Mathematics Notes

Note 102

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Solved Problems in Integral Equations - Part 1

By

Dr. D. V. Giri
Pro-Tech, 45 Cliff Road, Wellesley, MA 02481 USA
Dept. of ECE, University of New Mexico, Albuquerque, NM

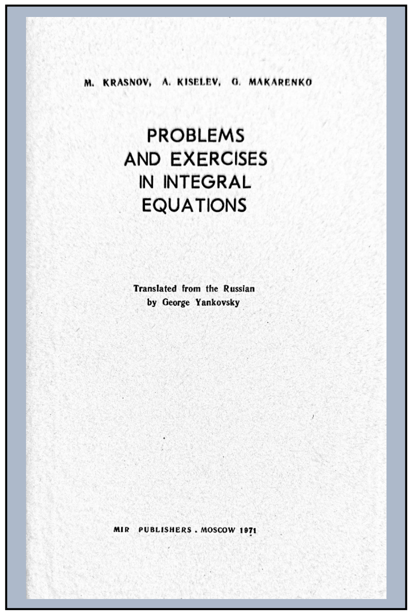
Abstract

An elegant book on integral equations is titled "***Problems and Exercises in Integral Equations by M. Krasnov, A. Kiselev and G. Makarenko.*** This book was originally published by Mir Publishers in Russian language in 1968 and an English language version became available in 1971. This book has over 300 problems with answers. These integral equation problems have analytic solutions. The authors also describe some approximate methods to solve integral equations, such as a) replacing the Kernel function by a simplified Kernel and b) method of successive approximation. Of course, we live in an age of fast computers and commercially available software routines to solve such problems. Finding analytical solutions could become a lost art. They do serve a function of bench-marking numerical solutions in some cases.

Over 80 problems in this book are fully solved and presented in this Note. Hopefully, one of the readers will be inspired to solve the remaining problems.

Introduction

We note that this book cited in the abstract is available free online at the following URL.
https://ia903006.us.archive.org/32/items/ProblemsAndExercisesInIntegralEquationsKrasnovKiselevMakarenko/problems-and-exercises-in-integral-equations-krasnov-kiselev-makarenko.pdf



M. KRASNOV, A. KISELEV, G. MAKARENKO

PROBLEMS AND EXERCISES IN INTEGRAL EQUATIONS

Translated from the Russian by George Yankovsky

MIR PUBLISHERS . MOSCOW 1971

Integral equations come in mainly two forms namely Volterra Integral Equations (VIE) and Fredholm Integral Equations (FIE), as written below.

$$\int_a^x K(x,t)\ \phi(t)\ dt = f(x) \qquad \text{(VIE of the first kind)}$$

$$\phi(x) = f(x) + \lambda \int_a^x K(x,t)\,\phi(t)\ dt \qquad \text{(VIE of the second kind)}$$

where $f(x)$ *and* $K(x,t)$ are known functions and λ is a parameter.
On the other hand,

$$\phi(x) - \lambda \int_a^b K(x,t)\ \phi(t)\ dt = f(x) \qquad \text{(FIE of the second kind)}$$

Note the difference in the limits of integration.
Both integral equations and integro-differential equations occur in many physical problems.

Acknowledgment

I am thankful to my colleague Dr. Eugene Tomer who provided guidance and helped in type-setting hand-written notes.

Problem 1.

Solution $$\phi(x) = \frac{x}{(1 + x^2)^{5/2}}$$

V.I.E. $$\phi(x) = \frac{3x + 2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x + 2x^3 - t}{(1 + x^2)^2} \phi(t)\, dt$$

Solution:

R.h.s. $$= \frac{3x + 2x^3}{3(1 + x^2)^2} - \frac{1}{(1 + x^2)^2} \int_0^x \frac{(3x + 2x^3 - t)\, t}{(1 + t^2)^{5/2}}\, dt$$

$$= \frac{1}{3(1 + x^2)^2} \int_0^x (3x + 2x^3 - t) \left\{\frac{d}{dt} \frac{1}{(1 + t^2)^{3/2}}\right\} dt$$

$$= \frac{3x + 2x^3}{3(1+x^2)^2} + \frac{1}{3(1+x^2)^2}\left[\frac{2x + 2x^3}{(1+x^2)^{3/2}} - (3x + 2x^3)\right] + \frac{1}{3(1+x^2)^2}\int_0^x \frac{1}{(1+t^2)^{3/}}$$

$$= \frac{2x}{3(1 + x^2)^{5/2}} + \frac{1}{3(1 + x^2)^2}\left\{\frac{t}{(1 + t^2)^{1/2}}\right\}_0^x$$

$$= \frac{2x}{3(1 + x^2)^{5/2}} + \frac{x}{3(1 + x^2)^{5/2}}$$

$$= \frac{x}{(1 + x^2)^{5/2}}$$

$$= \phi(x)$$

= L.h.s.

Problem 2.

Solution $\phi(x) = e^x\left[\cos(e^x) - e^x \sin(e^x)\right]$

V.I.E. $\phi(x) = (1 - xe^{2x}) \cos(1) - e^{2x}\sin(1) + \int_0^x \left[1 - (x-t)\, e^{2x}\right]\phi(t)\, dt$

Solution:

$$\text{R.h.s.} = \left\{(1 - xe^{2x}) \cos(1) - e^{2x} \sin(1)\right\} + \int_0^x \left[1 - (x-t)e^{2x}\right] e^t\left[\cos(e^t) - e^t \sin(e^t)\right] dt$$

With a change of variable $e^t = z$; $dt = dz/z$,

$$\text{R.h.s.} = \left\{\text{— " —}\right\} + \int_1^{e^x} \left[1 - (x - \ln z)\, e^{2x}\right] \left[\cos(z) - z \sin(z)\right] dz$$

$$= \left\{\text{— " —}\right\} + (1 - e^{2x}) \int_1^{e^x} \left[\cos(z) - z \sin(z)\right] dz + e^{2x} \int_1^{e^x} \ln(z)\left[\cos(z) - z \sin(z)\right] dz$$

$$= \left\{\text{— " —}\right\} + (1 - xe^{2x})\left\{z \cos(z)\right\}\Big|_1^{e^x} + e^{2x} \left\{z \cos(z) \ln(z) - \sin(z)\right\}_1^{e^x}$$

$$= (1 - xe^{2x}) \cos(1) - e^{2x} \sin(1) + (1 - xe^{2x})\, e^x \cos(e^x) - (1 - xe^{2x}) \cos(1)$$

$$+ e^{2x} \left\{e^x \cos(e^x)\, x - \sin(e^x) + \sin(1)\right\}$$

$$= e^x\left[\cos(e^x) - e^x \sin(e^x)\right]$$

$$= \phi(x)$$

$$\equiv \text{L.h.s.}$$

Problem 3.

Solution $\phi(x) = xe^x$

V.I.E. $\phi(x) = e^x \sin(x) + 2 \int_0^x \cos(x-t)\, \phi(t)\, dt$

Remark: We believe that the factor e^x in the first term of R.h.s. should be absent. ok

Solution:

$$\text{R.h.s.} = \sin(x) + 2 \int_0^x \cos(x-t)\, te^t\, dt$$

$$= \sin(x) + 2\cos(x) \int_0^x t\, e^t \cos(t)\, dt + 2\sin(x) \int_0^x te^t \sin(t)\, dt$$

$$= \sin(x) + 2\cos(x) \left[\frac{e^t}{2} \left\{ t\cos(t) + t\sin(t) - \sin(t) \right\}_0^x \right]$$

$$+ 2\sin(x) \left[\frac{e^t}{2} \left\{ t\sin(t) - t\cos(t) + \cos(t) \right\}_0^x \right]$$

$$= \sin x + e^x \left[x\cos^2(x) + x\sin(x)\cos(x) - \sin(x)\cos(x) \right]$$

$$+ e^x \left[x\sin^2(x) - x\sin(x)\cos(x) + \sin(x)\cos(x) \right]$$

$$- \sin x$$ ok

$$= xe^x \left[\cos^2(x) + \sin^2(x) \right]$$

$$= xe^x$$

$$= \phi(x)$$

$$\equiv \text{L.h.s.}$$

Problem 4.

Solution $\phi(x) = x - x^3/6$

V.I.E. $$\phi(x) = x - \int_0^x \sinh(x-t) \quad (t)\, dt$$

Solution:

$$\text{R.h.s.} = x - \int_0^x \sinh(x-t)(t - \frac{t^3}{5}\, dt$$

$$= x - \sinh(x)\left[\int_0^x (t - \frac{t^3}{6})\cosh(t)\, dt\right]$$

$$+ \cosh(x)\left[\int_0^x (t - \frac{t^3}{6})\sinh(t)\, dt\right]$$

$$= x + \sinh(x)\left[\frac{1}{6}\left\{t^3 \sinh(t) - 3t^2 \cosh(t)\right\}_0^x\right]$$

$$- \cosh(x)\left[\frac{1}{6}\left\{t^3 \cosh(t) - 3t^2 \sinh(t)\right\}_0^x\right]$$

$$= x + \sinh(x)\left[\frac{x^3}{6}\sinh(x) - \frac{x^2}{2}\cosh(x)\right] - \cosh(x)\left[\frac{x^3}{6}\cosh(x) - \frac{x^2}{2}\sinh(x)\right]$$

$$= x - \frac{x^3}{6}\left[\cosh^2(x) - \sinh^2(x)\right]$$

$$= x - \frac{x^3}{6}$$

$$= \phi(x)$$

$$= \text{L.h.s.}$$

Problem 5.

Solution $\phi(x) = (1 - x)$

V.I.E. $$\int_0^x e^{x-t} \phi(t)\, dt = x$$

Solution:

$$\text{L.h.s.} = \int_0^x e^{x-t} (1-t)\, dt$$

$$= e^x \int_0^x e^{-t}\, dt + e^x \int_0^x (-te^{-t})\, dt$$

$$= -e^x(e^{-x} - 1) + e^x \left\{te^{-t} + e^{-t}\right\}_0^x$$

$$= -1 + e^x + e^x\left[xe^{-x} + e^{-x} - 1\right]$$

$$= -1 + e^x + x + 1 - e^{-x}$$

$$= x$$

$$= \text{R.h.s.}$$

Problem 6.

Solution $\phi(x) = 3$

V.I.E. $$\int_0^x (x-t)^2 \phi(t)\, dt = x^3$$

Solution:

$$\text{L.h.s.} = 3 \int_0^x (x-t)^2\, dt$$

$$= - \left\{(x-t)^3\right\}_0^x$$

$$= x^3$$

$$\equiv \text{R.h.s.}$$

Problem 7.

Solution $\phi(x) = 1/2$

V.I.E. $$\int_0^x \frac{\phi(t)}{\sqrt{x-t}} = \sqrt{x}$$

Solution:

$$\text{L.h.s.} = \int_0^x \frac{1}{2\sqrt{x-t}}\, dt$$

$$= -\left\{\sqrt{x-t}\right\}_0^x$$

$$= \sqrt{x}$$

$$\equiv \text{R.h.s}$$

Problem 8.

Solution $\qquad \phi(x) = \dfrac{1}{\pi\sqrt{x}}$

V.I.E. $\qquad \displaystyle\int_0^x \frac{\phi(t)}{\sqrt{x-t}}\, dt = 1$

Solution:

$$\text{L.h.s.} = \int_0^x \frac{1}{\pi\sqrt{xt-t^2}}\, dt$$

$$= \frac{1}{\pi}\left\{\arcsin\left(\frac{2t-x}{x}\right)\right\}_0^x$$

$$= (2/\pi)\arcsin(1)$$

$$= 1$$

$$\equiv \text{R.h.s.}$$

Problem 9.

$y'' + y = 0$

with $y(0) = 0$ and $y'(0) = 1$ (1)

Solution:

Put $y'' = \phi(x)$ (2)

then, $\displaystyle y' = y'(0) + \int_0^x \phi(t)\, dt = 1 + \int_0^x (x-t)\,\phi(t)\, dt$ (3)

$$y = y(0) + y'(0)x + \int_0^x (x-t)\,\phi(t)\, dt$$

$$= x + \int_0^x (x-t)\,\phi(t)\, dt \qquad (4)$$

Substituting equations (2) and (4) in equation (1),

$$\phi(x) = -x + \int_0^x (t - x)\, \phi(t)\, dt \tag{5}$$

Equation (5) is the required V.I.E.

Problem 10.

$$y' - y = 0 \tag{1}$$

with $y(0) = 1$

Solution:

$$\text{Put } y' = \phi(x) \tag{2}$$

$$\text{then, } y = y(0) + \int_0^x \phi(t)\, dt = 1 + \int_0^x \phi(t)\, dt \tag{3}$$

Substituting equations (2) and (3) in equation (1),

$$\phi(x) = 1 + \int_0^x \phi(t)\, dt \tag{4}$$

Equation (4) is the required V.I.E.

Problem 11.

$$y'' + y = \cos(x) \tag{1}$$

with $y(0) = y'(0) = 0$

Solution:

$$\text{Put } y'' = \phi(x) \tag{2}$$

$$\text{then,}\quad y = y(0) + xy'(0) + \int_0^x (x-t)\,\phi(t)\,dt \tag{2}$$

$$= \int_0^x (x-t)\,\phi(t)\,dt \tag{3}$$

Substituting equations (2) and (3) in equation (1),

$$\phi(x) = \cos(x) - \int_0^x (x-t)\,\phi(t)\,dt \tag{4}$$

Equation (4) is the required V.I.E.

Problem 12.

$$y'' - 5y' + 6y = 0 \tag{1}$$

with $y(0) = 0$ and $y'(0) = 1$

Solution:

$$\text{Put}\quad y'' = \phi(x) \tag{2}$$

$$\text{then,}\quad y' = y'(0) + \int_0^x \phi(t)\,dt = 1 + \int_0^x \phi(t)\,dt \tag{3}$$

$$y = y(0) + xy'(0) + \int_0^x (x-t)\,\phi(t)\,dt$$

$$= x + \int_0^x (x-t)\,\phi(t)\,dt \tag{4}$$

Substituting equations (2), (3) and (4) in equaiton (1),

$$\phi(x) - 5 + 5\int_0^x \phi(t)\,dt + 6x + 6\int_0^x (x-t)\,\phi(t)\,dt = 0 \tag{5}$$

or,

$$\phi(x) = (5 - 6x) + \int_0^x [5 - 6(x-t)]\, \phi(t)\, dt \qquad (6)$$

Equation (6) is the required V.I.E.

Problem 13.

$y'' + y = \cos(x)$

with $y(0) = 0$ and $y'(0) = 1$ (1)

Solution:

Put $y'' = \phi(x)$ (2)

$$\text{then } y = y(0) + xy'(0) + \int_0^x (x-t)\, \phi(t)\, dt$$

$$= x + \int_0^x (x-t)\, \phi(t)\, dt \qquad (3)$$

Substituting equations (2) and (3) in equation (1),

$$\phi(x) = \cos(x) - x - \int_0^x (x-t)\, \phi(t)\, dt \qquad (4)$$

Equation (4) is the required V.I.E.

Problem 14.

$y'' - y' \sin(x) + e^x y = x$ (1)

with $y(0) = 1$ and $y'(0) = -1$

Solution:

Put $y'' = \phi(x)$ (2)

then, $y' = y'(0) + \int_0^x \phi(t)\,dt = -1 + \int_0^x \phi(t)\,dt$ (3)

$$y = y(0) + xy'(0) + \int_0^x (x-t)\,\phi(t)\,dt$$

$$= 1 - x + \int_0^x (x-t)\,\phi(t)\,dt \qquad (4)$$

Substituting equations (2), (3) and (4) in equation 1

$$\phi(x) = x - \sin(x) + \sin(x)\int_0^x \phi(t)\,dt - e^x(1-x) - e^x\int_0^x (x-t)\,\phi(t)\,dt$$

or,

$$\phi(x) = x - \sin(x) + e^x(x-1) + \int_0^x [\sin(x) - e^x(x-t)]\,\phi(t)\,dt \qquad (5)$$

Equation 5 is the required V.I.E.

Problem 15.

$y'' + (1 + x^2)y = \cos(x)$ (1)

with $y(0) = 0$, $y'(0) = 2$

Solution:

Put $y'' = \phi(x)$ (2)

then $y = y(0) + xy'(0) + \int_0^x (x-t)\,\phi(t)\,dt$

$$= 2x + \int_0^x (x-t)\,\phi(t)\,dt \qquad (3)$$

Substituting equaitons (2) and (3) in equation (1),

$$\phi(x) + (1+x^2)\,2x + (1+x^2)\int_0^x (x-t)\,\phi(t)\,dt = \cos(x)$$

or,

$$\phi(x) = \cos(x) - 2x(1+x^2) - \int_0^x (1+x^2)(x-t)\,\phi(t)\,dt \tag{4}$$

Equation (4) is the required V.I.E.

Problem 16.

$$y''' + xy'' + (x^2 - x)y = xe^x + 1 \tag{1}$$

with $y(0) = y'(0) = 1$ and $y''(0) = 0$

Solution:

Put $y''' = \phi(x)$ (2)

$$\text{then, } y'' = y''(0) + \int_0^x \phi(t)\,dt = \int_0^x \phi(t)\,dt \tag{3}$$

$$y' = y'(0) + xy''(0) + \int_0^x (x-t)\,\phi(t)\,dt$$

$$= 1 + \int_0^x (x-t)\,\phi(t)\,dt \tag{4}$$

$$y = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{1}{2}\int_0^x (x-t)^2\,\phi(t)\,dt$$

$$= 1 + x + \frac{1}{2}\int_0^x (x-t)^2\,\phi(t)\,dt \tag{5}$$

Substituting equations (2), (3), (4) and (5) in equation (1),

$$\phi(x) + x \int_0^x \phi(t)\, dt + (x^2 - x)((x+1) + \frac{(x^2 - x)}{2} \int_0^x (x-t)^2 \phi(t)\, dt = xe^x + 1$$

or

$$\phi(x) = xe^x + 1 - x(x^2 - 1) - \int_0^x \left[x + \tfrac{1}{2}(x^2 - x)(x-t)^2\right] \phi(t)\, dt \qquad (6)$$

Equation (6) is the required V.I.E.

Problem 17.

$$y''' - 2xy = 0 \qquad (1)$$

with $y(0) = \frac{1}{2}$, $y'(0) = y''(0) = 1$

Put $y''' = \phi(x)$ (2)

$$\text{then } y = y(0) + xy'(0) + \frac{x^2}{2} y''(0) + \frac{1}{2} \int_0^x (x-t)^2 \phi(t)\, dt$$

$$= \frac{1}{2} + x + \frac{x^2}{2} + \frac{1}{2} \int_0^x (x-t)^2 \phi(t)\, dt \qquad (3)$$

Substituting equaitons (2) and (3) in equation (1),

$$\phi(x) - \left[2x \ \frac{1}{2} + x + \frac{x^2}{2} + \frac{1}{2} \int_0^x (x-t)^2 \phi(t)\, dt\right] = 0$$

or

$$\phi(x) = x(x+1)^2 + \int_0^x x(x-t)^2 \phi(t)\, dt \qquad (4)$$

Equation (4) is the required V.I.E.

Problem 18.

Consider a linear differential equation with constant coefficients, given by

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}\frac{dy}{dx} + a_n\, y = F(x) \tag{1}$$

with initial conditions

$$y(0) = C_0, \quad y'(0) = C_1 \;, \ldots, \quad y^{(n-1)}(0) = C_{n-1} \tag{2}$$

Put $$\frac{d^n y}{dx^n} = \phi(x) \tag{3}$$

then $$\frac{d^{n-1}y}{dx^{n-1}} = C_{n-1} + \int_0^x (x-t)\ \phi(t)\ dt \tag{4}$$

$$\frac{d^{n-2}y}{dx^{n-2}} = C_{n-2} + xC_{n-1} + \frac{1}{2!}\int_0^x (x-t)^2\ \phi(t)\ dt \tag{5}$$

$$\frac{d^{n-3}y}{dx^{n-3}} = C_{n-3} + xC_{n-2} + \frac{x^2}{2!}C_{n-1} + \frac{1}{3!}\int_0^x (x-t)^3\ \phi(t)\ dt \tag{6}$$

$$\vdots$$

$$\frac{dy}{dx} = \sum_{i=1}^{n-1} C_i \frac{x^i}{i!} + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\ \phi(t)\ dt$$

$$y = \sum_{i=0}^{n-1} C_i \frac{x^i}{i!} + \frac{1}{n!}\int_0^x (x-t)^n\ \phi(t)\ dt$$

Substituting equations (3), (4) ... in equations (1),

$$\phi(x) + a_1 C_{n-1} + \int_0^x a_1(x-t)\,\phi(t)\,dt$$

$$+ a_2 C_{n-2} + a_2 x C_{n-1} + \frac{1}{2!}\int_0^x a_2(x-t)^2\,\phi(t)\,dt$$

$$+ a_3 C_{n-3} + a_3 x C_{n-2} + \frac{1}{2!} a_3 x^2 C_{n-1} + \frac{1}{3!}\int_0^x a_3(x-t)^3\,\phi(t)\,dt$$

$$+ \cdots$$

$$+ \sum_{i=1}^{n-1} a_{n-1} C_i \frac{x^i}{i!} + \frac{1}{(n-1)!}\int_0^x a_{n-1}(x-t)^{n-1}\,\phi(t)\,dt$$

$$+ \sum_{i=0}^{n-1} a_n C_i \frac{x^i}{i!} + \frac{1}{n!}\int_0^x (x-t)^n\,\phi(t)\,dt \equiv F(x)$$

or

$$\phi(x) = \left[F(x) - \sum_{i=1}^{n}\sum_{k=0}^{i-1} a_i C_{n-i+k}\frac{x^k}{k!} - \sum_{i=1}^{n}\int_0^x \frac{a_i}{i!}(x-t)^i\,\phi(t)\,dt\right]$$

This is a Volterra integral equation of the second kind where the kernels are dependent solely on the difference $(x-t)$ of arguments.

Problem 19.

$K(x, t) = (x-t)$

Solution

$K(x, t) = (x-t)$

The iterated kernels are given by,

$$K_1(x, t) = K(x, t) = (x-t)$$

$$K_2(x, t) = \int_t^x (x-z)(z-t)\, dz = \int_t^x \frac{(z-t)^2}{2!}\, dz = \frac{(x-t)^3}{3!}$$

$$K_3(x, t) = \int_t^x (x-z)\frac{(z-t)^3}{3!}\, dz = \int_t^x \frac{(z-t)^4}{4!}\, dz = \frac{(x-t)^5}{5!}$$

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$$K_n(x, t) = \int_t^x (x-2)\, K_{n-1}(z, t)\, dz = \int_t^x \frac{(x-z)(z-t)^{2n-3}}{(2n-3)!}$$

$$= \int_t^x \frac{(z-t)^{2n-2}}{(2n-2)!}\, dz = \frac{(x-t)^{(2n-1)}}{(2n-1-!}$$

Thus, by the definition of the resolvent kernel,

$$T(x, t; \lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, t) = \sum_{n=0}^{\infty} \lambda^n \frac{(x-t)^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{\sqrt{\lambda}} \sum_{n=0}^{\infty} \frac{[\sqrt{\lambda}\,(x-t)]^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{\sqrt{\lambda}} \sinh\,[\sqrt{\lambda}(x-t)] \quad \text{when } \lambda > 0.$$

Problem 20.

$K(x, t) = e^{(x-t)}$

Solution:

$K(x, t) = e^{(x-t)}$

The iterated kernels are given by,

$$K_1(x, t) = K(x, t) = e^{(x-t)}$$

$$K_2(x, t) = \int_t^x e^{(x-z)} e^{(z-t)} dz = e^{(x-t)} \int_t^x dz = (x-t) e^{(x-t)}$$

$$K_3(x, t) = \int_t^x e^{(x-z)}(z-t) e^{(z-t)} dz = e^{(x-t)} \int_t^x (z-t) dz = \frac{(x-t)^2}{2!} e^{(x-t)}$$

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$$K_n(x, t) = \int_t^x e^{(x-z)} \frac{(z-t)^{n-2}}{(n-2)!} e^{(z-t)} dz = \frac{e^{(x-t)}}{(n-2)!} \int_t^x (z-t)^{n-2} dz$$

$$= \frac{(x-t)^{n-1}}{(n-1)!} e^{(x-t)}$$

Thus, by the definition of the resolvent kernel,

$$R(x, t; \quad) = \sum_{n=0}^{\infty} {}^{n} K_{n+1}(x, t) = \sum_{n=0}^{\infty} \lambda^n \frac{(x-t)^n}{n!} e^{(x-t)}$$

$$= e^{(x-t)} e^{\lambda(x-t)}$$

$$= e^{(1+\lambda)(x-t)}.$$

Problem 21.

$K(x, t) = e^{(x^2-t^2)}$

Solution:

$K(x, t) = e^{(x^2-t^2)}$

The iterated kernels are given by,

$$K_1(x, t) = K(x, t) = e^{(x^2-t^2)}$$

$$K_2(x, t) = \int_t^x e^{(x^2-z^2)} e^{(z^2-t^2)} dz = e^{(x^2-t^2)} \int_t^x dz = (x-t) e^{(x^2-t^2)}$$

$$K_3(x, t) = \int_t^x e^{(x^2-z^2)} (z-t) e^{(z^2-t^2)} dz = e^{(x^2-t^2)} \int_t^x (z-t) dz = \frac{(x-t)^2}{2!} e^{(x^2-t^2)}$$

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$$K_n(x, t) = \int_t^x e^{(x^2-z^2)} \frac{(z-t)^{n-2}}{(n-2)!} e^{(x^2-t^2)} dz = \frac{e^{(x^2-t^2)}}{(n-2)!} \int_t^x (z-t)^{n-2} dz = \frac{(x-t)^{n-1}}{(n-1)!} e^{(x^2-t^2)}$$

Thus, by the definition of the resolvent kernel,

$$R(x, t; \lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1} (x, t) = \sum_{n=0}^{\infty} \lambda^n \frac{(x-t)^n}{n!} e^{(x^2-t^2)}$$

$$= e^{\lambda(x-t)} e^{(x^2-t^2)}.$$

Problem 22.

$K(x, t) = (1+x^2)/(1+t^2)$

Solution:

$K(x, t) = (1+x^2)/(1+t^2)$

The iterated kernels are given by

$$K_1(x, t) = K(x, t) = \frac{1+x^2}{1+t^2}$$

$$K_2(x, t) = \int_t^x \frac{1+x^2}{1+z^2}\,\frac{1+z^2}{1+t^2}\,dz = \frac{1+x^2}{1+t^2}\int_t^x dz = (x-t)\,\frac{(1+x^2)}{(1+t^2)}$$

$$K_3(x, t) = \int_t^x \frac{1+x^2}{1+z^2}\,(z-t)\,\frac{(1+z^2)}{(1+t^2)} = \frac{1+z^2}{1+t^2}\int_t^x (z-t)\,dz = \frac{(x-t)^2}{2!}\,\frac{(1+x^2)}{(1+t^2)}$$

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$$K_n(x, t) = \int_t^x \frac{1+x^2}{1+z^2}\,\frac{(z-t)^{n-2}}{n-2!}\,\frac{(1+z^2)}{(1+t^2)}\,dz = \frac{1+x^2}{1+t^2}\int_t^x \frac{(z-t)^{n-2}}{(n-2)!}\,dz =$$

$$= \frac{(x-t)^{n-1}}{(n-1)!}\,\frac{(1+x^2)}{(1+t^2)}$$

Thus, by the definition of the resolvent kernel,

$$R(x, t; \lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, t) = \sum_{n=0}^{\infty} \lambda^n \frac{(x-t)^n}{n!}\,\frac{1+x^2}{1+t^2}$$

$$= \left(\frac{1+x^2}{1+t^2}\right) e^{\lambda(x-t)}$$

Problem 23.

$K9x, t) = \frac{2+\cos(x)}{2+\cos(t)}$

Solution:

$$K(x, t) = \frac{2+\cos(x)}{2+\cos(t)}$$

The iterated kernels are given by

$$K_1(x, t) = K(x, t) = \frac{2+\cos(x)}{2+\cos(t)}$$

$$K_2(x, t) = \int_t^x \frac{2+\cos(x)}{2+\cos(z)} \frac{2+\cos(z)}{2+\cos(t)} dz = (x-t) \frac{2+\cos(x)}{2+\cos(t)}$$

$$K_3(x, t) = \int_t^x \frac{2+\cos(x)}{2+\cos(z)} (z-t) \frac{2+\cos(z)}{2+\cos(t)} dz = \frac{(x-t)^2}{2!} \frac{2+\cos(x)}{2+\cos(t)}$$

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$$K_n(x, t) = \int_t^x \frac{2+\cos(x)}{2+\cos(z)} \frac{(z-t)^{n-2}}{(n-2)!} \frac{2+\cos(z)}{2+\cos(t)} dz = \frac{(x-t)^{n-1}}{(n-1)!} \frac{2+\cos(x)}{2+\cos(t)}$$

Thus, by the definition of the resolvent kernel,

$$R(x, t; \lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, t) = \sum_{n=0}^{\infty} \lambda^n \frac{(x-t)^n}{n!} \frac{2+\cos(x)}{2+\cos(t)}$$

$$= \frac{2+\cos(x)}{2+\cos(t)} e^{\lambda(x-t)}$$

Problem 24.

$$K(x, t) = \frac{\cosh(x)}{\cosh(t)}$$

Solution:

$$K(x, t) = \frac{\cosh(x)}{\cosh(t)}$$

The iterated kernels are given by,

$$K_1(x, t) = K(x, t) = \frac{\cosh(x)}{\cosh(t)}$$

$$K_2(x, t) = \int_t^x \frac{\cosh(x)}{\cosh(z)} \frac{\cosh(z)}{\cosh(t)} dz = \frac{\cosh(x)}{\cosh(t)} (x-t)$$

$$K_3(x, t) = \int_t^x \frac{\cosh(x)}{\cosh(z)} (z-t) \frac{\cosh(z)}{\cosh(t)} dz = \frac{\cosh(x)}{\cosh(t)} \frac{(x-t)^2}{2!}$$

$\cdot$
$\cdot$
$\cdot$
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$$K_n(x, t) = \int_t^x \frac{\cosh(x)}{\cosh(z)} \frac{(z-t)^{n-2}}{(n-2)!} \frac{\cosh(z)}{\cosh(t)} dz = \frac{\cosh(x)}{\cosh(t)} \frac{(z-t)^{n-1}}{(n-1)!}$$

Thus, by the definition of the resolvent kernel,

$$R(x, t; \lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t) = \sum_{n=0}^{\infty} \lambda^n \frac{(z-t)^n}{n!} \frac{\cosh(x)}{\cosh(t)}$$

$$= \frac{\cosh(x)}{\cosh(t)} e^{\lambda(x-t)}$$

Problem 25.

$K(x, t) = a^{(x,t)} \qquad (a > 0)$

Solution:

$K(x, t) = a^{(x-t)}$

The iterated kernels are given by,

$$K_1(x, t) = K(x, t) = a^{(x-t)}$$

$$K_2(x, t) = \int_t^x a^{(x-z)}\, a^{(z-t)}\, dz = (x-t)\, a^{(x-t)}$$

$$K_3(x, t) = \int_t^x a^{(x-z)}\, (z-t)\, a^{(z-t)}\, dz = \frac{(x-t)^2}{2!}\, a^{(x-t)}$$

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$$K_n(x, t) = \int_t^x a^{(x-z)}\, \frac{(x-t)^{n-2}}{(n-2)!}\, a^{(z-t)}\, dz = \frac{(x-t)^{n-1}}{(n-1)!}\, a^{(x-t)}$$

Thus, by the definition of the resolvent kernel,

$$R(x, t; \lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t) = \sum_{n=0}^{\infty} \lambda^n \frac{(x-t)^n}{n!}\, a^{(x-t)}$$

$$= a^{(x-t)}\, e^{\lambda(x-t)}.$$

NOTE 1.

We now describe the necessary background material for solving the next four (i.e., 26 to 29) problems.

Suppose that the kernel $K(x, t)$ is a polynomial of degree $(n - 1)$ in t, so that it may be represented in the form

$$K(x, t) = a_0(x) + a_1(x)(x-t) + \dots + \frac{a_{n-1}(x)}{(n-1)!}(x-t)^{n-1} \tag{1}$$

and the coefficients $a_x(x)$ are continuous in $0,a$. If the function $g(x, t; \lambda)$ is defined as a solution ov the differential equation

$$\frac{d^n g}{dx^n} - \lambda\left[a_0(x)\frac{d^{n-1}}{dx^{n-1}}g + a_1(x)\frac{d^{n-2}}{dx^{n-2}}g \_ \dots + a_{n-1}(x)\, g\right] = 0 \tag{2}$$

satisfying the conditions,

$$g\Big|_{x=t} = \frac{dg}{dx}\Big|_{x=t} = \dots = \frac{d^{n-2}}{dx^{n-2}}g = 0; \quad \frac{d^{n-1}}{dx^{n-1}}g\Big|_{x=t} = 1 \tag{3}$$

and the resolvent kernel $R(x, t; \lambda)$ will be defined by the equality

$$R(x, t; \lambda) = \frac{1}{\lambda}\,\frac{d^n g(x, t; \lambda)}{dx^n} \tag{4}$$

and similarly when

$$K(x, t) = b_0(t) + b_1(t)(t-x) + \dots + \frac{b_{n-1}(t)}{(n-1)!}(t-x)^{n-1} \tag{5}$$

and resolvent kernel

$$F(x, t; \lambda) = -\frac{1}{\lambda}\,\frac{d^n g(t, x; \lambda)}{dt^n} \tag{6}$$

where $g(x, t; \lambda)$ is a solution of equation

$$\frac{d^n g}{dt^n} + \lambda\left[b_0(t)\frac{d^{n-1}}{dt^{n-1}}g + \dots + b_{n-1}(t)\, g\right] = 0 \tag{7}$$

satisfying the conditions,

$$g\Big|_{t=x} = \frac{dg}{dt}\Big|_{t=x} = \ldots = \frac{d^{n-2}}{dx^{n-2}}\, g\Big|_{t=x} = 0\,; \quad \frac{d^{n-1}}{dx^{n-1}}\, g\Big|_{x=t} = 1 \qquad (8)$$

Problem 26.

$K(x, t) = 2 - (x-t)$; $\lambda = 1$

Solution:

With reference to the equations in Note 1, we have

$$\lambda = 1, \qquad a_0(x) = 2$$
$$n = 2, \qquad a_1(x) = -1 \tag{1}$$

$$\frac{d^2g}{dx^2} - 2\frac{dg}{dx} + g = 0 \tag{2}$$

Solving the differential equation (2),

$$g(x, t; 1) = A(t)\, x\, e^x + B(t)\, e^x \tag{3}$$

$g\big|_{x=t} = 0$ and $\frac{dg}{dx}\Big|_{x=t} = 1$, give

$$A(t)\, te^t + B(t)\, e^t = 0 \tag{4}$$

$$A(t)\, te^t + A(t)\, e^t + B(t)\, e^t = 1 \tag{5}$$

Solving the coupled equations,

$$A(t) = e^{-t}, \quad B(t) = -te^{-t} \tag{6}$$

Therefore,

$$g(x, t; 1) = e^{(x-t)}(x-t) \tag{7}$$

and, consequently the resolvent kernel is given by

$$R(x, t; 1) = \frac{d^2g}{dx^2} = \frac{d}{dx}\left[e^{(x-t)}(x-t+1)\right]$$
$$= e^{(x-t)}\,(x-t+2).$$

Problem 27.

$K(x, t) = -2 + 3(x-t) \; ; \quad \lambda = 1$

Solution:

With reference to the equations in Note 1, we have

$$\lambda = 1\,, \qquad a_0(x) = -2 \qquad\qquad (1)$$
$$n = 2\,, \qquad a_1(x) = 3$$

$$\frac{d^2g}{dx^2} + 2\frac{dg}{dx} - 3g = 0 \qquad\qquad (2)$$

Solving the differential equation,

$$g(x, t; 1) = A(t)\, e^{x} + B(t)\, e^{-3x} \qquad\qquad (3)$$

$g\big|_{x=t} = 0$ and $\frac{dg}{dx}\Big|_{x=t} = 1$ gives

$$A(t)\, e^{t} + B(t)\, e^{-3t} = 0 \qquad\qquad (4)$$

$$A(t)\, e^{t} - 3B(t)\, e^{-3t} = 1 \qquad\qquad (5)$$

Solving the coupled equations,

$$A(t) = e^{t}/4, \quad B(t) = -e^{3t}/4 \qquad\qquad (6)$$

Therefore,

$$g(x, t; 1) = \left(\frac{e^{(x-t)} - e^{-3(x-t)}}{4}\right)$$

and, consequently the resolvent kernel is given by,

$$R(x, t; 1) = \frac{d^2g}{dx^2} = \left[\frac{1}{4}\, e^{(x-t)} - \frac{9}{4}\, e^{-3(x-t)}\right] .$$

Problem 28.

$K(x, t) = 2x$, $= 1$

Solution:

With reference to the equations in Note 1, we have

$$= 1, \qquad a_0(x) = 2x \tag{1}$$

$$\frac{dg}{dx} - 2x\,g = 0 \tag{2}$$

Solving the differential equation,

$$g(x, t; 1) = A(t)\, e^{x^2} \tag{3}$$

$$g(t, t; 1) = 1 \text{ gives, } A(t) = e^{-t^2}$$

Therefore,

$$g(x, t; 1) = e^{(x^2-t^2)}$$

and, consequently the resolvent kernel is given by,

$$R(x, t; 1) = \frac{dg}{dx}$$

$$= 2x\, e^{(x^2-t^2)} \;.$$

NOTE 2.

Applicable to problems 31 to 35.

In this note, a method of finding the resolvent kernel, when the given kernel is a function of $(x-t)$, will be described. This method entails the use of Laplace transformation.

Consider the V.I.E.,

$$\phi(x) = f(x) + \int_0^x K(x-t)\, \phi(t)\, dt \tag{1}$$

Define a Laplace transform pair, according as

$$\tilde{g}(s) = \int_0^\infty g(x)\, e^{-sx}\, dx \; ; \quad g(x) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \tilde{g}(s)\, e^{sx}\, ds \tag{2}$$

Now, taking the Laplace transform of equation (1) and using the product theorem (transform of a convolution), we have

$$\tilde{\phi}(s) = \tilde{f}(s) + \tilde{K}(x)\, \tilde{\phi}(s) \tag{3}$$

or,

$$\tilde{\phi}(s) = \frac{\tilde{f}(s)}{1 - \tilde{K}(s)} \tag{4}$$

Taking advantage of the results of problem 30, the solution of the given V.I.E., is

$$(x) = f(x) + \int_0^x R(x-t)\, f(t)\, dt \tag{5}$$

Once again, taking Laplace transform of both sides of equation (5), we get

$$\tilde{\phi}(s) = \tilde{f}(s) + \tilde{R}(s)\, \tilde{f}(s) \tag{6}$$

Therefore

$$\tilde{R}(s) = \frac{\tilde{\phi}(s) - \tilde{f}(s)}{\tilde{f}(s)} \tag{7}$$

Substituting for (s) from equation (4) into the above,

$$\tilde{R}(s) = \frac{\tilde{K}(s)}{1 - \tilde{K}(s)} \qquad (8)$$

and consequently the inverse transformation gives the resolvent kernel as,

$$R(x, t; 1) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \tilde{R}(s)\, e^{s(x-t)}\, es \qquad (9)$$

Problem 31.

$$K(x, t) = \sinh(x-t) \tag{1}$$

Solution:

For the given kernel $K(x, t)$, we have

$$\tilde{K}(s) = \frac{1}{s^2 - 1} \tag{2}$$

Therefore, using equation (8) of Note 2,

$$\tilde{R}(s) = \frac{\tilde{K}(s)}{1 - \tilde{K}(s)} = \frac{1}{s^2 - 2}$$

Consequently, the resolvent kernel is given by,

$$R(x, t; 1) = L^{-1}\left(\frac{1}{s^2 - 2}\right) = \frac{1}{\sqrt{2}} \; L^{-1}\left[\frac{\sqrt{2}}{s^2 - (\sqrt{2})^2}\right]$$

$$= \frac{1}{\sqrt{2}} \sinh\left[\sqrt{2}(x-t)\right] .$$

Problem 32.

$$K(x, t) = e^{-(x-t)} \tag{1}$$

Solution:

For the given kernel $K(x, t)$, we have

$$\tilde{K}(s) = \frac{1}{s+1} \tag{2}$$

Therefore, using equation (8) of Note 2,

$$\tilde{R}(s) = \frac{\tilde{K}(s)}{1 - \tilde{K}(s)} = \frac{1}{s} \tag{3}$$

Consequently, the resolvent kernel is given by,

$$R(x, t; 1) = L^{-1}\left(\frac{1}{s}\right) = 1.$$

Problem 33.

$$K(x, t) = e^{-(x-t)} \sin(x-t) \tag{1}$$

Solution:

For the given kernel $K(x, t)$, we have

$$\tilde{K}(s) = \frac{1}{(s+1)^2 + 1} \tag{2}$$

Therefore, using equation (8) of Note 2,

$$\tilde{R}(s) = \frac{\tilde{K}(s)}{1 - \tilde{K}(s)} = \frac{1}{(s+1)^2}$$

Consequently, the resolvent kernel is given by,

$$R(x, t; 1) = L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-(x-t)} L^{-1}\left(\frac{1}{s^2}\right)$$

$$= (x-t)\, e^{-(x-t)}$$

Problem 34.

$$K(x, t) = \cosh(x-t) \tag{1}$$

Solution:

For the given kernel $K(x, t)$, we have

$$\tilde{K}(s) = \frac{s}{s^2 - 1} \tag{2}$$

Therefore, using equation (8) of Note 2,

$$\tilde{R}(s) = \frac{\tilde{K}(s)}{1 - \tilde{K}(s)} = \frac{s}{s^2 - s - 1} \tag{3}$$

Consequently, the resolvent kernel is given by,

$$R(x, t; 1) = L^{-1}\left(\frac{s}{s^2 - s - 1}\right) = L^{-1}\left[\frac{s}{(s - 1/2)^2 - (5/4)}\right]$$

$$= L^{-1}\left[\frac{(s - 1/2)}{(s - 1/2)^2 - (5/4)} + \frac{(1/2)}{(s - 1/2)^2 - (5/4)}\right]$$

$$= e^{(x-t)/2}\, L^{-1}\left[\frac{s}{s^2 - (5/4)} + \frac{1}{\sqrt{5}}\,\frac{\sqrt{5}/2}{s^2 - (5/4)}\right]$$

$$= e^{(x-t)/2}\left[\cosh\left\{\frac{\sqrt{5}\,(x-t)}{2}\right\} + \frac{1}{\sqrt{5}}\,\sin\left\{\frac{\sqrt{5}}{2}\,(x-t)\right\}\right].$$

Problem 35.

$$K(x, t) = 2\cos(x-t) \tag{1}$$

Solution:

For the given kernel $K(x, t)$, we have

$$\tilde{K}(s) = \frac{2s}{(s^2+1)} \tag{2}$$

Therefore, using equation (8) of Note 2,

$$\tilde{R}(s) = \frac{\tilde{K}(s)}{1-\tilde{K}(s)} = \frac{2s}{(s-1)^2} \tag{3}$$

Consequently, the resolvent kernel is given by,

$$\begin{aligned} R(x, t; 1) &= L^{-1}\left[\frac{2s}{(s-1)^2}\right] \\ &= 2\,L^{-1}\left[\frac{1}{(s-1)} + \frac{1}{(s-1)^2}\right] \\ &= 2\,e^{(x-t)} + 2\,L^{-1}\left[\frac{1}{(s-1)^2}\right] \\ &= 2\,e^{(x-t)} + 2\,e^{(x-t)}\ {}^{-1}(1/s^2) \\ &= 2\,e^{(x-t)}\left[1 + (x-t)\right]. \end{aligned}$$

NOTE 3.

Applicable to problems 36 to 45.

Consider the V.I.E.,

$$\phi(x) = f(x) + \lambda \int_0^x K(x, t)\, \phi(t)\, dt \qquad (1)$$

After we determine, the resolvent kernel $R(x, t; \lambda)$, it can be shown that the solution of the above V.I.E. is given by

$$\phi(x) = f(x) + \lambda \int_0^x R(x, t; \lambda)\, f(t)\, dt \qquad (2)$$

Problem 36.

$$\phi(x) = e^x + \int_0^x e^{(x-t)} \phi(t)\, dt \tag{1}$$

Solution:

The required resolvent kernel $R(x, t; \lambda)$ may be written from the results of problem 20, as

$$R(x, t; 1) = e^{2(x-t)} \tag{2}$$

Using equation (2) of Note 3, the solution of the given V.I.E. can now be written as,

$$\begin{aligned}\phi(x) &= e^x + \int_0^x e^{2(x-t)} e^t\, dt \\ &= e^x + e^{2x} \int_0^x e^{-t}\, dt \\ &= e^x - e^{2x} (e^{-x} - 1) \\ &= e^{2x} .\end{aligned}$$

Problem 37.

$$\phi(x) = \sin(x) + 2\int_0^x e^{(x-t)}\,\phi(t)\,dt \qquad (1)$$

Solution:

The required resolvent kernel $R(x, t; 2)$ may be written from the results of problem 20, as

$$R(x, t; 2) = e^{3(x-t)} \qquad (2)$$

Using equation (2) of Note 3, the solution of the given V.I.E. can now be written as,

$$\phi(x) = \sin(x) + 2\int_0^x e^{3(x-t)}\sin(t)\,dt$$

$$= \sin(x) + 2\,e^{3x}\int_0^x e^{-3t}\sin(t)\,dt$$

$$= \sin(x) - \frac{2}{10}\,e^{3x}\left[e^{-3t}\left\{3\sin(t) + \cos(t)\right\}\right]_0^x$$

$$= \sin(x) - \frac{1}{5}\,e^{3x}\left[e^{-3x}\left\{3\sin(x) + \cos(x)\right\} - 1\right]$$

$$= \frac{1}{5}\,e^{3x} - \frac{1}{5}\cos(x) + \frac{2}{5}\sin(x).$$

Problem 38.

$$\phi(x) = x\,3^x - \int_0^x 3^{x-t}\,\phi(t)\,dt \tag{1}$$

Solution:

The required resolvent kernel $R(x, t; -1)$ may be written from the results of problem 25 as,

$$R(x, t; -1) = 3^{x-t}\,e^{-(x-t)} \tag{2}$$

Using equation (2) of Note 3, the solution of the given V.I.E. can now be written as

$$\phi(x) = x\,3^x - \int_0^x 3^{(x-t)}\,e^{-(x-t)}\,t\,3^t\,dt$$

$$= x\,3^x - 3^x e^{-x} \int_0^x t\,e^t\,dt$$

$$= x\,3^x - 3^x e^{-x}\,[e^x\,(x-1) + 1]$$

$$= 3^x\,(x - x + 1 - e^{-x})$$

$$= 3^x\,(1 - e^{-x}).$$

Problem 39.

$$\phi(x) = e^x \sin(x) + \int_0^x \frac{2 + \cos(x)}{2 + \cos(t)} \phi(t)\, dt \tag{1}$$

Solution:

The required resolvent kernel $R(x, t; 1)$ may be written from the results of problem 23, as

$$R(x, t; 1) = \frac{2 + \cos(x)}{2 + \cos(t)} e^{(x-t)} \tag{2}$$

Using equation (2) of Note 3 the solution of the given V.I.E. can now be written as,

$$\phi(x) = e^x \sin(x) + \int_0^x \frac{2 + \cos(x)}{2 + \cos(t)} e^{(x-t)} e^t \sin(t)\, dt$$

$$= e^x \sin(x) + e^x (2 + \cos x) \int_0^x \frac{\sin(t)}{2 + \cos(t)}\, dt$$

$$= e^x \sin(x) - (2 + \cos x)\, e^x \left[\ln(2 + \cos t) \Big|_0^x \right]$$

$$= e^x \sin(x) + (2 + \cos x)\, e^x \ln\left[\frac{3}{2 + \cos(x)}\right].$$

Problem 40.

$$\phi(x) = 1 - 2x - \int_0^x e^{x^2 - t^2} \phi(t)\, dt \tag{1}$$

Solution:

The required resolvent kernel $R(x, t, -1)$ may be written from the results of problem 21, as

$$R(x, t; -1) = e^{-(x-t)} e^{(x^2 - t^2)} \tag{2}$$

Using equation (2) of Note 3 the solution of the given V.I.E. can now be written as,

$$\phi(x) = (1 - 2x) - \int_0^x e^{-(x-t)} e^{(x^2 - t^2)} (1 - 2t)\, dt$$

$$= (1 - 2x) - e^{(x^2 - x)} \int_0^x (1 - 2t)\, e^{t - t^2}\, dt$$

$$= (1 - 2x) - e^{(x^2 - x)} \left[e^{(t - t^2)} \Big|_0^x \right]$$

$$= (1 - 2x) - e^{(x^2 - x)} \left[e^{(x - x^2)} - 1 \right]$$

$$= e^{(x^2 - x)} - 2x .$$

Problem 41.

$$\phi(x) = e^{(x^2+2x)} + 2\int_0^x e^{(x^2-t^2)}\,\phi(t)\,dt \qquad (1)$$

Solution:

The required resolvent kernel $R(x, t; 2)$ may be written from the results of problem 21, as

$$R(x, t; 2) = e^{2(x-t)}\,e^{(x^2-t^2)} \qquad (2)$$

Using equation (2) of Note 3, the solution of the given V.I.E. may be written as

$$\phi(x) = e^{(x^2+2x)} + 2\int_0^x e^{2(x-t)}\,e^{(x^2-t^2)}\,e^{(t^2+2t)}\,dt$$

$$= e^{(x^2+2x)} + 2\,e^{(x^2+2x)}\int_0^x dt$$

$$= e^{(x^2+2x)}\,[1 + 2x] \;.$$

Problem 42.

$$\phi(x) = (1 + x^2) + \int_0^x \frac{1+x^2}{1+t^2}\,\phi(t)\,dt \qquad (1)$$

Solution:

The required resolvent kernel $R(x, t; 1)$ may be written from the results of problem 22, as

$$R(x, t; 1) = \frac{1+x^2}{1+t^2}\,e^{(x-t)} \qquad (2)$$

Using equation (2) of Note 3, the solution of the given V.I.E. may be written as,

$$\phi(x) = (1 + x^2) + \int_0^x \frac{(1 + x^2)}{(1 + t^2)} e^{(x-t)} (1 + t^2)\, dt$$

$$= (1 + x^2) + (1 + x^2)\, e^x \int_0^x e^{-t}\, dt$$

$$= (1 + x^2) + (1 + x^2)\, e^x (1 - e^{-x})$$

$$= e^x (1 + x^2) .$$

Problem 43.

$$\phi(x) = \frac{1}{(1 + x^2)} + \int_0^x \sin(x-t)\, \phi(t)\, dt \qquad (1)$$

Solution:

We first need to find the resolvent kernel. This can be done by the method of Laplace transformation described in Note 2. We have

$$K(x, t) = \sin(x-t) \qquad (2)$$

Therefore

$$\tilde{K}(s) = \frac{1}{s^2 + 1}$$

and consequently, the Laplace transform $\tilde{R}(s)$ of the resolvent kernel is given by

$$\tilde{R}(s) = \frac{\tilde{K}(s)}{1 - \tilde{K}(s)} = \frac{1}{s^2} \qquad (3)$$

Therefore, the resolvent kernel is given by

$$R(x, t; 1) = (x-t) \tag{4}$$

Now, using equation (2) of Note 3, the solution of the given V.I.E. may be written as,

$$\phi(x) = \frac{1}{(1+x^2)} + \int_0^x \frac{(x-t)}{(1+t^2)}\, dt$$

$$= \frac{1}{(1+x^2)} + x\int_0^x \frac{1}{1+t^2}\, dt - \frac{1}{2}\int_0^x \frac{2t}{(1+t^2)}\, dt$$

$$= \frac{1}{(1+x^2)} + \arctan(x) - \frac{1}{2}\ln(1+x^2) .$$

Problem 44.

$$\phi(x) = x\, e^{(x^2/2)} + \int_0^x e^{-(x-t)}\, \phi(t)\, dt \tag{1}$$

Solution:

The required resolvent kernel $R(x, t; 1)$ may be written from the results of problem 32, as

$R(x, t; 1) = 1$

Using equation (2) of Note 3, the solution of the given V.I.E. may be written as,

$$\phi(x) = x\, e^{(x^2/2)} + \int_0^x t\, e^{(t^2/2)}\, dt$$

$$= x\, e^{(x^2/2)} + \int_0^x \frac{d}{dt}\left\{e^{(t^2/2)}\right\} dt$$

$$= x\, e^{(x^2/2)} + e^{(x^2/2)} - 1$$

$$= e^{(x^2/2)}\,(x + 1) - 1 \,.$$

Problem 45.

$$\phi(x) = e^{-x} + \int_0^x e^{-(x-t)} \sin(x-t)\, \phi(t)\, dt \qquad (1)$$

Solution:

The required resolvent kernel $R(x, t; 1)$ may be written from the results of problem 33, as

$$R(x, t; 1) = (x-t)\, e^{-(x-t)} \qquad (2)$$

Using equation (2) of Note 3, the solution of the given V.I.E. may be written as

$$\phi(x) = e^{-x} + \int_0^x (x-t)\, e^{-(x-t)}\, e^{-t}\, dt$$

$$= e^{-x} + e^{-x} \int_0^x (x-t)\, dt$$

$$= e^{-x}\left(\frac{x^2}{2} + 1\right) \,.$$

NOTE 4.

Applicable to problems 46 to 55.

Suppose we have a V.I.E. of the second kind

$$\phi(x) = f(x) + \lambda \int_0^x K(x, t)\, \phi(t)\, dt \tag{1}$$

We assume that $f(x)$ is continuous in $[0, a]$ and the kernel $K(x, t)$ is continuous for $0 \le x \le a$, $0 \le t \le x$.

Take some function $\phi_0(x)$ continuous in $[0, a]$. Putting the function $_0(x)$ into the right side of equation (1) in place of $\phi(x)$, we get

$$\phi_1(x) = f(x) + \lambda \int_0^x K(x, t)\, \phi_0(t)\, dt \tag{2}$$

Then $\phi_1(x)$ is also continuous in the interval $[0, a]$. Continuing the process, we obtain a sequence of functions

$$\phi_0(x),\ \phi_1(x),\ \ldots,\ \phi_n(x),\ \ldots.$$

where

$$\phi_n(x) = f(x) + \lambda \int_0^x K(x, t)\, \phi_{n-1}(t)\, dt \tag{3}$$

Under the assumptions made about $f(x)$ and $K(x, t)$, the sequence $\phi_n(x)$ converges as $n \to \infty$, to the solution $\phi(x)$ of the integral equation (1).

Problem 46.

$$\phi(x) = x - \int_0^x (x-t)\,\phi(t)\,dt \ ; \quad \phi_0(x) = 0$$

Solution:

Since $\phi_0(x) = 0$, it follows that $\phi_1(x) = x$. Then, using equation (3) of Note 4,

$$\phi_2(x) = x - \int_0^x (x-t)\,t\,dt = x - \frac{x^3}{3!}$$

$$\phi_3(x) = x - \int_0^x (x-t)\left(t - \frac{t^3}{3!}\right) dt = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\phi_4(x) = x - \int_0^x (x-t)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!}\right) dt = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

Obviously

$$\phi_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots + (-1)^{n-1}\,\frac{x^{(2n-1)}}{(2n-1)!}$$

Thus, $\phi_n(x)$ is the n^{th} partial sum of the sine series and consequently, the solution of the given V.I.E., is

$$\phi(x) = \sin(x)\ .$$

Problem 47.

$$\phi(x) = 1 - \int_0^x (x-t)\, \phi(t)\, dt \ ; \qquad \phi_0(x) = 0$$

Solution:

Since $\phi_0(x) = 0$, it follows that $\phi_1(x) = 1$. Then, using equation (3) of Note 4,

$$\phi_2(x) = 1 - \int_0^x (x-t)\, dt = 1 - \frac{x^2}{2!}$$

$$\phi_3(x) = 1 = \int_0^x (x-t)\left(1 - \frac{t^2}{2!}\right) dt = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\phi_4(x) = 1 - \int_0^x (x-t)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!}\right) dt = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

Obviously

$$\phi_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ \ldots + (-1)^{n-1} \frac{x^{(2n-2)}}{(2n-2)!}$$

Thus, $\phi_n(x)$ is the n^{th} partial sum of the cosine series and consequently, the solution of the given V.I.E. is,

$$\phi(x) = \cos(x) \ .$$

Problem 48.

$$\phi(x) \;+\; 1 + \int_0^x (x-t)\,\phi(t)\,dt\,, \qquad \phi_0(x) = 1$$

Solution:

Using equation (3) of Note 4,

$$\phi_1(x) = 1 + \int_0^x (x-t)\,dt = 1 + \frac{x^2}{2!}$$

$$\phi_2(x) = 1 + \int_0^x (x-t)\left(1 + \frac{t^2}{2!}\right)dt = 1 + \frac{x^2}{2!} + \frac{x^4}{2!}$$

$$\phi_3(x) = 1 + \int_0^x (x-t)\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!}\right)dt = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}$$

Obviously

$$\phi_n(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!}$$

Thus, $\phi_n(x)$ is the n^{th} partial sum of the hyperbolic cosine series and consequently, the solution of the given V.I.E. is,

$$\phi(x) = \cosh(x) \quad .$$

Problem 49.

$$\phi(x) = (x + 1) - \int_0^x \phi(t)\, dt;$$

a) $\phi_0(x) = 1$ and b) $\phi_0(x) = x + 1$

Solution:

a) $\phi_0(x) = 1$

Using equation (3) of Note 4,

$$\phi_1(x) = x + 1 - \int_0^x dt = 1$$

$$\phi_2(x) = x + 1 - \int_0^x dt = 1$$

$$\phi_3(x) = x + 1 - \int_0^x dt = 1$$

Obviously, this leads to a stationary sequence $\{\phi_n\}$ so that the solution of the given V.I.E. is $\phi(x) = 1$.

b) $\phi_0(x) = (1 + x)$

Using equation (3) of Note 4,

$$\phi_1(x) = x + 1 - \int_0^x (t + 1)\, dt = 1 - \frac{x^2}{2!}$$

$$\phi_2(x) = x + 1 - \int_0^x \left(1 - \frac{t^2}{2!}\right) dt = 1 + \frac{x^3}{3!}$$

$$\phi_3(x) = x + 1 - \int_0^x \left(1 + \frac{t^3}{3!}\right) dt = 1 - \frac{x^4}{4!}$$

$$\phi_5(x) = \frac{x^2}{2} + x - \int_0^x \left(t + \frac{t^4}{4!} - \frac{t^5}{5!} \right) dt = \left(x - \frac{x^5}{5!} + \frac{x^6}{6!} \right)$$

.
.
.
.

Continuing this process, we get

$$\phi_n(x) = x + \frac{(-x)^n}{n!} + \frac{(-x)^{n+1}}{(n+1)!}$$

The sequence $\{\phi_n\}$ tends to x as n becomes unbounded. Therefore, the solution of the given V.I.E. is $\phi(x) = x$.

b) $\phi_0(x) = x$

Using equation (3) of Note 4,

$$\phi_1(x) = \frac{x^2}{2} + x - \int_0^x t \, dt = x$$

$$\phi_2(x) = \frac{x^2}{2} + x - \int_0^x t \, dt = x$$

.
.
.

This leads to a stationary sequence $\{\phi_n\}$, so that the solution of the given V.I.E. is $\phi(x) = x$.

c) $\phi_0(x) = x + \frac{x^2}{2}$

Using equation (3) of Note 4,

$$\phi_5(x) = \frac{x^2}{2} + x - \int_0^x \left(t + \frac{t^4}{4!} - \frac{t^5}{5!} \right) dt = \left(x - \frac{x^5}{5!} + \frac{x^6}{6!} \right)$$

.
.
.
.

Continuing this process, we get

$$\phi_n(x) = x + \frac{(-x)^n}{n!} + \frac{(-x)^{n+1}}{(n+1)!}$$

The sequence $\{\phi_n\}$ tends to x as n becomes unbounded. Therefore, the solution of the given V.I.E. is $\phi(x) = x$.

b) $\phi_0(x) = x$

Using equation (3) of Note 4,

$$\phi_1(x) = \frac{x^2}{2} + x - \int_0^x t \, dt = x$$

$$\phi_2(x) = \frac{x^2}{2} + x - \int_0^x t \, dt = x$$

.
.
.

This leads to a stationary sequence $\{\phi_n\}$, so that the solution of the given V.I.E. is $\phi(x) = x$.

c) $\phi_0(x) = x + \frac{x^2}{2}$

Using equation (3) of Note 4,

Obviously

$$\phi_n(x) = 1 - \frac{(-x)^{(n+1)}}{(n+1)!}$$

We assume that $f(x)$ is continuous in $[0, a]$ and the kernel $K(x, t)$ is continuous for $0 \le x \le a$, $0 \le t \le x$. In view of this, the limit of the sequence $\{\phi_n\}$ is 1 and therefore the solution of the given V.I.E. is $\phi(x) = 1$.

Problem 50.

$$\phi(x) = \frac{x^2}{2} + x - \int_0^x \phi(t)\, dt$$

a) $\phi_0(x) = 1$, b) $\phi_0(x) = x$ and c) $\phi_0(x) = \frac{x^2}{2} + x$

Solution:

a) $\phi_0(x) = 1$

Using equation (3) of Note 4,

$$\phi_1(x) = \frac{x^2}{2} + x - \int_0^x dt = \frac{x^2}{2}$$

$$\phi_2(x) = \frac{x^2}{2} + x - \int_0^x \frac{t^2}{2}\, dt = x + \frac{x^2}{2!} - \frac{x^3}{3!}$$

$$\phi_3(x) = \frac{x^2}{2} + x - \int_0^x \left(\frac{t^2}{2} + t - \frac{t^3}{3!}\right) dt = x - \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$\phi_4(x) = \frac{x^2}{2} + x - \int_0^x \left(t - \frac{t^3}{3!} + \frac{t^4}{4!}\right) dt = x + \frac{x^4}{4!} - \frac{x^5}{5!}$$

$$\phi_1(x) = \frac{x^2}{2} + x - \int_0^x \left(t + \frac{t^2}{2}\right) dt = x - \frac{x^3}{3!}$$

$$\phi_2(x) = \frac{x^2}{2} + x - \int_0^x \left(t - \frac{t^3}{3!}\right) dt - x + \frac{x^4}{4!}$$

$$\phi_3(x) = \frac{x^2}{2} + x - \int_0^x \left(t + \frac{t^4}{4!}\right) dt = x - \frac{x^5}{5!}$$

$$\vdots$$

Continuing this process leads to

$$\phi_n(x) = x + \frac{(-x)^{n+2}}{(n+2)!}$$

This sequence $\{\phi_n\}$ tends to x as n becomes unbounded, so that the solution of the given V.I.E. is $\phi(x) = x$.

Problem 51.

$$\phi(x) = 1 + x + \int_0^x (x-t) \quad (t)\, dt \ ; \quad \phi_0(x) = 1$$

Solution:

Using equation (3) of Note 4,

$$\phi_1(x) = 1 + x + \int_0^x (x-t)\, dt = \left(1 + x + \frac{x^2}{2}\right)$$

$$\phi_2(x) = (1+x) + \int_0^x (x-t)\left(1+t+\frac{t^2}{2}\right) dt = 1+x+\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$\phi_3(x) = (1+x) + \int_0^x (x-t)\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}\right) dt = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}$$

Continuing this process, we find that ϕ_n is the n^{th} partial sum of the series for e^x. Consequently, the solution of the given V.I.E. is e^x.

Problem 52.

$$\phi(x) = 2x + 2 - \int_0^x \phi(t)\, dt \; ; \quad a) \;\; \phi_0(x) = 1 \, , \quad b) \;\; \phi_0(x) = 2$$

Solution:

a) $\phi_0(x) = 1$

Using equation (3) of Note 4,

$$\phi_1(x) = 2x + 2 - \int_0^x 1\, dt = 2 + x$$

$$\phi_2(x) = 2x + 2 - \int_0^x (t + 2)\, dt = 2 - \frac{x^2}{2!}$$

$$\phi_3(x) = 2x + 2 - \int_0^x \left(2 - \frac{t^2}{2}\right) dt = 2 + \frac{x^3}{3!}$$

$$\phi_4(x) = 2x + 2 - \int_0^x \left(2 + \frac{t^3}{3!}\right) dt = 2 - \frac{x^4}{4!}$$

Continuing this process, we find that

$$\phi_n = 2 + (-1)^{n+1} \frac{x^n}{n!}$$

The sequence $\{\phi_n\}$ tends to the limit 2 as $n \to \infty$, so that the solution for the given V.I.E. is $\phi(x) = 2$.

b) $\phi_0(x) = 2$

Using equation (3) of Note 4,

$$\phi_1(x) = 2x + 2 - \int_0^x 2\,dt = 2$$

$$\phi_2(x) = 2x + 2 - \int_0^x 2\,dt = 2$$

Obviously, this leads to a stationary sequence and hence the solution of the given V.I.E. is $\phi(x) = 2$.

Problem 53.

$$\phi_0(x) = 2x^2 + 2 - \int_0^x x\,\phi(t)\,dt \;;$$

a) $\phi_0(x) = 2$ b) $\phi_0(x) = x^2$

Solution:

a) $\phi_0(x) = 2$

Using equation (3) of Note 4,

$$\phi_1(x) = 2x^2 + 2 - \int_0^x 2x\,dt = 2$$

$$\phi_2(x) = 2x^2 + 2 - \int_0^x 2x\,dt = 2$$

Obviously, this leads to a stationary sequence, so that the solution of the given V.I.E. is $\phi(x) = 2$.

b) $\phi_0(x) = x^2$

Using equation (3) of Note 4,

$$\phi_1(x) = 2x^2 + 2 - \int_0^x x^3\,dt = 2 + 2x^2 - x^4$$

$$\phi_2(x) = 2x^2 + 2 - \int_0^x x(2 + 2t^2 - t^4)\,dt = 2 - \frac{2x^4}{3} + \frac{x^5}{5}$$

$$\phi_3(x) = 2x^2 + 2 - \int_0^x x\left(2 - \frac{2t^4}{3} + \frac{t^6}{5}\right) dt = 2 + \frac{2x^6}{15} - \frac{x^8}{35}$$

Clearly, the leading term in ϕ_n will be equal to 2 and the sequence $\{\phi_n\}$ will also tend to 2 as n becomes unbounded. Consequently, the solution of the given V.I.E. is $\phi(x) = 2$.

Problem 54.

$$\phi(x) = \frac{x^3}{3} - 2x - \int_0^x \phi(t)\,dt \;; \qquad \phi_0(x) = x^2$$

Solution:

$\phi_0(x) = x^2$

Upon using equation (3) of Note 4,

$$\phi_1(x) = \frac{x^3}{3} - 2x - \int_0^x t^2\,dt = -2x$$

$$\phi_2(x) = \frac{x^3}{3} - 2x + \int_0^x 2t\,dt = -2x + x^2 + \frac{x^3}{3}$$

$$\phi_3(x) = \frac{x^3}{3} - 2x - \int_0^x \left(-2t + t^2 + \frac{t^3}{3}\right) dt = -2x + x^2 - \frac{x^4}{12}$$

$$\phi_4(x) = \frac{x^3}{3} - 2x - \int_0^x \left(-2t + t^2 - \frac{t^4}{12}\right) dt = -2x + x^2 + \frac{2x^5}{5!}$$

Continuing this process, we will find

$$\phi_n = x^2 - 2x + 2(-1)^n \frac{x^{(n+1)}}{(n+1)!} \quad \text{for} \quad n \geq 2$$

Consequently, the solution of the given V.I.E. is given simply by

$$\phi(x) = x^2 - 2x.$$

Problem 57.

$$\phi(x) = 1 + \int_0^x [\phi^2(t) + t\,\phi(t) + t^2] \, dt \qquad (1)$$

Solution:

Let us choose $\phi_0(x) = 1$ (2)

Substituting $\phi_0(t)$ on the R.h.s., we get

$$\phi_1(x) = 1 + \int_0^x [1 + t + t^2] \, dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} \qquad (3)$$

Substituting for $\phi_1(t)$ on the R.h.s., we get

$$\phi_2(x) = 1 + \int_0^x \left[\left(1 + t + \frac{t^2}{2} + \frac{t^3}{3}\right)^2 + t\left(1 + t + \frac{t^2}{2} + \frac{t^3}{3}\right) + t^2\right] dt$$

$$= 1 + \int_0^x \left[\left(1 + 2t + 2t^2 + \frac{5t^3}{3} + \frac{11t^4}{12} + \frac{t^5}{3} + \frac{t^6}{9}\right)\right.$$

$$\left. + \left(t + t^2 + \frac{t^3}{2} + \frac{t^4}{3}\right) + t^2\right] dt$$

$$= 1 + \int_0^x \left[1 + 3t + 4t^2 + \frac{13t^3}{6} + \frac{5t^4}{4} + \frac{2t^5}{3} + \frac{t^6}{9}\right] dt$$

$$= \left[1 + x + \frac{3x^2}{2} + \frac{4x^3}{3} + \frac{13x^4}{24} + \frac{x^5}{4} + \frac{x^6}{18} + \frac{x^7}{63}\right].$$

Problem 58.

$$\phi(x) = \int_0^x [t \quad ^2(t) - 1] \quad dt \qquad (1)$$

Solution:

Let us choose $\phi_0(x) = 0$ (2)

Substituting for $\phi_0(t)$ on the R.h.s., we get

$$\phi_1(x) = \int_0^x -1 \quad dt = -x \qquad (3)$$

Substituting for $\phi_1(t)$ on the R.h.s., we get

$$\phi_2(x) = \int_0^x (t^3 - 1)\, dt = \frac{x^4}{4} - x \qquad (4)$$

Substituting for $\phi_2(t)$ on the R.h.s., we finally get

$$\phi_3(x) = \int_0^x \left[t\,\frac{t^4}{4} - t^{\,2} - 1 \right] dt$$

$$= \int_0^x \left[-1 + t^3 - \frac{t^6}{2} + \frac{t^9}{16} \right] dt$$

$$= \left[-x + \frac{x^4}{4} - \frac{x^7}{14} + \frac{x^{10}}{160} \right] .$$

Problem 59.

$$\phi(x) = e^x - \int_0^x e^{x-t}\,\phi(t)\,dt \tag{1}$$

Solution:

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we get

$$\tilde{\Phi}(s) = \frac{1}{s-1} - \frac{\tilde{\Phi}(s)}{s-1}$$

or

$$\tilde{\Phi}(s) = \frac{1}{s} \tag{2}$$

and hence, the solution is given by

$$\phi(x) = 1. \tag{3}$$

Problem 60.

$$\phi(x) = x - \int_0^x e^{x-t} \phi(t)\, dt \tag{1}$$

Solution:

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we get

$$\tilde{\phi}(s) = \frac{1}{s^2} - \frac{\tilde{\phi}(s)}{s-1} \tag{2}$$

or,

$$\tilde{\phi}(s) = \frac{1}{s^2} - \frac{1}{s^3} \tag{3}$$

Consequently, the solution of the given V.I.E. is

$$\phi(x) = \left(x - \frac{x^2}{2}\right). \tag{4}$$

Problem 61.

$$\phi(x) = e^{2x} + \int_0^x e^{(t-x)} \phi(t)\, dt \tag{1}$$

Solution:

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we get

$$\tilde{\phi}(s) = \frac{1}{s-2} + \frac{\tilde{\phi}(s)}{s+1} \tag{2}$$

or,

$$\tilde{\phi}(s) = \frac{1}{s - 2} + \frac{1}{s(s - 2)} \tag{3}$$

Consequently, the solution of the given V.I.E. is

$$\phi(x) = e^{2x} - \frac{(1 - e^{2x})}{2} = \frac{1}{2}(3e^{2x} - 1) .$$

Problem 62.

$$\phi(x) = x - \int_0^x (x-t)\,\phi(t)\,dt \tag{1}$$

Solution:

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1) and employing the convolution theorem, we get

$$\tilde{\phi}(s) = \frac{1}{s^2} - \frac{\tilde{\phi}(s)}{s^2} \tag{2}$$

or

$$\tilde{\phi}(s) = \frac{1}{s^2 + 1} \tag{3}$$

Consequently, the solution of the given V.I.E. is

$$\phi(x) = \sin(x) .$$

Problem 63.

$$\phi(x) = \cos(x) - \int_0^x (x-t)\cos(x-t)\,\phi(t)\,dt \qquad (1)$$

Solution:

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1) and employing the convolution theorem, we get

$$\tilde{\phi}(s) = \frac{s}{s^2+1} + \tilde{\phi}(s)\,\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{s}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}\,\tilde{\phi}(s) \qquad (2)$$

or,

$$\tilde{\phi}(s) = \frac{s^2+1}{s(s^2+3)} = \frac{s}{s^2+3} + \frac{1}{s(s^2+3)} \qquad (3)$$

Consequently, the solution of the given V.I.E. is

$$\phi(x) = \cos(\sqrt{3}\,x) + \frac{2}{3}\sin^2\left(\frac{\sqrt{3}}{2}\,x\right)$$

$$= \cos(\sqrt{3}\,x) + \frac{1}{3}\,[1 - \cos(\sqrt{3}\,x)]$$

$$= \frac{1}{3}\,[2\cos(\sqrt{3}\,x) + 1]\ .$$

Problem 64.

$$\phi(x) = 1 + x + \int_0^x e^{-2(x-t)} \phi(t)\, dt \qquad (1)$$

Solution:

We recognize the given V.I.E. as the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we get

$$\tilde{\phi}(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{\tilde{\phi}(s)}{s + 2} \qquad (2)$$

or

$$\tilde{\phi}(s) = \frac{1}{s + 1} \quad \frac{3}{s(s + 1)} \quad \frac{2}{s^2(s + 1)} \qquad (3)$$

Consequently, the solution of the given V.I.E. is

$$\phi(x) = e^{-x} + 3(1 - e^{-x}) + 2(x - 1 + e^{-x})$$

$$= (2x + 1).$$

Problem 65.

$$\phi(x) = x + \int_0^x \sin(x-t)\, \phi(t)\, dt \qquad (1)$$

Solution:

We recognize the given V.I.E. as the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we get

$$\tilde{\phi}(s) = \frac{1}{s^2} + \frac{\tilde{\phi}(s)}{s^2 + 1} \qquad (2)$$

or,

$$\phi(s) = \frac{s^2 + 1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \qquad (3)$$

Consequently, the solution of the given V.I.E. is

$$\phi(x) = x + \frac{x^3}{6} .$$

<u>Problem 66.</u>

$$\phi(x) = \sin(x) + \int_0^x (x-t)\, \phi(t)\, dt \qquad (1)$$

<u>Solution:</u>

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we find

$$\tilde{\phi}(s) = \frac{1}{s^2 + 1} + \frac{\tilde{\phi}(s)}{s^2} \qquad (2)$$

or,

$$\tilde{\phi}(s) = \frac{s^2}{(s^4 - 1)} \qquad (3)$$

Taking the inverse Laplace transform, the solution of the given V.I.E. is given by

$$\phi(x) = \frac{d^2}{dx^2}\left[L^{-1}\left\{\frac{1}{(s^4 - 1)}\right\}\right]$$

$$= \frac{d^2}{dx^2}\left[\frac{1}{2}\left\{\sinh(x) - \sin(x)\right\}\right]$$

$$= \frac{1}{2}\left[\sin(x) + \sinh(x)\right] .$$

Problem 67.

$$\phi(x) = x - \int_0^x \sinh(x-t)\, \phi(t)\, dt \tag{1}$$

Solution:

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we find,

$$\tilde{\phi}(s) = \frac{1}{s^2} - \frac{\tilde{\phi}(s)}{s^2 - 1} \tag{2}$$

or

$$\tilde{\phi}(s) = \frac{s^2 - 1}{s^4} = \frac{1}{s^2} - \frac{1}{s^4} \tag{3}$$

Taking the inverse Laplace transform of both sides of equation (3), we get the solution of the given V.I.E., as

$$\phi(x) = \left(x - \frac{x^3}{3}\right) .$$

Problem 68.

$$\phi(x) = 1 - 2x - 4x^2 + \int_0^x [3 + 6(x-t) - 4(x-t)^2]\, \phi(t)\, dt \tag{1}$$

Remark: Note that the sign of the $(x-t)$ term under the integral is misprinted in the book.

Solution:

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we get,

$$\tilde{\phi}(s) = \frac{1}{s} - \frac{2}{s^2} - \frac{8}{s^3} + \tilde{\phi}(s)\left[\frac{3}{s} + \frac{6}{s^2} - \frac{8}{s^3}\right] \qquad (2)$$

or

$$\tilde{\phi}(s)\left[1 - \frac{3}{s} - \frac{6}{s^2} + \frac{8}{s^3}\right] = \left[\frac{s^2 - 2s - 8}{s^3}\right] \qquad (3)$$

or

$$\tilde{\phi}(s) = \frac{s^2 - 2s - 8}{s^3 - 3s^2 - 6s + 8} = \frac{1}{(s-1)} \qquad (4)$$

Consequently, taking the inverse Laplace transform of both sides of equation (4), we get the solution of the given V.I.E. to be

$$\phi(x) = e^x.$$

Problem 69.

$$\phi(x) = \sinh(x) - \int_0^x \cosh(x-t)\,\phi(t)\,dt \qquad (1)$$

Solution:

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we get

$$\phi(s) = \frac{1}{s^2 - 1} - \frac{s\tilde{\phi}(s)}{s^2 - 1} \qquad (2)$$

or

$$\phi(s) = \frac{1}{s^2 + s - 1} = \frac{1}{\left(s + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2} \qquad (3)$$

Taking the inverse Laplace transform of both sides of equation (1), the solution of the given V.I.E. is given by

$$\phi(x) = e^{-(x/2)} L^{-1}\left\{ \frac{1}{s^2 - \left(\frac{\sqrt{5}}{2}\right)^2} \right\}$$

$$= \frac{2}{\sqrt{5}} \sinh\left(\frac{\sqrt{5}}{2} x\right) e^{-(x/2)} \quad .$$

Problem 70.

$$\phi(x) = 1 + 2 \int_0^x \cos(x-t)\, \phi(t)\, dt \qquad (1)$$

Solution:

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we get

$$\tilde{\phi}(s) = \frac{1}{s} + \frac{2s\tilde{\phi}(s)}{s^2 + 1} \qquad (2)$$

or

$$\tilde{\phi}(s) = \frac{s}{(s - 1)^2} + \frac{1}{s(s - 1)^2}$$

$$= e^x (1 + x) + [1 - (1-x)\, e^x]$$

$$= (1 + 2x\, e^x) \quad .$$

Problem 71.

$$\phi(x) = e^x + 2\int_0^x \cos(x-t)\,\phi(t)\,dt \qquad (1)$$

Solution:

We recognize the given V.I.E. to be of the convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we get

$$\tilde{\phi}(s) = \frac{1}{s-1} + \frac{2s\tilde{\phi}(s)}{s^2+1} \qquad (2)$$

or,

$$\tilde{\phi}(s) = \frac{s^2+1}{(s-1)^3} = \frac{s^2}{(s-1)^3} + \frac{1}{(s-1)^3} \qquad (3)$$

Consequently, taking the inverse Laplace transform of both sides of equation (3), we get the solution of the given V.I.E. to be

$$\phi(x) = e^x[1 + 2x + \frac{x^2}{2}] + \frac{x^2}{2}e^x$$

$$= e^x[1 + 2x + x^2]$$

$$= e^x(1+x)^2 \quad .$$

Problem 72.

$$\phi(x) = \cos x + \int_0^x \phi(t)\,dt \qquad (1)$$

Solution:

We may treat the given V.I.E. to be a convolution type. Therefore, taking the Laplace transform of both sides of equation (1), and employing the convolution theorem, we get

$$\tilde{\Phi}(s) = \frac{2}{s^2 + 1} + \frac{\tilde{\Phi}(s)}{s} \tag{2}$$

or,

$$\tilde{\Phi}(s) = \frac{s^2}{(s-1)(s^2+1)} = \frac{1}{2}\left[\frac{1}{s-1} \quad \frac{s}{s^2+1} \quad \frac{1}{s^2+1}\right] \tag{3}$$

Consequently, taking the inverse Laplace transform of both sides of equation (3), we get

$$\phi(x) = \left[\frac{e^x + \cos(x) + \sin(x)}{2}\right].$$

Problem 73.

$$\phi_1(x) = \sin(x) + \int_0^x \phi_2(t)\, dt \tag{1a}$$

$$\phi_2(x) = 1 - \cos(x) - \int_0^x \phi_1(t)\, dt \tag{1b}$$

Solution:

Taking the Laplace transform of the given system of V.I.E. and employing the convolution theorem, we get

$$\tilde{\Phi}_1(s) = \frac{1}{s^2 + 1} + \frac{\tilde{\Phi}_2(s)}{s} \tag{2a}$$

$$\tilde{\Phi}_2(s) = \frac{1}{s} - \frac{s}{s^2 + 1} - \frac{\tilde{\Phi}_1(s)}{s} \tag{2b}$$

Solving the above simultaneous equations, we get

$$\tilde{\Phi}_1(s) = \frac{1}{s^2 + 1} \tag{3a}$$

$$\tilde{\Phi}_2(s) = 0 \tag{3b}$$

Consequently, taking the inverse Laplace transform of equation (3), we get

$$\phi_1(x) = \sin(x) \quad ; \qquad \phi_2(x) = 0 \ .$$

Problem 74.

$$\phi_1(x) = e^{2x} + \int_0^x \phi_2(t)\, dt \tag{1a}$$

$$\phi_2(x) = 1 - \int_0^x e^{2(x-t)}\, \phi_1(t)\, dt \tag{1b}$$

Solution:

Taking the Laplace transform of the given system of V.I.E. and employing the convolution theorem, we get

$$\tilde{\Phi}_1(s) = \frac{1}{s - 2} + \frac{\tilde{\Phi}_2(s)}{s} \tag{2a}$$

$$\tilde{\Phi}_2(s) = \frac{1}{s} - \frac{\tilde{\Phi}_1(s)}{s - 2} \tag{2b}$$

Solving the above simultaneous system, we have

$$\tilde{\Phi}_1(s) = \frac{s^2 + s - 2}{s(s - 1)^2} = \frac{1}{s} + \frac{3}{s(s - 1)} \tag{3a}$$

$$\Phi_2(s) = \frac{1}{s} - \frac{1}{(s-1)(s-2)} - \frac{2}{s(s-1)(s-2)} \tag{3b}$$

Consequently, taking the inverse Laplace transforms in equation (3),

$$\phi_1(x) = 1 - 3(1 - e^x) = (3e^x - 2)$$

$$\phi_2(x) = 1 + (e^x - e^{2x}) - (1 - 2e^x + e^{2x}) = (3e^x - 2e^{2x}) .$$

Problem 75.

$$\phi_1(x) = e^x + \int_0^x \phi_1(t)\,dt - \int_0^x e^{(x-t)}\,\phi_2(t)\,dt \tag{1a}$$

$$\phi_2(x) = -x - \int_0^x (x-t)\,\phi_1(t)\,dt + \int_0^x \phi_2(t)\,dt \tag{1b}$$

Solution:

Taking the Laplace transform of the given system of V.I.E. and employing the convolution theorem, we get,

$$\Phi_1(s) = \frac{1}{s-1} + \frac{\Phi_1(s)}{s} - \frac{\Phi_2(s)}{s-1} \tag{2a}$$

$$\Phi_2(s) = -\frac{1}{s^2} - \frac{\Phi_1(s)}{s^2} + \frac{\Phi_2(s)}{s} \tag{2b}$$

Solving the simultaneous equations, we have

$$\Phi_1(s) = \frac{1}{s-2} \tag{3a}$$

$$\Phi_2(s) = -\frac{1}{s(s-2)} \tag{3b}$$

Consequently, taking the inverse Laplace transform of both sides of equation (3), we get the solution of the given system of V.I.E. as

$$\phi_1(x) = e^{2x} \quad , \qquad \phi_2(x) = \left(\frac{1 - e^{2x}}{2}\right) .$$

Problem 77.

$$\phi_1(x) = x + \int_0^x \phi_2(t)\, dt \tag{1a}$$

$$\phi_2(x) = 1 - \int_0^x \phi_1(t)\, dt \tag{1b}$$

$$\phi_3(x) = \sin(x) + \frac{1}{2}\int_0^x (x-t)\, \phi_1(t)\, dt \tag{1c}$$

Solution:

Taking the Laplace transform of the given system of V.I.E., we have

$$\tilde{\phi}_1(s) = \frac{1}{s^2} + \frac{\tilde{\phi}_2(s)}{s} \tag{2a}$$

$$\tilde{\phi}_2(s) = \frac{1}{s} - \frac{\tilde{\phi}_1(s)}{s} \tag{2b}$$

$$\tilde{\phi}_3(s) = \frac{1}{s^2+1} + \frac{1}{2}\,\frac{\tilde{\phi}_1(s)}{s^2} \tag{2c}$$

Solving the simultaneous equation (2), we get

$$\tilde{\phi}_1(s) = \frac{2}{s^2+1} \tag{3a}$$

$$\tilde{\phi}_2(s) = \frac{2s}{s^2+1} - \frac{1}{s} \tag{3b}$$

$$\tilde{\phi}_3(s) = \frac{1}{s^2} \tag{3c}$$

Considering the inverse Laplace transform of equation (3), the solution is given by,

$$\phi_1(x) = 2\sin(x) \tag{4a}$$

$$\phi_2(x) = 2\cos(x) - 1 \tag{4b}$$

$$\phi_3(x) = x \tag{4c}$$

Problem 78.

$$\phi_1(x) = 1 - \int_0^x \phi_2(t)\,dt \tag{1a}$$

$$\phi_2(x) = \cos(x) - 1 + \int_0^x \phi_3(t)\,dt \tag{1b}$$

$$\phi_3(x) = \cos(x) + \int_0^x \phi_1(t)\,dt \tag{1c}$$

Solution:

Taking the Laplace transform of the given system of V.I.E., we have

$$\tilde{\phi}_1(s) = \frac{1}{s} - \frac{\tilde{\phi}_2(s)}{s} \tag{2a}$$

$$\tilde{\phi}_2(s) = \frac{s}{s^2+1} - \frac{1}{s} + \frac{\tilde{\phi}_3(s)}{s} \tag{2b}$$

$$\tilde{\phi}_3(s) = \frac{s}{s^2+1} + \frac{\tilde{\phi}_1(s)}{s} \tag{2c}$$

Solving the above simultaneous equations,

$$\tilde{\phi}_1(s) = s/(s^2 + 1) \tag{3a}$$

$$\tilde{\phi}_2(s) = 1/(s^2 + 1) \tag{3b}$$

$$\tilde{\phi}_3(s) = \frac{s}{(s^2 + 1)} + \frac{1}{(s^2 + 1)} \tag{3c}$$

Considering the inverse Laplace transform of equation (3), the solution is given by,

$$\phi_1(x) = \cos(x) \tag{4a}$$

$$\phi_2(x) = \sin(x) \tag{4b}$$

$$\phi_3(x) = \cos(x) + \sin(x) \tag{4c}$$.

Problem 79.

$$\phi_1(x) = x + 1 + \int_0^x \phi_3(t)\, dt \tag{1a}$$

$$\phi_2(x) = -x + \int_0^x (x-t)\, \phi_1(t)\, dt \tag{1b}$$

$$\phi_3(x) = \cos x - 1 - \int_0^x \phi_1(t)\, dt \tag{1c}$$

Solution:

Taking the Laplace transform of the given system of V.I.E. we have

$$\tilde{\phi}_1(s) = \frac{1}{s^2} + \frac{1}{s} + \frac{\tilde{\phi}_3}{s} \tag{2a}$$

$$\tilde{\phi}_2(s) = -\frac{1}{s^2} + \frac{\tilde{\phi}_1(s)}{s^2} \tag{2b}$$

$$\tilde{\phi}_3(3) = \frac{s}{s^2+1} - \frac{1}{s} - \frac{\tilde{\phi}_1(s)}{s} \tag{2c}$$

Solving the simultaneous equation (2),

$$\tilde{\phi}_1(s) = \frac{s}{s^2+1} + \frac{s^2}{(s^2+1)^2} \tag{3a}$$

$$\tilde{\phi}_2(s) = \frac{-1}{s^2} + \frac{1}{s(s^2+1)} + \frac{1}{(s^2+1)^2} \tag{3b}$$

$$\tilde{\phi}_3(s) = \frac{s}{s^2+1} - \frac{1}{s} - \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2} \tag{3c}$$

Taking the inverse Laplace transform of equation (3c), the solution is given by

$$\phi_1(x) = \cos(x) + \frac{x}{2}\cos(x) + \frac{1}{2}\sin(x) \qquad (4a)$$

$$\phi_2(x) = -x + 2\sin^2(\tfrac{x}{2}) + \frac{1}{2}\sin(x) - \frac{x}{2}\cos(x)$$

$$= 1 - x + \frac{1}{2}\sin(x) - (1 + \tfrac{x}{2})\cos(x) \qquad (4b)$$

$$\phi_3(x) = \cos(x) - 1 - \sin(x) - \frac{x}{2}\sin(x) \qquad (4c).$$

Problem 80.

$$\phi''(x) + \int_0^x e^{2(x-t)}\,\phi'(t)\,dt = e^{2x} \qquad (1)$$

with $\phi(0) = 0$ and $\phi'(0) = 1$ (2)

Solution:

Considering the Laplace transformation of equation (1), we have

$$s^2\,\tilde{\phi}(s) - s\,\phi(0) - \phi'(0) + \frac{1}{s-2}\,[s\,\tilde{\phi}(s) - \phi(0)] = \frac{1}{s-2} \qquad (3)$$

Using the initial conditions of equation (2), equation (3) becomes,

$$\left(s^2 + \frac{s}{s-2}\right)\tilde{\phi}(s) = \frac{s-1}{s-2}$$

or,

$$\tilde{\phi}(s) = \frac{1}{s(s-1)} \qquad (4)$$

Taking the inverse Laplace transform of equation (4), we get the solution of the given integro-differential equation to be,

$$\phi(x) = e^{x} - 1 \qquad (5).$$

Problem 81.

$$\phi'(x) - \phi(x) + \int_0^x (x-t)\,\phi'(t)\,dt - \int_0^x \phi(t)\,dt = x \qquad (1)$$

with $\phi(0) = -1$ (2)

Solution:

Considering the Laplace transformation of equation (1), we have

$$s\,\tilde{\phi}(s) - \phi(0) - \tilde{\phi}(s) + \frac{1}{s^2}\,[s\,\phi(s) - \phi(0)] - \frac{\tilde{\phi}(s)}{s} = \frac{1}{s^2} \qquad (3)$$

Using the initial condition of equation (2), equation (3) becomes

$$(s-1)\;\tilde{\phi}(s) = -1$$

or

$$\tilde{\phi}(s) = -\frac{1}{s-1} \qquad (4)$$

Taking the inverse Laplace transform of equation (4), we get the solution of the given integro-differential equation to be,

$$\phi(x) = -e^{t} \quad x \qquad (5).$$

Problem 82.

$$\phi''(x) - 2\,\phi'(x) + \phi(x) + 2\int_0^x \cos\,(x-t)\;\phi''(t)\,dt \qquad (1)$$

$$+\,2\int_0^x \sin\,(x-t)\;\phi'(t)\,dt \;=\; \cos\,(x)$$

with $\phi(0) = \phi'(0) = 0$ (2)

Solution:

Considering the Laplace transformation of equation (1) with due attention to the initial conditions of equation (2), we have

$$s^2\,\tilde{\Phi}(s) - 2s\,\tilde{\Phi}(s) + \tilde{\Phi}(s) + \frac{2s^3}{s^2+1}\,\tilde{\Phi}(s)\;\frac{2s}{s^2+1}\;\tilde{\Phi}(s) = \frac{s}{s^2+1} \qquad (3)$$

or,

$$\left[s^2 - 2s + 1 + \frac{2s^3}{s^2+1} + \frac{2s}{s^2+1}\right]\tilde{\Phi}(s) = \frac{s}{s^2+1}$$

or,

$$\tilde{\Phi}(s) = \frac{s}{(s^2+1)^2} \qquad (4)$$

Taking the inverse Laplace transform of equation (4), the solution of the given integro-differential equation is given by

$$\phi(x) = \frac{1}{2}\,x\,\sin\,(x) \qquad (5).$$

Problem 83.

$$\phi''(x) + 2\,\phi'(x) - 2\int_0^x \sin(x-t)\,\phi'(t)\,dt = \cos(x) \tag{1}$$

$$\text{with} \quad \phi(0) = \phi'(0) = 0 \tag{2}$$

Solution:

Considering the Laplace transformation of equation (1) with due attention to the initial conditions of equation (2), we get

$$s^2\,\tilde{\phi}(s) + 2s\,\tilde{\phi}(s) - \frac{2s}{s^2+1}\,\tilde{\phi}(s) = \frac{s}{s^2+1} \tag{3}$$

or,

$$\tilde{\phi}(s) = \frac{1}{s(s+1)^2} \tag{4}$$

Taking the inverse Laplace transform of equation (4), the solution of the given integro-differential equation is given by,

$$\phi(x) = 1 - e^{-x} - x\,e^{-x} \tag{5}.$$

Problem 84.

$$\phi''(x) + \phi(x) + \int_0^x \sinh(x-t)\,\phi(t)\,dt$$

$$+ \int_0^x \cosh(x-t)\,\phi'(t)\,dt = \cosh(x) \qquad (1)$$

with $\phi(0) = \phi'(0) = 0$ (2)

Solution:

Considering the Laplace transformation of equation (1) with due attention to the initial conditions of equation (2), we have

$$s^2\,\bar{\phi}(s) + \bar{\phi}(s) + \frac{1}{s^2-1}\,\bar{\phi}(s) + \frac{s^2}{s^2-1}\,\bar{\phi}(s) = \frac{s}{s^2-1} \qquad (3)$$

or,

$$\left[(s^2+1) + \frac{s^2+1}{s^2-1}\right]\bar{\phi}(s) = \frac{s}{s^2-1}$$

or,

$$\bar{\phi}(s) = \frac{1}{s(s^2+1)} \qquad (4)$$

Taking the inverse Laplace transform of equation (4), the solution of the given integro-differential equation is given by

$$\phi(x) = 2\sin^2(x/2) = 1 - \cos(x) \qquad (5).$$

Problem 85.

$$\phi''(x) + (x) + \int_0^x \sinh(x-t)\,\phi(t)\,dt$$

$$+ \int_0^x \cosh(x-t)\,\phi'(t)\,dt = \cosh(x) \qquad (1)$$

with

$$\phi(0) = -1 \quad \text{and} \quad \phi'(0) = 1 \qquad (2)$$

Solution:

Considering the Laplace transformation of equation (1) we have

$$\left\{\left[s^2\,\tilde{\Phi}(s) - s\,\phi(0) - \phi'(0)\right] + \tilde{\Phi}(s) + \frac{1}{s^2-1}\,\tilde{\Phi}(s)\right.$$

$$\left. + \frac{s}{s^2-1}\left[s\,\tilde{\Phi}(s) - \phi(0)\right]\right\} = \frac{1}{s^2-1} \qquad (3)$$

Imposing the initial conditions of equation (2) into equation (3), we have

$$\left[(s^2+1) + \frac{s^2+1}{s^2-1}\right]\tilde{\Phi}(s) = \frac{s}{s^2-1} - s + 1 - \frac{s}{s^2-1}$$

or

$$\tilde{\Phi}(s) = \left[-\frac{(s-1)(s^2-1)}{s^2(s^2+1)}\right] = \left[\frac{1}{s} - \frac{1}{s^2} + \frac{2}{s^2+1} - \frac{2s}{s^2+1}\right] \qquad (4)$$

Taking the inverse Laplace transform of equation (4), the solution of the given integro-differential equation is given by,

$$\phi(x) = 1 - x + 2\sin(x) - 2\cos(x) \qquad (5).$$