Measurement Notes

Note 33

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EFFECTIVE RISE TIME RESULTING FROM A CONVOLUTION OF WAVEFORMS WITH EXPONENTIAL AND GAUSSIAN RISE CHARACTERISTICS

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Abstract

A fast rising pulse generator is often used as the source of energy in applied electromagnetic problems. In a typical pulse generator, various physical mechanisms contribute to the rise portion of the resulting waveform. The rise characteristics assoicated with these mechanisms could be different viz., exponential, integrated Gaussian etc. In this note, a frequency domain approach is used in defining and evaluating an effective rise time for the combined waveform. Such a definition is, by no means, a unique one. The illustrative examples worked out consist of combining, a) two exponentially rising waveforms and b) an exponentially rising waveform with an integrated Gaussian waveform.

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CONTENTS

I. Introduction

The objective of this note is to investigate how certain waveforms with known rise characteristics combine to produce the resulting waveform. For example, if a pulse generator is characterized by two rise-time contributing mechanisms, each of which produces an exponentially rising waveform, one wishes to know the effective rate of rise in the resulting waveform. Approximate estimates such as linear or quadrature addition of the two individual rise times have been suggested and used in the past. The present approach, however, consists of obtaining the resulting waveform as a convolution in time domain or simply a product of the individual transforms in the frequency domain. In the frequency domain, that is on the $j\omega$ axis of the complex frequency s-plane, one may look for the frequency at which the convolved spectrum is reduced from its ideal by the conventional factor of $(2^{-1/2})$ in the signal or the familiar half-power point. The reciprocal of this frequency then may be used as the effective rise time of the combined waveform. It is noted that the criterion used here for defining the effective rise time is by no means unique. Rise times are measurable in many ways [1] viz., 10-90%, e-fold, (peak value/maximum slope), and the present definition from a frequency domain approach.

In what follows, the process of convolution which is the basis for the present approach is briefly reviewed in Section II. Using the convolution approach, two combinations are considered in this note. Firstly, two exponentially rising waveforms are combined in Section III. In Section IV, an exponentially rising waveform is combined with an integrated Gaussian waveform. The normalized results are presented in tables and plots that could be useful in design applications. This note is concluded with a summarizing Section V, followed by a list of references.

II. Review of Convolution

A frequently encountered problem in system analysis is the determination of the output for a specified input. One method of obtaining the output employs the principle of superposition which in turn uses the technique of convolution [2,3]. The basic ideas behind convolution may be summarized in the following four steps [2].

1) the input is represented as a continuum of impulses
2) the output for a single impulse is determined
3) the output to each of the elementary impulses representing the input is computed
4) the total system output is obtained by superimposing the responses to all of the elementary impulses in the representation of the input.

The superposition required by step 4) above is achieved by means of a convolution integral. If x(t) and y(t) represent the input and output functions, and h(t) is the impulse response, the above four steps can be respectively written as the following four equations.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau \tag{1}$$

$$h(t) \equiv \text{causal output for a single impulse input } \delta(t) \tag{2}$$

$$h(t-\tau)\cdot x(\tau)\, d\tau \equiv \text{output for each elementary impulse in the input} \tag{3}$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{t} x(t-\tau)\, h(\tau)\, d\tau \tag{4}$$

Note that in (3) above $x(\tau)\, d\tau$ is the strength of each elementary impulse in the input. It is easily observed via (4) that the output y(t) is simply the convolution of x(t) and h(t), keeping in mind h(t) is the impulse response. In passing, one might also mention that continuous-time convolution is commutative, associative and distributive.

Occasionally, the step-function response $g(t)$ is known in place of the impulse response $h(t)$. Recognizing the following relationship between a step function $u(t)$ and the impulse function $\delta(t)$

$$\delta(t) = \frac{d}{dt} u(t) \tag{5}$$

One may rewrite (4) in terms of the step function response as

$$y(t) = \int_{-\infty}^{t} \left[\frac{d}{dt} x(t-\tau)\right] g(\tau)\, d\tau = \int_{-\infty}^{t} \left[\frac{d}{d\tau} g(\tau)\right] x(t-\tau)\, d\tau \tag{6}$$

Note that the lower limit of $-\infty$ in (4) and (6) may be replaced by zero if the input is zero for negative times.

The ease and benefit of this technique however, lies in the transform domain, defined as follows. If $f(t)$ is some arbitrary time function, we define the two-sided Laplace transform with an overhead symbol ~ as

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt \tag{7}$$

and the inverse transform is given by

$$f(t) = \frac{1}{2\pi j} \int_{\Omega_o - j\infty}^{\Omega_o + j\infty} \tilde{f}(s)\, e^{st}\, ds \tag{8}$$

where $s = \Omega + j\omega$ is the complex frequency. Throughout this note we adopt the above transform, noting that by setting $s = j\omega$, one obtains a Fourier transform as needed for numerical purposes. The contour for inverse transformation integral is the Bromwich contour defined by Re $\{s\} = \Omega_o$ such that all singularities of $\tilde{f}(s)$, in a complex variable sense lie to the left of this contour [4].

Using the above definition, equations (4), (5) and (6) above become in the transform domain

$$\tilde{y}(s) = \tilde{x}(s)\, \tilde{h}(s) \tag{9}$$

$$\tilde{\delta}(s) = s\ \tilde{u}(s) = 1 \tag{10}$$

$$\tilde{y}(s) = s\ \tilde{x}(s)\ \tilde{g}(s) \tag{11}$$

So, in the s-domain, knowing the impulse response $\tilde{h}(s)$ or the step response $\tilde{g}(s)$, the transform of the output $\tilde{y}(s)$ is easily constructed by a simple multiplication procedure. Inverse transformation of $\tilde{y}(s)$ can be performed to obtain the time domain output function $y(t)$.

This completes a brief review of convolution technique which is used in the following sections.

III. Combination of Exponentially Rising Waveforms

Consider an exponentially rising waveform given by

$$f_1(t) = \left[1 - e^{-\alpha_1 t}\right] u(t) = \left[1 - e^{-(t/t_1)}\right] u(t) \qquad (12)$$

with

$$\alpha_1 = (1/t_1) \qquad (13)$$

which is shown plotted in figure 1. Its Laplace transform is readily given by

$$\tilde{f}_1(s) = \left(\frac{1}{s} - \frac{1}{s+\alpha_1}\right) = \frac{1}{s}\left(\frac{\alpha_1}{s+\alpha_1}\right) \qquad (14)$$

Note that if the system were ideal, the response to an ideal step function is an ideal step function. Using (11), it is evident that one can define a quality factor $\tilde{Q}_1(s)$ given by

$$\tilde{Q}_1(s) = \frac{\tilde{f}_1(s)}{\tilde{u}(s)} = \frac{\alpha_1}{s+\alpha_1} \qquad (15)$$

which is indicative of the deviation of the output transform $\tilde{f}_1(s)$ from its ideal value of $u(s) = s^{-1}$. The magnitude of $|\tilde{Q}_1(j\omega)|$, plotted in figure 2, is given by

$$|\tilde{Q}_1(j\omega)| = \frac{1}{\sqrt{1+(\omega/\alpha_1)^2}} = \frac{1}{\sqrt{1+\omega^2 t_1^2}} \qquad (16)$$

It is evident that at a frequency $\omega = \alpha_1$, the Fourier transform $f_1(j\omega)$ is less than its ideal value in magnitude by an amount of $2^{-\frac{1}{2}} \approx 0.707$. Also, observe that for this exponentially rising waveform of $f_1(t)$, the various rise times are

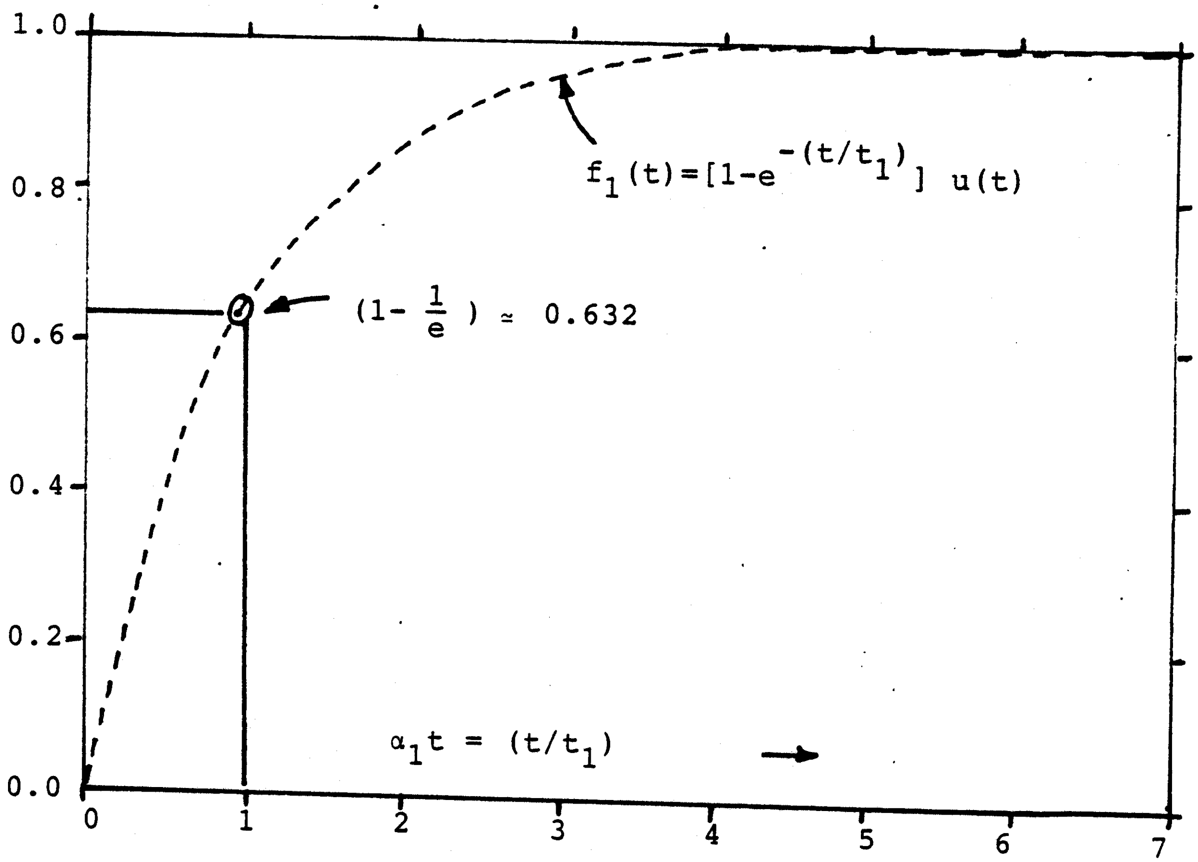


Figure 1. An exponentially rising waveform in time domain.

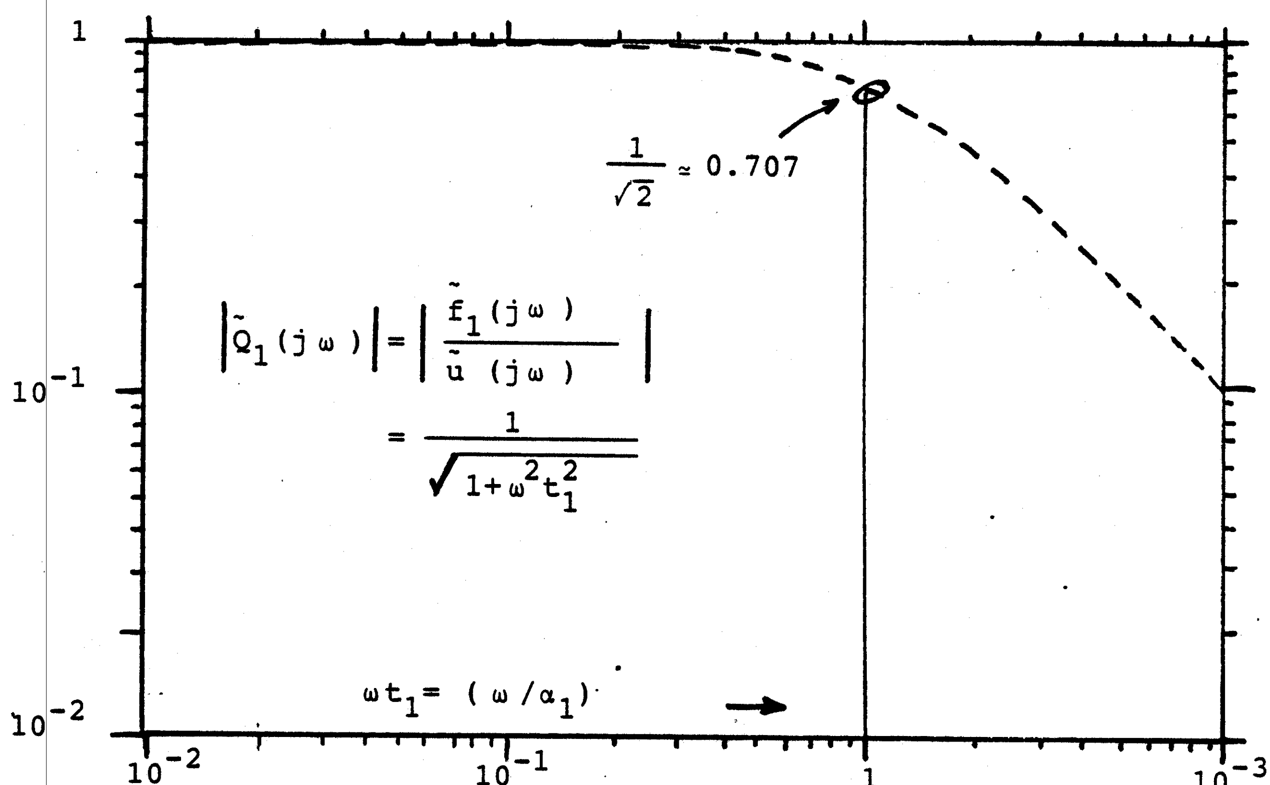


Figure 2. Magnitude spectrum of $f_1(t)$ normalized to the spectrum of an ideal step function.

$$t_{10-90} \equiv \text{(10-90\%) rise time} = \ln(9)\ t_1 \simeq 2.2t_1 \qquad (17)$$

t_{mr} ≡ rise time associated with the maximum rate of rise

$$= \frac{f_1(t)_{max}}{\left.\frac{df_1}{dt}\right|_{max}} = t_1 \qquad (18)$$

$$t_r \equiv \text{exponential rise time} = t_{mr} = t_1 \qquad (19)$$

Next, one may try to combine two exponentially rising waveforms as follows. Consider a system A which puts out $f_1(t)$ for an ideal-step-function input and, a system B which puts out a similar $f_2(t)$ for an ideal-step-function input. If an ideal step function is sequentially applied to systems A and B, (see figure 3), the resulting $f_{out}(t)$ is the desired output, given by the time convolution of $f_1(t)$ and $f_2(t)$, remembering that $f_1(t)$ and $f_2(t)$ are step responses and not impulse responses. In the frequency domain, use of (11) leads to

$$\begin{aligned}\tilde{f}_{out}(s) &= s\ \tilde{f}_1(s)\ \tilde{f}_2(s)\\ &= s\left[\frac{1}{s}\ \frac{\alpha_1}{s+\alpha_1}\right]\left[\frac{1}{s}\ \frac{\alpha_2}{s+\alpha_2}\right]\\ &= \frac{1}{s}\left[\frac{\alpha_1}{s+\alpha_1}\right]\left[\frac{\alpha_2}{s+\alpha_2}\right]\\ &= \frac{1}{s}\ \tilde{Q}_1(s)\ \tilde{Q}_2(s)\end{aligned} \qquad (20)$$

Once again, we recognize that

$$\tilde{Q}_2(s) = \frac{\alpha_2}{s+\alpha_2} \qquad (21)$$

and define

$$\tilde{Q}_{1,2}(s) \equiv \tilde{Q}_1(s)\ \tilde{Q}_2(s) \qquad (22)$$

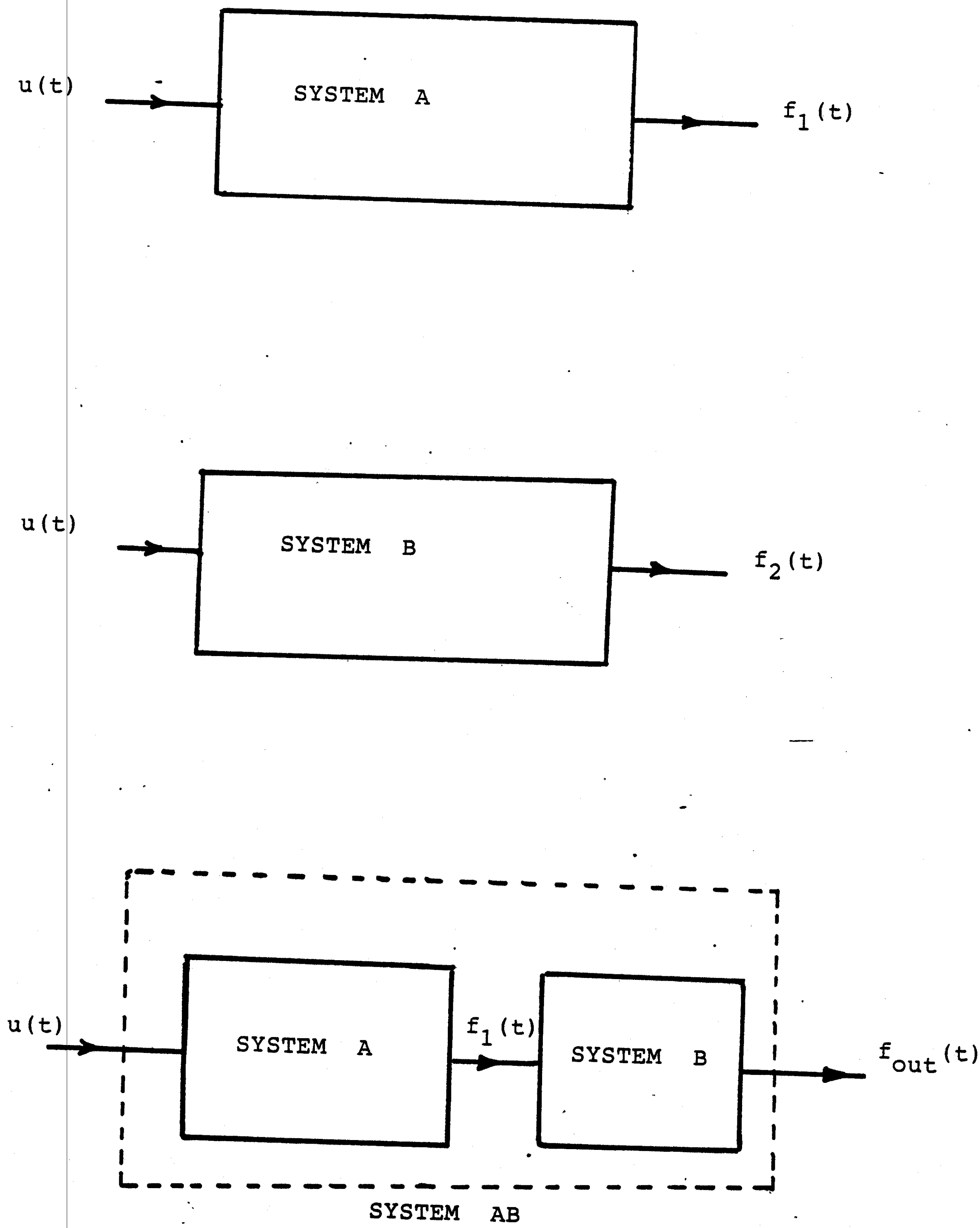


Figure 3. Combination of two exponentially rising waveforms $f_1(t)$ and $f_2(t)$ resulting in $f_{out}(t)$.

$\tilde{Q}_{1,2}(s)$ is indicative of the departure of the transform of the combined waveform from its ideal value of s^{-1}. In terms of the magnitude of the Fourier transform,

$$|\tilde{Q}_{1,2}(j\omega)| = \left[\frac{1}{\sqrt{1+(\omega/\alpha_1)^2}} \; \frac{1}{\sqrt{1+(\omega/\alpha_2)^2}}\right] \tag{23}$$

We now look for the effective frequency $\omega=\alpha_{out}$ at which the combined waveform has its Fourier transform magnitude reduced from its ideal by a factor of $2^{-\frac{1}{2}}$, leading to

$$\frac{1}{\sqrt{2}} = \left[\frac{1}{\sqrt{1+(\alpha_{out}/\alpha_1)^2}}\right]\left[\frac{1}{\sqrt{1+(\alpha_{out}/\alpha_2)^2}}\right] \tag{24}$$

which becomes

$$\alpha_{out}^4 + \alpha_{out}^2 \, (\alpha_1^2 + \alpha_2^2) - \alpha_1^2 \, \alpha_2^2 = 0 \tag{25}$$

with its solution

$$\alpha_{out}^2 = \left(\frac{\alpha_1^2 + \alpha_2^2}{2}\right)\left[\sqrt{1 + \frac{4\alpha_1^2 \, \alpha_2^2}{(\alpha_1^2 + \alpha_1^2)^2}} - 1\right] \tag{26}$$

In time domain, the effective rise time is given by

$$t_{eff}^2 = \frac{1}{\alpha_{out}^2} = \frac{2 \, t_1^2 \, t_2^2}{(t_1^2 + t_2^2)\left[\sqrt{1 + \frac{4 \, t_1^2 \, t_2^2}{(t_1^2 + t_2^2)^2}} - 1\right]} \tag{27}$$

or

$$t_{eff.} = \frac{\sqrt{t_1 t_2}}{\left[\sqrt{\frac{(t_1^2 + t_2^2)^2}{4 \, t_1^2 \, t_2^2} + 1} - \frac{(t_1^2 + t_2^2)}{2 \, t_1 t_2}\right]^{1/2}} \tag{28}$$

Equation (28) gives the effective rise time of the combined waveform, in terms of the e-fold rise times (=t_{mr} for exponentials) of the two individual waveforms.

For purposes of tabulating and plotting, we define the following

$$t_\ell \equiv \text{linear addition} = t_1 + t_2 \tag{29}$$

$$t_q \equiv \text{quadrature addition} = \sqrt{t_1^2 + t_2^2} \tag{30}$$

$$t_{eff} = \frac{t_g}{\left[\sqrt{A^2 + 1} - A\right]^{1/2}} \tag{31}$$

where

$$t_g \equiv \text{geometric mean} = \sqrt{t_1 t_2} \tag{32}$$

$$A = \frac{t_1^2 + t_2^2}{2t_1 t_2} = \frac{t_q^2}{2t_g^2} \equiv \text{a factor} \tag{33}$$

Although not obvious, it has been verified by taking the proper limits that

$$t_{eff} = t_1 + t_2 \quad \text{iff} \quad t_1 = 0 \text{ or } t_2 = 0 \tag{34}$$

The results are presented in a normalized format as follows. Given t_1 and t_2, all times are normalized with respect to the larger of the two times, say t_1, as follows

$$T_\ell = \frac{t_\ell}{t_1} = 1 + \left(\frac{t_2}{t_1}\right) = 1 + T \quad , \quad T \equiv (t_2/t_1) \tag{35}$$

$$T_q = \frac{t_q}{t_1} = \sqrt{1 + \left(\frac{t_2}{t_1}\right)} = 1 + T^2 \tag{36}$$

$$T_{eff} = \frac{t_{eff}}{t_1} = \frac{\sqrt{T}}{\left[\sqrt{\left(\frac{1 + T^2}{2T}\right)^2 + 1} - \left(\frac{1 + T^2}{2T}\right)\right]^{1/2}} \tag{37}$$

With the above normalization, T can be varied from 0 to 1 to cover all possibilities. Table 1 lists the computed values that are plotted in figure 4. The use of table 1 and figure 4 may be illustrated by an example.

t_1 = 5 ns and t_2 = 3 ns (say)

T = smaller/larger = 3 ns/5 ns = 0.6

$$\left.\begin{aligned} T_\ell &= 1.6000 \\ T_{eff} &= 1.2597 \\ T_q &= 1.1662 \end{aligned}\right\} \text{from table 1} \tag{38}$$

denormalizing

$$\begin{aligned} t_\ell &= 1.6000 \times 5\text{ ns} = 8.0000\text{ ns} \\ t_{eff} &= 1.2597 \times 5\text{ ns} = 6.2985\text{ ns} \\ t_q &= 1.1662 \times 5\text{ ns} = 5.8310\text{ ns} \end{aligned} \tag{39}$$

It is also interesting to note that the present method yields effective rise times that fall in between linear (upper value) and quadrature (lower value) estimates for the entire range.

$t_1 = t_>$	$t_2 = t_<$	T_q	T_{eff}	T_ℓ
1.0000	0.0000	1.0000	1.0000	1.0000
1.0000	0.0200	1.0002	1.0003	1.0200
1.0000	0.0400	1.0008	1.0016	1.0400
1.0000	0.0600	1.0018	1.0036	1.0600
1.0000	0.0800	1.0032	1.0063	1.0800
1.0000	0.1000	1.0050	1.0099	1.1000
1.0000	0.1200	1.0072	1.0141	1.1200
1.0000	0.1400	1.0098	1.0191	1.1400
1.0000	0.1600	1.0127	1.0247	1.1600
1.0000	0.1800	1.0161	1.0310	1.1800
1.0000	0.2000	1.0198	1.0379	1.2000
1.0000	0.2200	1.0239	1.0453	1.2200
1.0000	0.2400	1.0284	1.0533	1.2400
1.0000	0.2600	1.0332	1.0619	1.2600
1.0000	0.2800	1.0385	1.0709	1.2800
1.0000	0.3000	1.0440	1.0803	1.3000
1.0000	0.3200	1.0500	1.0902	1.3200
1.0000	0.3400	1.0562	1.1005	1.3400
1.0000	0.3600	1.0628	1.1111	1.3600
1.0000	0.3800	1.0698	1.1221	1.3800
1.0000	0.4000	1.0770	1.1334	1.4000
1.0000	0.4200	1.0846	1.1450	1.4200
1.0000	0.4400	1.0925	1.1568	1.4400
1.0000	0.4600	1.1007	1.1690	1.4600
1.0000	0.4800	1.1092	1.1813	1.4800
1.0000	0.5000	1.1180	1.1939	1.5000
1.0000	0.5200	1.1271	1.2067	1.5200
1.0000	0.5400	1.1365	1.2197	1.5400
1.0000	0.5600	1.1461	1.2329	1.5600
1.0000	0.5800	1.1560	1.2462	1.5800
1.0000	0.6000	1.1662	1.2597	1.6000
1.0000	0.6200	1.1766	1.2734	1.6200
1.0000	0.6400	1.1873	1.2872	1.6400
1.0000	0.6600	1.1982	1.3011	1.6600
1.0000	0.6800	1.2093	1.3152	1.6800
1.0000	0.7000	1.2207	1.3294	1.7000
1.0000	0.7200	1.2322	1.3437	1.7200
1.0000	0.7400	1.2440	1.3581	1.7400
1.0000	0.7600	1.2560	1.3726	1.7600
1.0000	0.7800	1.2682	1.3873	1.7800
1.0000	0.8000	1.2806	1.4020	1.8000
1.0000	0.8200	1.2932	1.4168	1.8200
1.0000	0.8400	1.3060	1.4317	1.8400
1.0000	0.8600	1.3189	1.4467	1.8600
1.0000	0.8800	1.3321	1.4618	1.8800
1.0000	0.9000	1.3454	1.4769	1.9000
1.0000	0.9200	1.3588	1.4922	1.9200
1.0000	0.9400	1.3724	1.5075	1.9400
1.0000	0.9600	1.3862	1.5228	1.9600
1.0000	0.9800	1.4001	1.5383	1.9800
1.0000	1.0000	1.4142	1.5538	2.0000

Table 1. Combining two exponentially rising waveforms using the quadrature, the convolution and linear methods.

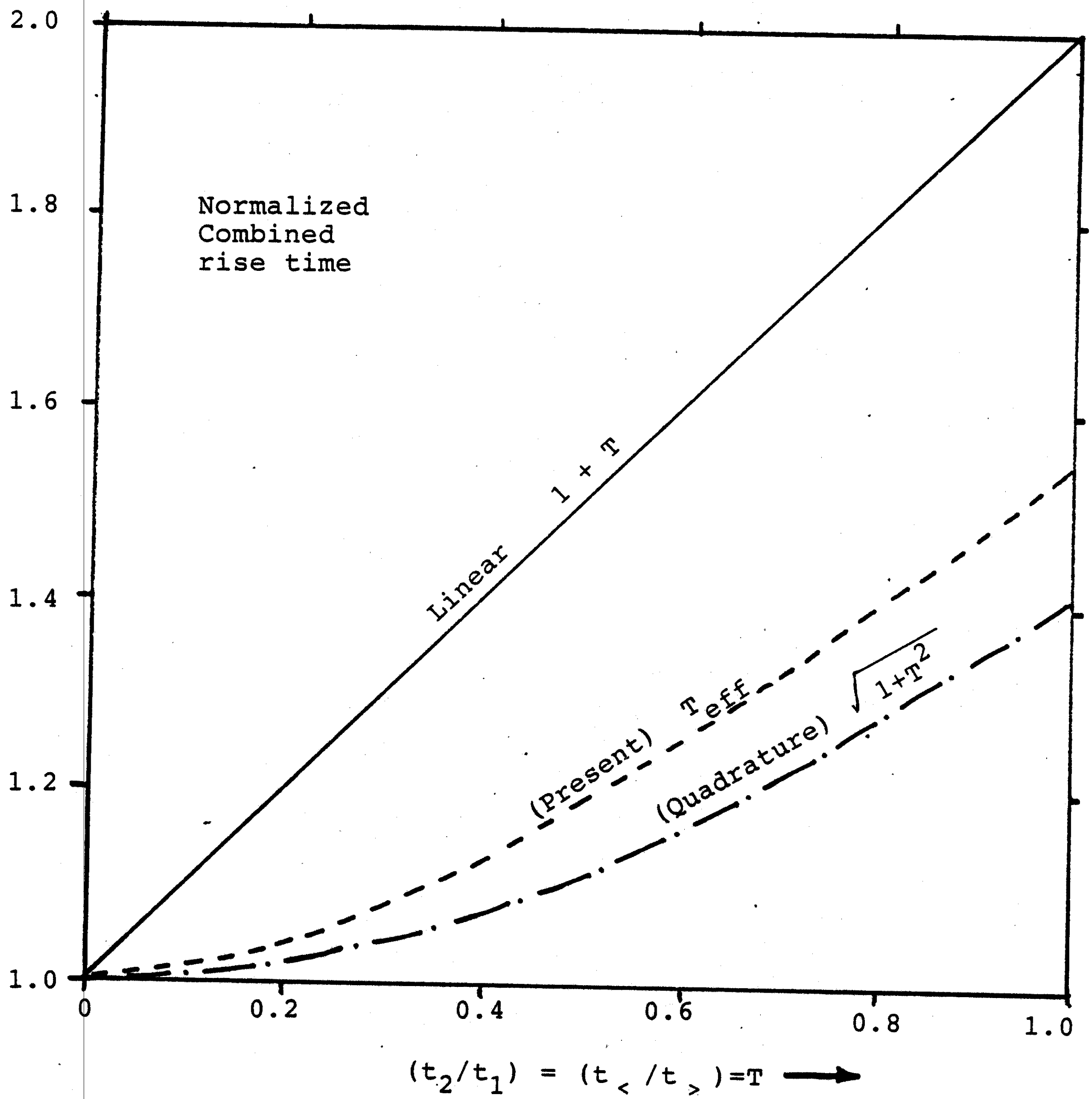


Figure 4. Various estimates of the effective rise time of a combination of two exponentially rising waveforms.

IV. Combining an Exponentially Rising Waveform with an Integral Gaussian Waveform

Recall that the exponentially rising waveform that we have been using, is given by

$$f_1(t) = [1-e^{-\alpha_1 t}]\, u(t) = [1 - e^{-(t/t_1)}]\, u(t) \tag{40}$$

and its Laplace transform by

$$\tilde{f}_1(s) = \frac{1}{s}\left(\frac{\alpha_1}{s+\alpha_1}\right) \tag{41}$$

To the above waveform, we now wish to add an integrated Gaussian waveform via the convolution procedure. Consider a normal or Gaussian distribution function with zero mean and a standard deviation of t_3 as given by

$$G(t) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{t_3}\,\exp\left(-\frac{1}{2}\,\frac{t^2}{t_3^2}\right) \tag{42}$$

The reason for choosing the above form for the Gaussian, as opposed to defining a characteristic time $t_4 = 2\sqrt{t_3}$ to obtain a simpler exponent in (42) will be made clear later. If G(t) is an voltage signal, it is measurable in units of volts/second and is shown plotted in figure 5. We desire combining the integrated Gaussian waveform with the exponential and hence the integrated Gaussian may be written as

$$F(t) = \int_{-\infty}^{t} G(\tau)\; d\tau$$

$$= \frac{1}{t_3\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{1}{2}\,\frac{\tau^2}{t_3^2}\right) d\tau = \left[\frac{1}{2} + \frac{1}{2}\ \mathrm{erf}\left(\frac{t}{\sqrt{2}\ t_3}\right)\right] \tag{43}$$

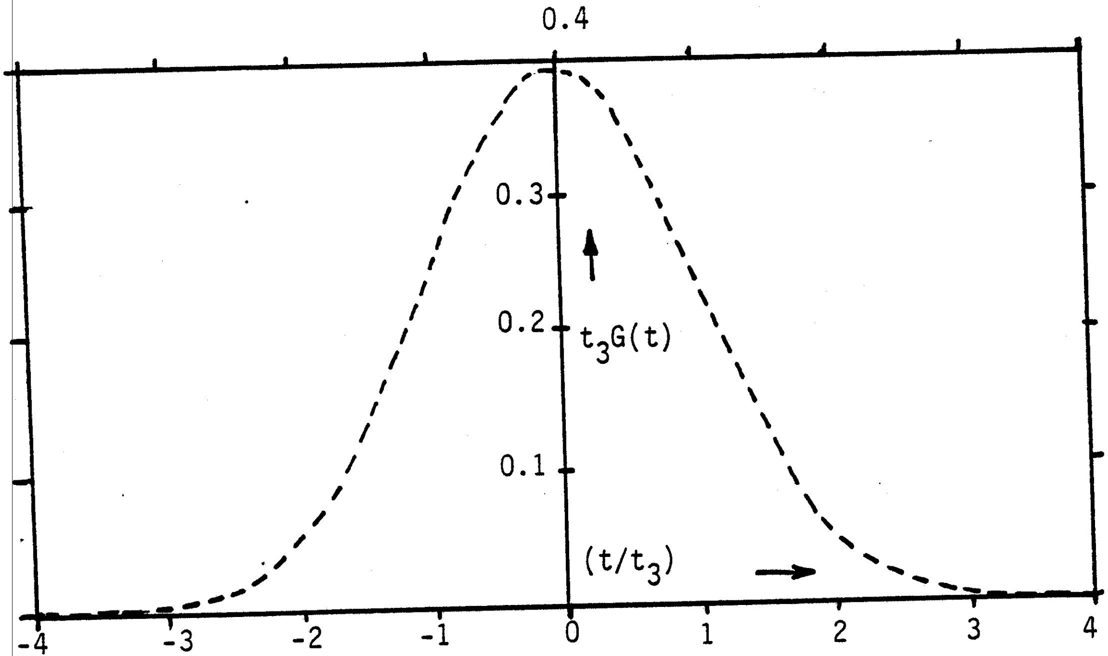


Figure 5. Normalized Gaussian waveform with zero mean and a standard deviation of t_3.

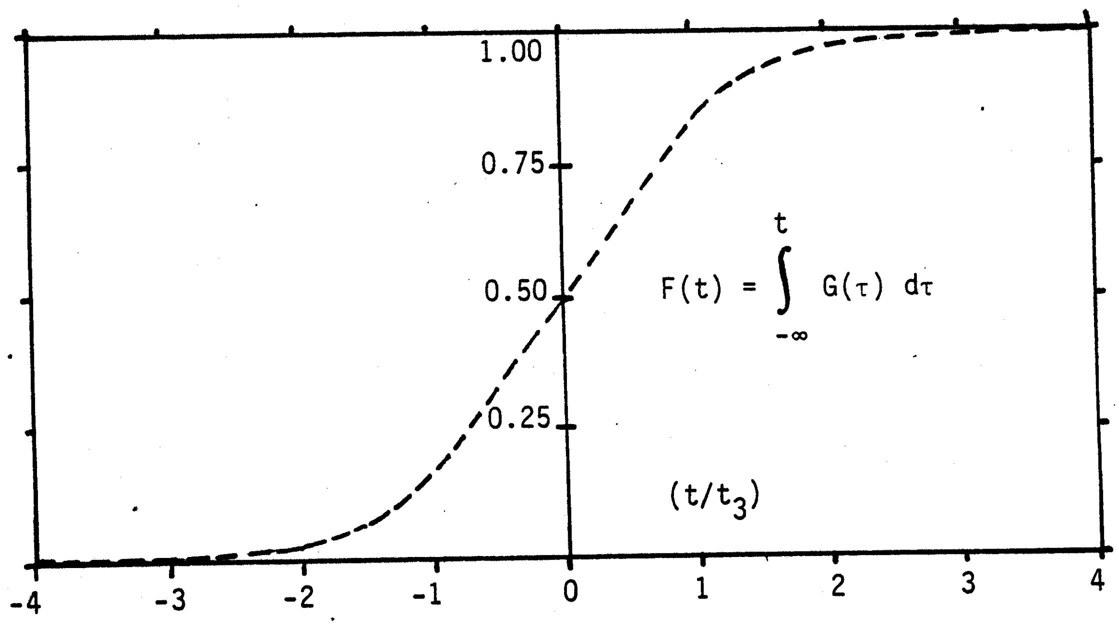


Figure 6. Integrated Gaussian waveform.

F(t) is an s-shaped curve and is shown plotted in figure 6. It has values of 0, 0.5 and 1 at times $-\infty$, 0 and $+\infty$ respectively.

The normalized Gaussian i.e., $t_3\, G(t)$, and the integrated waveform F(t) are also tabulated in Table 2, as functions of normalized time $T_n = (t/t_3)$ ranging from -5 to 5, using the odd symmetry property of erf(x) appearing in (43).

A couple of observations about the integrated Gaussian waveform seem to be in order. One can estimate the maximum slope at the origin as

$$\left.\frac{dF(t/t_3)}{d(t/t_3)}\right|_{max} = \left.\frac{dF(t/t_3}{d(t/t_3)}\right|_{t=0} = t_3 G(0) = 0.39894 \qquad (44)$$

leading to

$$t_{mr} \equiv \text{rise time associated with the maximum slope} = (0.39894)^{-1}$$
$$\cong 2.50752\, t_3 \qquad (45)$$

It is also seen from interpolating table 2, that the 10-90% rise time of the integrated Gaussian waveform is

$$t_{10-90} \cong 2.56384 \quad t_3 \qquad (46)$$

Next, we may look at the effective rate of rise of the integrated Gaussian waveform by itself, using the present frequency domain approach. The frequency spectrum of the Gaussian and the integrated Gaussian are given by

$$\tilde{G}(s) = \exp\left(\frac{1}{2}\; s^2\; t_3^2\right) \qquad (47)$$

$$\tilde{F}(s) = \frac{1}{s}\; \exp\left(\frac{1}{2}\; s^2\; t_3^2\right) \qquad (48)$$

$\frac{t}{t_3}$	$t_3G(t)$	$F(t)$	$\frac{t}{t_3}$	$t_3G(t)$	$F(t)$
0.00	0.39894	0.50000	2.55	0.01545	$0.9^{2}4614$
0.05	0.39844	0.51994	2.60	0.01358	$0.9^{2}5339$
0.10	0.39695	0.53983	2.65	0.01191	$0.9^{2}5975$
0.15	0.39448	0.55962	2.70	0.01042	$0.9^{2}6533$
0.20	0.39104	0.57926	2.75	$0.0^{2}9094$	$0.9^{2}7020$
0.25	0.38667	0.59871	2.80	$0.0^{2}7915$	$0.9^{2}7445$
0.30	0.38139	0.61791	2.85	$0.0^{2}6873$	$0.9^{2}7814$
0.35	0.37524	0.63683	2.90	$0.0^{2}5953$	$0.9^{2}8134$
0.40	0.36827	0.65542	2.95	$0.0^{2}5143$	$0.9^{2}8411$
0.45	0.36053	0.67364	3.00	$0.0^{2}4432$	$0.9^{2}8650$
0.50	0.35207	0.69146	3.05	$0.0^{2}3810$	$0.9^{2}8856$
0.55	0.34294	0.70884	3.10	$0.0^{2}3267$	$0.9^{3}0324$
0.60	0.33322	0.72575	3.15	$0.0^{2}2794$	$0.9^{3}1836$
0.65	0.32297	0.74215	3.20	$0.0^{2}2384$	$0.9^{3}3129$
0.70	0.31225	0.75804	3.25	$0.0^{2}2029$	$0.9^{3}4230$
0.75	0.30114	0.77337	3.30	$0.0^{2}1723$	$0.9^{3}5166$
0.80	0.28969	0.78814	3.35	$0.0^{2}1459$	$0.9^{3}5959$
0.85	0.27798	0.80234	3.40	$0.0^{2}1232$	$0.9^{3}6631$
0.90	0.26609	0.81594	3.45	$0.0^{2}1038$	$0.9^{3}7197$
0.95	0.25406	0.82894	3.50	$0.0^{3}8727$	$0.9^{3}7674$
1.00	0.24197	0.84134	3.55	$0.0^{3}7317$	$0.9^{3}8074$
1.05	0.22988	0.85314	3.60	$0.0^{3}6119$	$0.9^{3}8409$
1.10	0.21785	0.86433	3.65	$0.0^{3}5105$	$0.9^{3}8689$
1.15	0.20594	0.87493	3.70	$0.0^{3}4248$	$0.9^{3}8922$
1.20	0.19419	0.88493	3.75	$0.0^{3}3526$	$0.9^{4}1158$
1.25	0.18265	0.89435	3.80	$0.0^{3}2919$	$0.9^{4}2765$
1.30	0.17137	0.90320	3.85	$0.0^{3}2411$	$0.9^{4}4094$
1.35	0.16038	0.91149	3.90	$0.0^{3}1987$	$0.9^{4}5190$
1.40	0.14973	0.91924	3.95	$0.0^{3}1633$	$0.9^{4}6092$
1.45	0.13943	0.92647	4.00	$0.0^{3}1338$	$0.9^{4}6833$
1.50	0.12952	0.93319	4.05	$0.0^{3}1094$	$0.9^{4}7439$
1.55	0.12001	0.93943	4.10	$0.0^{4}8926$	$0.9^{4}7934$
1.60	0.11092	0.94520	4.15	$0.0^{4}7263$	$0.9^{4}8338$
1.65	0.10226	0.95053	4.20	$0.0^{4}5894$	$0.9^{4}8665$
1.70	0.09405	0.95543	4.25	$0.0^{4}4772$	$0.9^{4}8931$
1.75	0.08628	0.95994	4.30	$0.0^{4}3854$	$0.9^{5}1460$
1.80	0.07895	0.96407	4.35	$0.0^{4}3104$	$0.9^{5}8193$
1.85	0.07206	0.96784	4.40	$0.0^{4}2494$	$0.9^{5}4587$
1.90	0.06562	0.97128	4.45	$0.0^{4}1999$	$0.9^{5}5706$
1.95	0.05959	0.97441	4.50	$0.0^{4}1598$	$0.9^{5}6602$
2.00	0.05399	0.97725	4.55	$0.0^{4}1275$	$0.9^{5}7318$
2.05	0.04879	0.97982	4.60	$0.0^{4}1014$	$0.9^{5}7888$
2.10	0.04398	0.98214	4.65	$0.0^{5}8047$	$0.9^{5}8340$
2.15	0.03955	0.98422	4.70	$0.0^{5}6370$	$0.9^{5}8699$
2.20	0.03547	0.98610	4.75	$0.0^{5}5030$	$0.9^{5}8983$
2.25	0.03174	0.98778	4.80	$0.0^{5}3961$	$0.9^{6}2067$
2.30	0.02833	0.98928	4.85	$0.0^{5}3112$	$0.9^{6}3827$
2.35	0.02522	$0.9^{2}0613$	4.90	$0.0^{5}2439$	$0.9^{6}5208$
2.40	0.02239	$0.9^{2}1802$	4.95	$0.0^{5}1907$	$0.9^{6}6289$
2.45	0.01984	$0.9^{2}2857$	5.00	$0.0^{5}1487$	$0.9^{6}7133$
2.50	0.01753	$0.9^{2}3790$			

Note: $0.0^{2}9094 = 0.009094$ $0.9^{3}0324 = 0.9990324$

Table 2. Normalized Gaussian ($t_3G(t)$) and integral of the Gaussian ($F(t)$) as a function of normalized time (from [5]).

As before, the above spectram after normalization by an ideal step function spectrum are given by

$$\tilde{Q}_1(s) = s\tilde{f}_1(s) = \frac{1}{1+st_1} = \left[1 - st_1 + s^2t_1^2 - s^3t_1^3 + O\,(s^4t_1^4)\right] \text{ as } s \to 0 \qquad (49)$$

$$\tilde{Q}_3(s) = s\tilde{F}(s) = \exp\,(\tfrac{1}{2}\,s^2t_3^2) = \left[1 + \tfrac{1}{2}\,s^2t_3^2 + O\,(s^4t_1^4)\right] \text{ as } s \to 0 \qquad (50)$$

In writing $\tilde{F}(s)$ above, use has been made of the property that the integration in time domain becomes a division by the complex frequency s in the frequency domain. Equation (48) can now be used with the $(2)^{-\frac{1}{2}}$ criterion to yield the effective frequency ω_{eff} after setting $s = j\omega$,

$$\frac{1}{\sqrt{2}} = \exp\left(-\frac{1}{2}\,\omega_{eff}^2\,t_3^2\right)$$

or

$$t_{eff} = \frac{1}{\omega_{eff}} = \frac{t_3}{\sqrt{\ln(2)}} \simeq 1.20112\ t_3 \qquad (51)$$

One can summarize the different rise times of the integrated Gaussian waveform above as follows .

$$\left.\begin{aligned} t_{10-90} &\simeq 2.56384\ t_3 \\ t_{mr} &\simeq 2.50752\ t_3 \\ t_{eff} &\simeq 1.20112\ t_3 \end{aligned}\right\} \qquad (52)$$

Before proceding to combine the integrated Gaussian with an exponentially rising waveform, let us display the Fourier transform of the integrated Gaussian in comparison with the exponential waveform. The quantities plotted in Figure 7 are

$$\left|\tilde{Q}_1(j\omega)\right| = 1\Big/\left(\sqrt{1+\omega^2 t_1^2}\right) = \left[1 - \frac{1}{2}\ \omega^2 t_1^2 + O\ (\omega^4 t_1^4)\right] \quad \text{as } \omega \to 0$$

$$\left|\tilde{Q}_3(j\omega)\right| = \exp\ (-\frac{1}{2}\ \omega^2 t_3^2) = \left[1 - \frac{1}{2}\ \omega^2 t_3^2\ O\ (\omega^4 t_3^4)\right] \quad \text{as } \omega \to 0 \tag{53}$$

The above equation explains the particular choice of Gaussian waveform G(t) used here, leading to the same low-frequency asymptotic behavior (in magnitude) in the integrated Gaussian and the exponential waveforms.

We can now combine the integrated Gaussian with an exponential. As before, the transform $\tilde{F}_{out}(s)$ of the combined waveform is given by

$$\begin{aligned}\tilde{F}_{out}(s) &= s\ \tilde{f}_1(s)\ \tilde{F}\ (s) \\ &= s\left[\frac{1}{s}\ \frac{\alpha_1}{s+\alpha_1}\right]\left[\frac{1}{s}\ \exp\ (\frac{1}{2}\ s^2\ t_3^2)\right] \\ &= \frac{1}{s}\left[\frac{\alpha_1}{s+\alpha_1}\right]\ \exp\ (\frac{1}{2}\ s^2\ t_3^2)\end{aligned} \tag{54}$$

Once again, the deviation of the above from its ideal s^{-1} is given, after setting $s = j\omega$, by

$$\tilde{Q}_{1,3}(j\omega) = \left[\frac{\alpha_1}{j\omega+\alpha_1}\right]\ \exp\ (-\frac{1}{2}\ \omega^2\ t_3^2) \tag{55}$$

By imposing the present criterion in frequency domain

$$\left|\tilde{Q}_{1,3}(j\omega_{eff})\right| = \frac{1}{\sqrt{2}} \tag{56}$$

we get

$$\left[1 + \frac{\omega_{eff}^2}{\alpha_1^2}\right] = 2 \quad \exp\left[-\ \omega_{eff}^2\ t_3^2\right] \tag{57}$$

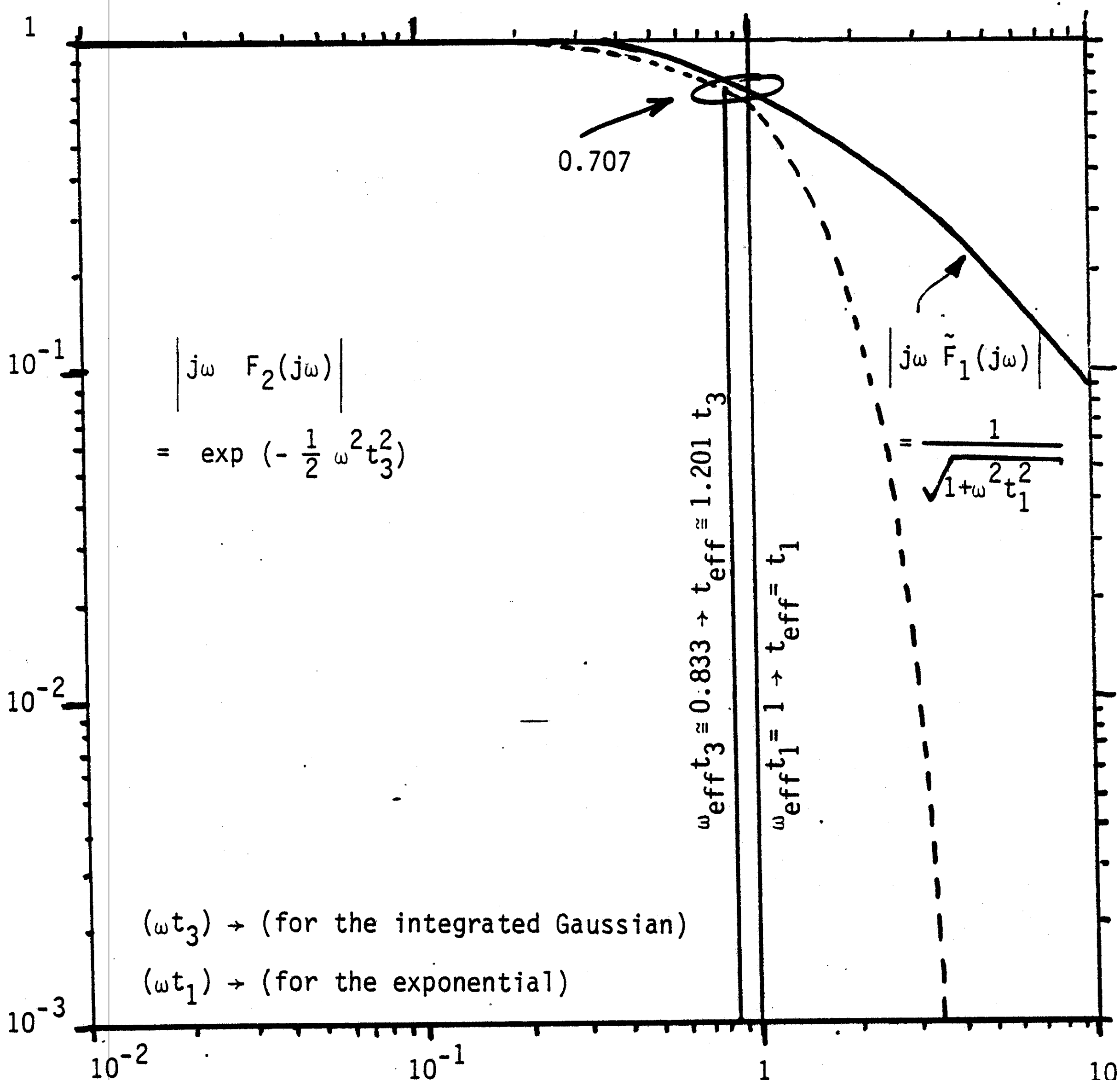


Figure 7. Comparing the magnitude spectra of an integrated Gaussian with that of an ideal step function.

or

$$1+\frac{t_1^2}{t_{eff}^2} = 2\ e^{-(t_3^2/t_{eff}^2)} \qquad (58)$$

Two choices are possible for normalization of all times. They are t_3 and t_1. The transcendental equation above is numerically solved for the following 2 cases.

case 1. Normalize all times with respect to $(1.201\ t_3)$

$T_{n1} = t_{eff}/(1.201t_3)$

for $0 \leq t_1/(1.201t_3) \leq 1$

case 2. Normalize all times with respect to t_1

$T_{n2} = t_{eff}/t_1$

for $0 \leq 1.201t_3/t_1 \leq 1$

The above computations are listed in Tables 3 and 4 and the results are plotted in figures 8a and 8b respectively. If t_1 is smaller than $1.201t_3$, case 1 normalization is useful. If t_1 is larger than $1.201t_3$, case 2 normalization is useful.

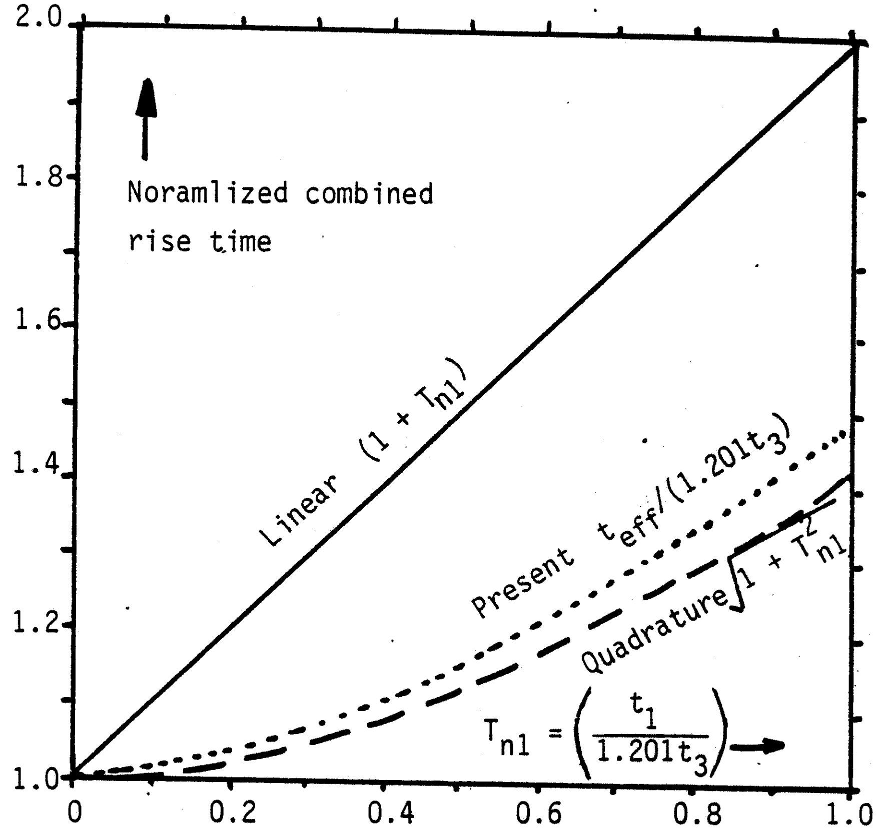
In both cases of normalization, linear and quadrature estimates are also included in both the tables and figures for comparison. As before, it is found that the present estimate via frequency domain falls in between linear and quadrature estimates.

t_3	$\frac{t_1}{t_3}$	$T_{n1} = \frac{t_1}{1.201}$	$\frac{t_{eff}}{t_3}$	$\frac{t_{eff}}{1.201 t_3}$	Linear	Quadrature
					$1 + T_{n1}$	$\sqrt{1 + T_{n2}^2}$
1.000	0.000	0.000	1.201	1.000	1.000	1.000
1.000	0.050	0.042	1.202	1.001	1.042	1.001
1.000	0.100	0.083	1.207	1.005	1.083	1.003
1.000	0.150	0.125	1.214	1.011	1.125	1.008
1.000	0.200	0.167	1.224	1.019	1.167	1.014
1.000	0.250	0.208	1.237	1.030	1.208	1.021
1.000	0.300	0.250	1.252	1.043	1.250	1.031
1.000	0.350	0.291	1.270	1.057	1.291	1.042
1.000	0.400	0.333	1.289	1.073	1.333	1.054
1.000	0.450	0.375	1.311	1.091	1.375	1.068
1.000	0.500	0.416	1.334	1.110	1.416	1.083
1.000	0.550	0.458	1.359	1.131	1.458	1.100
1.000	0.600	0.500	1.385	1.153	1.500	1.118
1.000	0.650	0.541	1.412	1.176	1.541	1.137
1.000	0.700	0.583	1.441	1.200	1.583	1.157
1.000	0.750	0.624	1.471	1.224	1.624	1.179
1.000	0.800	0.666	1.501	1.250	1.666	1.202
1.000	0.850	0.708	1.533	1.277	1.708	1.225
1.000	0.900	0.749	1.566	1.304	1.749	1.250
1.000	0.950	0.791	1.599	1.331	1.791	1.275
1.000	1.000	0.833	1.633	1.359	1.833	1.301
1.000	1.050	0.874	1.668	1.389	1.874	1.328
1.000	1.100	0.916	1.703	1.418	1.916	1.356
1.000	1.150	0.958	1.739	1.448	1.958	1.385
1.000	1.200	0.999	1.775	1.478	1.999	1.414
1.000	1.201	1.000	1.776	1.479	2.000	1.414

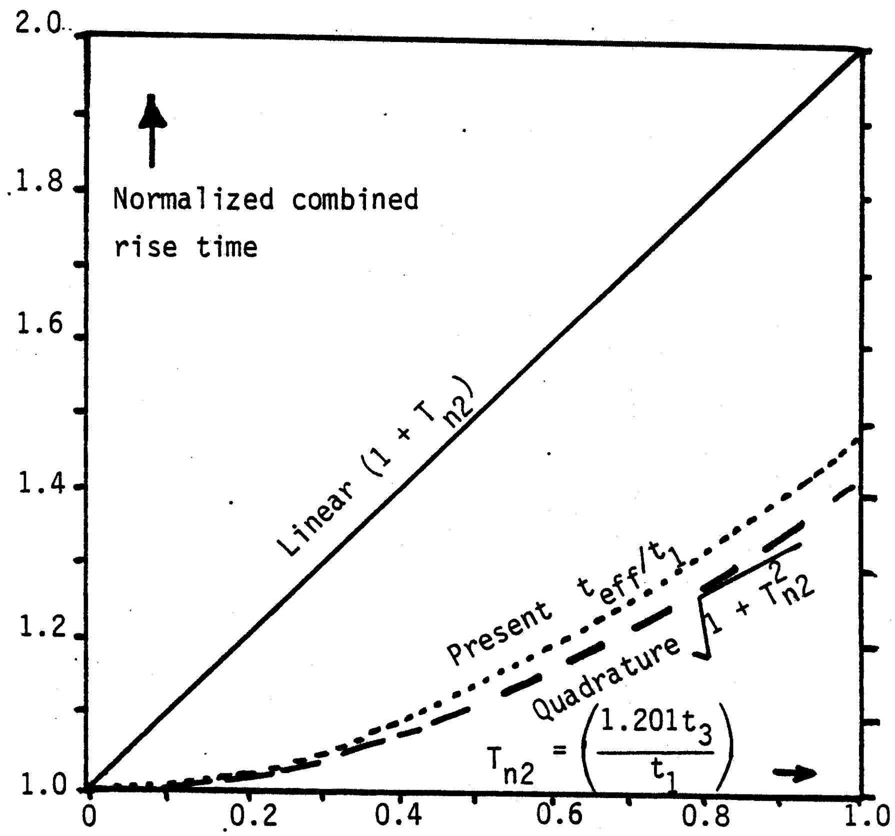
Table 3. Combination of an exponential waveform of rise time t_1 with an integrated Gaussian of standard deviation t_3;
Case 1 Normalization.

t_1	$\frac{t_3}{t_1}$	$T_{n2} = \frac{1.201}{t_1}$	$\frac{t_{eff}}{t_1}$	Linear $1 + T_{n2}$	Quadrature $\sqrt{1 + T_{n2}^2}$
1.000	0.000	0.000	1.000	1.000	1.000
1.000	0.042	0.050	1.001	1.050	1.001
1.000	0.083	0.100	1.006	1.100	1.005
1.000	0.125	0.150	1.015	1.150	1.011
1.000	0.167	0.200	1.026	1.200	1.020
1.000	0.208	0.250	1.041	1.250	1.031
1.000	0.250	0.300	1.058	1.300	1.044
1.000	0.291	0.350	1.078	1.350	1.059
1.000	0.333	0.400	1.100	1.400	1.077
1.000	0.375	0.450	1.124	1.450	1.097
1.000	0.416	0.500	1.150	1.500	1.118
1.000	0.458	0.550	1.178	1.550	1.141
1.000	0.500	0.600	1.207	1.600	1.166
1.000	0.541	0.650	1.238	1.650	1.193
1.000	0.583	0.700	1.269	1.700	1.221
1.000	0.624	0.750	1.302	1.750	1.250
1.000	0.666	0.800	1.336	1.800	1.281
1.000	0.708	0.850	1.370	1.850	1.312
1.000	0.749	0.900	1.406	1.900	1.345
1.000	0.791	0.950	1.442	1.950	1.379
1.000	0.833	1.000	1.478	2.000	1.414

Table 4. Combination of an exponential waveform of rise time t_1 with an integrated Gaussian of standard deviation t_3; Case 2 Normalization.



a) Case 1 normalization of all times w.r.t.$(1.201t_3)$



b) Case 2 normalization of all times w.r.t. t_1

Figure 8. Rise times of the combination of an exponential and an integrated Gaussian waveforms.

V. Summary

The problem of combining waveforms with exponential and Gaussian rise characteristics has been addressed in this note. For example, if one wishes to combine two exponentially rising waveforms, linear addition or quadrature addition of individual rise times (e-folding or maximum slope) can give estimates of the combined waveform's rate of rise. The method used here consists of convolving the waveforms to determine the frequency spectrum of the combined waveform. Then the frequency at which this spectrum is reduced in magnitude by an amount of $(2)^{-1/2}$ from that of an ideal-step-function spectrum is determined. The reciprocal of this frequency then is defined as the effective rate of rise of the combined output waveform.

Two examples have been worked out. One consists of combining two exponentially rising waveforms. The second example is that of combining an exponentially rising waveform with that of an integrated Gaussian distribution. In the case of two exponentials, the present method gives results that fall in between linear and quadrature estimates. This is also true of the combination of an exponential with an integrated Gaussian. For this combination, it has been determined that the effective rise time of an integrated Gaussian is 1.201 times the standard deviation of the Gaussian Waveform. This factor has been incorporated into the normalization procedures while presenting the results.

It is observed that for the case of two exponentials, the resulting rise time is available in closed form, whereas for the second case of exponential and integrated Gaussian waveforms, one needs to numerically solve a transcendental equation involving algebraic and exponential functions.

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