

As long as we stay in the electrically-small region for both antennas and equate $\vec{p}(t)$ and $\vec{p}_{eq}(t)$ for all time, we also have the same fields. How now can we, away from the source regions, tell the two cases apart? Is there anything inherent in the Lorenz potentials, as distinguished from other potentials, such as derived from the electric gauge which can be measured to make this distinction?

IX. Concluding Remarks

So now we have the question posed in an interesting form. There are two quite different kinds of antennas, both of which produce electric-dipole fields, but different Lorenz potentials, one emphasizing the vector potential and the other the scalar potential. In a classical electromagnetic sense, one cannot distinguish these two cases by measurements of the fields (the measurable quantities) at distances away from the source region. The gauge invariance of QED implies the same in quantum sense.

Note that as discussed in previous sections, under static conditions, these two antennas give no fields. In going between two static conditions, one can have the same fields at intermediate times, but a change in the electric impulse, this being related to a change in the Lorenz vector potential or to a non-zero time integral of the gradient of the Lorenz scalar potential. However, with no fields, the vector potential has zero curl, which in a QED sense is not measurable.

One can modify the two-antenna experiment in various ways, if one wishes, to give other kinds of antennas. For example, one could enclose each antenna in a conducting shield, perhaps with high permeability as well. One merely redefines the antenna to include the shield as part of it. Under initial and final static conditions, the ideal toroidal coil has no external fields and its shield has no currents, and any residual magnetization is assumed negligible (assumed linear materials). The static Lorenz vector potential is then the same, although it may take more time to achieve static conditions due to the required time for the shield currents and magnetization to decay to zero. The electric dipole inside a shield has its excitation modified so that the electric-dipole-moment time history including the charges induced on the shield have the previously specified form giving the desired electric impulse. Note that the return of the scalar potential to zero is associated with the dipole moment returning to zero and the decay to zero of the exterior charges on the conducting shield, even with charges allowed to remain on the interior antenna (and shield interior surface as well).

So our choice of the two antennas is not unique for separately emphasizing the Lorenz vector and scalar potentials. All that is required is for the two to have the same exterior fields (say electric-dipole fields, or more general multipole fields) with different potentials (related by the gauge condition). In a classical electromagnetic sense, these antennas cannot be distinguished by exterior measurements. This is a classical non-uniqueness of sources. In a QED sense, the same is the case due to gauge invariance in its currently accepted form.

Appendix A. The Toroidal Antenna

To analyze the properties of the toroidal antenna, consider it first as a receiver. As in Figure A.1, let the antenna be a body of revolution with respect to the z axis with the usual coordinates. With the incident electric field $\vec{E}^{(inc)}$ taken initially parallel to the z axis let the antenna be electrically small. Neglect the field distortion due to the antenna conductors, or equivalently consider the antenna (as in Section VII) as a set of distributed sources in space specified by a surface current density $\vec{J}_s$ with

$$\vec{J}_s(\vec{r}_s,t)\cdot\vec{1}_\phi = 0 \tag{A.1}$$

From

$$\oint_C \vec{H}\cdot d\vec{l} = \frac{\partial}{\partial t}\int_S \vec{D}\cdot d\vec{S} \tag{A.2}$$

take the contour on constant Ψ, z (a circle) giving

$$\Psi\int_0^{2\pi} H_\phi^{(inc)}(\vec{r},t)d\phi = \pi\Psi^2\varepsilon_o \frac{\partial}{\partial t}E_z^{(inc)} \tag{A.3}$$

where $\vec{E}^{(inc)}$ is taken as uniform over the antenna. The average (over ϕ) magnetic flux density is

$$B_\phi^{(avg)} = \mu_o H_\phi^{(avg)} = \frac{\Psi}{2}\mu_o\varepsilon_o\frac{\partial}{\partial t}E_z^{(inc)} \tag{A.4}$$

The magnetic flux in the toroid is just the integral of $B_\phi^{(avg)}$ over the toroidal cross section at constant ϕ giving a flux (per turn) as

$$\Upsilon = w\int_{\Psi_1}^{\Psi_2} B_\phi^{(avg)} d\Psi = \frac{\Psi_2^2-\Psi_1^2}{4} w\mu_o\varepsilon_o\frac{\partial}{\partial t}E_z^{(inc)} \tag{A.5}$$

giving a total flux for an N-turn toroidal antenna as $N\Upsilon$.

The actual design of such an antenna has many possibilities ranging from the typical Rogowski coil (large N) to other forms (such as the CPM type) involving various parallel arrangements for high-frequency performance [2,19]. This antenna has some similarity to the FMM type of sensor for measuring vertical current density (displacement and, if present, conduction) [1,19]. In any event, the geometry of the various turns is made to assure an accurate averaging over the incident magnetic field so that (A.5) applies.

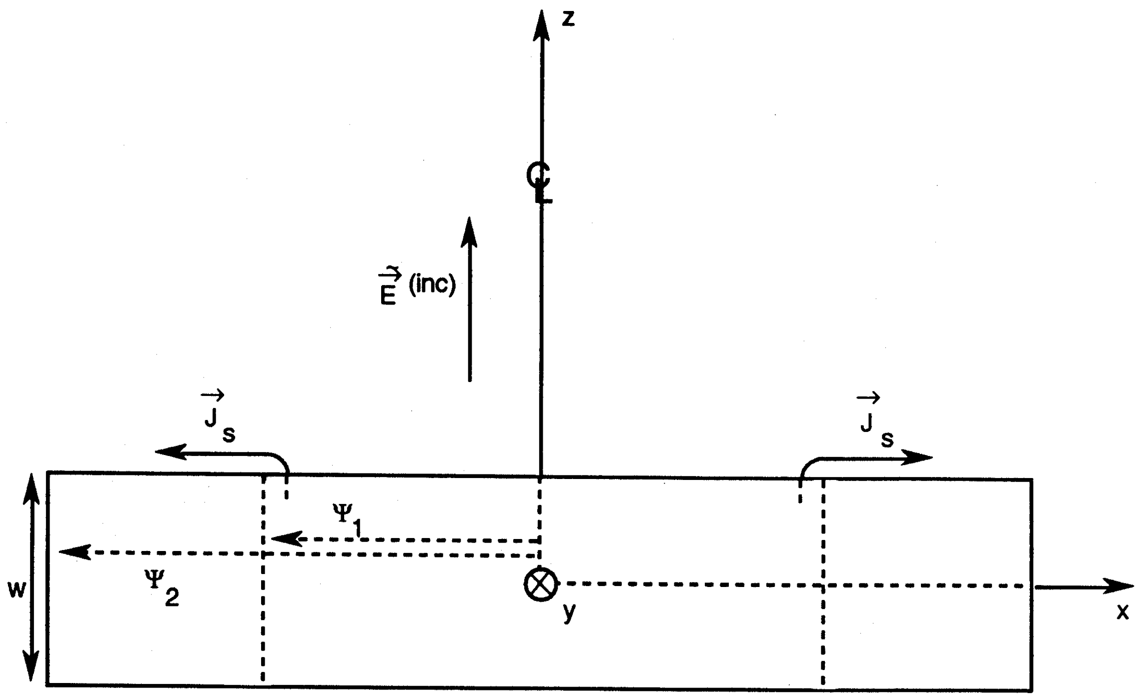

Fig. A.1. Toroidal Antenna as a Receiver

Such toroidal antennas are also used to measure current, say I positive along the z axis. In this case the magnetic field falls off as Ψ^{-1} and the open-circuit voltage is [2,19]

$$V_{o.c.} = M \frac{dI}{dt} \tag{A.6}$$

$$M = \frac{\mu_o N w}{2\pi} \ell n \left(\frac{\Psi_2}{\Psi_1} \right) \equiv \text{ mutual inductance}$$

Note there is a question of sign convention at the antenna port. By comparison (A.5) gives

$$V_{o.c.} = \frac{d}{dt}(N\Upsilon) = \frac{\Psi_2^2 - \Psi_1^2}{4} w N \mu_o \varepsilon_o \frac{\partial^2 E_z^{(inc)}}{\partial t^2} \tag{A.7}$$

$$= \frac{\Psi_2^2 - \Psi_1^2}{4} w N \mu_o \frac{\partial^2 D_z^{(inc)}}{\partial t^2}$$

The difference is associated with the fact that displacement current density $\partial D_z^{(inc)} / \partial t$ is allowed in the region $\Psi_1 < \Psi < \Psi_2$. For Ψ_1 near Ψ_2 one can assign an area $\pi \Psi_1^2$ or $\pi \Psi_2^2$ to multiply by the displacement current density to give a current so that we can write

$$V_{o.c.} = M \frac{\partial}{\partial t} \left[\vec{A}_e \cdot \frac{\partial}{\partial t} \vec{D}^{(inc)} \right] \tag{A.8}$$

where now

$$M\vec{A}_e = \frac{\Psi_2^2 - \Psi_1^2}{4} w \mu_o N \vec{1}_z \tag{A.9}$$

If desired we can separate these as

$$\vec{A}_e = \pi \Psi_e^2 \vec{1}_z \equiv \text{ equivalent area}$$

$$\Psi_e \equiv \text{ equivalent radius}$$
$$= \text{ average of } \Psi_1 \text{ and } \Psi_2 \text{ in some chosen sense} \tag{A.10}$$

$$M = \frac{\Psi_2^2 - \Psi_1^2}{4\pi \Psi_e^2} w \mu_o N$$

In another form we can write

$$V_{o.c.} = \frac{\Psi_2^2 - \Psi_1^2}{4} wN \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \tilde{E}_2^{(inc)} \tag{A.11}$$

so that this is a second-time-derivative electric-field sensor. However, the usual electric-field sensor has the open-circuit voltage proportional to the incident electric field (no time derivatives). So, except for the second time derivative (or factor of s^2), this can be regarded as an electric dipole, except that it does not have any electric dipole moment [17]

$$\vec{p} = \int_V \rho(\vec{r}) \vec{r} \, dV = Q \vec{h}_e \tag{A.12}$$

$\vec{h}_e \equiv$ equivalent height

$Q =$ charge delivered to antenna port (in transmission)

since (7.6) gives zero charge density. One way to look at this is to consider the free-space wave equation for the incident electric field as

$$\nabla \times \left[\nabla \times \tilde{E}^{(inc)} \right] = -\nabla^2 \tilde{E}^{(inc)} = \frac{1}{c^2} \frac{\partial^2 \tilde{E}^{(inc)}}{\partial t^2} \tag{A.13}$$

The second time derivative is related to $\nabla \times \nabla \times$. In the antenna one curl corresponds to converting $\tilde{E}_z^{(inc)}$ to B_ϕ, the other curl corresponds to converting B_ϕ by a surface integral to a line integral of an electric field for the antenna voltage.

So we have an equivalent electric dipole to which we can assign the equivalent area $\vec{A}_e$ as discussed above. We can also think of this as an electric dipole of equivalent area $\vec{A}_e$ coupled to the output via a transformer of mutual inductance *M*. In terms of the usual antenna reception properties [5] we have

$$V_{oc}(s) = \tilde{\vec{h}}_V(s) \cdot \tilde{\vec{E}}^{(inc)}(s)$$

$$\tilde{\vec{h}}_V(s) = \Xi s^2 \vec{1}_z \tag{A.14}$$

$$\Xi = \frac{\Psi_2^2 - \Psi_1^2}{4} \frac{wN}{c^2}$$

This last term is an effective sensitivity parameter for the toroidal antenna in reception.

Now consider such an antenna in transmission by reciprocity. An electric dipole produces fields [3]

$$\tilde{\vec{E}}(\vec{r},s) = e^{-\gamma r}\left\{-\frac{\mu_o}{4\pi r}s^2\vec{1}_r + \frac{Z_o}{4\pi r^2}s\left[3\vec{1}_r\vec{1}_r - \vec{1}\right] + \frac{1}{4\pi\varepsilon_o r^3}\left[3\vec{1}_r\vec{1}_r - \vec{1}\right]\right\}\cdot\tilde{\vec{p}}\ (s) \tag{A.15}$$

For our case let us restrict

$$\lambda\!\!\!^{-} = \frac{c}{\omega} >> \Psi_1, \ell \tag{A.16}$$

$$r >> \Psi_1, \ell$$

and assign an equivalent electric dipole moment $\vec{p}_{eq}$ to go in (A.16) and describe the radiation properties of the antenna. Clearly by symmetry

$$\vec{p}_{eq}(t) = p_{eq}(t)\vec{1}_z \tag{A.17}$$

i.e. parallel to $\vec{A}_e$.

The far field from such an equivalent electric dipole is

$$\tilde{\vec{E}}_f(\vec{r},s) = -e^{-\gamma r}\frac{\mu_o}{4\pi r}s^2\vec{1}_r\cdot\vec{1}_z\tilde{p}_{eq}(s)$$

$$\equiv \frac{e^{-\gamma r}}{r}\tilde{\vec{F}}_V(\vec{1}_r,s)\tilde{V}(s)$$

$$\equiv \frac{e^{-\gamma r}}{r}\tilde{\vec{F}}_I(\vec{1}_r,s)\tilde{I}(s) \tag{A.18}$$

$V(t) =$ voltage at antenna port
$I(t) =$ current out of antenna port

$\tilde{Z}_{in}(s) = \frac{1}{\tilde{Y}_{in}(s)} = \frac{\tilde{V}(s)}{\tilde{I}(s)}$ (in transmission)

$\equiv$ antenna input impedance

This follows the conventions discussed in [5]. Then we have the transmission functions for the far field as

$$\tilde{\vec{F}}_V(\vec{1}_r,s) = -\frac{\mu_o}{4\pi}\frac{s^2}{\tilde{V}(s)}\vec{1}_r\cdot\vec{1}_z\tilde{p}_{eq}(s)$$

$$\tilde{\vec{F}}_I(\vec{1}_r,s) = -\frac{\mu_o}{4\pi}\frac{s^2}{\tilde{I}(s)}\vec{1}_r\cdot\vec{1}_z\tilde{p}_{eq}(s) \tag{A.19}$$

$$\tilde{\vec{F}}_I(\vec{1}_r,s) = \tilde{Z}_{in}(s)\tilde{\vec{F}}_V(\vec{1}_r,s)$$

In reception antennas can be characterized by effective height for voltage and a related parameter for current [5] with wave incident in direction $\vec{1}_i$ as

$$\tilde{V}_{o.c.}(s) \equiv \tilde{\vec{h}}_V(\vec{1}_i,s)\cdot\tilde{\vec{E}}^{(inc)}(s) \equiv \text{ open circuit voltage} \tag{A.20}$$

$$\tilde{I}_{s.c.}(s) \equiv \tilde{\vec{h}}_I(\vec{1}_i,s)\cdot\tilde{\vec{E}}^{(inc)}(s) \equiv \text{ short circuit current}$$

$$\tilde{\vec{h}}_I(\vec{1}_i,s) = -\tilde{Y}_{in}(s)\tilde{\vec{h}}_V(\vec{1}_i,s)$$

where as before the incident field is evaluated at $\vec{r}=\vec{0}$. Since the antenna is electrically small then only the z component is relevant and

$$\tilde{V}_{o.c.}(s) \equiv \tilde{h}_V(s)\tilde{E}_z^{(inc)}(s)$$

$$\tilde{I}_{s.c.}(s) \equiv \tilde{h}_I(s)\tilde{E}_z^{(inc)}(s) \tag{A.21}$$

$$\tilde{\vec{h}}_V(\vec{1}_i,s) = \tilde{h}_V(s)\vec{1}_z$$

$$\tilde{\vec{h}}_I(\vec{1}_i,s) = \tilde{h}_I(s)\vec{1}_z$$

The reciprocity between transmission and reception now establishes [5]

$$\tilde{\vec{F}}_I(\vec{1}_r,s) = -s\frac{\mu_o}{4\pi}\vec{1}_r\cdot\tilde{\vec{h}}_V(-\vec{1}_r,s)$$

$$= -s\frac{\mu_o}{4\pi}\vec{1}_r\cdot\vec{1}_z\tilde{h}_V(s) \tag{A.22}$$

$$\tilde{\vec{F}}_V(\vec{1}_r,s) = s\frac{\mu_o}{4\pi}\vec{1}_r\cdot\tilde{\vec{h}}_I(-\vec{1}_r,s)$$

$$= s\frac{\mu_o}{4\pi}\vec{1}_r\cdot\vec{1}_z\tilde{h}_I(s)$$

Combining with (A.18) this gives

$$\tilde{p}_e(s) = \frac{1}{s}\tilde{h}_V(s)\tilde{I}(s) \tag{A.23}$$

$$= -\frac{1}{s}\tilde{h}_I(s)\tilde{V}(s)$$

and various other combinations.

The equivalent dipole moment in transmission is then using (A.14)

$$\tilde{p}_{eq}(s) = \Xi s\tilde{I}(s)$$

$$= \Xi \frac{s}{\tilde{Z}_{in}(s)} \tilde{V}(s) \tag{A.24}$$

Such an antenna is clearly inductive, and for low frequencies has

$$\tilde{Z}_{in}(s) = sL_{in} \tag{A.25}$$

$L_{in} \simeq NM$ from (A.6)

$$= \frac{\mu_o N^2 w}{2\pi} \ell n\left(\frac{\Psi 2}{\Psi 1}\right)$$

Actually the inductance is slightly larger than this since the N discrete windings only approximate a continuous ϕ-independent surface current density.

In terms of a low-frequency voltage drive we then have

$$\tilde{p}_{eq}(s) = \frac{\Xi}{L_{in}} \tilde{V}(s) \tag{A.26}$$

This is like the usual electric dipole except that the current does not go to zero as $s \to 0$, but rather diverges as $1/s$. On the other hand consider a current drive. Then at low frequencies (A.24) indicates that $\tilde{p}_{eq} \to 0$ (as well as $\tilde{V}(s) \to 0$).

In resolving the apparent paradox of how an antenna with no charge (in free space) can have an electric-dipole moment, one can go back to definitions. In [17] the fields from a current distribution are evaluated by expanding

$$\tilde{\vec{A}}(\vec{r},s) = \mu_o \int_V \frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \tilde{\vec{J}}(\vec{r}',s)dV' \tag{A.27}$$

$$\simeq \mu_o \frac{e^{-\gamma r}}{4\pi r} \int_V \tilde{\vec{J}}(\vec{r}',s)dV'$$

Then evaluating the volume integral of the current density (with no current crossing s, the boundary of V) we have [3,17]

$$\int_V \tilde{\vec{J}}(\vec{r}\,',s)dV' = s\int_V \vec{r}\,'\tilde{\rho}(\vec{r}\,',s)dV'$$

$$= s\tilde{\vec{p}}(s) \qquad \text{(A.28)}$$

This shows that a divergenceless current distribution gives a zero electric dipole moment in this sense.

However, the expansion in (A.27) as a leading term at low frequencies is only an approximation. Since, with an assumed divergenceless current distribution, this is zero, then we need to include higher order terms. This involves higher order terms in the expansion of $e^{-\gamma|\vec{r}-\vec{r}\,'|+\gamma r}$ around s=0 (for an electrically small antenna). By the previous derivation such higher-order terms can give non-zero electric-dipole-like fields. See the factor of s^2 that enters in the results, clearly a higher-order term.

In the usual texts there is also introduced a multipole expansion involving spherical Bessel functions and spherical vector harmonics [7,8,18,21]. The fields from electric and magnetic dipoles correspond to the lowest order terms (n=1) in the expansion. If we define dipole by this expansion then our toroidal antenna is an electric dipole. In any event the fields away from the source are the same. This is perhaps a matter of consistency in definitions.

References

1. C.E. Baum, Two Types of Vertical Current Density Sensors, Sensor and Simulation Note 33, February 1967.

2. C.E. Baum, Some Considerations for Inductive Current Sensors, Sensor and Simulation Note 59, July 1968.

3. C.E. Baum, Some Characteristics of Electric and Magnetic Dipole Antennas for Radiating Transient Pulses, Sensor and Simulation Note 125, January 1971.

4. Y.G. Chen, R. Crumley, C.E. Baum, and D.V. Giri, Field-Containing Enductors, Sensor and Simulation Note 287, July 1985, and IEEE Trans. EMC, 1988, pp. 345-350.

5. C.E. Baum, General Properties of Antennas, Sensor and Simulation Note 330, July 1991.

6. C.E. Baum and H.N. Kritikos, Symmetry in Electromagnetics, Physics Note 2, December 1990.

7. J.A. Stratton, Electromagnetic Theory, McGraw Hill, 1941.

8. P.M. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw Hill, 1953.

9 J. Schwinger, On Gauge Invariance and Vacuum Polarization, paper 20 in J. Schwinger (ed.), Selected Papers on Quantum Electrodynamics, Dover, 1958.

10. S.S. Schwever, An Introduction to Relativistic Quantum Field Theory, Row, Peterson, and Company, 1961.

11. R.P. Feynman, Quantum Electrodynamics, W.A. Benjamin, 1962.

12. R.P. Feynman, R.B. Leighton, and M. Sands, The Feynman Lectures on Physics, Vol. II, Addison-Wesley, 1964.

13. J.J. Sakurai, Advanced Quantum Mechanics, Addison-Wesley, 1967.

14. P. Roman, Introduction to Quantum Field Theory, John Wiley and Sons, 1969.

15. V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, Relativistic Quantum Theory, Addison-Wesley, 1971.

16. C.-T. Tai, Dyadic Green's Functions in Electromagnetic Theory, Intext Educational Publishers, 1971.

17. J.D. Jackson, Classical Electrodynamics, 2nd ed. John Wiley & Sons, 1975.

18. J. Van Bladel, Electromagnetic Fields, Hemisphere Publishing Corp., 1985.

19. C.E. Baum, Electromagnetic Sensors and Measurement Techniques, pp.73-144, in J.E. Thompson and L.H. Luessen (eds.), Fast Electrical and Optical Measurements, Vol I, Martinus Nijhoff, Dordrecht, 1986.

20. C.H. Papas, Theory of Electromagnetic Wave Propagation, Dover, 1988.

21. W.R. Smythe, Static and Dynamic Electricity, 3rd ed., Hemisphere Publishing Corp., 1989.

22. J. Schwinger, private communication, 27 September 1991. (Unfortunately R.P. Feynman is not currently available for consultation.)