Probability and Statistics Notes

Note 7

Effects of Supplemental Information on Confidence and Reliability

by
Chris Ashley

22 September 1973

Air Force Weapons Laboratory

Abstract

A brief discussion is presented of the effects of alternatives to the maximum ignorance assumption on the theory developed in Probability and Statistics Notes 1 and 2. Some values of population reliability confidence calculated assuming different initial states of information from sources other than testing are presented as examples.

Contents

Some Notes on Effects of Supplemental Information on Confidence and Reliability.

1. (The maximum ignorance assumption.) In the Appendix the reader will find reproduced the first four paragraphs of PSN (Probability and Statistics Note) 1. In paragraph 4 of this Appendix (and of PSN 1) there is a proviso which should receive careful attention: "you have no idea what the number of defective missiles is before you begin to test." The same proviso appears in PSN 2 (paragraph 8): "In this paper we shall assume . . . that the distribution of f is unknown and that the only information the evaluator of the data can get about it is from a finite number L of measurements." Thus, in PSNs 1 and 2 (and also in PSNs 5 and 6 and SDAN 13), except for independence of failure among population members and except for the dichotomy imposed by the experimenter, *there was no assumption made regarding the distribution of the population* prior to the acquisition of the test data. That is, *no information was assumed about the population from any other source* than the given experiment. Thus the axiom set from which the theory was developed was in some sense minimal. However, this very lack of assumptions may itself be phrased as an assumption: *it was assumed that no information about the population was available from any source* except the given experiment. So it might be said that when one assumes as little as possible about the population he is making an assumption of maximum ignorance.

2. (Strengths and weaknesses.) The maximum ignorance assumption has certain virtues to recommend it. For example, if one makes other, stronger assumptions about the population then one presumably has to be prepared to defend them. It may be more difficult generally to prove from miscellaneous scraps of information that only a restricted class of distributions of the population is possible than simply to plead ignorance and consider all kinds of populations impartially. Also, the confidence in reliability of a well designed and constructed system which one will calculate from test results using the maximum ignorance assumption will in general be a lower bound ("worst case") on the confidence one would calculate using stronger (and therefore harder to defend), more optimistic assumptions. On the other hand there are certain drawbacks to using the maximum ignorance assumption. Most importantly, it is seldom true that one has absolutely no other information from which confidence about the population distribution could be deduced prior to testing. Therefore the maximum ignorance assumption, though easy to defend and apply and though it gives useful results, is seldom the most accurate model of the real world.[1] Another drawback is that failure to include important information among one's assumptions can result in needing a larger (and more expensive) number of tests in order to achieve a given level of confidence in the actual character of the population.

3. (Alternatives to the maximum ignorance assumption: Homogeneity.) What other kinds of information might one have than that from testing? One example is, it might be known that the population is quite homogeneous. This is something that could be deduced from knowldge of stringent QC (quality control) procedures in force during production of a significant fraction of the population. (Such QC need govern only characteristics relevant to the particular dichotomy which has

been imposed by the experimenter or system analyst. If the concern is whether the population elements will or will not perform properly electrically, then it is usually irrelevant whether or not they are all the same color.) Or it might be possible to deduce homogeneity from the successful production of a theoretical model of relevant characteristics of population members. For *if* whoever produced the model has also verified it against a significant fraction of the population, then for members of that fraction to all be like the model they must also be like one another.

4. (Optimism.) It has been suggested[2] that in most real cases testing is not even considered until one has achieved assurance from some other source that the population would do well in the testing. To give a specific example, consider a population each element of which is itself a simple system involving a few distinct components. It could be that one has already completed exhaustive tests on each kind of individual component employed in the construction of all population elements. If all components have performed outstandingly in these earlier tests, and if the population elements in addition are each strongly parallel in design[3], then analysis of component test results might have resulted in one's being completely confident that the fraction of population elements which would pass the coming test, assembled, is at least, say, 50%. This then would be an example of à priori (with respect to tests of the assembled elements) confidence in 50% population reliability. Another kind of information which could lead to this sort of à priori optimism is knowledge of high quality in design and manufacture, as evidenced for example by past successes of those responsible for design and manufacture of members of the present population. Such information might give one higher confidence, prior to testing, that the population has only a small fraction defective than that it has a large fraction defective. To give an example of this kind of optimism, one might, prior to testing, have not just higher confidence for higher probabilities of success but in fact have confidence[4] linearly proportional to those probabilities (with a positive coefficient of proportionality). Yet another example would be the case in which available information made one quite sure, prior to testing, that his confidence in a given probability of success should be directly proportional to the square (or other higher power) of that probability (again, with a positive coefficient of proportionality). If one has information from which such à priori confidence can be deduced, then taking account of it would be more realistic than employing the maximum ignorance assumption.[5]

5. (Pessimism.) The example and remarks in the preceding paragraph also apply when information is available from which inverse or negative proportionality can be deduced.

6. (The finite population case: $N < \infty$.) Granting then that information on hand prior to testing may warrant one of a variety of alternative assumptions, making the maximum ignorance assumption a relatively poor model, how do we adjust our theory to use such new information or assumptions? To answer this question, let us look first at how the maximum ignorance assumption affected our equations. Equation (4) of PSN 1 is

$$C(R) = \frac{\sum_{I=M}^{N(1-R)} \left[\binom{I}{M} \binom{N-I}{L-M} \right]}{\sum_{I=M}^{N-L+M} \left[\binom{I}{M} \binom{N-I}{L-M} \right]} ,$$

where C is confidence, R is population reliability (by which we mean a lower bound on the fraction of the population which is "good"), L and M are defined[6] in paragraph 4 of the Appendix to the present note, and N is the (finite) number of members (i.e., cardinality) of the population. This equation covers the case of independent sampling without replacement from a finite population, assuming maximum ignorance. Let us, for the sake of figures which will appear later in this note, here change the definition of M. In PSN 1 M represented the number of *un*satisfactory elements in the sample. In PSN 2, in contrast, M was used to represent the number in the sample which were *satisfactory*. Let us here shift from the convention of PSN 1 to that of PSN 2. It can be shown, by reasoning analogous to that in PSN 1, that the above equation then becomes

$$C(R) = \frac{\sum_{I=R*N}^{N-L+M} \left[\binom{I}{M} \binom{N-I}{L-M} \right]}{\sum_{I=M}^{N-L+M} \left[\binom{I}{M} \binom{N-I}{L-M} \right]} \qquad (1) .$$

(Note that in this expression the index of summation, I, represents the number "good" in a population which is a candidate for the source of the given sample, whereas the indices of summation in equation (4) of PSN 1 represented the possible number "bad".)

7. (Generalization to allow for supplemental information.) Equation (1) arose from letting each of the N+1 possible kinds of populations (or Minuteman fleets, in the case of PSN 1), with fractions good from $\frac{0}{N}$, $\frac{1}{N}$, $\frac{2}{N}$, ... , to $\frac{N-1}{N}$, $\frac{N}{N}$, be represented equally as possibilities for the population from which the random data was actually taken.[7] Therefore each of these possible kinds of populations is represented by an equal number of terms (viz., one) in the denominator of the expression.[8] Similarly, of all the possible kinds of populations which have fractions good greater than or equal to R, each is represented by an equal number of terms (viz., one) in the numerator of equation (1)[8]. That the maximum ignorance assumption thus led us to represent each possible kind of

population impartially, i.e., by exactly the same number of terms, viz., by exactly *one* term, in the numerator and denominator of equation (1), can be stated mathematically by saying

$$A_I = 1 \quad \forall I \tag{2}$$

in a more general formulation of equation (1), viz.,

$$C(R) = \frac{\sum_{I=R*N}^{N-L+M} \left[A_I \binom{I}{M} \binom{N-I}{L-M} \right]}{\sum_{I=M}^{N-L+M} \left[A_I \binom{I}{M} \binom{N-I}{L-M} \right]} \qquad (3)^9 \; .$$

Thus equation (2) is a mathematical statement of the maximum ignorance assumption. It permits us, effectively, to factor out the A_I in both the numerator and the denominator of equation (3), and then to cancel them to derive equation (1) as a special case of the more general reliability-confidence relation given in equation (3). However, if maximum ignorance had *not* been assumed then A_I would not have been constant. In that case the factoring out and cancelling of the A_I are no longer legitimate and we must use equation (3) directly, instead of using the simplified version of it given as equation (1). To use equation (3) directly we need to know the values of the A_I. Without the maximum ignorance assumption we cannot use equation (2) to obtain those values. How do we get them?

8. To see how to obtain the values of the A_I in equation (3), let us return to the discussion of Sam and his marbles in PSN 1 (reproduced in part as an Appendix to the present note). In paragraph 3 of PSN 1 we calculated a confidence value of $66\frac{2}{3}$ % . This calculation was done by dividing the probability of acquiring the given experimental results from the white barrel by the sum of the probabilities of achieving those results from each of all possible kinds of barrels (cf. the Appendix). Let's rewrite that sentence in a more concise notation. Let P denote the (unknown) fraction of marbles which are white in the barrel. Since we know (from paragraph 1 in PSN 1) that every marble in the barrel is either black or white, it follows immediately that the fraction of marbles which are black in the barrel is 1-P . Similarly, if L marbles are drawn and M of them are white then L-M of the sample are black. Let $P_W(M|N,L,P)$ denote the probability of drawing M white marbles from a total of N marbles in L independent random draws, given that the fraction of the marbles which are white in the

barrel is P. Let $P_b(B|N,L,P)$ denote the probability of drawing B black marbles, where the other parameters are given exactly the same definitions as above. These remarks imply

$$P_b(B|N,L,P) \equiv P_b(L-M|N,L,P) \equiv$$

$$\equiv P_w(M|N,L,P) \quad .$$

Since "the probability of getting a white marble in a single random draw" *means* the fraction of marbles which are white in the barrel, and since P is being used to denote this fraction, another immediate consequence of these definitions is

$$P_w(1|N,1,P) \equiv P \quad .$$

Let C(R) denote our confidence that $P \geq R$ (R for "reliability"). Then the calculation which is stated verbally in paragraph 3 of PSN 1 may now be written:

$$C(1.) = \frac{P_w(1|100000,1,1)}{P_w(1|100000,1,1) + P_w(1|100000,1,.5)} =$$

$$= \frac{1}{1+.5} = \frac{1}{3/2} = \frac{2}{3} =$$

$$= 66\tfrac{2}{3}\ \% \qquad (4) \ .$$

This equation, paragraph 3 in PSN 1, and paragraph 3 in the Appendix to the present note are all equivalent. These three equivalent statements at no point in their development assumed maximum ignorance. What is assumed instead is the information provided in paragraph 1 of PSN 1, viz., that only two barrels of marbles were to be considered, that for one of them P = 1, and that for the other P = .5 . Therefore equation (3) was applied, and the supplementary information led us to set

$$A_I = \begin{cases} 1 & \text{for } I = 50{,}000 \text{ or } 100{,}000 \\ 0 & \text{otherwise} \end{cases} \qquad (5) \ .$$

This equation is simply a mathematical statement of the assumption that the barrel from which we have randomly drawn a sample of one white marble may contain either 50,000 white marbles out of 100,000 or else 100,000 white marbles out of 100,000; we have no reason to prefer one of these alternatives over the other; and we know that no other alternatives are possible for the barrel in front of us. So equation (5) is a mathematical representation of a particular state of supplementary information. As such it is an alternative to equation (2), and in fact a more realistic (and therefore preferable) alternative for the problem proposed in the opening paragraphs of PSN 1.

9. Equations (5) and (3), then, are the general statement of equation (4). They are what produce the number $66\frac{2}{3}$ % in paragraph 3 of PSN 1. (Note that equation (1), which is equivalent to equation (4) in PSN 1, which was derived using the maximum ignorance assumption, would *not* produce that number for the confidence that the sample was from the white barrel. The maximum ignorance assumption would in fact imply a confidence of virtually zero for R = 1, N = 100,000, L = 1, and M = 1.[10]) Thus PSN 1 in fact contained calculations involving an alternative to the maximum ignorance assumption. Where did the particular supplementary information represented by equation (5) come from? Paragraph 1 of PSN 1 didn't say, but we could for example have learned from Sam, who had perhaps examined the contents of the barrels in detail and then told us some of his findings. (If this was our source of supplementary information then PSN 1 also assumed implicitly that we could place absolute confidence in anything Sam says, at least regarding these barrels. "Absolute" may be too strong a word to constitute a perfectly accurate model, and yet may be sufficiently accurate to provide a usable approximation.) In the present note we are concerned not so much with the details of finding and justifying such supplementary information for a particular population as we are in studying the effects of it on reliability-confidence relationships after it has been found and justified.

10. (Summary so far.) Let us summarize to this point. *Equation (3) of this note is the general statement of the reliability-confidence relationship for finite populations* (dichotomized and sampled independently without replacement). Equation (2) is a statement of the maximum ignorance assumption. These two equations taken together generate equation (1). Equation (1) therefore gives the reliability-confidence relationship for finite populations in the special case where maximum ignorance is assumed. This equation appears as equation (4) of PSN 1 (with appropriate adjustments for the definition of M). Equation (5) of the present note is a statement of one alternative to the maximum ignorance assumption. Equation (3) and equation (5) taken together generate equation (4), which is a particular numerical example presented verbally in paragraph 3 of PSN 1.

11. Before proceeding to develop mathematical models of other alternatives to the maximum ignorance assumption it might be helpful to consider graphical representations of the two supplementary information states already discussed. The maximum ignorance assumption, given by equation (2), is shown in Figure 1.

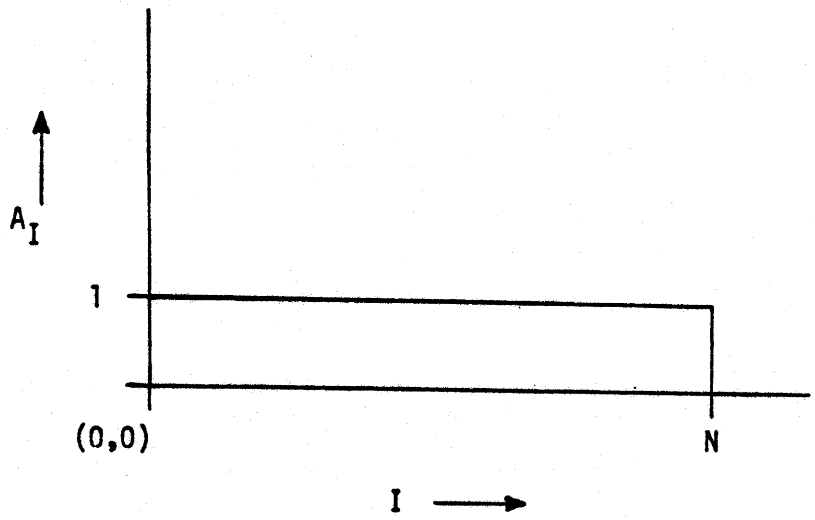

Figure 1. The maximum ignorance assumption.

The state of supplementary knowledge assumed in the particular example discussed in the opening paragraphs of PSN 1, given by equation (5) in the present note, is shown in Figure 2.

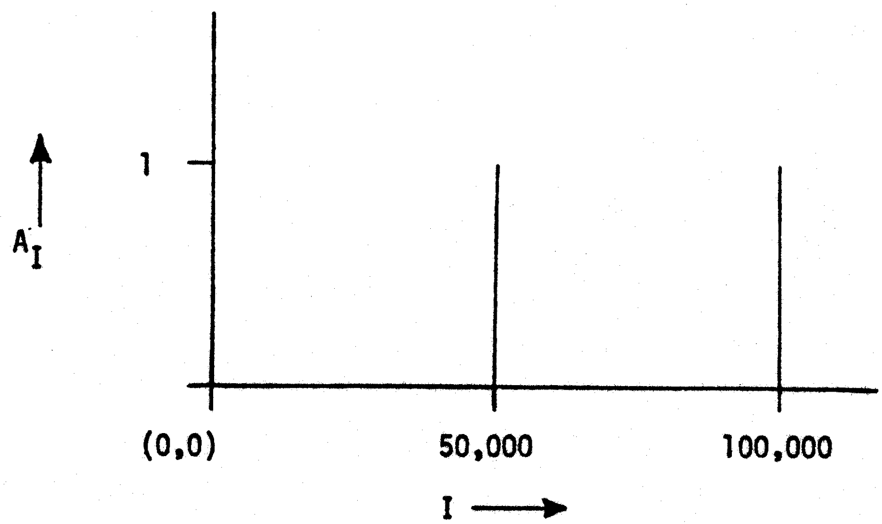

Figure 2. An alternative to the maximum ignorance assumption.

12. (Homogeneity.) Let us now develop values for the A_I in equation (3) to correspond to some other alternatives to the maximum ignorance assumption. We will use some of the specific examples already discussed in paragraphs 3 and 4, above. Consider for example the case of a population which is known, from

whatever source, to be at least 100f% homogeneous (cf. paragraph 3, above). If at least a fraction f of the population is completely alike in the respects of interest, say with respect to those features of its design and construction which are relevant to its being able or not being able to operate successfully in a certain harsh (e.g., EMP) environment, then either

a. the fraction of the population which will be able to operate successfully in the harsh environment is greater than or equal to f, or else

b. the fraction of the population which will fail in the harsh environment is greater than or equal to f.

The second of these two conditions may be restated in the following equivalent form:

b′. the fraction of the population which will be able to operate successfully in the harsh environment is less than or equal to 1-f.

Let P now denote the (unknown) fraction of the population which actually will be able to operate successfully in the harsh environment. Then the assumption, or supplementary information, of population homogeneity in this respect is seen from conditions b′ and a to imply that

$$P \leq 1 - f \quad \vee \quad f \leq P \quad .$$

Consequently, in the presence of non-trivial homogeneity assurances we need not consider any population with the property that $1-f < P < f$.[11] Recall that the quantity I in equation (3) represents the number of satisfactory elements in a population from which the sample of L might have been taken. Therefore the fraction satisfactory in a population represented by any single term in equation (3) is I/N. Thus the homogeneity assumption tells us that we need not consider any terms in equation (3) with I such that $1-f < I/N < f$, that is, with I such that $(1-f)N < I < fN$. Therefore we set $A_I = 0$ for those terms. On the other hand, a homogeneity assumption by itself does not give us any reason to favor any one of the remaining population candidates over any other. Therefore we give A_I a constant (non-zero) value for all remaining terms. In conclusion, therefore, we can represent the homogeneity assumption of paragraph 3, above, with the following mathematical expression:

$$A_I = \begin{cases} 0 & \text{if } (1-f)N < I < fN \\ 1 & \text{otherwise} \end{cases} \qquad (6)^{11,12} .$$

Graphically, therefore, the homogeneity assumption appears as in Figure 3.

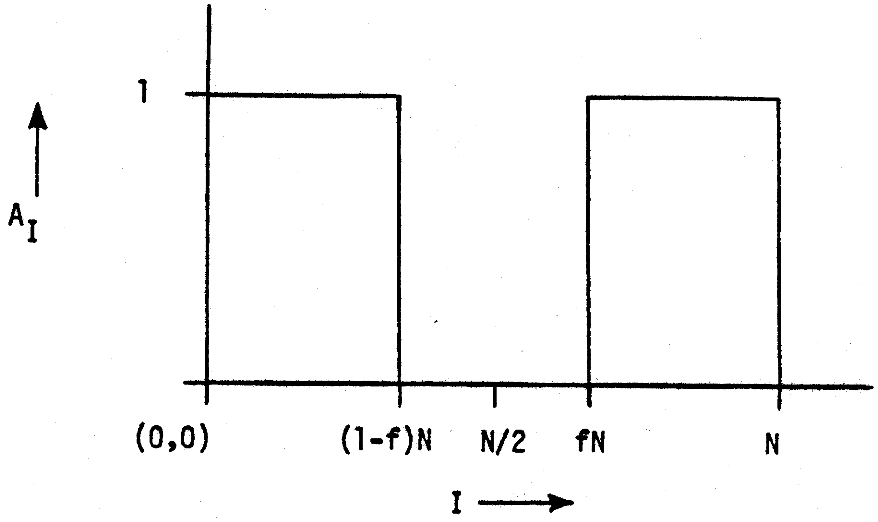

Figure 3. The homogeneity assumption.

13. (Step optimism.) In paragraph 8 of PSN 1 we said we would use the term "reliability" to mean a lower bound on the fraction P of the population elements which will perform their missions successfully. In paragraph 4 of the present note, above, we discussed the possibility of having information, from some other source than direct full system testing, which might make us want to incorporate in our calculations an *assumption* of population reliability. Let R_0 denote this value of à priori population reliability. Then R_0 is the greatest number such that we know, from information available from some other source than the direct full system testing which is to furnish the values of L and M in equation (3), that

$$R_0 \leq P \qquad (7) .$$

(Since $0 \leq P \leq 1$, this definition of R_0 tells us immediately that $0 \leq R_0$ no matter how weak or trivial the supplementary information may be.) Therefore we need not consider to be a possible source for the test sample any candidate population in which $P < R_0$. Therefore when evaluating equation (3) with this kind of supplemental information we can neglect all terms in which $I/N < R_0$, that is, terms for which $I < R_0 N$. Therefore we set $A_I = 0$ for those terms. If no information is available which would give us reason to favor any one of the remaining population candidates over any other then we give A_I a constant (non-zero) value for all terms representing those remaining candidate populations. Thus we can represent supplementary reliability information mathematically by setting

$$A_I = \begin{cases} 0 & \text{if } I < R_0 N \\ 1 & \text{otherwise} \end{cases} \qquad (8)^{12} .$$

Graphically, therefore, such an assumption looks like this:

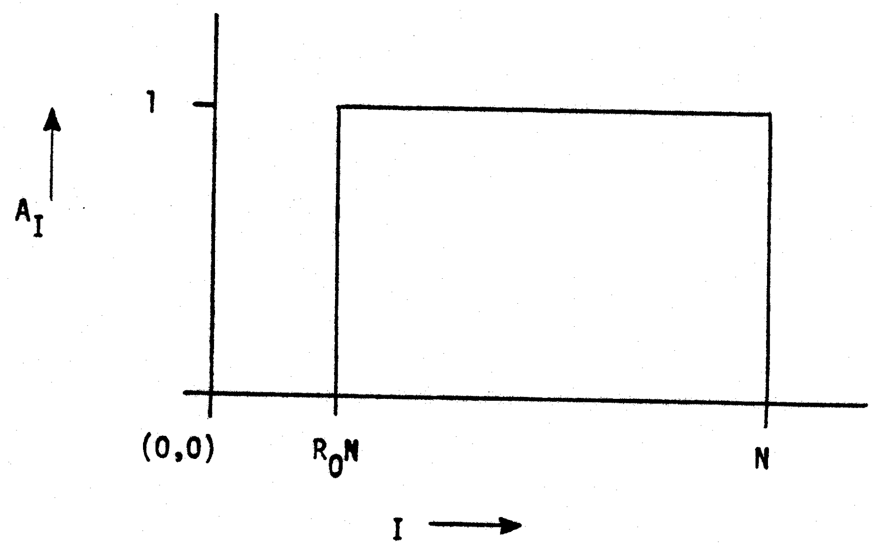

Figure 4. The step optimism assumption.

14. (Linear optimism.) In the last two paragraphs we have developed mathematical models for two of the supplementary information states mentioned in paragraphs 3 and 4, above. Let's model just one more of those example states, from paragraph 4. Assume that we have information, from some other source than the full system testing, from which we deduce a confidence in P which is directly proportional to the value of P. Thus if $0 < P_1 < P_2 \leq 1$, and if in fact $P_2 = k*P_1$, then we are here assuming that the supplemental information is such that we have, à priori, k times more confidence[4] that $P = P_2$ than we have that $P = P_1$ (for values of P_1 and P_2 compatible with $N < \infty$). To grasp this situation intuitively, consider again Sam and his marbles (cf. the Appendix). This time grant that we know he has three barrels, viz., the two he had before plus one new one of 100,000 solely white marbles. So the new arrangement is, he has *two* barrels for which $P = 1$ and still only one for which $P = .5$ (for definition of P in Sam's case, cf. paragraph 8, above); N = 100,000 for all three barrels. The reader can probably see already how the argument will go analogously to that in the opening paragraphs of PSN 1. The details are therefore relegated to footnote 13. Here we will instead pursue the consequences for equation (3) of this particular supplementary information. We see that it requires that candidate source populations containing 100,000 white marbles have twice as much representation as those containing 50,000 white marbles. That is, we should consider twice as often in equation (3) any term giving the probability of obtaining the given test results from an all-white-marbles population as we do any term giving the probability of obtaining the given test results from a half-white-marbles population. Therefore the supplementary information requires

$A_{100,000} = 2*A_{50,000}$. And we still need no representation at all for populations containing any other number of white marbles. We can satisfy these requirements by setting

$$A_I = \begin{cases} 2 & \text{for } I = 100{,}000 \\ 1 & \text{for } I = 50{,}000 \\ 0 & \text{otherwise} \end{cases} \qquad (9)^{12} .$$

Graphically this looks like:

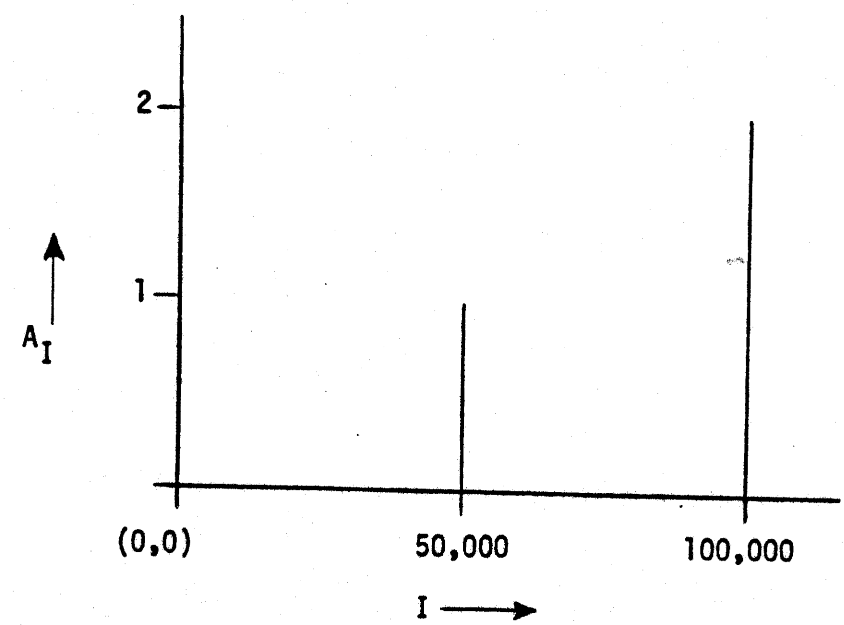

Figure 5. Another alternative to the maximum ignorance assumption.

It is easy to extend this example to the case in which the prior information is that the barrel of marbles before us came from a class of barrels (the rest of which need not actually exist at the time of the sampling: cf. footnote 13) among which were exactly I barrels containing I white marbles each (the remaining marbles in each barrel being black) for all integers I such that $0 \le I \le N$. Thus we can represent an assumption of linear optimism (with zero constant[14]) mathematically in equation (3) by setting

$$A_I = I \quad \forall I \qquad (10)^{12} .$$

The graph of this is:

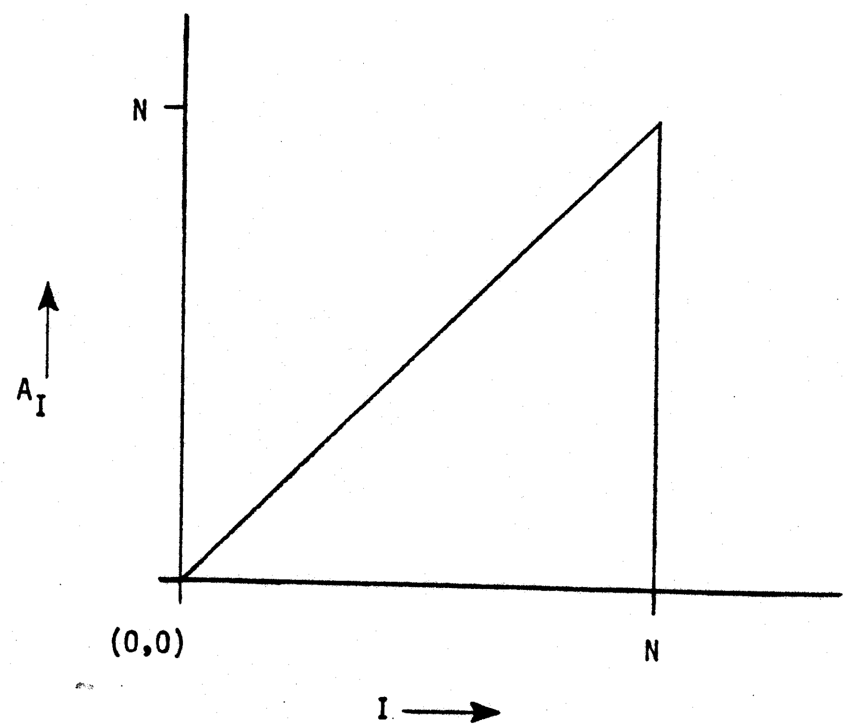

Figure 6. The linear optimism assumption (with zero constant).

It would be quite easy now to continue with this example, extending it to the case in which $A_I = I+K$ (for any non-negative integer value of K,[14] for all I such that $0 \leq I \leq N$), or to the case of linear pessimism ($A_I = -I-K$), or to the cases of non-integer and non-linear relationships between A_I and I. However, we should by now have enough examples to make the idea clear. Let us therefore proceed instead to illustrate the case in which there is more than one kind of supplementary information of which we wish to take account.

15. (Homogeneity *and* step optimism.) An example of a combination of kinds of supplementary information might be a situation in which information is available, from some other source than the system test, which à priori guarantees some fraction f of population homogeneity, and other information is also available which (independently) assures us of some value R_0 of à priori population reliability. Recall from paragraph 12: "Thus the homogeneity assumption tells us that we need not consider any terms in equation (3) with I such that . . . $(1-f)N < I < fN$." And from paragraph 13: "Therefore when evaluating equation (3) with . . . [step optimism] supplemental information we can

neglect all terms in which . . . $I < R_0 N$." Therefore, if information is available which warrants making both the homogeneity and the step optimism assumptions simultaneously, we set $A_I = 0$ for all I such that $(1-f)N < I < fN$ *and* for all I such that $I < R_0 N$. On the other hand, these two assumptions by themselves do not give us any reason to favor any one of the possible source populations represented by the remaining terms over any of the others. Therefore, as before, we give A_I some constant (non-zero) value for all remaining terms. Thus we can represent an assumption of both à priori reliability (i.e., step optimism) *and* homogeneity mathematically by setting

$$A_I = \begin{cases} 0 & \text{if} \quad (1-f)N < I < fN \quad \textit{or} \quad I < R_0 N \\ 1 & \text{otherwise} \end{cases} \qquad (11)^{11,12} .$$

Thus there are three possible graphs of this combination assumption, one for each of the three possible cases, viz., $R_0 N$ below, in, or above the interval $((1-f)N, fN)$. For the case in which $R_0 N \leq (1-f)N$ the graph is:

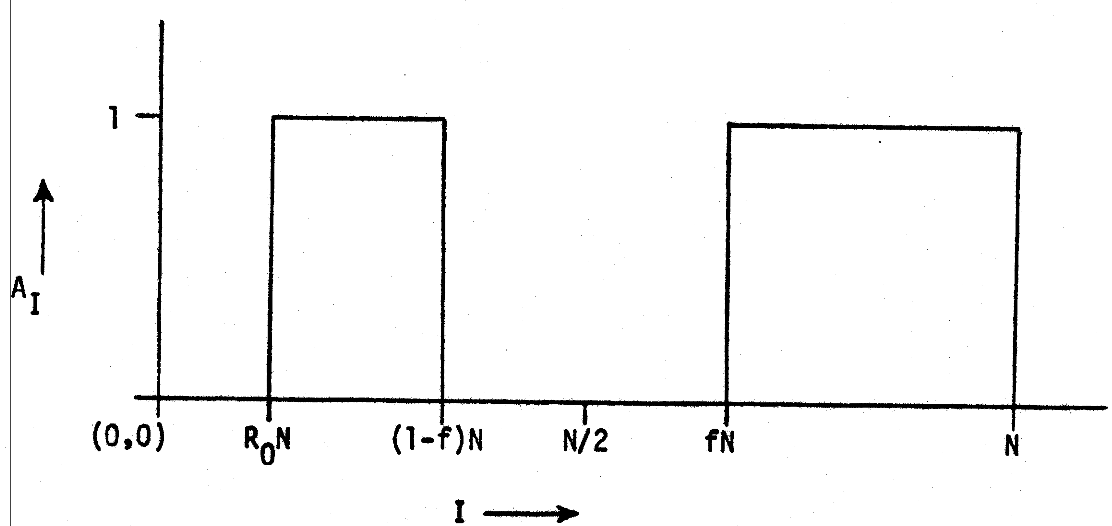

Figure 7. The assumption of homogeneity *and* step optimism, for $R_0 + f \leq 1$.

For the case in which $(1-f)N < R_0N \leq fN$ the graph is:

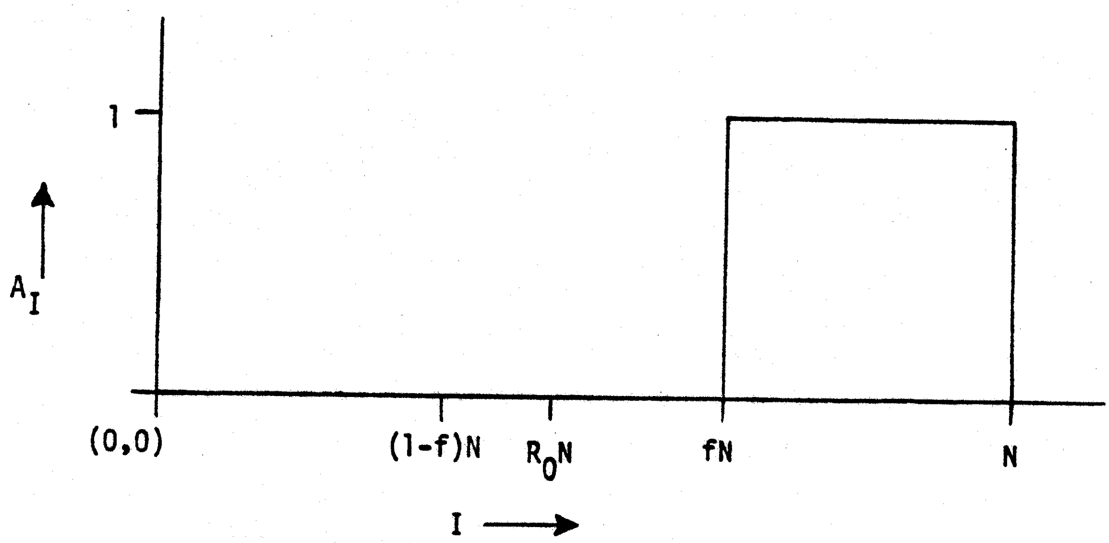

Figure 8. The assumption of homogeneity *and* step optimism, for $1-f < R_0 \leq f$.[11]

And for $fN < R_0N \leq N$ the graph is:

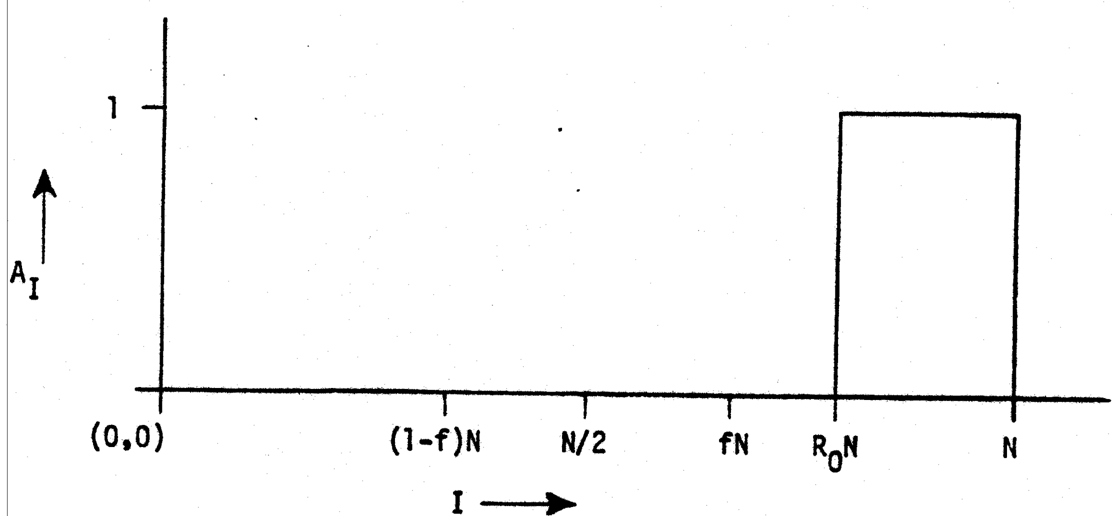

Figure 9. The assumption of homogeneity *and* step optimism, for $f < R_0$.

It should be noted that under the conditions of this last graph the homogeneity assumption is weaker than, and is in fact implied by, the step optimism assumption. Therefore if the supplementary information is such that $f \leq R_0$, and if only homogeneity and step optimism are being considered, then we might as well disregard all supplemental information except that which warrants the step optimism since only this latter will affect the values of the A_I in equation (3).

16. (Step optimism *and* linear optimism.) Another example of a situation in which there is more than one kind of supplementary information to take into account is the case in which the supplementary information warrants both step optimism at R_0 and, independently, that large fractions satisfactory are more likely than small fractions satisfactory. For this latter kind of information we might as well use again the example of linear optimism (with zero constant) discussed in paragraph 14, above. Then these two kinds of supplementary information, if both are present together, correspond to the situation in which the barrel of marbles before us came from a class of barrels (the rest of which need not actually exist at the time of the sampling of L marbles from one barrel: cf. footnote 13) among which there were exactly I barrels containing I white marbles each, the remaining marbles in each barrel being black, for all I such that $R_0N \le I \le N$, and no other barrels. Thus we can represent an assumption of both step optimism *and* linear optimism (with zero constant) mathematically in equation (3) by setting

$$A_I = \begin{cases} 0 & \text{if } I < R_0N \\ I & \text{otherwise} \end{cases} \qquad (12)^{12} .$$

The graph of this is:

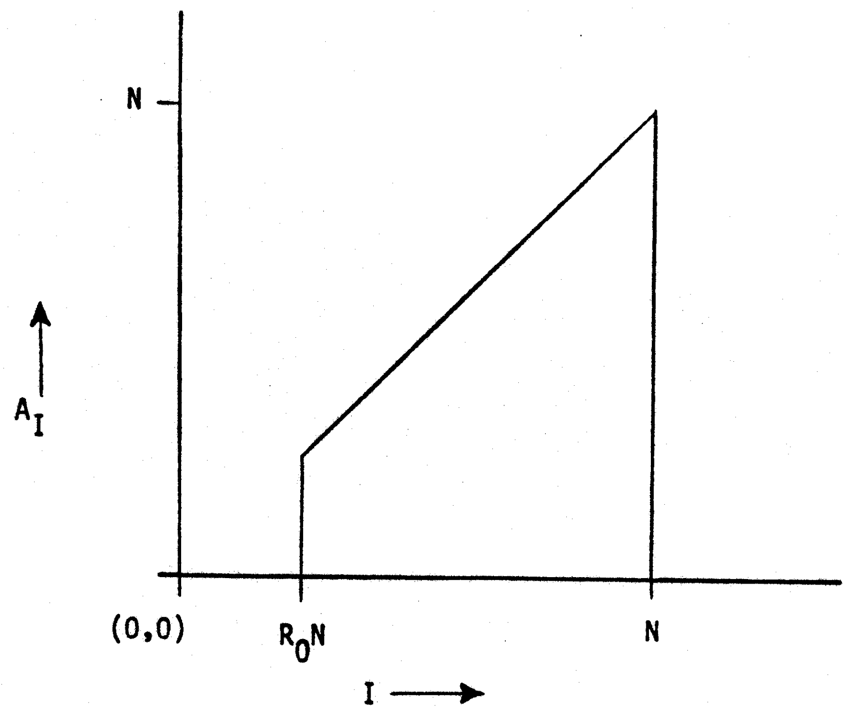

Figure 10. The assumption of step optimism *and* linear optimism (with zero constant).

17. (Homogeneity *and* linear optimism.) In the foregoing paragraphs we have developed in detail several examples of how to incorporate supplementary information into the mathematical reliability-confidence model represented by equation (3). We have looked at assumptions associated with such supplementary information both singly and in combination. Let us conclude our consideration of how to construct such supplementary information models by presenting the models for two other combinations of assumptions, and then get on to some calculated results. (It is assumed that by now the reader is well able to supply the arguments necessary for developing these models.) If supplementary information is available which warrants *both* the assumptions discussed in paragraphs 12 and 14, above, and only those, then the values of A_I to be used in equation (3) are given by:

$$A_I = \begin{cases} 0 & \text{if } (1-f) < I < fN \\ I & \text{otherwise} \end{cases} \qquad (13)^{11,12}.$$

The graph of this is:

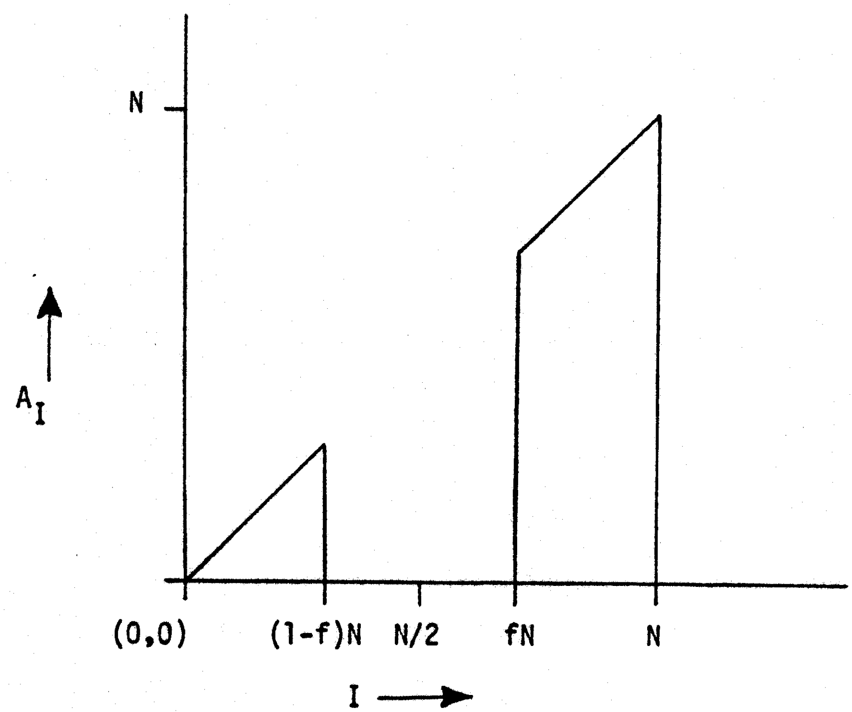

Figure 11. The assumption of homogeneity *and* linear optimism (with zero constant).

18. (Homogeneity *and* step optimism *and* linear optimism.) If supplementary information is available which warrants *all three* of the assumptions discussed in paragraphs 12, 13, and 14, above, and only those, then the values of A_I to be used in equation (3) are indicated by Figure 12.

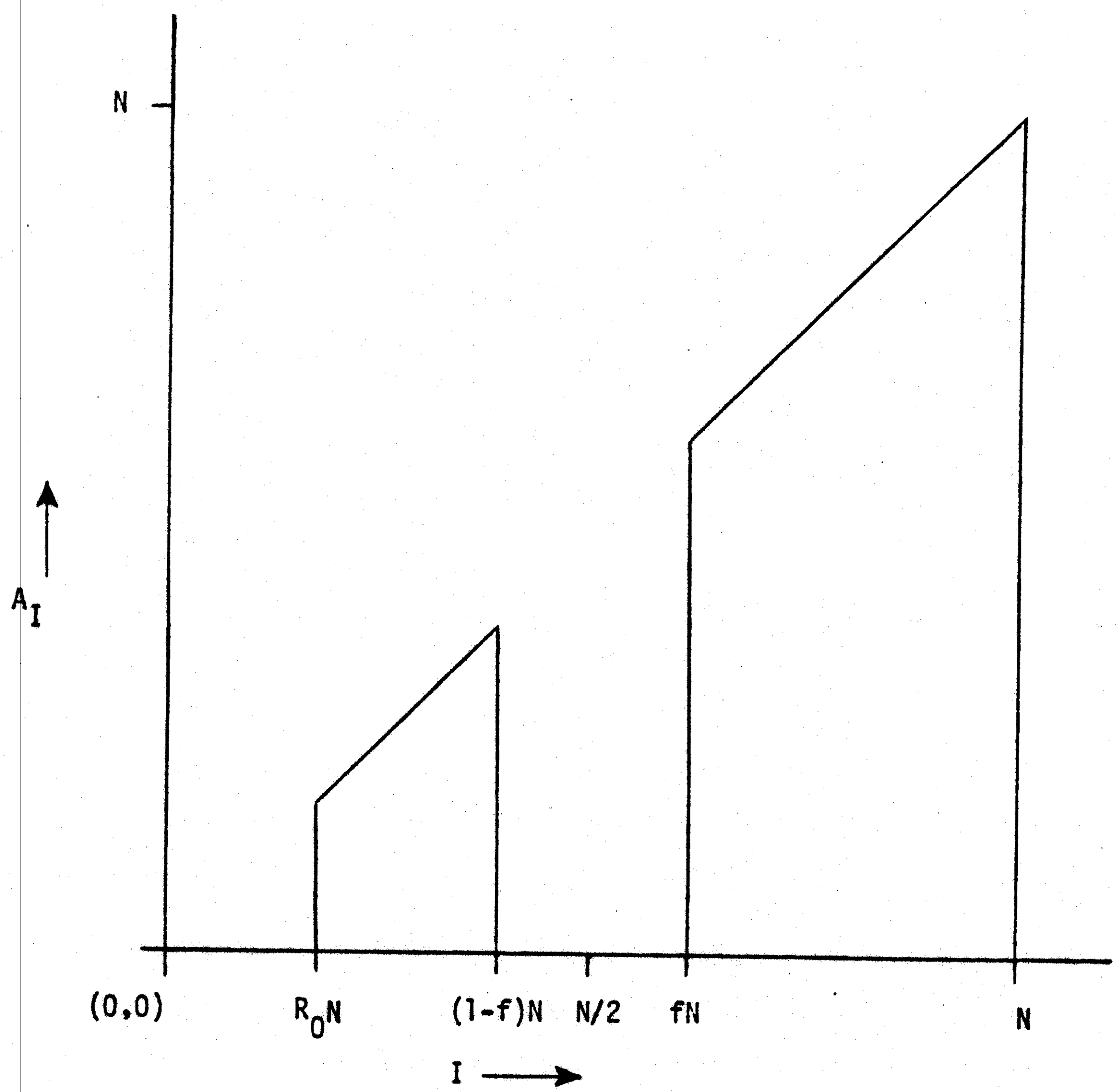

Figure 12. The assumption of homogeneity, step optimism (for $R_0+f \le 1$), *and* linear optimism (with zero constant).

19. (Calculated results.) So far in this note we have considered the maximum ignorance assumption (paragraphs 1 and 2), possible sources of supplementary information which could warrant various alternatives to the maximum ignorance assumption (paragraphs 3 through 5), and how some of these example alternative assumptions can be modeled mathematically in the case of finite populations (paragraphs 6 through 18). Let us now compare the quantitative effects of such alternative assumptions on the reliability-confidence relationship. In order to make this comparison we will select eighteen specific examples of possible alternative assumptions and see what numbers they, and the maximum ignorance assumption, produce for a few kinds of possible test results, i.e., for a few sets of possible values of N, L, and M. Note that in the development of the theory up to this point we have not made use of any large sample or large population assumptions or approximations (except in one example which is worked out in footnote 10). Therefore the theory which we have developed is as valid for the smallest non-negative values of $M \leq L \leq N$ as for the largest. Table 1 assigns Roman numeral names to the various specific assumptions chosen as examples for comparison. Since this theory covers a problem in many dimensions (cf. paragraph 10 of PSN 1), it is difficult to present an exhaustive comparison on a few two-dimensional sheets of paper. Therefore we will concentrate on cases in which very little full system test data is available, i.e., cases in which L is very small, since it is under these circumstances that any supplementary information at all is most valuable. Figures 13 and 14 and Table 2 present this selection of calculated results.

20. (The infinite population case: $N = \infty$.) PSN 2 treats confidence and reliability in infinite populations (or finite populations sampled with replacement; sampling is independent in either case), subject to the maximum ignorance assumption. (Cf. the preface, and also the last sentences in paragraphs 8 and 9, of that note.) Let us define C(R) for infinite populations in the language of PSN 2, thus:

$$C(R) \triangleq C(R,1,L,M)$$

This definition conforms to the usage up to this point in the present note, and also to that in PSN 1 (allowing that M is defined here to be the number of *successful* tests). Using this definition, then, a generalized statement of equation (4) in PSN 2 is:

$$C(R) = \frac{\int_R^1 f(p)\, p^M (1-p)^{L-M}\, dp}{\int_0^1 f(p)\, p^M (1-p)^{L-M}\, dp} \qquad (14),$$

where f(p) is the weighting factor for an infinite population corresponding to A_I

Table 1. Some examples of alternative assumptions upon which the reliability-confidence relationship might be based.

Roman numeral name	Verbal statement of the assumption	Number of the equation which is the mathematical model of the assumption	Value of any parameters of the assumption
I.	Maximum ignorance. (I.e., the population is dichotomized by the analyst, and population elements "fail" independently of one another, but *no assumption is made about population distribution.)*	(2)	--------
II.	Homogeneity. (I.e., a fraction of at least f of the population is *alike* in the respects of interest, good *or* bad.)	(6)	f = 75%
III.	Homogeneity.	(6)	f = 80%
IV.	Homogeneity.	(6)	f = 85%
V.	Homogeneity.	(6)	f = 87.5%
VI.	Homogeneity.	(6)	f = 90%
VII.	Step optimism. (I.e., R_0 is known, prior to consideration of results of full system testing, to be a lower bound on P.)	(8)	R_0 = 50%
VIII.	Step optimism.	(8)	R_0 = 65%
IX.	Step optimism.	(8)	R_0 = 70%
X.	Step optimism.	(8)	R_0 = 75%
XI.	Linear optimism (with zero constant). (I.e., there exists some -- any -- real number $a > 0$ such that $\tilde{C}(p) = ap$ for $0 \le p \le 1$.[14])	(10)	--------
XII.	Step optimism *and* linear optimism (with zero constant).	(12)	R_0 = 50%
XIII.	Step optimism *and* linear optimism (with zero constant).	(12)	R_0 = 65%
XIV.	Step optimism *and* linear optimism (with zero constant).	(12)	R_0 = 70%
XV.	Step optimism *and* linear optimism (with zero constant).	(12)	R_0 = 75%
XVI.	Homogeneity *and* linear optimism (with zero constant).	(13)	f = 75%
XVII.	Homogeneity *and* linear optimism (with zero constant).	(13)	f = 80%
XVIII.	Homogeneity *and* linear optimism (with zero constant).	(13)	f = 85%
XIX.	Homogeneity *and* linear optimism (with zero constant).	(13)	f = 87.5%

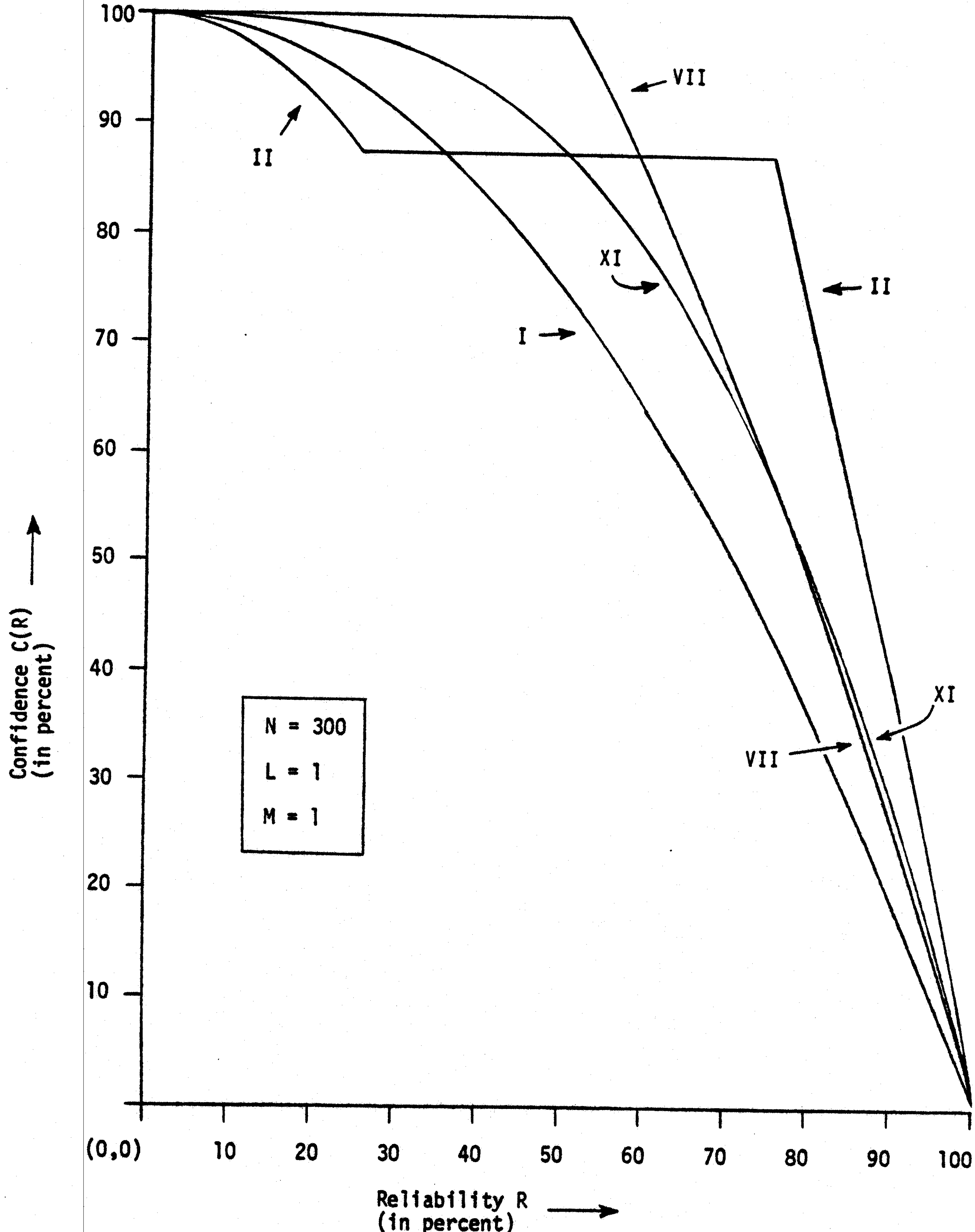

Figure 13. Confidence as a function of reliability for four different states of prior information, with N = 300, L = 1, and M = 1.

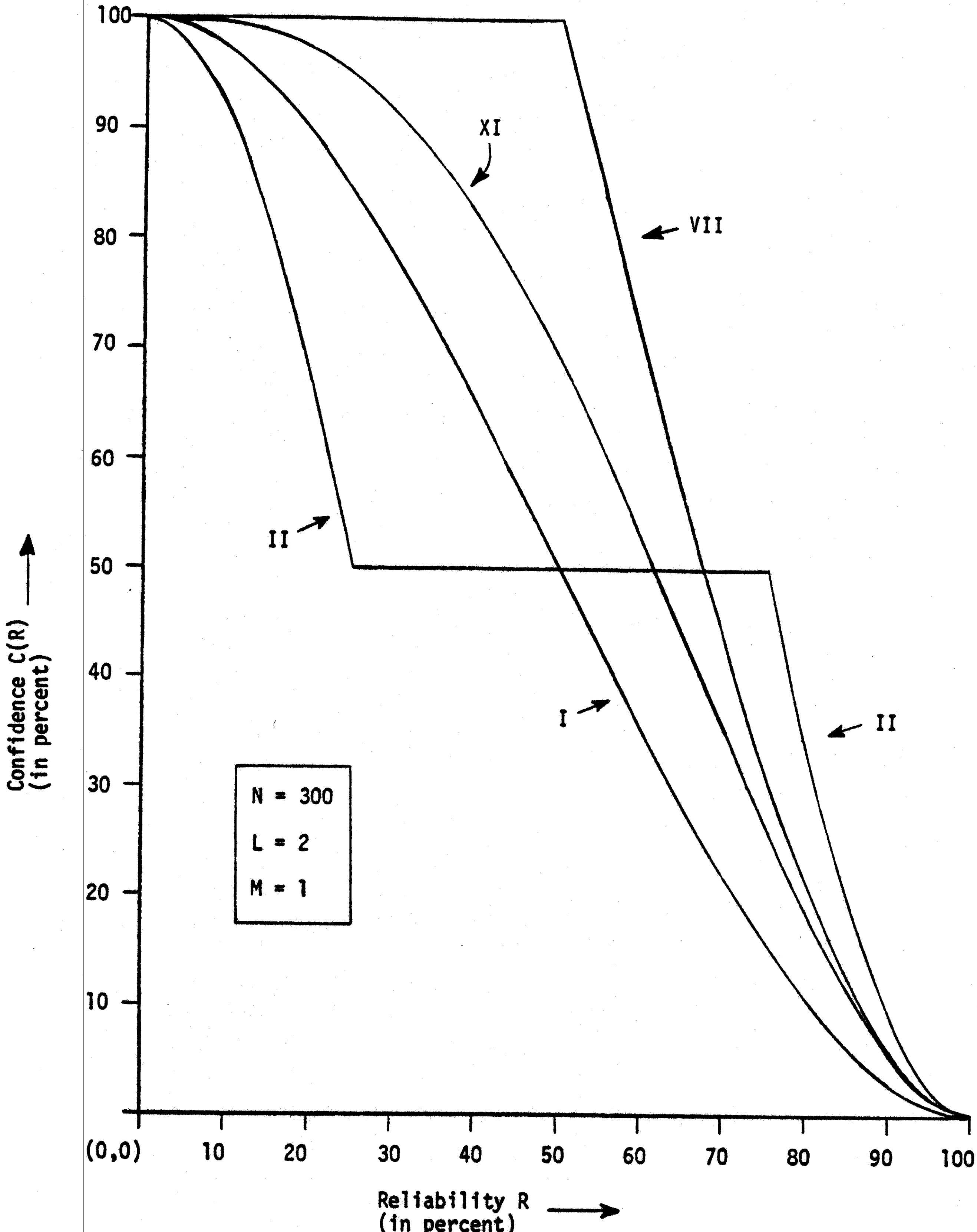

Figure 14. Confidence as a function of reliability for four different states of prior information, with N = 300, L = 2, and M = 1.

Table 2.

Number of uniformly successful tests (i.e., L with M = L) required for a confidence of at least C(R) in reliability R, assuming population size N = 250 *and* :

Assumption:	R = 80% C(R) = 80%	R = 80% C(R) = 85%	R = 80% C(R) = 90%	R = 85% C(R) = 80%	R = 85% C(R) = 85%	R = 85% C(R) = 90%	R = 90% C(R) = 80%	R = 90% C(R) = 85%	R = 90% C(R) = 90%
I.	6	8	9	9	11	13	14	16	20
VII.	6	8	9	9	11	13	14	16	20
XI.	6	7	8	8	10	12	13	15	19
VIII.	5	7	9	9	11	13	14	16	20
XII.	5	7	8	8	10	12	13	15	19
IX.	4	6	8	8	10	13	14	16	20
XIII.	4	6	8	8	10	12	13	15	19
X.	4	5	8	8	10	13	14	16	20
XIV.	3	5	7	7	9	12	13	15	19
XV.	3	4	7	7	9	12	13	15	19
II.	2	3	5	7	9	12	13	16	20
XVI.	1	2	4	6	8	11	12	15	19
III.	1	1	1	3	6	9	12	15	19
XVII.	1	1	1	2	5	8	11	14	18
IV.	1	1	1	1	1	1	8	11	16
XVIII.	1	1	1	1	1	1	7	10	15
V.	1	1	1	1	1	1	2	4	10
XIX.	1	1	1	1	1	1	1	3	9
VI.	1	1	1	1	1	1	1	1	1

in equation (3) of the present note for a finite population. The discussions of PSN 2, throughout which $f(p) = 1$, can be easily adapted to those alternatives to the maximum ignorance assumption cited in the present note. For example, assumptions of homogeneity or of step optimism are taken account of simply by appropriate adjustments to the limits of integration in equation (14), above. Since $f(p)$ for those assumptions is given by

$$f(p) = \begin{cases} 0 & \text{if } (1-f) < p < f \text{ or } p < R_0 \\ 1 & \text{otherwise} \end{cases}$$

(compare with equations (11), above), the case $R_0 \le R \le 1-f$ immediately makes equation (14) become:

$$C(R) = \frac{\int_R^{1-f} p^M (1-p)^{L-M} \, dp + \int_f^1 p^M (1-p)^{L-M} \, dp}{\int_{R_0}^{1-f} p^M (1-p)^{L-M} \, dp + \int_f^1 p^M (1-p)^{L-M} \, dp} .$$

The other cases for homogeneity and step optimism produce similar variations of equation (14). As a concluding example, the assumption of linear optimism (with zero constant) is handled equally easily:

$$f(p) = p$$

(compare with equation (10)) yields

$$C(R) = \frac{\int_R^1 p \, p^M (1-p)^{L-M} \, dp}{\int_0^1 p \, p^M (1-p)^{L-M} \, dp} = \frac{\int_R^1 p^{M+1} (1-p)^{L-M} \, dp}{\int_0^1 p^{M+1} (1-p)^{L-M} \, dp} .$$

Footnotes

1. Another problem in using the maximum ignorance assumption is the question of just how one should go about using it. This question has been hotly debated in the past, and will be debated again in the future. It is the question of how one should behave when one has no data at all about the population, other than an adequate specification of it (i.e., an ultimately extensive, or experimental, definition of it). This series of notes has used a widely accepted answer to the question, viz., that in the case of maximum ignorance no suggested value of population dichotomization fraction (or equal length interval of possible fractions) should be accorded any greater confidence than any other, i.e., that all possible values of that fraction (or equal length intervals of possible values of it) merit equal confidence under maximum ignorance. It is not the purpose of the present note to defend this answer, nor how we have been applying the maximum ignorance assumption up to now; for the present we will simply let the arguments of the notes speak for themselves. However, explicit treatment of the issue is to be included in a future Probability and Statistics Note. References will be cited as part of that treatment. (Cf. also footnote 9, below.)

2. By Stephen M. Pollock, Associate Professor, Department of Industrial and Operations Engineering, University of Michigan, on 9 August 1973 in an Operations Research seminar, for example.

3. Cf. PSN 6.

4. As defined in PSNs 1 and 2.

5. The word "deduced" is used here to emphasize stongly that one should be able to justify such stronger assumptions quantitatively with rigorous logic from available hard data. Honest science prohibits employing such an assumption when applying confidence theory just because one "has a feeling things are that way" ("subjective" or "personal" probability to the contrary notwithstanding).

6. These variable names were chosen for convenience in FORTRAN computer programming of equation (3), since these variables have integer ("fixed point") values. Mnemonics are: "L" for "little x" (in the notation of many textbooks in probability and statistics); "M" for "mistakes" or "mishaps". In view of the redefinition of M which occurs in PSN 2 and also later in paragraph 6 of the text of the present note, perhaps a better mnemonic for that variable name would now be "M" for "memorable".

7. Cf. paragraph 4 of PSN 1, which is reproduced as paragraph 4 of the Appendix to the present note.

8. Except that for the sake of efficiency in calculation the limits of summation were selected to exclude those terms which can be seen beforehand to be forced to zero by the input parameter values. Cf. paragraphs 7 and 8 of PSN 1.

9. The reader may recognize this generalized reliability-confidence expression as the special case of Bayes' Theorem for finite populations where sampling is without replacement. Cf. A. Bruce Clarke and Ralph L. Disney, *Probability and Random Processes for Engineers and Scientists*, p. 38, equation (2.14). The quantity $Pr[A|B_i]$ in that equation is provided by expression (2.20) on p. 43 of that reference, and the quantity $Pr[B_i]$ is a submultiple of A_I in equation (3) of the present note. (Cf. also footnote 13 of this note.) To see a more general statement of equation (3), cf. Athanasios Papoulis, *Probability, Random Variables, and Stochastic Processes*, p. 111, equation (4-75); it is the quantity f(p) in that equation which is the submultiple of A_I in equation (3) of the present note. (Cf. also paragraph 20 of the present note, especially equation (14).) -- Incidentally, Figure 4-21 on p. 113 of this Papoulis reference also treats the same general subject as the present note. And on pp. 112 to 114 of this reference the reader can also find some discussion of the case *against* using equation (3) with the maximum ignorance assumption (equation (2), or $f(p) = 1 \ \forall p$). The same argument is also presented, from another point of view, by William Feller in *An Introduction to Probability Theory and Its Applications*, Volume I, the note on pp. 124, 125. A more detailed treatment of this argument, among others, is presented by John Maynard Keynes in *A Treatise on Probability*, especially in Chapter IV. The core of all these objections is that *one must have some rationale for assigning values to the* A_I *in equation (3) of this note, or to* f(p) *in equation (14), before he can evaluate the equation.* The present author agrees with the objection, stated in this form. However, it is to a considerable degree irrelevant to the discussion in this note, which is devoted to *alternatives* to the maximum ignorance assumption, since we are here explicitly assuming that supplementary information is available which provides values for the A_I (or f(p)): cf. footnote 5, above. Regarding the debate in general, cf. footnote 1, above.

10. Cf. the first four lines of paragraph 21 in PSN 2, which give us that $C(1-Q,1,1,1) = 2Q-Q^2$. Letting R = 1 and Q = 0 therefore permits us to write $C(R) = C(1) = C(1-Q) = C(1-Q,1,1,1) = 2Q-Q^2 = 2\cdot 0 - 0^2 = 0$, provided $N = \infty$. That this result gives an adequate approximation for the case in which N = 100,000 may be seen by considering paragraph 24 of PSN 2.

11. In the text f is defined to be a known lower bound either on P or on 1-P. Since P is a fraction in the interval [0,1] we know that $.5 \le P$ or $.5 \le 1-P$. Therefore it is trivially true that .5 is a lower bound on P or on 1-P. Consequently we might as well consider only f such that $.5 \le f$. This much we know from definitions, without recourse to any supplementary information or assumptions. Therefore supplementary information which assures more than trivial homogeneity (in the respects of interest) in the population must assure something stronger than $.5 \le f$. That is, we might as well consider only homogeneity information or assumptions which assure

$$.5 < f .$$

An immediate consequence of this is that $1-f < f$.

12. It is true that this way of satisfying the requirements for A_I is not unique. In fact, if we use $\tilde{A}_I$ in equation (3) in place of A_I, where $\tilde{A}_I \overset{\Delta}{=} r*A_I$ $\forall$ I, for any value of $r \neq 0$, then the requirements will still be satisfied. However, the quantity r may then be factored out of both the numerator and the denominator in equation (3) and cancelled. Consequently computational results are wholly unaffected by such substitutions. These comments apply to *all* mathematical models (given by the values of A_I) of supplementary information states, including of course the others developed as examples in this note.

13. As before, Sam then puts only one of these barrels in front of us. In this new experiment let's have Sam now destroy the other barrels before anything else happens. This makes the situation even more analogous to real life, since the next steps then occur with only the single barrel of marbles in existence which we have before us. Next we select one marble "randomly from the well stirred" barrel presented to us, and observe that it is white. This time we know we could have taken one white marble from one of the two white barrels (which between them contain 200,000 white marbles) in *four* times as many ways as we could have from the one barrel containing only 50,000 white marbles. We conclude that we should be four times more confident that we have an all-white barrel before us than that this barrel is only half white. Numerically, then, the supplementary information plus the given experimental results lead us to be 80% confident that all the marbles are white in this, now the only existing, barrel of marbles (and 20% confident that only half of them are white). Computation will show that this result agrees with the figures yielded by equations (3) and (9), with N = 100,000, L = 1, and M = 1.

14. Since we are, in this part of the note, considering only finite populations, P can have one of only a finite set S of discrete values, viz., $S = \{ \frac{0}{N}, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, \frac{N}{N} \}$. Let $\tilde{C}(p)$ denote our confidence, prior to consideration of the results of full system testing, that $P = p$. Then a general statement of linear optimism would be that there exist real numbers $a > 0$ and $b \geq 0$ such that

$$\tilde{C}(p) = \begin{cases} ap + b & \text{if } p \in S \\ 0 & \text{otherwise} \end{cases} .$$

Let $C'(I)$ denote our confidence, prior to consideration of the results of full system testing, that $NP = I$. Then

$$\tilde{C}(p) = C'(Np) = C'(I) .$$

Since the index I of summation is defined such that $I = Np$, we have that $p = \frac{I}{N}$. Consequently the last equation may be rewritten

$$C'(I) = \tilde{C}(\frac{I}{N}) .$$

$0 \leq p \leq 1 \Leftrightarrow 0 \leq I \leq N$. Using these facts we can now write the general statement of linear optimism, above, in an alternate form, viz.,

$$C'(I) = \begin{cases} a*\frac{I}{N} + b & \text{for integers } I \ni 0 \leq I \leq N \\ 0 & \text{otherwise} \end{cases} .$$

Finally, note that $A_I = C'(I)$. Or, by footnote 12, we can as well let $A_I = \frac{N}{a} C'(I)$. Then

$$A_I = \frac{N}{a} (a*\frac{I}{N} + b) = I + \frac{Nb}{a} .$$

-- All the cases of supplementary information involving linear optimism which are developed as examples in this note happen to have the constant b in the equations above set to zero. It is for this reason that the phrase "with zero constant" appears repeatedly in the text. The physical interpretation of this is simply that $\tilde{C}(0) = 0$.

Appendix

From Probability and Statistics Note 1:

Some Notes on Confidence and Reliability in a Finite Population.

1. Your friend Sam has two barrels which contain 100,000 marbles each, all identical except for color. You know that all the marbles in one of these two barrels are white, and that the other barrel is evenly divided between 50,000 white marbles and 50,000 black marbles, all stirred up well together. Sam puts one of these barrels in front of you, but he doesn't tell you which one. You reach in without looking and pull out a great big double handful of marbles. If this double handfull of well stirred marbles, say 100 of them, was solid white, didn't contain a single black marble, then you'd be pretty convinced you'd had to deal with the barrel that had nothing in it but white marbles.

2. Another example which is even more obvious is two barrels of sand, in one of which all the grains are white and in the other of which only half are white and the other half are black, but otherwise identical. If the mixed barrel is thoroughly stirred, so that any grain you choose blindfold is truly random, and the barrel as a whole has a salt and pepper appearance, then a double handful which was perfectly white, showing not a single black grain would pretty much persuade you that you'd been offered the homogeneous barrel to choose from.

3. Let's quantify. Recall the barrels of marbles. If you drew out only one marble and it was white, then you'd take that as a more mild indication that it came from the completely white barrel. The reason is that there are twice as many ways for you to pull a single white marble out of that barrel as out of the mixed barrel, so it would be twice as likely you'd draw a white marble from the white barrel. (Read that sentence again.) In fact, we say your confidence in having had the white barrel set in front of you was in this case precisely twice as great as your confidence that the mixed barrel has been. Numerically, you'd be $66\frac{2}{3}\%$ confident that you'd had to deal with the white barrel.

4. Now consider Minuteman. Assume there are exactly 1000 missiles in the fleet, and you have no idea what the number of defective missiles is before you begin to test. So you pretend that someone has 1001 different fleets of Minutemen (each fleet analogous to a barrel in the examples above), and that in each of his fleets there is a different number defective, from 0 through 1000. And you pretend that he has put one of these fleets in front of you, so that the number defective in the fleet you have to deal with could equally likely be any number from 0 to 1000. You conduct a test of L sites and discover that M of these L are [non-] defective.

Definitions of Symbols

Symbol	Meaning
$\triangleq$	is defined to mean
$\Leftrightarrow$	implies and is implied by
$\vee$	logical "OR" (weak disjunction)
$\in$	is a member of
$\forall$	for all
$\ni$	such that
*	"times" (i.e., ordinary multiplication; this operation is also indicated in some places by juxtaposition)