Measurement Notes

Note 47

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Transient Skin Effect in Cables

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Abstract

Skin-effect losses limit the high-frequency performance of cables (TEM transmission lines). In time domain this limits the rise time that one can obtain for a pulse propagating along such a cable. This paper extends the model to cases other than coaxial cables to allow for nonuniform surface current density on the conductors. This shows that there is a significant increase in these losses for cases in which one (or both) of the conductors has one or more knife edges, as in the case of a thin conducting plate.

1. Introduction

In propagating fast-rising pulses along some length of cable, the rise time may be increased by the skin effect in the cable conductors. This introduces losses which increase with frequency. A well-known model for this which is consistent with measurement is given in [4, 5]. This model is based on the assumption of two circular coaxial conductors separated by a lossless dielectric and skin depths in the conductors small compared to conductor thicknesses and radii of curvature.

Recently it has been experimentally observed that the rise-time degradation is larger than predicted by this model in some transmission-line configurations including small-angle TEM horns [11]. This raised the obvious question of how to model and reduce these losses.

Note that the geometries of interest involved one or more thin metal plates with relatively sharp edges (small radii of curvature at the edges compared to the plate width). Idealizing these edges as knife edges the surface current density J_s (parallel to the edge) is proportional to $D^{-1/2}$ where D is the distance from the edge. If one integrates the magnetic energy $\mu H^2/2$, there is a differential-area factor $DdDd\phi$ in the integrand (surface integral over cross section) canceling the D^{-1} near the edge giving a finite value for any finite volume. This is a special case of the edge condition [9]. However, if one integrates $\tilde{J}_s^2\tilde{Z}_s'$ along the surface of the conductor, where $\tilde{Z}_s'$ is some skin impedance (assumed independent of position), then one has an integral of the form $\int D^{-1}dD$ which blows up when taken to the edge. This implies large losses and provides the basis for extending the model to include conductors with curvatures varying around their circumferences.

2. Coaxial Cable

As indicated in fig. 2.1, consider a coaxial cable with inner conductor of radius Ψ_1 and outer conductor of inner radius Ψ_2. If we first approximate the conductors as perfect we have

$$
\begin{aligned}
f_g &= \frac{1}{2\pi} \ell n\left(\frac{\Psi_2}{\Psi_1}\right) \equiv \text{geometrical impedance factor} \\
L' &= \mu_0 f_g \equiv \text{inductance per unit length} \\
C' &= \frac{\varepsilon}{f_g} \equiv \text{capacitance per unit length} \\
Z_{c0} &= \left[\frac{L'}{C'}\right]^{\frac{1}{2}} = \left[\frac{\mu_0}{\varepsilon}\right]^{\frac{1}{2}} f_g \equiv \text{characteristic impedance} \\
\tilde{\gamma}_0 &= s\left[L'C'\right]^{\frac{1}{2}} = s\left[\mu_0 \varepsilon\right]^{\frac{1}{2}} \equiv \text{propagation constant} \\
\sim &\equiv \text{two-sided Laplace transform over time } t \\
s &\equiv \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency}
\end{aligned}
\tag{2.1}
$$

Note that these depend only on the parameters of the dielectric medium with permittivity ε and permeability μ_0.

With non-perfect conductors we have an additional series impedance per unit length

$$
\begin{aligned}
\tilde{Z}_s' &\simeq s^{\frac{1}{2}} \Xi \\
\Xi &= \frac{1}{2\pi\Psi_1}\left[\frac{\mu_1}{\sigma_1}\right]^{\frac{1}{2}} + \frac{1}{2\pi\Psi_2}\left[\frac{\mu_2}{\sigma_2}\right]^{\frac{1}{2}}
\end{aligned}
\tag{2.2}
$$

Here we have neglected the permittivities of the conductors, assuming that they are dominated by the conductivities for frequencies of interest. The skin depths have been assumed much smaller than the appropriate radii and conductor thicknesses so that a skin-depth approximation of the fields in the conductors can be used. Then we have

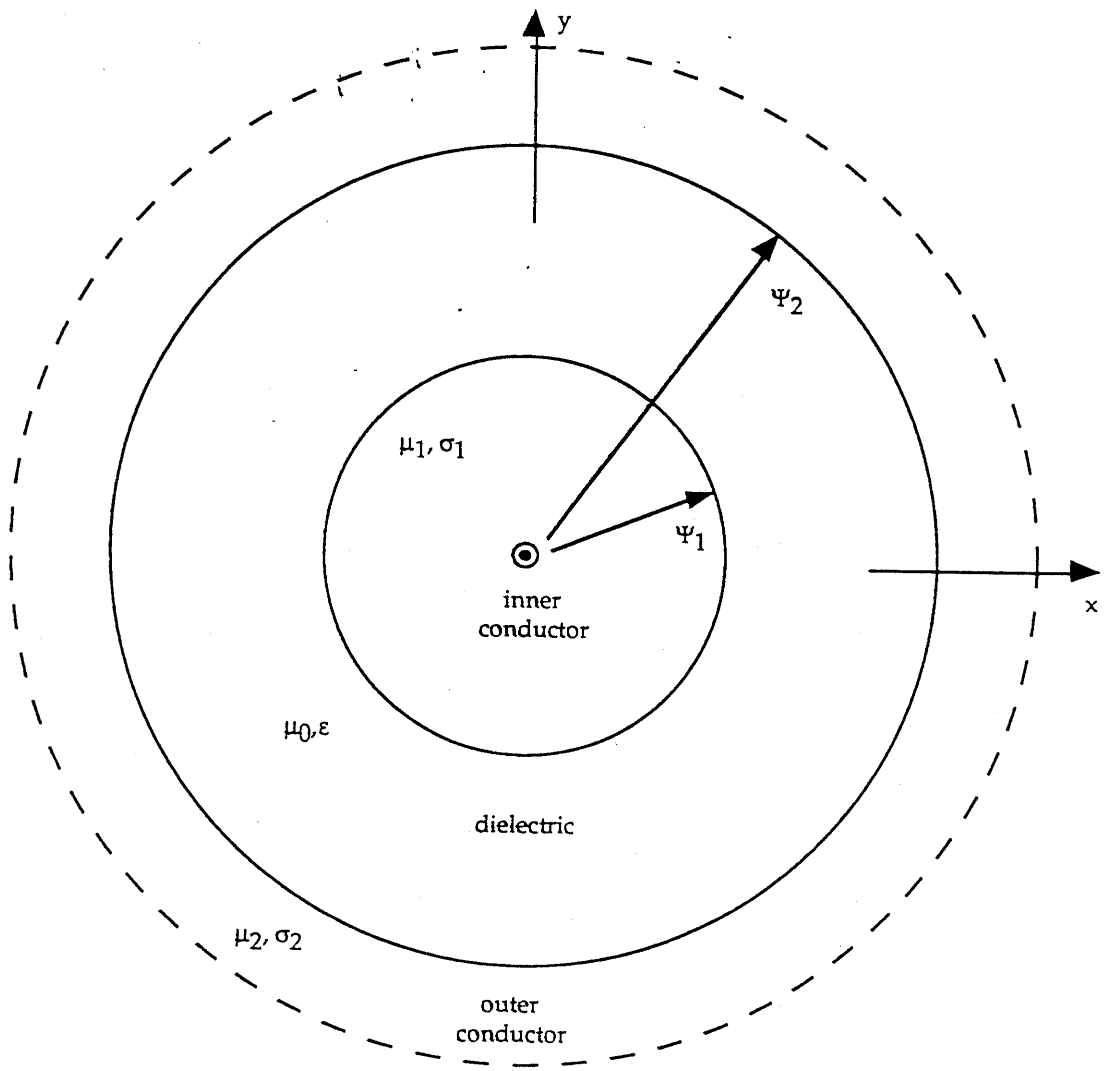

Fig. 2.1. Coaxial Cable

$\tilde{Z}' = sL' + \tilde{Z}'_s \equiv$ impedance per unit length (longitudinal).

$\tilde{Y}' = sC' \equiv$ admittance per unit length (transverse)

$$\tilde{Z} = \left[\frac{\tilde{Z}'}{\tilde{Y}'}\right]^{\frac{1}{2}} = \left[\frac{SL' + \tilde{Z}'_s}{sC'}\right]^{\frac{1}{2}} \equiv \text{characteristic impedance} \tag{2.3}$$

$$\tilde{\gamma} = \left[\tilde{Z}'\tilde{Y}'\right]^{\frac{1}{2}} = \left[\left(sL' + \tilde{Z}'_s\right)sC'\right]^{\frac{1}{2}} \equiv \text{propagation constant}$$

At this point we can exhibit the well-known approximate transient solution [4, 5]. Expand the propagation constant for the case that $\left|\tilde{Z}'_s\right| << |sL'|$ as

$$\begin{aligned}
\tilde{\gamma}\left[\left(sL' + \tilde{Z}'_s\right)sC'\right]^{\frac{1}{2}} &= s[L'C']^{\frac{1}{2}}\left[1 + \frac{\tilde{Z}'_s}{sL'}\right]^{\frac{1}{2}} \\
&= s[L'C']^{\frac{1}{2}}\left[1 + \frac{1}{2}\frac{\tilde{Z}'_s}{sL'} + O\left(\left(\frac{\tilde{Z}'_s}{sL'}\right)^2\right)\right] \\
&= s[L'C']^{\frac{1}{2}} + \frac{1}{2}\frac{\tilde{Z}'_s}{Z_{c0}} + O\left(\left(\frac{\tilde{Z}'_s}{sL'}\right)^2\right) \quad \text{as} \quad \frac{\tilde{Z}'_s}{sL'} \to 0
\end{aligned} \tag{2.4}$$

As one can see, this is a high-frequency approximation since $\tilde{Z}'_s$ is proportional to $s^{1/2}$. The cable transfer function over some distance z is

$$\tilde{T}(s) = e^{-\tilde{\gamma}z} \simeq \underbrace{e^{-s[L'C']^{\frac{1}{2}}z}}_{\text{delay}} e^{-\frac{z}{2}\frac{\tilde{Z}'_s}{Z_{c0}}} \tag{2.5}$$

Ignoring the delay the other term can be put in the form

$$\begin{aligned}
\frac{z}{2}\frac{\tilde{Z}'_s}{Z_{c0}} &= [4s\tau]^{\frac{1}{2}} \\
\tau &= \frac{z^2}{16}\frac{\Xi^2}{Z_{c0}^2} = \left[\frac{z}{4}\frac{\Xi}{Z_{c0}}\right]^2
\end{aligned} \tag{2.6}$$

The step response is given by

$$\frac{1}{s}e^{-2[s\tau]^{\frac{1}{2}}} \to \operatorname{erfc}\left(\left[\frac{\tau}{t}\right]^{\frac{1}{2}}\right)u(t) \tag{2.7}$$

If desired the delay can be included by substituting retarded time t_r for t with

$$t_r = t - [L'C']^{\frac{1}{2}} z = t - [\mu_0 \varepsilon]^{\frac{1}{2}} z \tag{2.8}$$

Note that [7]

$$\begin{aligned} \operatorname{erfc}(y) &= 1 - \operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-y'^2} dy' \\ \operatorname{erfc}(0) &= \operatorname{erf}(\infty) = 1 \\ \operatorname{erfc}(\infty) &= \operatorname{erf}(0) = 0 \end{aligned} \tag{2.9}$$

The important point is that the step response and hence rise time is characterized by the time-constant τ. The smallest rise time is obtained by minimizing τ. Note that the largest contribution to Ξ (and hence τ) typically comes from the inner conductor due to its small radius. However, as indicated in (2.2) the result easily generalizes to include both conductors. These can have different conductivities (say copper inner conductor and aluminum outer conductor), and for good performance one also takes

$$\mu_1 = \mu_2 = \mu_0 \tag{2.10}$$

One can also decrease τ by increasing the cable size, but one needs to be concerned about introducing higher order modes. This is given by a cutoff wavelength [8]

$$\lambda_c \simeq \begin{cases} \frac{2}{n}[\Psi_2 - \Psi_1] & \text{for } E \text{ modes} \\ \frac{\pi}{n}[\Psi_2 + \Psi_1] & \text{for } H \text{ modes} \end{cases} \tag{2.11}$$

$$n = 1, 2, 3, \ldots$$

Unless one takes special precautions (e.g., no bends or use of special lenses) to avoid such other modes then the first H mode with λ_c about one circumference at the average radius can give a high-frequency limitation. This can be used to choose a cable size (for a given length) which matches the bandwidths for the two effects: skin losses and overmoding.

3. Cable with One Inner and One Outer Conductor

Let us now generalize the results for a coax to a cable of arbitrary cross section as indicated in fig. 3.1. For this discussion consider the TEM mode as described by a conformal transformation given by

$$w(\zeta) \equiv u(\zeta) + jv(\zeta) \equiv \text{ complex potential}$$
$$\zeta \equiv x + jy = \Psi e^{j\phi} \equiv \text{ complex coordinate} \tag{3.1}$$

The two conductors (inner one enclosed by outer one, but not necessarily so) are given by constant values of u with index 1 referring to the inner conductor and index 2 referring to the outer conductor. In general we have

$$f_g = \frac{\Delta u}{\Delta v}$$
$$\Delta u \equiv u_2 - u_1 \equiv \text{ change in } u \text{ between conductors} \tag{3.2}$$
$$\Delta v \equiv \text{ change in } v \text{ in passing around (circumnavigating) center conductor}$$

This can be substituted for f_g in (2.1) to obtain the cable parameters for the case that the conductors are perfect.

As discussed in [2, 10] let us regard the cable cross section as a jacket (small Δu) comprised of N ducts with boundaries between adjacent ducts given by constant values of v. For this purpose define

$$\Delta' \equiv \frac{\Delta v}{N}$$
$$v_n = v_0 + n\Delta' \quad , \quad n = 0,1,2,\ldots,N \tag{3.3}$$
$$v_N - v_0 = \Delta v$$

where v_N and v_0 are the two values of v corresponding to the same boundary which we take as the branch cut in the conformal transformation. The nth duct is then between v_{n-1} and v_n.

For each duct define appropriate transmission-line parameters

$$L'_n = \mu_0 \frac{\Delta u}{\Delta'} \equiv \text{ inductance per unit length}$$
$$C'_n = \varepsilon \frac{\Delta'}{\Delta u} \equiv \text{ capacitance per unit length} \tag{3.4}$$
$$\tilde{Z}'_{s_n} \equiv \text{ skin impedance per unit length}$$

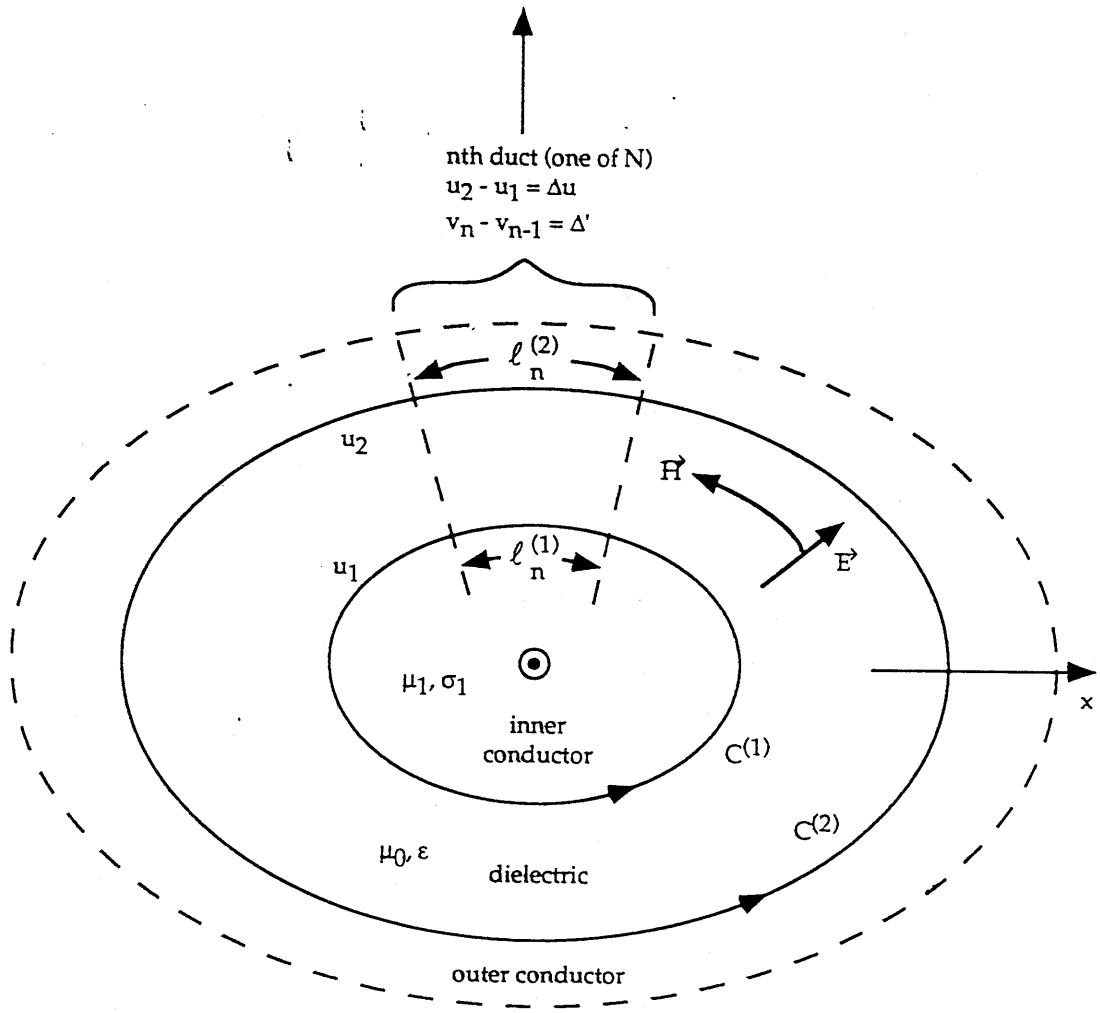

Fig. 3.1. Cable with General Cross Section Divided into Ducts

To evaluate $\tilde{Z}'_{s_n}$ we need the width of each duct on the two conductors. The line element for the conformal transformation is [1]

$$
\begin{aligned}
(d\ell)^2 &= (dx)^2 + (dy)^2 = |d\zeta|^2 \\
&= h_w^2\left[(du)^2 + (dv)^2\right] = h_w^2\,|dw|^2 \\
h_w &= \left|\frac{d\zeta}{dw}\right|
\end{aligned}
\tag{3.5}
$$

Then we have for small duct widths

$$
\begin{aligned}
&\frac{\ell_n^{(m)}}{\Delta'} \simeq h_w^{(m)} \\
&m = \begin{cases} 1 \Rightarrow \text{inner conductor} \\ 2 \Rightarrow \text{outer conductor} \end{cases}
\end{aligned}
\tag{3.6}
$$

with the approximation becoming exact in the limit of small Δ'. With this we have

$$
\begin{aligned}
\tilde{Z}'_{s_n} &= \tilde{Z}'^{(1)}_{s_n} + \tilde{Z}'^{(2)}_{s_n} \\
\tilde{Z}'^{(m)}_{s_n} &= s^{\frac{1}{2}}\,\ell_n^{(m)^{-1}}\left[\frac{\mu_m}{\sigma_m}\right]^{\frac{1}{2}}
\end{aligned}
\tag{3.7}
$$

In combining the duct parameters to form the jacket or cable parameters consider the parallel combination of the parameters in (3.4). First we have

$$
\begin{aligned}
L' &= \left[\sum_{n=1}^{N} L_n'^{-1}\right]^{-1} = \mu_0 \Delta u \left[\sum_{n=1}^{N} \Delta'\right]^{-1} = \mu_0 \frac{\Delta u}{\Delta v} \\
C' &= \sum_{n=1}^{N} C_n' = \frac{\varepsilon}{\Delta u}\sum_{n=1}^{N} \Delta' = \varepsilon \frac{\Delta v}{\Delta u}
\end{aligned}
\tag{3.8}
$$

which is in agreement with (3.2) as one expects for the case of perfect conductors. In this case all the ducts have the same propagation constant $\tilde{\gamma}_0$ as in (2.1). Note that equal currents (equal Δ') are flowing through each L_n' and equal voltages (common Δu) are across each C_n, thereby justifying the simple parallel combination above.

Now consider the problem of the additional series impedance per unit length for non-perfect conductors. We first assume that the field distribution in the dielectric is very nearly that of the ideal

TEM mode. As the $\tilde{Z}'_{s_n}$ vary from duct to duct this can introduce higher order modes, so the high-frequency restriction in (2.11) should be recalled. Now the restriction of the cutoff wavelength can have the largest λ_c (giving lowest cutoff frequency) estimated as one mean circumference in the dielectric.

With $J_{s_n}^{(m)}$ as the surface current densities and $I_n^{(m)}$ as the current for each duct on inner and outer surfaces we have (signs taken positive)

$$I_n^{(1)} = I_n^{(2)} = I_n = J_{s_n}^{(1)} \ell_n^{(1)} = J_{s_n}^{(2)} \ell_n^{(2)} = \frac{I}{N} \tag{3.9}$$

The longitudinal voltage drop per unit length for each tube inner and outer surface is

$$V_n'^{(m)} = \tilde{Z}_{s_n}'^{(m)} I_n = s^{\frac{1}{2}} \left[\frac{\mu_m}{\sigma_m}\right]^{\frac{1}{2}} J_{s_n}^{(m)} = \frac{s^{\frac{1}{2}}}{\ell_n^{(m)}} \left[\frac{\mu_m}{\sigma_m}\right]^{\frac{1}{2}} \frac{I}{N} \tag{3.10}$$

which is nonuniform around the boundaries in the general case, thereby making a small perturbation in the TEM mode as discussed previously. Seeing that $V_n'^{(m)}$ is a function of n, it raises a question of how to combine the $\tilde{Z}_{s_n}'^{(m)}$ (not simply in parallel) to form the $\tilde{Z}_s'^{(m)}$ for the per-unit-length model of the cable (jacket).

One way around this difficulty is to look at the power associated with the skin effect and sum over the powers associated with the $\tilde{Z}_{s_n}'^{(m)}$ to form the $\tilde{Z}_s'^{(m)}$. So define

$$\begin{aligned} \tilde{P}_{s_n}^{(m)} &\equiv V_n'^{(m)} I_n = \tilde{Z}_{s_n}'^{(m)} I_n^2 = \tilde{Z}_{s_n}'^{(m)} J_{s_n}^{(m)^2} \ell_n^{(m)^2} \\ &= s^{\frac{1}{2}} \left[\frac{\mu_m}{\sigma_m}\right]^{\frac{1}{2}} \frac{I_n^2}{\ell_n^{(m)}} \end{aligned} \tag{3.11}$$

This is a complex power of the form $\tilde{V}(s)\tilde{I}(-s)$ (or like $\vec{\tilde{E}}(s) \times \vec{\tilde{H}}(-s)$ [3]). Intuitively one can see how to add real powers for $s = j\omega$. The above formula includes an equal inductive part as well, so this includes the stored magnetic energy which we can also sum. One can also view (3.11) as an analytic form of power which is associated with an impedance function which is conjugate symmetric corresponding to a Laplace transform of a real-value time function, and which is analytic in the right half s-plane, as well as a positive real function due to passivity.

Now form

$$
\begin{aligned}
\tilde{P}_s^{(m)} &= \lim_{N\to\infty} \sum_{n=1}^{N} \tilde{P}_s^{(n)} = \lim_{N\to\infty} s^{\frac{1}{2}} \left[\frac{\mu_m}{\sigma_m}\right] \left(\frac{I}{N}\right)^2 \sum_{n=1}^{N} \ell_n^{(m)^{-1}} \\
&= \lim_{N\to\infty} s^{\frac{1}{2}} \left[\frac{\mu_m}{\sigma_m}\right]^{\frac{1}{2}} \left(\frac{I}{N}\right)^2 \sum_{n=1}^{N} \Delta'^{-2} h_w^{(m)^{-2}} \ell_n^{(m)} \\
&= s^{\frac{1}{2}} \left[\frac{\mu_m}{\sigma_m}\right]^{\frac{1}{2}} I^2 X^{(m)} \\
X^{(m)} &= \lim_{N\to\infty} (\Delta v)^{-2} \sum_{n=1}^{N} h_w^{(m)^{-2}} \ell_n^{(m)} \\
&= (\Delta v)^{-2} \oint_{C^{(m)}} h_w^{(m)^{-2}} d\ell = (\Delta v)^{-2} \oint_{C^{(m)}} \left|\frac{dw^{(m)}}{d\zeta}\right|^2 |d\zeta| \\
&= (\Delta v)^{-2} \oint_{C^{(m)}} h_w^{(m)^{-1}} |dw| = (\Delta v)^{-2} \oint_{C^{(m)}} h_w^{(m)^{-1}} |dv|
\end{aligned}
\tag{3.12}
$$

where $C^{(m)}$ is the contour around the inner or outer conductor as indicated in fig. 3.1. Note that $h_w^{(m)}$ and $w^{(m)}$ are also given to indicate evaluation on these two contours. Then setting

$$
\tilde{P}_s^{(m)} \equiv \tilde{Z}_s'^{(m)} I^2 \tag{3.13}
$$

we have

$$
\tilde{Z}_s'^{(m)} = s^{\frac{1}{2}} \left[\frac{\mu_m}{\sigma_m}\right]^{\frac{1}{2}} X^{(m)} \tag{3.14}
$$

This gives our skin impedance per unit length as

$$
\begin{aligned}
\tilde{Z}_s' &= \tilde{Z}_s'^{(1)} + \tilde{Z}_s'^{(2)} = s^{\frac{1}{2}} \Xi \\
\Xi &= \left[\frac{\mu_1}{\sigma_1}\right]^{\frac{1}{2}} X^{(1)} + \left[\frac{\mu_2}{\sigma_2}\right]^{\frac{1}{2}} X^{(2)}
\end{aligned}
\tag{3.15}
$$

With Ξ one can now go to (2.6) for τ as used in the cable transfer function and step response as discussed in Section 2.

As a simple example let us revisit the coax discussed in Section 2. An appropriate conformal transformation is

$$w(\zeta) = \ell n\left(\frac{\zeta}{\Psi_1}\right) = \ell n\left(\frac{\Psi}{\Psi_1}\right) + j\phi$$
$$\Delta u = \ell n\left(\frac{\Psi_2}{\Psi_1}\right) \quad , \quad \Delta v = 2\pi \tag{3.16}$$

in agreement with (2.1). Then (3.12) gives

$$h_w^{(m)} = \left|\frac{dw^{(m)}}{d\zeta}\right|^{-1} = \Psi_m$$

$$X^{(m)} = (\Delta v)^{-2} \oint_{C^{(m)}} h^{(m)^{-1}} |dv| = [2\pi\Psi_m]^{-1} \tag{3.17}$$

$$\Xi = \left[\frac{\mu_1}{\sigma_1}\right]^{\frac{1}{2}} X^{(1)} + \left[\frac{\mu_2}{\sigma_2}\right]^{\frac{1}{2}} X^{(2)}$$
$$= [2\pi\Psi_1]^{-1}\left[\frac{\mu_1}{\sigma_1}\right]^{\frac{1}{2}} + [2\pi\Psi_2]^{-1}\left[\frac{\mu_2}{\sigma_2}\right]^{\frac{1}{2}}$$

in agreement with (2.2). So the $X^{(m)}$ represent some kind of inverse effective circumference of the respective conductors, i.e.,

$$X^{(m)^{-1}} \equiv \text{effective circumference of mth conductor}$$
$$= \frac{\oint_{C^{(m)}} J_s^2 d\ell}{\left\{\oint_{C^{(m)}} J_s d\ell\right\}^2} \tag{3.18}$$

where the conformal transformation derivative $dw/d\zeta$ as in (3.12) can be interpreted as above in terms of the z-directed surface current density (or electric or magnetic field at the conducting boundaries).

We can define

$$\ell^{(m)} \equiv \oint\limits_{C^{(m)}} d\ell = \oint\limits_{C^{(m)}} h_w^{(m)} |dv|$$

$$\equiv \text{circumference of mth conductor (at interface with dielectric)} \tag{3.19}$$

and then define

$$v^{(m)} \equiv \left[\ell^{(m)} X^{(m)}\right]^{-1} \tag{3.20}$$

as some efficiency for the shape of the mth conductor. For the coax (circular $C^{(m)}$) these are unity; for other shapes we expect these to be less than one due to the nonuniform surface current densities around the contours.

4. Elliptical Cable

Consider the conformal transformation [6]

$$\zeta = -ja\sinh(w) = -a\sin(j\omega) = x + jy \quad , \quad a > 0$$

$$h_w = \left|\frac{d\zeta}{dw}\right| \quad , \quad w = u + jv$$

$$\frac{d\zeta}{dw} = -ja\cosh(w) = -ja\cos(jw)$$

$$= -ja\left[1+\sinh^2(w)\right]^{\frac{1}{2}} = -ja\left[1-\frac{\zeta^2}{a^2}\right]^{\frac{1}{2}} \tag{4.1}$$

with appropriate attention to the choice of branch-cut location. Expand this into real and imaginary parts [7]

$$x + jy = -ja\sinh(u + jv)$$

$$= -ja\left[\sinh(u)\cos(v) + j\cosh(u)\sin(v)\right]$$

$$x = a\cosh(u)\sin(v)$$

$$y = -a\sinh(u)\cos(v) \tag{4.2}$$

Surfaces of constant u (which include the conductor surfaces) are now found as satisfying

$$\left[\frac{x}{a\cosh(u)}\right]^2 + \left[\frac{y}{a\sinh(u)}\right]^2 = 1 \tag{4.3}$$

which is the equation of an ellipse. Letting $u \to +\infty$ this becomes a circle of radius $ae^u/2$. For $u \to 0$ we have

$$y = 0 \quad , \quad x = a\sin(v) \tag{4.4}$$

which gives a straight line of length 2a on the x axis noting that v varies from 0 to 2π for a contour circumscribing the inner conductor (for one convenient choice of the branch cut). Choosing two convenient values of u to define the conductor surfaces, we have as before

$$f_g = \frac{\Delta u}{\Delta v} = \frac{u_2 - u_1}{2\pi} \tag{4.5}$$

The scale factor is [7]

$$
\begin{aligned}
h_w &= a\left|\cosh(w)\right| = a\left|\cosh(u+jv)\right| \\
&= a\left|\cosh(u)\cos(v) + j\sinh(u)\sin(v)\right| \\
&= a\left[\cosh^2(u)\cos^2(v) + \sinh^2(u)\sin^2(v)\right]^{\frac{1}{2}} \\
&= a\left[\sinh^2(u) + \cos^2(v)\right]^{\frac{1}{2}} \\
&= a\left[\cosh^2(u) - \sin^2(v)\right]^{\frac{1}{2}}
\end{aligned}
\tag{4.6}
$$

The circumferences are

$$
\begin{aligned}
\ell^{(m)} &= \oint_{C^{(m)}} h_w^{(m)}\left|dv\right| = 4ap^{(m)^{-\frac{1}{2}}} \int_0^{\frac{\pi}{2}} \left[1 - p^{(m)}\sin^2(v)\right]^{\frac{1}{2}} dv \\
&= 4ap^{(m)^{-\frac{1}{2}}} E\left(p^{(m)}\right) \\
p^{(m)} &\equiv \cosh^{-2}(u_m) \quad , \quad 0 < p^{(m)} \leq 1
\end{aligned}
\tag{4.7}
$$

where $p^{(m)}$ is called the parameter of the elliptic integral [7]. Note the use of symmetry to reduce the integration interval to $0 \leq v \leq \pi/2$. We then also have

$$
\begin{aligned}
X^{(m)} &= (\Delta v)^{-2} \oint_{C^{(m)}} h_w^{(m)^{-1}}\left|dv\right| = \frac{p^{(m)^{\frac{1}{2}}}}{\pi^2 a} \int_0^{\frac{\pi}{2}} \left[1 - p^{(m)}\sin^2(v)\right]^{-\frac{1}{2}} dv \\
&= \frac{p^{(m)^{\frac{1}{2}}}}{\pi^2 a} K\left(p^{(m)}\right) \\
v^{(m)} &= \frac{\pi^2}{4}\left[E\left(p^{(m)}\right)K\left(p^{(m)}\right)\right]^{-1}
\end{aligned}
\tag{4.8}
$$

Now look at what happens concerning the inner conductor. For large u_1 we have

$$
p^{(1)} = 4e^{-2u_1}\left[1 + O\left(e^{-2u_1}\right)\right] \to \left(\frac{a}{\Psi_1}\right)^2 \quad \text{as } u_1 \to \infty
$$

$$
E\left(p^{(1)}\right) = \frac{\pi}{2}\left[1 + O\left(e^{-2u_1}\right)\right] \to \frac{\pi}{2} \quad \text{as } u_1 \to \infty
$$

$$K\left(p^{(1)}\right)=\frac{\pi}{2}\left[1+O\left(e^{-2u_1}\right)\right]\to\frac{\pi}{2}\ \text{as}\ u_1\to\infty$$

$$\ell_1=2\pi a\frac{e^{u_1}}{2}\left[1+O\left(e^{-2u_1}\right)\right]\to 2\pi\Psi_1\ \text{as}\ u_1\to\infty \tag{4.9}$$

where the conductor is approximately circular (radius Ψ_1 in this limit). The same then applies to the outer conductor since $u_2 > u_1$ (and hence $\Psi_2 > \Psi_1$), thereby giving a simple coaxial cable.

For small u_1 we have for the circumference

$$p_1^{(1)} \equiv 1-p^{(1)} \equiv \text{complementary parameter for elliptic integrals}$$

$$p_1^{(1)}=\tanh^2(u_1)=u_1^2+O\left(u_1^4\right)\ \text{as}\ u_1\to 0$$

$$E\left(p^{(1)}\right)=E'\left(p_1^{(1)}\right)=1+O\left(p_1^{(1)}\ell n\left(p_1^{(1)}\right)\right)\ \text{as}\ p_1^{(1)}\to 0$$

$$=1+O\left(u_1^2\ell n(u_1)\right)\ \text{as}\ u_1\to 0$$

$$\to 1\ \text{as}\ u_1\to 0$$

$$\ell^{(1)}=4a\left[1+O\left(u_1^2\ell n(u_1)\right)\right]\ \text{as}\ u_1\to 0 \tag{4.10}$$

$$\to 4a\ \text{as}\ u_1\to 0$$

which is just the perimeter of a strip of width 2a. For the effective circumference we have

$$K\left(p^{(1)}\right)=K'\left(p_1^{(1)}\right)=\frac{1}{2}\ell n\left(\frac{16}{p_1^{(1)}}\right)\left[1+O\left(p_1^{(1)}\right)\right]$$

$$=\ell n\left(\frac{4}{u_1}\right)\left[1+O\left(u_1^2\right)\right]\ \text{as}\ u_1\to 0$$

$$X^{(1)}=\left[\pi^2 a\right]^{-1}\ell n\left(\frac{4}{u_1}\right)\left[1+O\left(u_1^2\right)\right] \tag{4.11}$$

$$\to\infty\ \text{as}\ u_1\to 0$$

$$v^{(1)}=\frac{\pi^2}{4}\left[E'\left(p_1^{(1)}\right)K'\left(p_1^{(1)}\right)\right]^{-1}$$

$$=\frac{\pi^2}{4\ell n\left(\frac{4}{u_1}\right)}\left[1+O\left(u_1^2\ell n(u_1)\right)\right]$$

$$\to 0\ \text{as}\ u_1\to 0$$

From (4.3) we have the semi-minor axis (half thickness) of the thin ellipse as

$$y_{\max} = a\sinh(u_1) = au_1\left[1 + O\left(u_1^2\right)\right] \text{ as } u_1 \to 0 \tag{4.12}$$

which can in turn be used in (4.11) to show the effects of a thin sheet which in the limit has two knife edges. As expected from physical arguments the $X^{(m)}$ and hence Ξ and τ blow up in this limit.

5. Concluding Remarks

As the elliptical center conductor of the cable goes toward a strip with two knife edges, we can see in (4.11) that the losses increase logarithmically. This is similar to an integral of the form $\int D^{-1} dD$ where D is not taken to zero, but to some distance of the order of the conductor thickness $2au_1$ from (4.12).

Keep in mind the approximations used in developing this model. These in turn limit the domain of validity. The assumption of a TEM mode limits the highest frequencies to those which do not allow other modes to propagate as discussed in Section 2, due to variation in the $\tilde{Z}'_{s_n}$ in the various ducts around the cable cross section as discussed in Section 3. Furthermore, the skin-effect model assumes that the skin depth is small compared to the local radius of curvature, an assumption which is violated at a knife edge with zero radius of curvature. In addition, the skin-depth has been assumed small compared to conductor thicknesses, an assumption which is violated for a zero-thickness conducting sheet. So one cannot apply these results in the case of ideal zero-thickness sheets with two knife edges. However, they do indicate what happens as one approaches such a case, namely a significant increase in loss and rise time.

Considering various transmission-line cross sections it is apparent then that a coaxial circular-cylindrical structure is good from a low-loss standpoint. However, various applications, such as for some kinds of antennas, require other geometries such as two conductors with neither surrounding the other. In such cases one may need to consider rounding the edges to a sufficiently large minimum radius of curvature.

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