Physics Notes

Note 1

September 1989

A Note on the Casimir Effect in a Uniformly Accelerated Reference Frame

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ABSTRACT

Maxwell's equations are established for the free electromagnetic field in two-dimensional space-times. In Minkowski space they are solved for a pair of uniformly accelerated "plates". These solutions are quantized and are used to express the regularized energy-momentum tensor of the electromagnetic field. With the aid of the regularized vacuum expectation value of the energy-momentum tensor we derive (as a new result) the Casimir force in an accelerated reference frame.

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I. Introduction

The inclusion of gravity in a unifying grand gauge theory of the fundamental forces in nature is still an open and challenging problem. Therefore, every step and every attempt to represent quantum fields in curved space times or in accelerated reference frames and to describe their interaction with real measuring instruments is an important contribution to the illumination and to the better understanding of the outstanding problems.

In [1], Unruh describes the construction of uniformly accelerated model-detectors which move through Minkowski space-time and detect (i.e., they are coupled to) a linear free quantum field. This interaction with the quantum field in its ground (vacuum) state leads to an excitation of the detectors as if they were in contact with a heat reservoir of temperature

$$T_a = \hbar a / (2\pi k_B c) = 2.5 \cdot 10^{-20} a\left(ms^{-2}\right) K \qquad (1.1)$$

Here the quantities $\hbar$, c, and k_B denote Planck's reduced constant, the speed of light, and Boltzmann's constant, respectively.

Another interesting effect was derived by Casimir. In his paper [2] he considers two parallel perfectly conducting plates (squares of side length L) separated by a distance d from each other and at rest in an inertial system in an otherwise empty Minkowski space. According to Casimir the fluctuations of the quantized electromagnetic field in vacuum lead to an attractive force of strength

$$\left|\vec{F}_C\right| = \pi^2 \hbar c L^2 / \left(240 d^4\right) \qquad (1.2)$$

between these plates. Indeed this prediction was verified to be in agreement with the experiment [3]. Moreover, in a following step, temperature corrections to the above force were included [4], assuming the electromagnetic field in a thermal state of some temperature T.

The result of the present paper arose when we tried to replace Unruh's detectors by Casimir's two parallel plates, but in addition accelerated them. The question came up whether the Casimir force between these accelerated plates approaches (at least perhaps for "very late times") a value which is similar to (1.1). This paper[1] deals with the answer to this question.

1 For related "moving mirror" problems which mainly consider the "particle production" caused by the mirrors consider Ref. [5] and in particular the references cited in the corresponding paragraph of Ref. [6]. In Ref. [7], the author suggests to compute the energy-momentum tensor beyond an acceleration barrier in four dimensional Minkowski space.

The outline of our paper is as follows: In Section II we establish the "free electromagnetic field" in certain two dimensional space-times and derive the corresponding field equations. These field equations are solved under appropriate boundary and initial value conditions (for the electromagnetic field in the comoving reference frame of the accelerated plates), and the solutions are quantized. Section III is devoted to the determination of the regularized expectation value of the energy-momentum tensor belonging to the "electromagnetic field" in vacuum at the upper plate. This value serves to derive the Casimir force between the two plates. Finally, in Section IV, we conclude our paper with a short discussion.

II. A Special Quantized Solution of Maxwell's Equations in a Two-Dimensional Space-Time

Formally, it is straight forward to derive Maxwell's equations in a two-dimensional space-time. On the basis of physics however, it becomes questionable whether solutions of these equations have a reasonable interpretation in real nature. Nevertheless, we will deal with a special electromagnetic problem in two dimensions, and we tacitly assume that the outcome at least gives us some qualitative clue for the corresponding result in the four dimensional Minkowski space.

The calculus of differential forms [8, 9] is most appropriate to formulate the electromagnetic field equations in a coordinate independent manner. In addition, for the convenience of the reader, we also present a local description. Let (M, g, μ) be a two dimensional oriented and simply connected pseudo-riemannian C^∞- manifold with metric g and volume element μ. Moreover, may $x=\left(x^o,x^3\right):D(x)\to \boldsymbol{R}^2$ be a positive C^∞– chart for M. We then assume the electromagnetic field $F\left(:M\to T^*M\right)$ to be an exact (C^1-) one-form and the corresponding Lagrange - density to be given by

$$L(dA)=-(8\pi c)^{-1}F\wedge *F \tag{2.1a}$$

(global representation)

$$L(dA)\Big|_{D(x)}=-(8\pi c)^{-1}g^{ij}F_iF_j\sqrt{-|g|}dx^o\wedge dx^3 \tag{2.1b}$$

(local representation)

Here the quantity $A\left(\in C^2(M,\boldsymbol{R})\right)$ denotes the potential for F (F=dA), $*$ is the Hodge - $*$ -operator, and throughout the paper the Einstein sum-convention is used.

The field equations are derived from (2.1) and become

$$dF=0;\quad -*d*F=0 \tag{2.2a,b}$$

(global)

$$\partial_oB+\partial_3E=0;\quad \partial_i\left(\sqrt{-|g|}g^{ij}F_j\right)/\sqrt{-|g|}=0 \tag{2.3a,b}$$

(local)

where the fields E and B are defined by the equation

$$F_{|D(x)} = E dx^o - B dx^3 \tag{2.4}$$

For later purposes we need to have the energy-momentum tensor T (corresponding to F) and its conservation law. Assuming that (M, g) admits a (differentiable) one-parameter group of isometries and $\xi : M \to TM$ is the corresponding generating (Killing-) vector field then we find the conserved current (C^1-one-form) $\tau_\xi : M \to T^*M$ with the following properties:

$$\tau_\xi = i(\xi)T; \quad T := (4\pi)^{-1}\left[F \otimes F + \frac{1}{2}(*(F \wedge *F))g\right] \tag{2.5a,b}$$

$$-*d*\tau_\xi = 0 \tag{2.6}$$

or in the local representation:

$$\tau_{\xi\,|D(x)} = \xi^i T_{ij} dx^j; \quad T_{ij} = (4\pi)^{-1}\left(F_i F_j - \frac{1}{2} g_{ij} g^{k\ell} F_k F_\ell\right) \tag{2.7a,b}$$

$$(i, j \in \{0,3\})$$

$$\partial_j\left(\sqrt{-|g|}\, g^{ij}\left(\tau_\xi\right)_i\right) / \sqrt{-|g|} = 0 \tag{2.8}$$

Here $i(\xi)$ indicates the interior product which is associated with ξ (see e.g. [9]).

In the next step it is our aim to find a solution of the field equations (2.2/3) for a certain physical arrangement. Consider two infinitely conducting "plates" (these are in reality two points) in the two-dimensional Minkowski space-time (M, g, μ) which are constantly accelerated with respect to an inertial system $\left(x^o, x^3\right)$. The electromagnetic fields inside and outside these "plates" shall be determined. Now the manifold $M = \boldsymbol{R}^2$, the metric has a very simple form

$$g = dx^o \otimes dx^o - dx^3 \otimes dx^3\,, \tag{2.9}$$

the volume element is expressed by

$$\mu = dx^o \wedge dx^3 \tag{2.10}$$

and $D(x) = \boldsymbol{R}^2$.

Due to their acceleration the "plates" move on the following trajectories:

$$\text{lower plate} \begin{cases} x^3\left(c^2/a\right) \text{for } x^o \le 0 \\ x^3 = \left[\left(x^o\right)^2 + \left(c^2/a\right)^2\right]^{1/2} \text{ for } x^o > 0 \end{cases} \tag{2.11}$$

$$\text{upper plate} \begin{cases} x^3 = d + \left(c^2/a\right) \text{for } x^o \le 0 \\ x^3 = \left[\left(x^o\right)^2 + \left(d + \left(c^2/a\right)\right)^2\right]^{1/2} \text{ for } x^o > 0 \end{cases} \tag{2.12}$$

The quantity $d \in (0, \infty)$ designates the distance (with respect to x) of the plates for times $x^o \le 0$, and the magnitude of the proper acceleration of the lower and the upper plate is $a \in (0, \infty)$ and $a/\left(1 + ad/c^2\right)$, respectively. The above trajectories of the plates are chosen in such a way that both plates are at rest (for $x^o \ge 0$) in the comoving (hyperbolic) system indicated by the local chart $\xi = \left(\xi^o, \xi^3\right) : \left\{x \in \boldsymbol{R}^2 : x^3 > \left|x^o\right|\right\} \to \boldsymbol{R}^2$ of Minkowski space which is defined by the following change of charts:

$$x \circ \xi^{-1} : \mathrm{R} \times \left(-c^2/a, \infty\right) \to \left\{x \in \mathrm{R}^2 : x^3 > \left|x^o\right|\right\}$$

$$\left(\xi^o, \xi^3\right) \mapsto \left(\xi^3 + c^2/a\right)\left(\sinh\left(a\xi^o/c^2\right), \cosh\left(a\xi^o/c^2\right)\right) \tag{2.13}$$

The properties of the chart ξ are well-known [10, 11]. In particular the metric is now expressed by

$$g_{|D(\xi)} = \left(1 + a\xi^3/c^2\right)^2 d\xi^o \otimes d\xi^o - d\xi^3 \otimes d\xi^3 \tag{2.14}$$

With respect to ξ the plates are at rest (for $x^o \ge 0$) at $\xi^3 = d$ (upper plate) and $\xi^3 = 0$ (lower plate).

Finally, the canonical quantization for the field E does not cause any difficulty for times $x^o \leq 0$. One has to observe the usual boundary conditions for E on infinitely conducting plates (i.e., E has to vanish on each plate in its instantaneous rest system), and the following requirements[2] have equally well to be met.

$$E = \sum_{\ell=1}^{\infty} \sqrt{\pi\hbar\omega_\ell}\left[a(k_\ell)E_\ell + a^*(k_\ell)E_\ell^*\right] \tag{2.15}$$

$$B = \sum_{\ell=1}^{\infty} \sqrt{\pi\hbar\omega_\ell}\left[a(k_\ell)B_\ell + a^*(k_\ell)B_\ell^*\right] \tag{2.16}$$

with the definitions

$$E_\ell := (4/d)^{1/2} \sin\left(k_\ell\left(x^3 - \left(c^2/a\right)\right)\right)\exp\left(-i\omega_\ell x^o/c\right) \tag{2.17}$$

$$B_\ell := -i(4/d)^{1/2} \cos\left(k_\ell\left(x^3 - \left(c^2/a\right)\right)\right)\exp\left(-i\omega_\ell x^o/c\right) \tag{2.18}$$

The fields E_ℓ^* and B_ℓ^* are the complex conjugate of E_ℓ and B_ℓ, respectively. The dispersion relation

$$\omega_\ell = k_\ell c \tag{2.19}$$

holds, and the boundary conditions enforce the equation

$$k_\ell = \pi\ell/d \tag{2.20}$$

Now, the Fock space corresponding to the field F is constructed as usual from the vacuum state Ω_o (which satisfies $a(k_\ell)\Omega_o = 0$ for all $\ell \in \mathbf{N}_*$) with the aid of the "annihilators" $a(k_\ell)$ and their adjoints (the "creators") $a^*(k_\ell)$ which fulfill the following commutation relations:

$$a(k_\ell)a^*(k_{\ell'}) - a^*(k_{\ell'})a(k_\ell) = \delta_{\ell\ell'} \tag{2.21}$$

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2 Strictly speaking there are three (ad hoc unrelated) quantizations for the field F. They are on the closures of the regions $(-\infty, 0) \times (-\infty, 0)$, $(-\infty, 0) \times (0, d)$, and $(-\infty, 0) \times (d, \infty)$ (with respect to x). Since E_ℓ vanishes at $(x^0, x^3) \in (-\infty, 0] \times d.\mathbf{Z}$ (for all $\ell \in \mathbf{N}_*$) the above quantizations on $(-\infty, 0] \times (-\infty, 0]$ and $(-\infty, 0] \times [d, \infty)$ only become physically after performing the limit $d \to \infty$.

$$a(k_\ell)a^*(k_{\ell'}) - a^*(k_{\ell'})a(k_\ell) = \delta_{\ell\ell'} \tag{2.21}$$

$$a(k_\ell)a(k_{\ell'}) - a(k_{\ell'})a(k_\ell) = a^*(k_\ell)a^*(k_{\ell'}) - a^*(k_{\ell'})a^*(k_\ell) = 0 \tag{2.22}$$

(for all $\ell, \ell' \in \boldsymbol{N}_*$)

Next we have to extend our solutions for times $x^o \geq 0$, i.e., we have to consider the time evaluation for the operator $\hat{\mathrm{F}}$ in the regions

$\Re_1 := \xi^{-1}(\boldsymbol{R} \times (0, d))$ (inside the plates)
and
$\Re_2 := \xi^{-1}(\boldsymbol{R} \times (d, \infty))$ (outside the plates)

of the Minkowski space. $\Re_1$ and $\Re_2$ have the trajectory of the upper plate as a common boundary. We assume the time evolution of $\hat{\mathrm{F}}$ to be governed by the field equations (2.2/3) under observation of the above stated boundary conditions for F. Then for times $x^o \geq 0$ $\hat{F}$ is obtained via equations (2.15/16) where we have to replace the functions E_ℓ and B_ℓ by (operator-) solutions $\hat{E}_\ell^{(\alpha)}$ and $\hat{B}_\ell^{(\alpha)}$ which are required to be solutions of the following initial-value and boundary-value problem (for convenience now stated with respect to ξ):

(a) $\hat{E}_\ell^{(\alpha)}$ and $\hat{B}_\ell^{(\alpha)}$ are elements from $C^o(\overline{\Re}_\alpha, \boldsymbol{C})$ and their restrictions to $\Re_\alpha(\alpha \in \{1,2\})$ belong to $C^1(\Re_\alpha, \boldsymbol{C})$

(b) They are constructed out of the solutions of the field equations

(A more elaborate study on wave propagation in certain media and non-initial systems can be found in the paper of Tse Chin Mo [12]).

$$\left(\frac{\partial E}{\partial \xi^o}\right) + \left(1 + \frac{a\xi^3}{c^2}\right)\frac{\partial}{\partial \xi^3}\left(\left(1 + \frac{a\xi^3}{c^2}\right)B\right) = 0 \tag{2.23}$$

$$\frac{\partial}{\partial \xi^o}\left(\left(1 + \frac{a\xi^3}{c^2}\right)B\right) + \left(1 + \frac{a\xi^3}{c^2}\right)\left(\frac{\partial E}{\partial \xi^3}\right) = 0 \tag{2.24}$$

(c) $\hat{E}_\ell^{(\alpha)}$ and $\hat{B}_\ell^{(\alpha)}$ satisfy the conditions:

$$(c_1)\ \hat{E}_\ell^{(1)}\left(0,\xi^3\right)=(4/d)^{1/2}\left(1+\left(a\xi^3/c^2\right)\right)\sin\left(k_\ell\xi^3\right) \tag{2.25}$$

$$\hat{B}_\ell^{(1)}\left(0,\xi^3\right)=-i(4/d)^{1/2}\cos\left(k_\ell\xi^3\right) \tag{2.26}$$

for every $\xi^3\in[0,d]$

$$\hat{E}_\ell^{(2)}\left(0,\xi^3\right)=(4/d)^{1/2}\left(1+\left(a\xi^3/c^2\right)\right)\sin\left(k_\ell\xi^3\right) \tag{2.27}$$

$$\hat{B}_\ell^{(2)}\left(0,\xi^3\right)=-i(4/d)^{1/2}\cos\left(k_\ell\xi^3\right) \tag{2.28}$$

for every $\xi^3\in[d,\infty)$

(initial-value conditions)

$$(c_2)\ \hat{E}_\ell^{(1)}\left(\xi^o,0\right)=\hat{E}_\ell^{(1)}\left(\xi^o,d\right)=0 \tag{2.29}$$

and

$$\hat{E}_\ell^{(2)}\left(\xi^o,d\right)=0 \tag{2.30}$$

for all $\xi^o\in[0,\infty)$

(boundary conditions)

The Cauchy problem corresponding to equations (2.23/24) can easily be solved (for C^1 - data on $\{0\}\times(-(c^2/a),\infty)$. We present slightly generalized solutions

$$E,B\in C^1\left(\boldsymbol{R}\times(-(c^2/a),\infty),\boldsymbol{C}\right)$$

of (2.23/24):

$$E\left(\xi^o,\xi^3\right)=\frac{1}{2}[f_1\left(\left(\xi^3+\left(c^2/a\right)\right)\exp\left(-a\xi^o/c^2\right)-\left(c^2/a\right)\right)+$$

$$+f_2\left(\left(\xi^3+\left(c^2/a\right)\right)\exp\left(a\xi^o/c^2\right)-\left(c^2/a\right)\right)\Big] \tag{2.31}$$

$$B\left(\xi^o, \xi^3\right) = \left(2\left(1+\left(a\xi^3 / c^2\right)\right)\right)^{-1}\left[f_1\left(\left(\xi^3+\left(c^2/a\right)\right)\exp\left(-a\xi^o/c^2\right)-\left(c^2/a\right)\right)\right.$$

$$\left. - f_2\left(\left(\xi^3+\left(c^2/a\right)\right)\exp\left(a\xi^o/c^2\right)-\left(c^2/a\right)\right)\right] \tag{2.32}$$

$\left(\xi^o, \xi^3\right) \in [0, \infty) \times (-(c^2/a), \infty)$

with the initial-values

$$\mathrm{E}\left(0, \xi^3\right) = \frac{1}{2}\left[f_1\left(\xi^3\right) + f_2\left(\xi^3\right)\right] \tag{2.33}$$

$$B\left(0, \xi^3\right) = \left[2\left(1+\left(a\xi^3/c^2\right)\right)\right]^{-1}\left(f_1\left(\xi^3\right) - f_2\left(\xi^3\right)\right) \tag{2.34}$$

where the functions f_1 and f_2 are arbitrary elements from $C^1((-c^2/a, \infty)\ \boldsymbol{C})$. Of course, these solutions have to fulfill the conditions (2.25) - (2.30). This is established by the usual method [13] choosing the data inside and outside the plates such that the boundary conditions are satisfied. This implies e.g. that the initial values for $\hat{E}^{(\alpha)}$ and $\hat{B}^{(\alpha)}$ are obtained from (2.33/34) with -otherwise arbitrary- on $[0, d]$ continuous complex-valued functions f_1, f_2. These functions are expected to be continuously differentiable on $(0, d)$ (respectively on $[d, \infty)$) and to satisfy the consistency condition

$$f_1(0) + f_2(0) = f_1(d) + f_2(d) = 0 \tag{2.35}$$

(for $\alpha = 1$)

or

$$f_1(d) + f_2(d) = 0 \tag{2.36}$$

(for $\alpha = 2$)

With the aid of the functions f_1 and f_2 we construct (continuously differentiable) functions $\hat{f}_1 \in C^o\left((-c^2/a, d], \boldsymbol{C}\right)$ and $\hat{f}_2 \in C^o([0,\infty))$ which are defined on a more extended domain. In the case of $\alpha = 1$

$$\hat{f}_1\left(\xi^3\right) := \begin{cases} f_1\left(\left(S_2S_1\right)^n\left(\xi^3\right)\right) \text{for } \xi^3 \in \left[\left(S_1S_2\right)^n(0), \left(S_1S_2\right)^n(d)\right] \\ -f_2\left(S_1\left(S_2S_1\right)^n\left(\xi^3\right)\right) \text{for } \xi^3 \in \left[S_1\left(S_2S_1\right)^n(d), S_1\left(S_2S_1\right)^n(d)\right] \end{cases}$$

$n \in \boldsymbol{N}$ and $\xi^3 \in (-c^2 / a, d]$ (2.37)

$$\hat{f}_2(\xi^3) := \begin{cases} f_2\left((S_1S_2)^n(\xi^3)\right) for\ \xi^3 \in \left[(S_2S_1)^n(0), (S_2S_1)^n(d)\right] \\ -f_1\left(S_2(S_1S_2)^n(\xi^3)\right) for\ \xi^3 \in \left[S_2(S_1S_2)^n(d), S_2(S_1S_2)^n(0)\right] \end{cases}$$

$n \in \boldsymbol{N}$ and $\xi^3 \in [0, \infty)$ (2.38)

The functions S_1 and S_2 are the following (C^∞-) diffeomorphisms:

$$S_1 := \begin{pmatrix} \left(-c^2 / a, \infty\right) \to \left(-c^2 / a, \infty\right) \\ \xi^3 \mapsto \left(c^2 / a\right)\left[\left(1 + \left(a\xi^3 / c^2\right)\right)^{-1} - 1\right] \end{pmatrix} \tag{2.39}$$

$$S_2 := \begin{pmatrix} \left(-c^2 / a, \infty\right) \to \left(-c^2 / a, \infty\right) \\ \xi^3 \mapsto \left(c^2 / a\right)\left[\left(1 + \left(ad / c^2\right)\right)^2 \left(1 + \left(a\xi^3 / c^2\right)\right)^{-1} - 1\right] \end{pmatrix} \tag{2.40}$$

In the case where $\alpha = 2$ it is sufficient to extend f_1 to a function $\hat{f}_1 \in C^o\left(\left(-c^2 / a, \infty\right), \boldsymbol{C}\right)$ which is continuously differentiable on $\left(-c^2 / a, \infty\right) \setminus \{d\}$.

$$\hat{f}_1(\xi^3) := -f_2\left(S_2(\xi^3)\right),\ \ for\ \xi^3 \in \left(-c^2 / a, d\right] \tag{2.41}$$

Now we are prepared to estimate the (operator-) field $\hat{F}$, especially at the position of the upper plate.

III. Regularized Energy-Momentum Tensor and Casimir - Force on the Upper Plate

Since we are interested in the Casimir-force acting on the upper plate it is sufficient to know the $\hat{B}_\ell^{(\alpha)}$ at the position of the upper plate. These quantities can be derived applying equations (2.31/32), (2.37/38), and (2.41). We find (for all $\ell \in \boldsymbol{N}_*$)

$$\left|\hat{B}_\ell^{(1)}\left(\xi^0,d\right)\right|^2 = \left(\frac{4}{d}\right)\begin{cases} \exp\left(-\left(2a/c^2\right)\left(\xi^0 - \eta c\tau_o\right)\right) \\ for\ \xi^0 \in \left[nc\tau_o, \left(n+\frac{1}{2}\right)c\tau_o\right], n \in \boldsymbol{N} \\ \exp\left(\left(2a/c^2\right)\left(\xi^0 - (n+1)c\tau_o\right)\right) \\ for\ \xi^0 \in \left[\left(n+\frac{1}{2}\right)c\tau_o, (n+1)c\tau_o\right] n \in \boldsymbol{N} \end{cases} \tag{3.1}$$

$$\left|\hat{B}_\ell^{(2)}\left(\xi^0,d\right)\right|^2 = \left(\frac{4}{d}\right)\exp\left(2a\xi^o/c^2\right),\ \ \xi^0 \geq 0 \tag{3.2}$$

Here the time constant τ_o is defined by

$$\tau_o := 2(c/a)\ell n\left(1+\left(ad/c^2\right)\right) \tag{3.3}$$

Now, we assume the field $\hat{F}$ to be initially in the vacuum state on both sides of the upper plate (compare the second footnote) and obtain for the vacuum expectation value of the energy-momentum tensor at the upper plate the formal sum (depending on the region $\Re_\alpha$)

$$\left(\Omega_0\left|T_{ij}^{(\alpha)}\left(\xi^0,d\right)\right|\Omega_o\right) = \left(\frac{1}{8}\right)\left[\sum_{\ell=1}^{\infty}\hbar\omega_\ell\left|\hat{B}_\ell^{(\alpha)}\left(\xi^0,d\right)\right|^2\right]g_{ij}$$

$$i,j \in \{0,3\}\ ,\ \xi^0 \geq 0 \tag{3.4}$$

where the brackets (|) denote the scalar product corresponding to the Fock space of the field $\hat{F}$. Next we have to regularize the divergent formal sum

$$" \sum_{\ell=1}^{\infty} \ell " \quad \text{by} \quad \zeta(-1) = -\left(\frac{1}{12}\right) \tag{3.5}$$

(ζ denotes the riemannian zeta function) and eventually derive from (3.1/2) and (3.4) the final result

$$\left(\Omega_0 \left| T_{ij}^{(\alpha)}\left(\xi^0, d\right) \right| \Omega_0\right)_{reg} = -\frac{\pi\hbar c}{24 d^2}\left(\frac{d}{4}\left|\hat{B}_1^{(\alpha)}\left(\xi^0, d\right)\right|^2\right) g_{ij} \tag{3.6}$$

$$\left(i, j \in \{0,3\}, \alpha \in \{1,2\}, \; and \; \xi^0 \geq 0\right)$$

The magnitude $\left|\hat{B}_1^{(\alpha)}\left(\xi_1^0, d\right)\right|^2$ is given by (3.1/2). Using (3.6) we find that the upper plate experiences in its instantaneous rest system a (proper time-dependent) Casimir force $\vec{F}_C(\tau)$ which is directed towards the lower plate and has the magnitude ($\tau \geq 0$)

$$\left|\vec{F}_C(\tau)\right| = \left(\frac{\pi\hbar c}{24 d^2}\right) \begin{cases} \exp\left(-(2a/c)\left(1+\left(ad/c^2\right)\right)^{-1}\left(\tau - n\tau_d\right)\right) \\ for \; \tau \in \tau_d \cdot \left[n, n+\frac{1}{2}\right] \\ \exp\left((2a/c)\left(1+\left(ad/c^2\right)\right)^{-1}\left(\tau-(n+1)\tau_d\right)\right) \\ for \; \tau \in \tau_d \cdot \left[n+\frac{1}{2}, n+1\right], n \in \boldsymbol{N} \end{cases} \tag{3.7}$$

where we have introduced the abbreviation

$$\tau_d := \left(1 + ad/c^2\right)\tau_o \tag{3.8}$$

Equation (3.7) represents the central result of our paper. We will briefly discuss it in the next section.

IV. Discussion of the Result

The Casimir force (3.7) in the accelerated system has some special features. As expected $|\vec{F}_C(\tau)|$ depends on τ and varies periodically with the period τ_d. Moreover, it decreases exponentially in the first half of the period from the value $\pi\hbar c/(24d^2)$ to $\pi\hbar c/(24d^2(1+ad/c^2)^2)$ and increases to the former value in the second half of the period. The time constant of these decreases and increases turns out to be $c(1+ad/c^2)/(2a)$. In the case that we choose $d = 1\mu m$ and $a = 10\ m/s^2$ we obtain for the oscillation period $\tau_d = 10^{-14} s$ and for the time constant $1.5 \cdot 10^7 s = 0.5$ year.

Furthermore one can easily show that τ_d is the proper time which elapses at the upper plate between emission and reception of a light signal sent out from the upper plate, being reflected at the lower plate and finally again received at the upper plate. Therefore the periodicity of $|\vec{F}_C(\tau)|$ probably is caused by the special choice of the (in magnitude different) proper accelerations of the plates.

An interesting case derived from (3.7) can be obtained replacing the quantity a by the new acceleration $b := a/(1 + ad/c^2)$ (which is the absolute value of the proper acceleration of the upper plate) and then taking the limit $d \to c^2/b$ $\left(\approx 1 \text{ light year for } b = 10\ m/s^2\right)$. These replacements correspond to the situation where the acceleration of the lower plate becomes infinity and, moreover, τ_d also approaches infinity. (But still the boundary conditions for the electromagnetic field at the lower plate influence the Casimir force at the upper plate). Performing the above mentioned limits we find

$$|\vec{F}_C(\tau)| \to \pi\hbar b^2/\left(24c^3\right)\exp(-2b\tau/c) \quad , \quad \tau \geq 0 \tag{4.1}$$

Hence, in this case the Casimir force (at the upper plate) exponentially decreases with increasing proper time τ to zero with the time constant $c/(2b)$.

Acknowledgements

The author would like to thank F. W. Hehl for his constant support and for helpful discussions. He is grateful to C. E. Baum for the invitation to the Weapons Laboratory and for the shown warm hospitality. Moreover, he wants to express his sincere gratitude to L. Boehmer and Major E. Thebo for their friendly help and their continuous support during his stay at the Weapons Laboratory.

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