Mathematics Notes

Note 101

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Solved Problems in Fourier Transforms - Part 1

by

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Abstract

Fourier transforms and Laplace transforms have fundamental value to electrical engineers in solving many problems. Waves are ubiquitous or found everywhere. Perhaps the most basic wave is a harmonic or a sinusoidal wave. Mathematical description of any type of wave was recognized early on, to be a combination of sinusoidal waves. The Fourier series is a way of mathematically expressing a periodic time domain waveform. For aperiodic or transient waveforms, we use Fourier or Laplace transforms noting that Laplace transforms is a better option for transient problems. Fourier and Laplace were contemporary mathematicians in France. One of the popular books on Fourier Transforms is titled "***Fourier Integrals and its Applications***" by Athanasios Papoulis published by McGraw-Hill Book Companies in 1962. This book has 40 odd problems with some hints to solve these problems analytically. The trend nowadays is to use numerical codes to get Fourier transforms. The art of putting pencil-to-paper and solving such problems is going out of style. This note by a septuagenarian is an attempt to walk a nostalgic path and analytically solve Fourier transform problems. Half of the problems in this book are fully solved and presented in this note. Hopefully, one of the readers will be inspired to solve the remaining half.

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Fourier Transform Problems and Solutions

Problem (1)

Show that if $e^{j\phi(t)} \leftrightarrow F(\omega)$ and $\phi(t)$ is real, then

$$\cos\phi(t) \leftrightarrow \frac{[F(\omega) + F(-\omega)]}{2}; \quad \sin\phi(t) \leftrightarrow \frac{[F(\omega) - F(-\omega)]}{2j}$$

Solution: we know that, if

$$f(t) \leftrightarrow F(\omega)$$

then

$$\tilde{f}(t) \leftrightarrow \tilde{F}(-\omega)$$

∴ if

$$e^{j\phi(t)} \leftrightarrow F(\omega)$$

then

$$e^{-j\phi(t)} \leftrightarrow \tilde{F}(-\omega)$$

also

$$\cos\phi(t) = \frac{e^{j\phi(t)} + e^{-j\phi(t)}}{2}$$

and

$$\sin\phi(t) = \frac{e^{j\phi(t)} - e^{-j\phi(t)}}{2j}$$

Taking the Transform, and making use of the Linearity property, one can write.

$$\cos\phi(t) \leftrightarrow \frac{F(\omega) + \tilde{F}(-\omega)}{2}$$

and

$$\sin\phi(t) \leftrightarrow \frac{F(\omega) - \tilde{F}(-\omega)}{2j}$$

Problem (2)

Prove that if $f(t)$ is causal, then the real part $R(w)$ of its Fourier Transform Satisfies

$$R(\omega) = \frac{2}{\pi}\int_0^\infty \int_0^\infty R\,(y)\cos yt\,\cos\omega t\; dy\; dt.$$

Solution: For a causal function, we have,

$$f(t) = \frac{2}{\pi}\int_0^\infty R\,(y)\cos yt\; dy \rightarrow 1 \quad and \quad R(\omega) = \int_0^\infty f\,(t)\cos\omega t\; dt \rightarrow (2)$$

Substituting for $f(t)$ from (1) into (2), we have

$$R(\omega) = \frac{2}{\pi}\int_0^\infty \int_0^\infty R\,(y)\cos yt\,\cos\omega t\; dy\; dt$$

.

Problem (3)

Prove that

$$\int_{-\infty}^{\infty} \frac{\sin^3 t}{t^3}\,dt = \frac{3\pi}{4}; \quad \int_{-\infty}^{\infty} \frac{\sin^4 t}{t^4}\,dt = \frac{2\pi}{3}$$

Solution: we know

$$\frac{\sin t}{t} \leftrightarrow \pi p_1(\omega), \quad \frac{\sin^2 t}{t^2} \leftrightarrow \pi q_2(\omega)$$

and Parseval's relation

$$\int_{-\infty}^{\infty} f_1(t)\, f_2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(-\omega)\, F_2(\omega)\, d\omega \Big)$$

Using above result

$$\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} \cdot \frac{\sin t}{t} dt = \frac{\pi^2}{2\pi} \int_{-\infty}^{\infty} q_2\,(\omega)\; p_1(\omega)\; d\omega$$
$$= \frac{\pi}{2} \int_{-1}^{1} q_2\,(\omega)\; d\omega = \frac{\pi}{2} \int_{-1}^{1} \left(1 - \frac{|\omega|}{2}\right) d\omega$$
$$= 3\pi/4$$

Hence,

$$\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} \cdot \frac{\sin^2 t}{t^2} dt = \frac{\pi^2}{2\pi} \int_{-\infty}^{\infty} q_2\,(\omega)\; q_2(\omega\,) d\omega$$
$$= \frac{\pi}{2} \int_{-2}^{+2} \left(1 - \frac{|\omega|}{2}\right)^2 d\omega = \frac{2\pi}{3}$$

Problem (4)

Show that if $f(t)$ is band limited

$$F(w) = 0 \text{ for } |w| > \Omega, \text{ then}$$
$$f(t) * \frac{\sin at}{\pi t} = f(t)$$

Solution: From the band limitation on $F(w)$ it follows that

$$F(w)\; p_a(w) = F(w) \quad \text{for } a > \Omega$$

we further know

$$\frac{\sin at}{\pi t} \leftrightarrow \mathrm{p_a}(\omega)$$

we have the Convolution Theorem Saying

$$f_1(t) * f_2(t) \leftrightarrow F_1(\omega)\; F_2(\omega)$$
$$\therefore\; f(t) * \frac{\sin at}{\pi t} \leftrightarrow F(\omega)\; p_a(\omega)$$
$$\leftrightarrow F(\omega) \text{ for } a > \Omega.$$
$$\text{clearly } f(t) * \frac{\sin at}{\pi t} = f(t) \text{ for } a > \Omega.$$

Problem (5a)

Show that

$$e^{j(at^2+bt+c)} \leftrightarrow \sqrt{\frac{\pi j}{a}} \exp\left[j\left\{c - \frac{(b-\omega)^2}{4a}\right\}\right]$$

Solution: Let $F(\omega)$ represent the required Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} e^{j(at^2+bt+c)}\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \exp\left(at^2 + \overline{b-\omega}t + c\right) dt$$

writing $$(at^2 + (b-\omega)t + c) = \left[\sqrt{a}t + \frac{b-\omega}{2\sqrt{a}}\right]^2 + \left[c - \frac{(b-\omega)^2}{4a}\right]$$

$$F(\omega) = \exp j\left[c - \frac{(b-\omega)^2}{4a}\right] \int_{-\infty}^{\infty} \exp j\left[\sqrt{a}t + \frac{b-\omega}{2\sqrt{a}}\right\}^2 dt$$

putting $j\left\{\sqrt{at} + \frac{b-\omega}{2\sqrt{a}}\right\}^2 = -x$

$$\text{Differentiating } dt = \frac{-dx}{2j\sqrt{a}\sqrt{xj}} = \frac{dx}{2\sqrt{x}}\sqrt{\frac{j}{a}}$$

$$\therefore \quad F(w) = \exp j\left[c - \frac{(b-\omega)^2}{4a}\right] 2\int_0^{\infty} \frac{e^{-x}}{2\sqrt{x}}\sqrt{\frac{j}{a}}\, dx$$

$$= \exp j\left[c - \frac{(b-\omega)^2}{4a}\right] \sqrt{\frac{j}{a}} \int_0^{\infty} \frac{e^{-x}}{\sqrt{x}}\, dx$$

This integral by defn. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\therefore F(w) = \sqrt{\frac{\pi j}{a}} \exp\left[j\{c - \frac{(b-\omega)^2}{4a}\}\right]$$

Problem (5b)

Show that

$$\cos(at^2 + bt + c) \leftrightarrow \sqrt{\frac{\pi}{a}} \cos\left(\frac{b^2}{4a} + \frac{\omega^2}{4a} - c - \frac{\pi}{4}\right) e^{j\left(\frac{b\omega}{2a}\right)}$$

Solution: Using the result proved in prob 1

$$\cos(at^2+bt+c) \leftrightarrow \frac{1}{2}\left[\sqrt{\frac{\pi j}{a}}e^{j\left[c-\frac{b^2}{4a}-\frac{w^2}{4a}+\frac{bw}{2a}\right]}+\sqrt{\frac{-\pi j}{a}}e^{-j\left[c-\frac{b^2}{4a}-\frac{w^2}{4a}-\frac{b\omega}{2a}\right]}\right]$$

writing $\sqrt{j}$ as $e^{j\frac{\pi}{4}}$ and $\sqrt{-j}$ as $e^{-j\pi/4}$

$$\leftrightarrow \sqrt{\frac{\pi}{a}}e^{j\left(\frac{b\omega}{2a}\right)}\cos\left(\frac{b^2}{4a}+\frac{\omega^2}{4a}-c-\frac{\pi}{4}\right)$$

Problem (5c)

Show that

$$\sin(at^2+bt+c) \leftrightarrow \sqrt{\frac{\pi}{a}}\sin\left(c+\frac{\pi}{4}-\frac{b^2}{4a}-\frac{\omega^2}{4a}\right)e^{j\left(\frac{b\omega}{2a}\right)}$$

Solution: Again, using the result proved in prob 1

$$\sin(at^2+bt+c)$$

$$\leftrightarrow \frac{1}{2j}\left[\sqrt{\frac{\pi j}{a}}e^{j\left(c-\frac{b^2}{4a}-\frac{\omega^2}{4a}+\frac{b\omega}{2a}\right)}-\sqrt{-\frac{\pi j}{a}}e^{-j\left(c-\frac{b^2}{4a}-\frac{\omega^2}{da}+\frac{h\omega}{2a}\right)}\right]$$

writing $\sqrt{j}$ as $e^{j\pi/4}$ and $\sqrt{-j}$ as $e^{-j\pi/4}$

$$\leftrightarrow \sqrt{\frac{\pi}{a}}e^{j\left(\frac{b\omega}{2a}\right)}\frac{\left[e^{j\left(c+\frac{\pi}{4}-\frac{b^2}{4a}-\frac{\omega^2}{4a}\right)}-e^{-j\left(c+\frac{\pi}{4}-\frac{b^2}{4a}+\frac{\omega^2}{4a}\right)}\right]}{2j}$$

$$\leftrightarrow \sqrt{\frac{\pi}{a}}e^{j\left(\frac{b\omega}{2a}\right)}\ \sin\left(c+\frac{\pi}{4}-\frac{b^2}{4a}-\frac{\omega^2}{4a}\right)$$

Problem (5d)

Show that

$$\sin\omega_0 t\cos\alpha t^2 \leftrightarrow \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\left[e^{j\pi/4}\cos\frac{(\omega+\omega_0)^2}{4\alpha}+e^{-i\frac{\pi}{4}}\cos\frac{(\omega-\omega_0)^2}{4\alpha}\right]$$

Solution: We know

$$\sin\omega_0 t \ \ \cot\alpha t^2 = \frac{1}{2}[\sin(\omega_0 t + \alpha t^2) + \sin(\omega_0 t - \alpha t^2)]$$

Fourier Transform of L.H.S in the sum of individual Fourier Transforms of the two terms on R.H.S.
we know

$$\sin(\omega_0 t + \alpha t^2) \leftrightarrow \frac{1}{2j}\left[\sqrt{\frac{\pi j}{\alpha}}\, e^{j\left\{-\frac{(\omega_0-\omega)^2}{4\alpha}\right\}} - \sqrt{\frac{-\pi j}{\alpha}}\, e^{-j\left\{\frac{(\omega_0+\omega)^2}{4\alpha}\right\}}\right]$$

$$\leftrightarrow \sqrt{\frac{\pi}{\alpha}}\frac{1}{2}\left[e^{-j\frac{\pi}{4}} e^{j\left[-\frac{(\omega_0-\omega)^2}{4\alpha}\right]} + e^{j\frac{\pi}{4}} e^{-j\left[-\frac{(\omega_0+\omega)^2}{4\alpha}\right\}}\right]$$

Hence $\sin(\omega_0 t - \alpha t^2)$

$$\leftrightarrow \sqrt{\frac{\pi}{\alpha}}\frac{1}{2}\left[e^{-j\frac{\pi}{4}} e^{j\left\{\frac{(\omega_0-\omega)^2}{4\alpha}\right\}} + e^{j\frac{\pi}{4}} e^{-j\left\{\frac{(\omega_0+\omega)^2}{4\alpha}\right\}}\right]$$

$\therefore \sin\omega_0 t\cos\alpha t^2$

$$\leftrightarrow \sqrt{\frac{\pi}{\alpha}}\frac{1}{2}\left[e^{-j\frac{\pi}{4}} e^{j\left[-\frac{(\omega_0-\omega)^2}{4\alpha}\right]} + e^{j\frac{\pi}{4}} e^{-j\left[-\frac{(\omega_0+\omega)^2}{4\alpha}\right\}}\right] + \sqrt{\frac{\pi}{\alpha}}\frac{1}{2}\left[e^{-j\frac{\pi}{4}} e^{j\left\{\frac{(\omega_0-\omega)^2}{4\alpha}\right\}} + e^{j\frac{\pi}{4}} e^{-j\left\{\frac{(\omega_0+\omega)^2}{4\alpha}\right\}}\right]$$

$$\leftrightarrow \sqrt{\frac{\pi}{\alpha}}\frac{1}{2}\left[e^{-j\frac{\pi}{4}}\, \frac{e^{j\frac{(\omega_0-\omega)^2}{4\alpha}} + e^{-j\frac{(\omega_0-\omega)^2}{4\alpha}}}{2} + e^{j\frac{\pi}{4}}\, \frac{e^{j\frac{(\omega_0+\omega)^2}{4\alpha}} + e^{-j\frac{(\omega_0+\omega)^2}{4\alpha}}}{2}\right]$$

$$\leftrightarrow \sqrt{\frac{\pi}{\alpha}}\frac{1}{2}\left[e^{j\frac{\pi}{4}}\cos\frac{(\omega+\omega_0)^2}{4\alpha} + e^{-j\frac{\pi}{4}}\cos\frac{(\omega-\omega_0)^2}{4\alpha}\right]$$

Problem (6)

Snow that if $f(t)$ is a solution of the differential equation $\frac{d^2x(t)}{dt^2} - t^2x(t) = \lambda[x(t)]$ then its transform $F(w)$ is a solution of the same equation.

Solution: use is made of the following properties.

if $f(t) \leftrightarrow F(w)$

then $\frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n \; F(\omega) \quad \therefore \quad \frac{d^2 f(t)}{dt^2} \leftrightarrow (j\omega)^2 \; F(\omega)$

and $\frac{d^n F(\omega)}{d\omega^n} \leftrightarrow (-jt)^n f(t) \quad \therefore (-jt)^2 f(t) \leftrightarrow \frac{d^2 F(\omega)}{d\omega^2}$

The given differential equn can be written as

$$\frac{d^2x(t)}{dt^2} + (-jt)^2 x(t) = \lambda x(t)$$

Taking Fourier Transforms of both sides,

$$(j\omega)^2 F(\omega) + \frac{d^2F(\omega)}{d\omega^2} = \lambda F(\omega)$$

$$\text{or } \frac{d^2F(\omega)}{d\omega^2} - \omega^2 F(\omega) = \lambda F(\omega)$$

This means $F(w)$ is a solution of the same differential equation.

Problem (7)

Show that if $x(t)$ is a solution of

$$\frac{d^2x(t)}{dt^2} - t^2x(t) = -(2n+1)x(t) \text{ where } n \text{ is an}$$

integer. then except for a constant factor, $x(t)$ is its own transform.
Hint. Try a solution of the form $x(t) = \mathrm{H_n}(t)e^{-t^2/2}$ where $H_n(t)$ is a polynomial. show that there is only one soln and use the result of problem 1.

Solution: Let $x(t) \leftrightarrow X(\omega)$

we need to show, $x(t) = (\text{const})\, X(t)$ ...

$$\frac{d^2x(t)}{dt^2} - t^2x(t) = -(2n+1)x(t)$$

we try a Soln of the form $x(t) = H_n(t)e^{-t^2/2}$ where

$$H_n(t) \text{ is a polynomial } = \sum_{n=0}^{n} a_n\, t^n$$

$$\therefore \frac{dx(t)}{dt} = H_n(t)\, e^{-t^2/2}(-t) + e^{-\frac{t^2}{2}}\dot{H}_n(t)$$

$$\therefore \frac{d^2x(t)}{dt^2} = e^{-t^2/2}\left[\ddot{H}_n(t) - 2t\dot{H}_n(t) + t^2H_n(t) - H_n(t)\right].$$

substituting into the given differential equn.

$$\begin{cases} e^{-t^2/2}\left[\ddot{H}_n(t) - 2t\dot{H}_n(t) + t^2H_n(t) \quad -H_n(t) - t^2H_n(t)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad = -(2n+1)H_n(t)e^{-t^2/2} \end{cases}$$

This reduces to

$$\left[\ddot{H}_n(t) - 2t\dot{H}_n(t) + 2nH_n(t)\right] = 0 \rightarrow \text{Hermite equation}$$

we had $H_n(t) = \sum_{n=0}^{n} a_n\, t^n$

$$\therefore \dot{H}_n(t) = \sum_{n=0}^{n} n\, a_n t^{n-1}$$

$$\ddot{H}_n(t) = \sum_{n=0}^{n} n\,(n-1)a_n t^{n-2}$$

substituting into the Hermite equn.

$$\sum_{n=0}^{n} n\,(n-1)a_n t^{n-2} - 2\sum_{n=0}^{n} n\, a_n t^n + 2n\sum_{n=0}^{n} a_n\, t^n = 0$$

equating the coeffn of t^k on either side we get the recursion

$$(k+2)(k+1)a_{k+2} - 2ka_k + 2na_k = 0$$
$$\therefore a_{k+2} = \frac{2(k-n)a_k}{(k+1)(k+2} \text{ for all k}$$

Thus $x(t) = H_n(t)e^{-t^2/2}$ is an unique soln of the given differential equ. We also know, using the result of Prob 6 that $x(t)$ is also a Soln. $[x(t) \leftrightarrow X(w)]$. since $x(t)$ is an unigue soly. &$X(t)$ is also a soln, we conclude that they differ only by a constant, that is

$$x(t) = (\text{const})\ X(t)$$

Problem (8)

The Fourier Transform of a function f(t) is an unit step $F(\omega) = U(\omega)$: find $f(t)$?

Solution:

$$U(\omega) = \begin{cases} 1 & \text{for} \quad \omega > 0 \\ 0 & \text{for} \quad \omega < 0 \end{cases}$$

Let $f(t) \leftrightarrow U(\omega)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U\,(\omega)e^{+j\omega t}d\omega$$
$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{j\omega t}\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{j\omega t}\, d\omega - \frac{1}{2\pi}\int_{-\infty}^{0} e^{2\omega^2}\, d\omega$$
$$= \delta(t) - \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-\infty}^{0}$$
$$= \delta(t) - \frac{1}{2\pi jt}$$
$$\therefore f(t) = \quad \delta(t) + \frac{j}{2\pi t}$$

Problem (9)

The function $f(t)$ is real and its Fourier Transform is $F(w)$ with

$$\frac{1}{\pi}\int_{0}^{\infty} F\,(\omega)\, e^{j\omega t}\, d\omega = f_1(t) + j\, f_2(t)$$

show that $f_1(t) = f(t)$, $f_2(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{(t-\tau)}\, d\tau$

Solution: The Fourier transform of $f_1(t) + jf_2(t)$ equals $2F(w)U(w)$ where $U(w)$ is an unit step.

$$f(t) \leftrightarrow F(\omega)$$
$$2U(\omega) \leftrightarrow \left\{\delta(t) + \frac{j}{\pi t}\right\} \quad \text{from problem 8.}$$

also

$$\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \leftrightarrow F_1(\omega)F_2(\omega)$$

$$\int_{-\infty}^{\infty} f_1(\tau)\left\{\delta(t-\tau) + \frac{j}{\pi(t-\tau)}\right\} d\tau \leftrightarrow F_1(\omega)\, 2\, U(\omega)$$

But

$$f_1(t) + jf_2(t) \leftrightarrow F_1(\omega)\, 2\, U(\omega)$$

$$\therefore f_1(t) + jf_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\left\{\delta(t-\tau) + \frac{j}{\pi(t-\tau)}\right\} d\tau \quad = \int_{-\infty}^{\infty} f_1(\tau)\delta(t-\tau)\, d\tau + \frac{j}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{(t-\tau)}\, d\tau$$

$$\therefore f_1(t) + jf_2(t) = f(t) + \frac{j}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{(t-\tau)}\, d\tau$$

Equating real and imaginary parts,

$$\therefore \quad f_1(t) = f(t)$$
$$\text{and } f_2(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{(t-\tau)}\, d\tau.$$

Problem (10)

Find the Transform of a finite pulse train = $\sum_{k=0}^{n-1} \delta\,(t - kr)\, -j\omega KT$

Solution: we know that $\delta(t - kT) \leftrightarrow e^{-j\omega kT}$

$\therefore$ using the linearity property

$$\sum_{k=0}^{n-1} \delta\,(t-kr) \leftrightarrow \sum_{k=0}^{n-1} e^{-j\omega kT}$$

$$F(\omega) = 1 + e^{-j\omega T} + e^{-j\omega 2T} + \cdots + e^{-j\omega(n-1)T}$$

$$\therefore F(\omega)e^{-j\omega T} = e^{-j\omega T} + e^{-j\omega 2T} + e^{-j\omega 3T} + \cdots + e^{-j\omega nT}$$

subtracting

$$F(\omega)\left[1 + e^{-j\omega T}\right] = 1 - e^{-j\omega nT}$$

$$F(\omega) = \frac{1 - e^{-j\omega nT}}{1 + e^{-j\omega T}} = \frac{e^{-j\omega nT/2}}{e^{-j\omega T/2}} \frac{e^{j\omega nT/2} - e^{-j\omega nT/2}}{e^{j\omega T/2} - e^{-j\omega T/2}}$$

$$= \left\{e^{-j(n-1)\omega T/2} \quad \frac{\sin n\omega T/2}{\sin \omega T/2}\right\}$$

Problem (11)

The Fourier spectrum of a sequence of pulses

$$x^*(t) = \sum_{-\infty}^{\infty} x_n\, \delta(t - nT) \quad \text{is } A^*(\omega)\text{. we form}$$

the fn. $y^*(t) = \sum_{-\infty}^{\infty} y_n\, \delta(t-nT)$ where

$y_n = \left\{\frac{1}{N}\sum_{k=n-N+1}^{n} x_k\right\}$ is the average of N consecutive values of x_n. Show that the spectrum of $y^*(t)$ is given by

$$\frac{1}{N} A^*(\omega) \left\{\frac{\sin nT\omega/2}{\sin \omega T/2}\right\}$$

Solution:

$$x^*(t) = \sum_{-\infty}^{\infty} x_n\, \delta(t-nT) \leftrightarrow A^*(\omega)$$

$$A^*(\omega) = \sum_{-\infty}^{\infty} x_n\, e^{-j\omega nT} = e^{-j(n-1)T\omega/2} x_n \frac{\sin \frac{nT\omega}{2}}{\sin \frac{T\omega}{2}}$$

using the result of prob.10.
We now form the fo: $y^*(t)$

$$y^*(t) = \sum_{-\infty}^{n}\left[\frac{1}{N}\sum_{k=n-N+1}^{n} x_k\right]\delta(t-nT)$$
$$y^*(t) = \frac{1}{N}\sum_{-\infty}^{\infty}\left[\sum_{n-N+1}^{n} x_k\right]\delta(t-nT)$$

using the convolution Th, $f_1(t) * f_2(t) \Leftrightarrow F_1(\omega)F_2(\omega)$
In place of integrals, we have infinite summations which are essentially the same because of multiplication by a delta function

$\therefore$ The spectrum of $y^*(t)$

$$= \frac{1}{N}[\text{ Spectrum of } \sum x_n \delta(t-nT)]$$
$$= \frac{1}{N} A^*(\omega)$$

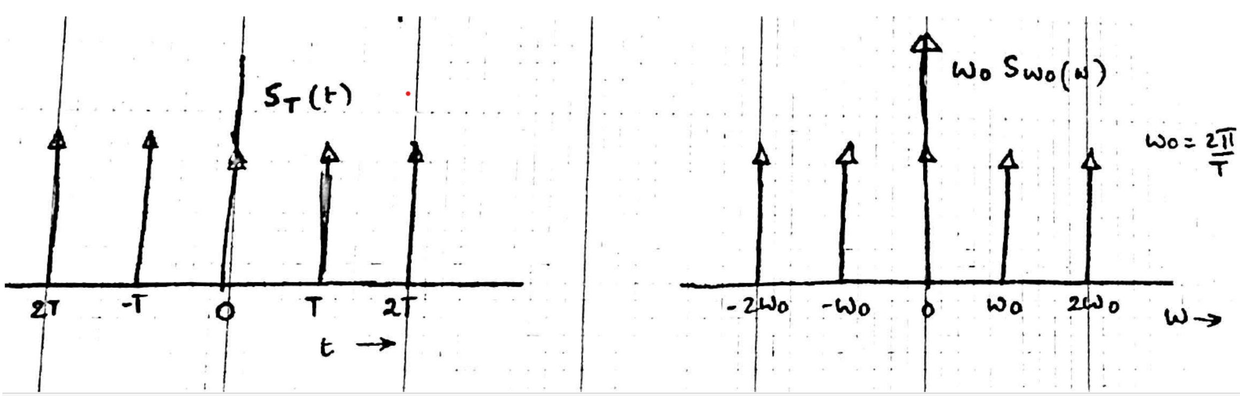
Problem (12)

Show that the Transform of a train of doublets is a train of impulses

$$\sum_{n=-\infty}^{\infty} \delta'(t-nT) = -\frac{4\pi}{T^2}\sum_{n=1}^{\infty} n \sin\frac{2\pi nt}{T} \quad \omega_0 = \frac{2\pi}{T}$$

Solution.: we know that

$$S_T(t) \leftrightarrow \omega_0\, S_{\omega_0}(\omega)$$



also

$$\frac{df(t)}{dt} \leftrightarrow (jw)\, F(\omega)$$

we have

$$\sum_{-\infty}^{\infty} \delta\,(t - nT) \leftrightarrow \omega_0\, S_{\omega_0}(\omega)$$

$$\therefore \quad \sum_{-\infty}^{\infty} \delta'\,(t - nT) \leftrightarrow (j\omega_0)\, \omega_0\, S_{\omega_0}(\omega)$$

$$\leftrightarrow j\omega_0^2 \sum_{-\infty}^{\infty} \omega\, \delta(\omega - n\omega_0)$$

$$\leftrightarrow j\omega_0 \sum_{-\infty}^{\infty} n\, \omega_0 \delta(\omega - n\omega_0)$$

$$\leftrightarrow j\omega_0^2 \sum_{-\infty}^{\infty} n\, \delta(\omega - n\omega_0)$$

But the inverse transform of $\sum n\delta(\omega - n\omega_0)$ is $\frac{1}{2\pi} e^{jn\omega_0 t}$

$$\therefore \quad \sum_{-\infty}^{\infty} \delta'\,(t - nT) = \frac{j\omega_0^2}{2\pi} \sum_{-\infty}^{\infty} n\, e^{jn\omega_0 t}$$

$$= \frac{j2\pi}{T^2} \sum_{-\infty}^{\infty} n\, \cos n\omega_0 t + nj \sin n\omega_0 t$$

$$= \quad -\frac{2\pi}{T^2} \sum_{-\infty}^{\infty} n\, \sin \frac{2\pi nt}{T}$$

Problem (13)

Using Poisson's Sum formula, prove that

$$\sum_{n=\infty}^{\infty} \frac{\sin a(t + nT)}{(t + nT)} = \frac{\omega_C \sin(2N + 1)\, \omega_c t}{\sin \omega_c t}$$

Where $\omega_c = \frac{\pi}{T}$ and N is such that

$$N < \frac{aT}{2\pi} < N + 1$$

Solution: Poisson's sum formula: if $f(t)$ is an arbitrary fn. and $F(\omega)$ its Fourier Transform, then the following identity is true. $(\omega_0 = 2\pi/T)$.

$$\sum_{n=-\infty}^{\infty} f\,(t+nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}\,F(n\omega_0)$$

we know

$$\frac{\sin at}{t} \leftrightarrow \pi P_a(\omega), \quad a = \omega_c = \frac{\pi}{T}, 2\omega_c = \frac{2\pi}{T}$$

Using Poisson's sum formula, we write

$$\sum_{n=-\infty}^{\infty} \frac{\sin a(t+nT)}{(t+nT)} = \frac{\pi}{T}\sum_{n=-\infty}^{\infty} e^{in2\omega_c t}\,p_a(2n\omega_c) \rightarrow (1)$$

but

$$p_n(2n\omega_c) = 0 \quad \text{for} \quad |2n\omega_c| > a$$

ie when $|2n\omega_c| = \left|2n\frac{\pi}{T}\right| = a$

ie when $|n| \geq \frac{aT}{2\pi}$ if $\frac{aT}{2\pi}$ is an integer

or when $n \geq N$ such these $N < \frac{aT}{2\pi} < N+1 \rightarrow (2)$

The summation on the RHS of equn (1) is nonzero for n=-N to N. outside this rang $p_a(2n\omega_c) = 0$ & hence R.H.S $= 0$

$$\therefore \sum_{n=-\infty}^{\infty} \frac{\sin a(t+nT)}{(t+nT)} = \frac{\pi}{T}\sum_{n=-N}^{N} e^{jn2\omega_c t}$$
$$= \left\{\omega_c \frac{\sin(2N+1)\,\omega_c t}{\sin\omega_c t}\right\} \text{ for N described by (2)}$$

We get the required result by writing $\frac{\pi}{T} = \omega_c$ and summing up the Geometric series.

Problem (14)

The input $f(t)$ to a linear system $H(w) = \frac{1}{\alpha + j\omega}$ is a time limited function [$f(t) = 0$ for $|t| > T$] of total energy E. Determine $f_0(t)$ so that the energy of the output is maximum. Try a soln. of the form

$$f_0(t) = \begin{cases} a\cos\omega t & \text{for } |t| < T \\ 0 & \text{for } |t| > T. \end{cases}$$

Solution: The optimum input $f_0(t)$ that maximizes the energy of the output, must satisfy the integral equn.

$$\mu f_0(t) = \int_{-T}^{T} f_0(\tau)\, L(t-\tau) d\tau \quad \to (1) \quad \text{where} \quad L(t) \leftrightarrow |H(\omega)|^2$$

and μ is the maximum eigen value

$$H(\omega) = \frac{1}{\alpha + j\omega} \quad \therefore |H(\omega)| = \frac{1}{\sqrt{\alpha^2 + \omega^2}} \quad \text{and} \quad |H(\omega)|^2 = \frac{1}{\alpha^2 + \omega^2}$$

$$L(t) \quad \text{is the inverse transform of} \quad |H(\omega)|^2 = \frac{e^{-\alpha(t)}}{2\alpha} \quad \text{from} \quad (1)$$

.

$$\therefore 2\alpha\lambda f_0(t) = \int_{-T}^{T} f_0(\tau) e^{|-\alpha(t-\tau)|}\, d\tau = \int_{-T}^{t} f_0(\tau) e^{-\alpha(t-\tau)}\, d\tau + \int_{t}^{T} f_0(\tau) e^{\alpha(t-\tau)}\, d\tau \quad \text{for} \quad |t| < T$$

we now try a soln. $f(t) = a\cos\omega t$ for $|t| < T$

$$2\alpha\lambda \cos\omega t = \int_{-T}^{t} a\cos\omega\tau\, e^{-\alpha(t-\tau)}\, d\tau + \int_{t}^{T} a\cos\omega\tau\, e^{\alpha(t-\tau)}\, d\tau \quad \text{for} \quad |t| < T$$

$$\text{or } 2\alpha\lambda\cos\omega t = \frac{2\alpha\cos\omega t}{\alpha^2 + \omega^2} + \frac{\omega\sin\omega T - \alpha\cos\omega T}{\alpha^2 + \omega^2} e^{-\alpha T}(e^{\alpha t} + e^{-\alpha t})$$

For this to be true for every $|t| < T$ we have

$$2\alpha\lambda\cos\omega t = \frac{2\alpha\cos\omega t}{\alpha^2 + \omega^2} \quad \text{or} \quad \lambda_0 = \frac{1}{\alpha^2 + \omega^2}$$

$=$ max eigen value that maximizes output energy

and $\omega\sin\omega T - \alpha\cos\omega T = 0$ or $\tan\omega T = \alpha/\omega$
w is the smallest root ω_0 of this equation.
$\therefore$ soln is $f_0(t) = a\cos\omega t$ for $|t| < T$
'a' can be found from the energy.

Problem (15)

Show that a System with the following Properties is causal a) it is linear b) if the input $f(t)$ is specified for $t < t_0$, then the output $g(t)$ is uniquely determined for $t < t_0$.

Solution: suppose that $f(t) = 0$ for $t < t_0$. From the Linearity property of the system, it follows that the response to $2f(t)$ is $2g(t)$. For $t < t_0$ we have $f(t) = 0 = 2f(t)$ and from the property (6) we conclude that for $t < t_0 \quad g(t) = 2g(t)$ which is possible only if $g(t) = 0$. This means the system is causal.

Problem (16)

Show that if $f(t)$ is causal and the real part $R(w)$ of its Fourier transform decreases monotonically to zero

$$R(\omega) \geq 0 \quad \frac{dR}{dw} \leq 0, \quad R(\infty) = 0,$$

then

$$|f(t)| \leq \frac{2R(0)}{\pi t}$$

Solution: we know, if $f(t)$ is causal and possesses a Transform $F(\omega) = R(\omega) + jX(\omega)$, then

$$f(\omega) = \frac{2}{\pi}\int_0^\infty R\,(\omega)\cos\omega t \;\; d\omega = -\frac{2}{\pi}\int_0^\infty x\,(\omega)\sin\omega t \;\; d\omega \to (1)$$

further, if $f(t)$ is causal, then $tf(t)$ which is also causal will have its transform $= j\frac{dF(w)}{dw}$

$$\therefore \quad f(t) \quad \to F(\omega)$$

$$tf(t) \quad \leftrightarrow j\frac{dF}{d\omega} = -X'(\omega) + jR'(\omega)$$

Now, from (1),

$$tf(t) = -\frac{2}{\pi}\int_0^\infty R'\,(\omega)\sin\omega t\, d\omega$$

$$tf(t) \leq -\frac{2}{\pi}\int_0^\infty R'\,(\omega)\, d\omega \quad \to \quad \because R'(\omega) \cdot \frac{dR}{d\omega} \leq 0 \quad \text{(given)}$$

$$tf(t) \leq -\frac{2}{\pi}[R(\infty) - R(0)] \quad \text{and} \quad R(\infty) = 0$$

$$tf(t) \leq \frac{2R(0)}{\pi} \qquad \therefore f(t) \leq \frac{2R(0)}{\pi t}$$

Problem (17)

Prove that if a real linear System is causal and the imaginary part $X(\omega)$ of its system function $H(\omega) = R(\omega) + jX(\omega)$ is negative, then its step response $a[t]$ satisfies the following inequality.

$$a(0) \leq a(t) \leq 2R(0) - a(0)$$

Solution: the response to a unit step $U(t)$ is denoted by $a(t) \leftrightarrow A(\omega)$.

$$\begin{aligned} A(\omega) &= U(\omega) \times H(\omega) \\ &= \left[\pi\delta(\omega) + \frac{1}{j\omega}\right][R(\omega) + jX(\omega)] \\ \text{Aise} \quad &= \left\{\pi\delta(\omega)R(0) + \frac{X(\omega)}{\omega}\right\} - j\left\{\frac{R(\omega)}{\omega}\right\} \\ A(\omega) &= R'(\omega) + jX'(\omega) \end{aligned}$$

$a(t)$ is causal and for a causal fn.

$$\therefore a(t) = \frac{2}{\pi}\int_0^\infty \left[\pi\delta(\omega)R(0) + \frac{X(\omega)}{\omega}\right]\cos\omega t \; d\omega$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \pi\,\delta(\omega)R(0)\cos\omega t \; d\omega \int_{-\infty}^{\infty} \frac{X(\omega)}{\omega}\cos\omega t \, d\omega\right]$$

$$= \int_{-\infty}^{\infty} \delta\,(\omega)R(0)\cos\omega t \, d\omega + \frac{2}{\pi}\int_0^\infty \frac{X(\omega)}{\omega}\cos\omega t \, d\omega$$

$$= R(0) + \frac{2}{\pi}\int_0^\infty \frac{X(\omega)}{\omega}\cos\omega t \, d\omega$$

$$|a(t) - R(0)| \leq \frac{-2}{\pi}\int_0^\infty \frac{X(\omega)}{\omega} d\omega \quad \text{since} \quad X(\omega) \quad \text{is given to be -ve}$$

$$\leq [a(t)]_0^\infty$$

$$\leq a(\infty) - a(0)$$

$$\leq R(0) - a(0)$$

We have

$$|a(t) - R(0)| \leq R(0) - a(0)$$

$$\text{or} \quad |a(t)| \leq 2R(0) - a(0)$$

$$\text{or} \quad a(0) \leq |a(t)|$$

putting the above inequalities into one, we have

$$a(0) \leq |a(t)| \leq 2R(0) - a(0)$$

Problem (18)

The input to an ideal tow pass filter

$$H(w) = \begin{cases} A_0 e^{-j\omega t_0} & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

is a train of pulses $f^*(t) = Tf(t)\sum_{n=-\infty}^{\infty} \delta\,(t - nT)$

whose envelope $f(t)$ has a band limited spectrum

$|F(\omega)| = 0$ for $|\omega| > \omega_c$. prove that if $T < \frac{\pi}{\omega_c}$, then the response $g(t)$ of the filter is given by $g(t) = A_0 f(t - t_0)$

Solution: The Fourier Transform of the output is given by

$$G(\omega) = F^*(\omega)H(\omega) \text{ where } f^*(t) \leftrightarrow F^*(\omega)$$

Input to the filter.

$$= f^*(t) = Tf(t)\sum_{-\infty}^{\infty} \delta\,(t - nT)$$

$$\therefore F^*(\omega) = T\sum_{-\infty}^{\infty} f\,(nT)e^{-jn\omega T} \rightarrow (2)$$

A Special form of Poisson's sum formula (see prob 13) may be stated as follows

$$\sum_{n=-\infty}^{\infty} f\,(nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F\,(n\omega_0) \quad \omega_0 = \frac{2\pi}{T}$$

$$\therefore \sum_{n=-\infty}^{\infty} f\,(nT)e^{-j\omega nT} = \frac{1}{T}\sum_{-\infty}^{\infty} F\left(\omega + \frac{2\pi}{T}n\right) \rightarrow (3)$$

Substituting equn(3) in equn(2), we get the following

$$F^*(\omega) = \sum_{n=\infty}^{\infty} F\left(\omega + \frac{2\pi n}{T}\right)$$

$$\therefore \mathbf{G}(\omega) = F^*(\omega)H(\omega)$$

$$= \left\{\sum_{n=-\infty}^{\infty} F\left(\omega + \frac{2\pi}{T}n\right)\right\} \quad \{A_0 e^{-j\omega t_0} p_{\omega_c}(\omega)\}$$

$$= \begin{cases} F(\omega)A_0 e^{-j\omega t_0} & \text{for} \quad T < \dfrac{\pi}{\omega_c} \\ 0 & \text{for other values} \end{cases}$$

$$\text{or} \quad G(\omega) = A_0 F(\omega) e^{-j\omega t_0} \text{ for } T < \frac{\pi}{\omega_c}$$

Taking the inverse transform

$$g(t) = A_0 f(t - t_0)$$

Problem (19)

Band Limited Interpolation: Given an arbitrary fn. $f(t)$ and a constant T, find a function $g(t)$ with a band ltd. spectrum $\left[G(\omega) = 0 \text{ for } |\omega_0| > \frac{\pi}{T} = \omega_c\right]$ such that $g(nT) =$

$f(nT)$ for all integer n.

Solution: we know

$$\frac{\sin w_c t}{\pi t} \leftrightarrow p_{\omega_c}(\omega)$$

$$\therefore \qquad \sin\frac{\omega_c(t - nT)}{\pi(t - nT)} \leftrightarrow p_{\omega_c}(\omega) e^{-j\omega nT}$$

Multiply both sides by a constant $Tf(nT)$

$$\therefore Tf(nT)\frac{\sin(\omega_c t-\omega_c nT)}{\pi(t-nT)} \leftrightarrow Tf(nT)p_{\omega_c}(\omega)e^{-j\omega nT}$$

$$\text{or}\quad Tf(nT)\frac{\sin(\omega_c t-n\pi)}{\omega_c t(t-nT)} \leftrightarrow Tf(nT)p_{\omega_c}(\omega)e^{-j\omega nT}$$

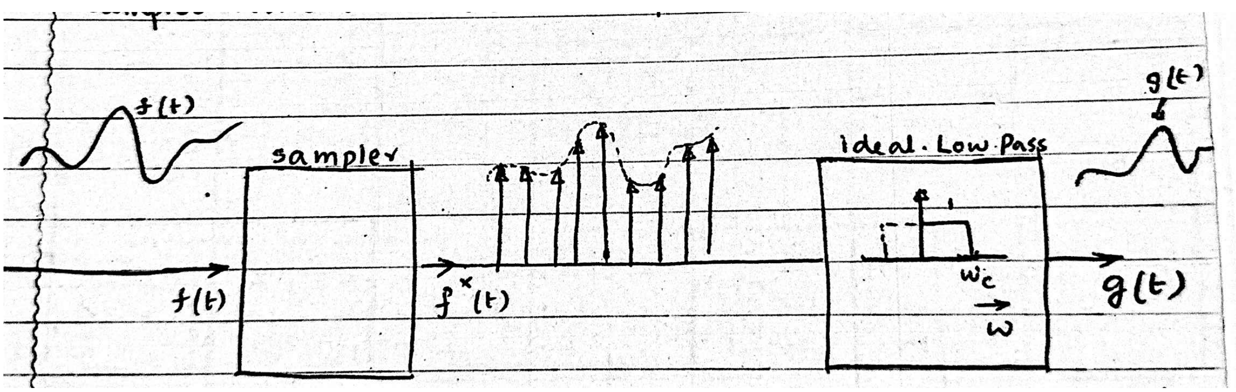
$$\text{or}\quad f(nT)\frac{\sin(\omega_c t-n\pi)}{(\omega_c t-n\pi)} \leftrightarrow Tf(nT)p_{\omega_c}(\omega)e^{-j\omega nT}$$

$$\therefore \sum_{n=-\infty}^{\infty} f\,(nT)\frac{\sin(\omega_c t-n\pi)}{(\omega_c t-n\pi)} \leftrightarrow \sum_{n=-\infty}^{\infty} T\,f(nT)p_{\omega_c}(\omega)e^{-jn\omega T}$$

$$g(t)=\sum_{n=-\infty}^{\infty} f\,(nT)\frac{\sin(\omega_c t-n\pi)}{(\omega_c t-n\pi)} \rightarrow \; (1)$$

satisfies the requirements that, 1)$g(nT)=f(nT)$
and 2)$G(\omega)=0$ for $|\omega|>\omega_c$

A sampler S is a linear (but not a time-invariant) device such that its output to a given input $f(t)$ is given by $f^*(t)=T\sum_{n=-\infty}^{\infty} f\,(nT)\delta(t-nT)$. Show that the above fn. $g(t)$ can be obtained by cascading a sampler with an ideal low-pass filter.



Strictly speaking, we need to take an infinite no of samples to reconstruct the signal. However with a large no of samples $g(t)$ will very nearly follow $f(t)$. When $f(t)$ is fed into the sampler, we get $f^*(t)$ aS the output

$$f^*(t)=T\sum_{n\to\infty} f\,(nT)\,\delta(t-nT)$$

We know that the response of an ideal low-pass filter

$$\text{to} \quad \delta(t-nt) = \frac{\sin[\omega_c(t-nT)]}{\pi(t-nT)}$$

∴ response to $f^*(T)$

$$= T\sum_{-i_0}^{\infty} f\,(nT)\sin\frac{(\omega_c t - \omega_c nT)}{\omega_c T(t-nT)}$$

$$= \sum_{n=-\infty}^{\infty} f\,(nt)\frac{\sin(\omega_c t - n\pi)}{(\omega_c t - n\pi)}$$

$$= g(t) \quad \text{from (1)}$$

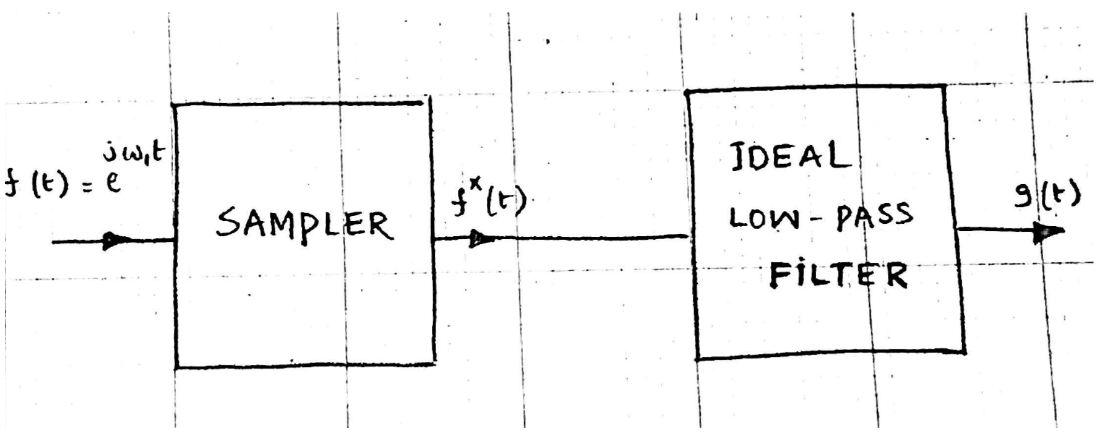
This proves that $g(t)$ can be obtained by cascading a Sampler with an ideal low pass filter.

Problem (20)

Frequency Shifting:

The input to the system of the previous problem is an exponential $f(t) = e^{j\omega_1 t}$ where $(2m-1)\omega_c < \omega_1 < (2m+1)\omega_c$ and m is an integer. Show that the output is given by $g(t) = e^{j\omega_0 t}$ where $\omega_0 = \omega_1 - 2m\omega_c$ Thus a band limited interpolation of an exponential is an exponential of a lower frequency.

Solution:



Input to the Sampler: $f(t) = e^{j\omega_1 t}$
Transform of the input: $F(\omega) = \pi\delta(\omega - \omega_1) \rightarrow (1)$
The output of the sampler = $f^*(t)$ and

$$f^*(t) = T \sum_{n=-\infty}^{\infty} f\,(nT)\delta(t - nT) \quad \text{Let } f^*(t) \leftrightarrow F^*(\omega)$$

we have shown in Problem 18,

$$\begin{aligned} F^*(\omega) &= \sum_{n=-\infty}^{\infty} F\left(\omega + \frac{2\pi n}{T}\right) \\ &= \sum_{n=-\infty}^{\infty} \pi\,\delta\left(\omega - \omega_1 + \frac{2\pi n}{T}\right) \quad \text{using(1).} \\ &= \sum_{n=-\infty}^{\infty} \pi\,\delta(\omega - \omega_1 + 2n\omega_c) \rightarrow \text{ (2).} \end{aligned}$$

we are given $(2m\omega_c - \omega_c) < \omega_1 < (2m\omega_c + \omega_c)$

∴ when $f^*(t)$ $[\leftrightarrow F^*(\omega)]$ is passed through an ideal lowpass filter, the only frequency comp. of the input that lies in the passband of the filter is $\pi\delta(\omega - \omega_1 + 2m\omega_c)$.
Therefore $G(\omega)$ is the Transform of the output of the ideal lowpass filter is $= \pi\delta(\omega - \omega_1 + 2m\omega_c)$

or $G(\omega) = \pi\delta\left(\omega - \overline{\omega_1 - 2m\omega_c}\right)$
Taking inverse Transform

$$\begin{aligned} \therefore g(t) &= \exp[j(\omega_1 - 2m\omega_c)] \\ g(t) &= e^{j\omega_0 t} \text{ where } \omega_0 = \omega_1 - 2m\omega_c. \end{aligned}$$

This result establishes the fact that the Band Limited interpolation of an exponential function is again an exponential of a lower frequency.

Problem (21a)

Rectification:
A full wave rectifier is a device such that its response $g(t)$ to an input $f(t)$ is given ty $g(t) = |f(t)|$. Show that if $f(t)$ is amplitude modulated $f(t) = \phi(t)\sin i\omega_0 t$, with positive envelope $\phi(t) \geq 0$, then the Transform $G(\omega)$ of the output is given by

$$G(\omega) = -\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{\Phi(\omega - 2n\omega_0)}{4n^2 - 1}$$

Solution: Input to the Rectifier: $f(t) = \phi(t)\ \sin i\omega_0 t$

output of the rectifier: $g(t) = |f(t)| = |\phi(t)\ \sin\omega_0 t|$

$$\text{or } g(t) = \phi(t)|\sin\omega_0 t| \quad \because \phi(t) \geq 0 \rightarrow (1)$$

we now expand $|\sin\omega_0 t|$ into a Fourier series and then take the inverse transform to obtain $G(\omega)$.

Let $|\sin\omega_0 t| = |\sin\theta| = \sum_{m=-\infty}^{\infty} C_m\, e^{jmt\omega_0}$

$$\text{Period of } |\sin\theta| = \pi$$

$$\therefore C_m = \frac{1}{\pi}\int_0^T \sin\theta e^{-jm\omega_0 t}\ d\theta = \frac{1}{\pi}\int_0^\pi \sin\theta e^{-jm\theta}\, d\theta$$

$$= \frac{1}{\pi}\left[\left.\frac{e^{-jm\theta}\sin\theta}{-jm}\right\}_0^\pi + \frac{1}{jm}\int_0^\pi e^{-jm\theta}\cos\theta\ \ d\theta\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{jm}\left\{\frac{e^{-jm\theta}}{-jm}\cos\theta\right\}_0^\pi - \frac{1}{jm}\frac{1}{jm}\int_0^\pi e^{-jm\theta}\sin\theta\ \ d\theta\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{m^2}(-\cos m\pi - 1)\right] + \frac{1}{m^2}\frac{1}{\pi}\int_0^\pi e^{-jm\theta}\sin\theta\ \ d\theta\bigg]$$

$$C_m = -\frac{1}{\pi m^2}(\cos m\pi + 1) + \frac{C_m}{m^2}$$

$$\therefore \mathrm{C_m}\left(1 - \frac{1}{m^2}\right) = \frac{-1}{\pi m^2}(\cos m\pi + 1)$$

$$\text{or} \quad C_m = \frac{-1}{\pi(m^2 - 1)}(\cos m\pi + 1)$$

$\therefore$ when m is odd $\quad C_m = 0$

$$\text{when } m \text{ is even} = 2n, \quad C_m = \frac{-2}{\pi(4n^2 - 1)}$$

$$|\sin\omega_0 t| = \sum_{m=-\infty}^{\infty} C_m\, e^{-jm\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} -\frac{2}{\pi}\frac{e^{-j2n\omega_0 t}}{(4n^2 - 1)}$$

$$g(t) = \phi(t)|\sin\omega_0 t|$$

$$= \phi(t)\sum_{n=-\infty}^{\infty} -\frac{2}{\pi}\frac{e^{-j2n\omega_0 t}}{(4n^2 - 1)}$$

Taking the Transform & using the Linearity property,

$$G(\omega) = -\frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{\Phi(\omega - 2n\omega_0)}{(4n^2-1)} \rightarrow (2) \text{ where } \phi(t) \leftrightarrow \Phi(\omega)$$

Problem (21b)

Detection: The output $g(t)$ of the above rectifier is inserted into an ideal lowpass filter of cut off freq ω_0. Show that if $\Phi(\omega) = 0$ for $|\omega| > \omega_0$, then the output $g_1(t)$ of the filer is of the same form an $\phi(t)$,

$$g_1(t) = \frac{2A_0}{\pi}\phi(t - t_0)$$

Solution: Input to the filter: $g(t) \leftrightarrow G(\omega)$ given by (2)

System fn.: $H(\omega) = A_0 e^{-j\omega t_0}\, p_{\omega_0}(\omega)$
output of the filter: $g_1(t) \leftrightarrow G_1(\omega)$
we know $G_1(\omega) = G(\omega)\, H(\omega)$.

$$G_1(\omega) = \left\{\frac{-2}{\pi}\sum_{n=-\infty}^{\infty}\frac{\Phi(\omega - 2n\omega_0)}{(4n^2-1)}\right\}\left\{A_0 e^{-j\omega t_0} p_{\omega_0}(\omega)\right\}$$

we are given $\Phi(\omega) = 0$ for $|\omega| > \omega_0$
The product is nonzero only for $n = 0$

$$\therefore G_1(\omega) = -\frac{2}{\pi}\Phi(\omega)\, A_0 e^{-j\omega t_0}$$
$$= -\frac{2A_0}{\pi}\Phi(\omega) e^{-j\omega t_0}$$

$\therefore$ Taking the inverse Transform,

$$g_1(t) = -\frac{2A_0}{\pi}\phi(t - t_0)$$

Problem (22a)

The input to an ideal low pass filter $H(w) = A_0 e^{-j\omega t_0} p_{\omega_c}(\omega)$ is a step modulated signal $f(t) = E\cos\omega_0 t U(t)$. Show that its response is given by

$$g(t)=\frac{EA_0}{2\pi}[\pi+S_i(\omega_c-\omega_0)(t-t_0)+S_i(\omega_c+\omega_0)(t-t_0)]\cos\omega_0(t-t_0)$$
$$+\frac{EA_0}{2\pi}[C_i(\omega_c-\omega_0)(t-t_0)-C_i(\omega_c+\omega_0)(t-t_0)]\sin\omega_0(t-t_0)$$
$$\text{where}\quad S_i(x)=\int_0^x\frac{\sin y}{y}dy\quad\text{and}\quad C_i(x)=-\int_x^\infty\frac{\cos y}{y}dy$$

Solution: The fitter is characterized by $H(\omega)$

$$H(\omega)=A_0e^{-j\omega t_0}p_{\omega_c}(\omega)\rightarrow(1)$$

Taking the inverse transform

$$\begin{aligned}h(t)&=A_0\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-j\omega t_0}\,p_{\omega_c}(\omega)e^{j\omega t}d\omega\\&=\frac{A_0}{2\pi}\int_{-\omega_c}^{\omega_c}e^{j\omega(t-t_0)}\,d\omega\end{aligned}$$

$$=\frac{A_0}{2\pi}\left[\frac{e^{j\omega(t-t_0)}}{j(t-t_0)}\right]_{-\omega_c}^{+\omega_c}$$
$$=\frac{A_0}{\pi(t-t_0)}\left[\frac{e^{j\omega_c(t-t_0)}-e^{-j\omega_c(t-t_0)}}{2j}\right]$$
$$=A_0\frac{\sin\omega_c(t-t_0)}{\pi(t-t_0)}$$

We further know that the indent response $g(t)$ of the system to any arbitrary input $f(t)$ can be expressed in terms of the impulse response only as follows:

$$\begin{aligned}g(t)&=\int_{-\infty}^{\infty}f\,(\tau)h(t-\tau)\,d\tau\\&=\int_{-\infty}^{\infty}f\,(t-\tau)h(\tau)\,d\tau.\end{aligned}$$

For the step-modulated input $[=E\cos\omega_0tU(t)]$
We have the response given by

$$g(t)=\int_{-\infty}^{\infty}E\cos\omega_0(t-\tau)U(t-\tau).A_0\frac{\sin\omega_c\tau}{\pi\tau}d\tau$$

not considering the small time delay t_0 in $h(t)$, which is a constant and can be plugged in later.

$$g(t)=\frac{EA_0}{\pi}\int_{-\infty}^{t}\cos\omega_0(t-\tau)\frac{\sin\omega_c\tau}{\tau}d\tau$$

$$\therefore \frac{\pi g(t)}{EA_0} = \int_{-\infty}^{t} (\cos\omega_0 t \ \ \cos\omega_0\tau + \sin\omega_0 t \ \ \sin\omega_0\tau)\frac{\sin\omega_c\tau}{\tau} d\tau$$

$$= \cos\omega_0 t \int_{-\infty}^{t} \frac{\cos\omega_0\tau\sin\omega_0\tau}{\tau} d\tau + \sin\omega_0 t \int_{-\infty}^{t} \frac{\sin\omega_0\tau\sin\omega_c\tau}{\tau} d\tau$$

$$\therefore \frac{2\pi g(t)}{EA_0} = \cos\omega_0 t \int_{-\infty}^{t} \frac{\sin(\omega_c - \omega_0)\tau + \sin(\omega_c + \omega_0)\tau}{\tau} d\tau$$

$$+\sin\omega_0 t \int_{-\infty}^{t} \frac{\cos(\omega_c - \omega_0)\tau - \cos(\omega_c + \omega_0)\tau}{\tau} d\tau$$

$$= \cos\omega_0 t \left[\int_{-\infty}^{0} \frac{\sin(\omega_c - \omega_0)\tau}{\tau} d\tau + \int_{0}^{t} \frac{\sin(\omega_c - \omega_0)\tau}{\tau} d\tau\right.$$

$$\left. + \int_{-\infty}^{0} \frac{\sin(\omega_c + \omega_0)\tau}{\tau} d\tau + \int_{0}^{t} \frac{\sin(\omega_c + \omega_0)\tau}{\tau} d\tau\right]$$

$$+\sin\omega_0 t \int_{-\infty}^{t} \frac{\cos(\omega_c - \omega_0)\tau - \cos(\omega_c + \omega_0)\tau}{\tau} d\tau$$

Use is made of a change of variable

$$1. \ (\omega_c - \omega_0)\tau = u$$

$$\therefore d\tau = \frac{du}{(\omega_c - \omega_0)}$$

$$2. \ (\omega_c + \omega_0)\tau = V$$

$$\therefore d\tau = \frac{dV}{(\omega_c + \omega_0)}$$

$$= \cos\omega_0 t \left[\int_{-\infty}^{0} \frac{\sin(u)}{u} du + \int_{0}^{(\omega_c - \omega_0)t} \frac{\sin(u)}{u} du + \int_{-\infty}^{0} \frac{\sin(V)}{V} dV + \int_{0}^{(\omega_c + \omega_0)t} \frac{\sin(V)}{V} dV\right]$$

$$+\sin\omega_0 t \int_{-\infty}^{(\omega_c - \omega_0)t} \frac{\cos u}{u} du - \int_{-\infty}^{(\omega_c + \omega_0)t} \frac{\cos V}{V} dV$$

$$\frac{2\pi g(t)}{EA_0} = \cos\omega_0 t \begin{Bmatrix} \frac{\pi}{2} + S_i(\omega_c - \omega_0)t \\ + \frac{\pi}{2} + S_i(\omega_c + \omega_0)t \end{Bmatrix}.$$

$$+\sin\omega_0 t\{C_i(\omega_c - \omega_0)t - C_i(\omega_c + \omega_0)t\}$$

$$= \frac{EA_0\cos\omega_0 t}{2\pi}\{\pi + S_i(\omega_c - \omega_0)t + S_i(\omega_c + \omega_0)t\}$$

$$+ \frac{EA_0}{2\pi}\cos\omega_0 t\{C_i(\omega_c - \omega_0)t - C_i(\omega_c + \omega_0)t\}$$

Inserting the time delay factor, [t is $(t-t_0)$]

$$g(t)=\frac{EA_0}{2\pi}\cos\omega_0(t-t_0)[\pi+S_i(\omega_c-\omega_0)(t-t_0)+S_i(\omega_c+\omega_0)(t-t_0)]$$
$$+\frac{EA_0}{2\pi}\sin\omega_0(t-t_0)[C(\omega_c-\omega_0)(t-t_0)-C_i(\omega_c+\omega_0)(t-t_0)]$$

Problem (22b)

Show that

$$\lim_{t\to\infty}g(t)=\begin{cases}EA_0\cos\omega_0(t-t_0) & \omega_0<\omega_c\\ 0 & \omega_0>\omega_c\end{cases}$$

Solution:

for $t\to\infty$, the fn. $S_i ax\to\pi/2 \quad (a>0)$ and $\to-\pi/2 \quad (a<0)$
$\&C_i(\infty)=0$
$\therefore$ for $t\to\infty$ and $(\omega_0<\omega_c)$

$$g(t)=\frac{EA_0}{2\pi}\cos\omega_0(t-t_0)[\pi+\pi/2+\pi/2]=EA_0\cos\omega_0(t-t_0)\text{ for }(\omega_0>\omega_C)$$
$$=\frac{EA_0}{2\pi}\cos\omega_0(t-t_0)[\pi-\pi/2-\pi/2]=0$$

Problem (23)

The system function of a linear filter is given by

$$H(\omega)=\begin{cases}A_0e^{-j\theta_0} & w>0\\ A_0e^{j\theta_0} & w<0\end{cases}$$

Determine its impulse response.

Solution:

$$H(\omega)=\begin{cases}A_0(\cos\theta_0-j\sin\theta_0) & \text{for }\omega>0\\ A_0(\cos\theta_0+j\sin\theta_0) & \text{for }\omega<0\end{cases}$$

we can write

$$H(\omega) = A_0(\cos\theta_0 - j\sin\theta_0 \text{Sgn}\omega) \rightarrow (1)$$

$$\text{we know} \quad \text{Sgn}t \leftrightarrow \frac{2}{j\omega}$$

$$\therefore \text{Sgn}(-\omega) \leftrightarrow \frac{2}{jt} \times \frac{1}{2\pi}$$

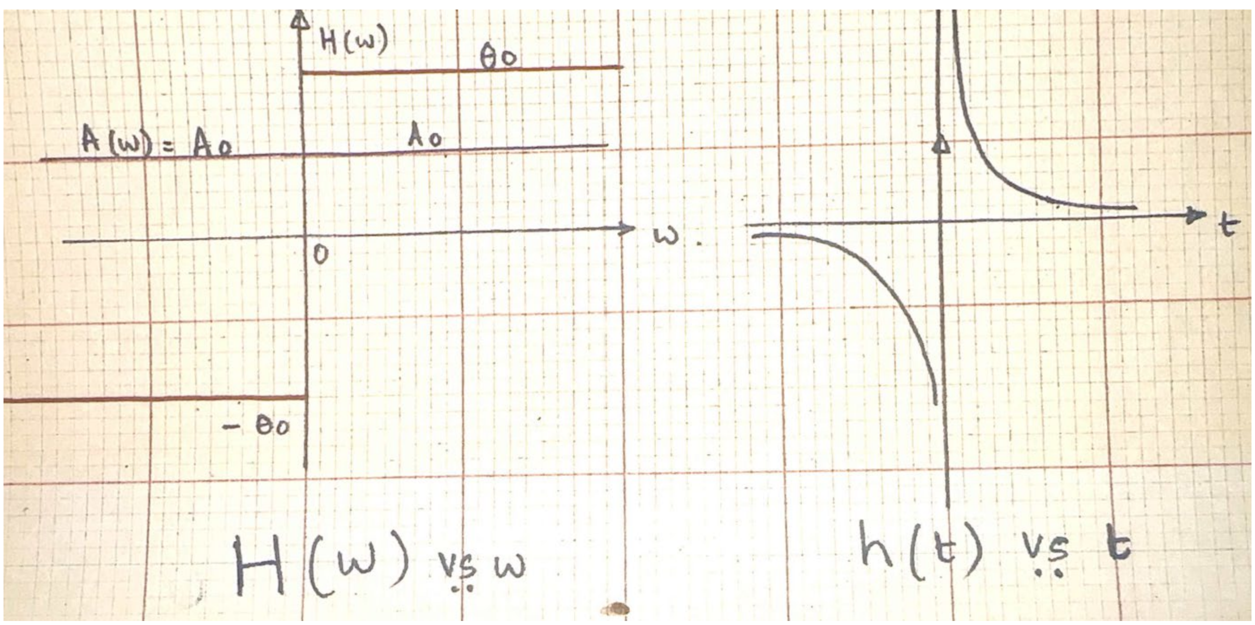
$$\text{or Sgn}(\omega) \leftrightarrow -\frac{1}{\pi jt}$$

$$\text{also} \quad A_0\cos\theta_0 \leftrightarrow A_0\cos\theta_0\delta(t)$$

Taking inverse transform of (1)

$$h(t) = A_0\cos\theta_0\delta(t) - A_0 j\sin\theta_0\left(\frac{-1}{\pi jt}\right)$$

$$= A_0\cos\theta_0\delta(t) + \frac{A_0\sin\theta_0}{\pi t}$$



Problem (24)

Given an all-pass filter $|H(j\omega)| = 1$ with monotone increasing phase lag as in fig. The input $f(t)$ is a signal whose Fourier Spectrum decreases with increasing $\omega \frac{d|F(\omega)|}{dw} \leq 0 \rightarrow (1)$ to is any constant. Show that the RMS error E between the output $g(t)$ and the delayed input $f(t-t_0)$

$$E=\int_{-\infty}^{\infty}[g(t)-f(t-t_0]^2dt \rightarrow (2) \text{ is minimum}$$

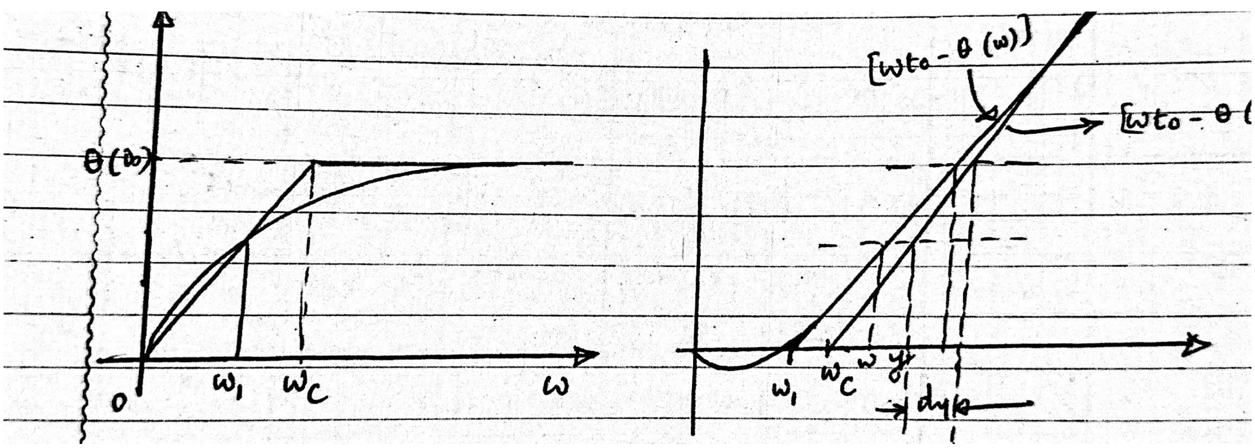
if $\theta(\omega)$ increases linearly to is final value $\theta(\infty)$

$$\theta(\omega)=\begin{cases}\omega t_0 & |\omega|<\omega_c\\ \theta(\infty) & \omega>\omega_c\end{cases} \rightarrow (3) \quad \omega_c=\frac{\theta(\infty)}{t_0}$$

Solution: The error $= g(t)-f(t-t_0)$

The Fourier transform of the error= $F(\omega)\left[e^{-j\theta(\omega)}-e^{-j\omega t_0}\right]$
$\therefore$ to prove (2) it is sufficient to show that

$\int_0^{\infty}|F(\omega)|^2\left|e^{j[\theta(\omega)-\omega t_0]}-1\right|^2d\omega$ is minimum for $\theta(\omega)$ as in (3).



We need to show

$$\therefore \quad \int_0^{\infty}|F(\omega)|^2\left|e^{j[\theta(\omega)-\omega t_0}-1\right|^2d\omega>\int_{\omega_c}^{\infty}|F(y)|^2\left|e^{j[\theta(\infty)-yt_0]}-1\right|^2dy$$

To a given y, we find the corresponding ω such that $\omega t_0-\theta(\omega)=yt_0-\theta(\infty)$. It is teary to see from the given figure that for $\omega<y$, with their corresponding incremental values $d\omega, \& dy$, we have $d\omega>dy$

$$\text{because } [t_0-\theta'(\omega)]\,d\omega=t_0\,dy$$

$$\text{and since } \theta'(\omega) \geq 0 \quad \therefore d\omega > dy.$$

we are also given that $\frac{\mathrm{d}|F(\omega)|}{d\omega} \leq 0$.
ie. $|F(\omega)| > |F(y)|$
The inequality to be proved is the following:

$$\int_0^\infty |F(\omega)|^2 \, e^{i[\theta(\omega)-\omega t_0]} \, d\omega > \int_{\omega_c}^\infty |F(y)^2| \, e^{j[\theta(\infty)-y t_0]} \, dy \rightarrow (4)$$

One can readily see that this is true.

$$\because \text{ we have } \quad |F(\omega)| > |F(y)|$$
$$d\omega > dy$$
$$\therefore \quad |F(\omega)|^2 e^{j[\theta(\omega)-\omega t_0]} d\omega > |F(y)^2| e^{j[\theta(\infty)-y t_0]} dy.$$

$\therefore$ (4) is true which is the requirement to show that the r.m.s error E is minimum

$$\theta(\omega) = \begin{cases} \omega t_0 & |\omega| < \omega_c \\ \theta(\infty) & \omega > \omega_c \end{cases}$$

Problem (25)

Show that if the frequency characteristics of a Low pass fitter are (fig 1)

$$A(\omega) = \begin{cases} A_0 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c. \end{cases}$$

$$\theta(w) = t_0 w + \alpha w^2 \quad w > 0$$

$\theta(-\omega) = -\theta(\omega)$, then its impulse response is given by

$$h(t) = \frac{A_0}{\sqrt{2\pi\alpha}} \Big\{ \big[C\big(\omega_c\sqrt{2\alpha/\pi} - \tau\big) + C(\tau)\big] \cos\frac{\pi}{2}\tau^2 + \big[S\big(\omega_c\sqrt{2\alpha/\pi} - \tau\big) + S(\tau)\big] \sin\frac{\pi}{2}\tau^2 \Big\}.$$

where $\tau = \frac{t-t_0}{\sqrt{2\pi\alpha}}$ and

$$C(\tau) = \int_0^\tau \cos\left(\frac{\pi}{2}y^2\right) dy \qquad S(\tau) = \int_0^\tau \sin\left(\frac{\pi}{2}y^2\right) dy$$

are the Fresnel integrals.

Solution:

$$H(\omega) = A_0 e^{-j\theta(\omega)} p_{\omega_c}(\omega)$$

$$\begin{aligned} \therefore h(t) &= \frac{A_0}{2\pi}\int_{-\infty}^{\infty} e^{-j\theta(\omega)}\, p_{\omega c}(\omega) e^{j\omega t} d\omega \\ &= \frac{A_0}{\pi}\int_0^{\omega_c} \cos\left[\omega t - \theta(\omega)\right] d\omega \quad \text{substituting for } \theta(\omega) \\ &= \frac{A_0}{\pi}\int_0^{\omega_c} \cos\left[\omega(t - t_0) - \alpha\omega^2\right] d\omega \rightarrow (1) \end{aligned}$$

Now let $\tau = \frac{t-t_0}{\sqrt{2\pi\alpha}}$ and let $\omega\sqrt{\alpha} - \tau\sqrt{\frac{\pi}{2}} = x\sqrt{\frac{\pi}{2}}$

(1) can be written as

$$h(t) = \frac{A_0}{\pi}\int_0^{\omega_c} \cos\left[\left(\omega\sqrt{\alpha} - \frac{(t-t_0)}{2\sqrt{\alpha}}\right)^2 - \left(\frac{t-t_0}{2\sqrt{\alpha}}\right)^2\right] d\omega$$

Now with

$$\omega\sqrt{\alpha} - \tau\sqrt{\pi/2} = x\sqrt{\pi/2}$$

$$\left\{d\omega = \sqrt{\pi/2}\,\frac{dx}{\sqrt{\alpha}}\right\}$$

$$\omega\sqrt{\alpha} - \sqrt{\frac{\pi}{2}}\tau = \sqrt{\frac{\pi}{2}}x$$

when $\omega = 0, \quad x = -\tau$

when $\omega = \omega_c, \quad x = \omega_c\sqrt{\alpha} \times \sqrt{\frac{2}{\pi}} - \tau$

$$= \omega_c \sqrt{\frac{2\alpha}{\pi}} - \tau$$

$$\therefore h(t) = \frac{A_0}{\sqrt{2\pi\alpha}} \int_{-\tau}^{\omega_c \sqrt{\frac{2\alpha}{\pi}} - \tau} \cos\frac{\pi}{2}(x^2 - \tau^2)\ dx$$

$$= \frac{A_0}{\sqrt{2\pi\alpha}} \int_{-\tau}^{\omega c \sqrt{\frac{2\alpha}{\pi}} - \tau} \left[\cos\frac{\pi}{2}x^2 \cdot \cos\frac{\pi}{2}\tau^2 + \sin\frac{\pi}{2}x^2 \sin\frac{\pi}{2}\tau^2\right] dx$$

$$= \frac{A_0}{\sqrt{2\pi\alpha}} \cos\frac{\pi}{2}\tau^2 \left[\int_{-\tau}^{0} \cos\left(\frac{\pi}{2}x^2\right) dx + \int_{0}^{\omega c \sqrt{\frac{2\alpha}{\pi}} - \tau} \cos\left(\frac{\pi}{2}x^2\right) dx\right]$$

$$+ \frac{A_0}{\sqrt{2\pi\alpha}} \sin\frac{\pi}{2}\tau^2 \left[\int_{-\tau}^{0} \sin\left(\frac{\pi}{2}x^2\right) dx + \int_{0}^{\omega_0 \sqrt{\frac{2\alpha}{\pi}} - \tau} \sin\left(\frac{\pi}{2}x^2\right) dx\right]$$

$$= \frac{A_0}{\sqrt{2\pi\alpha}} \left\{ \begin{array}{l} \left[-C(\tau) + C\left(\omega_c \sqrt{\frac{2\alpha}{\pi}} - \tau\right)\right] \cos\left(\frac{\pi^2}{2}\right) \\ + \left[-S(\tau) + S\left(\omega_c \sqrt{\frac{2\alpha}{\pi}} - \tau\right)\right] \sin\left(\frac{\pi}{2}\tau^2\right) \end{array} \right\}$$

where $C(\tau), S(\tau)$ are Fresnel integrals

Problem (26)

The phase distortion of the filter of Prob-25. is small. $\alpha\omega_c^2 << 1$. Show that its step response is given by

$$a(t) \approx a_0(t) + \frac{\alpha A_0}{\pi}\left[\frac{2\sin^2[\omega_c(t - t_0)/2]}{(t - t_0)^2} - \frac{\omega_c \sin\omega_c(t - t_0)}{(t - t_0)}\right]$$

Where $a_0(t)$ is the step response of the same fitter without phase distortion.

Solution: If $\Delta\theta(\omega)$ is a small ($<< 1$) phase distortion,

$\Delta\theta(\omega) = \theta(\omega) - \omega t_0 << 1$ in the pass band

Since $e^{-j\Delta\theta(\omega)} \doteq [1 - j\Delta\theta(\omega)]$ for small $\Delta\theta(\omega)$
we have.

$$\begin{aligned} H(\omega) &= A(\omega)e^{-j\theta(\omega)} \\ &\simeq A(\omega)e^{-j\omega t_o}[1 - j\Delta\theta(\omega)] \\ &\simeq H_0(\omega)[1 - j\Delta\theta(\omega)] \end{aligned}$$

where $H_0(\omega)$ is the system fn. with no distortion.

If $a(t)$ is the step response with distortion. $a_0(t)$ is the step response without distortion,

then the Fourier Transform of $a(t) - a_0(t)$ is given by

$$\left[\pi\delta(\omega) + \frac{1}{j\omega}\right][H_0(\omega)(-j)\ \Delta\theta(\omega)]$$

$$= \pi\delta(\omega)H_0(\omega)(-j)\Delta\theta(\omega) - \frac{H_0(\omega)\ \Delta\theta(\omega)}{\omega}$$

$$= -\frac{H_0(\omega)\Delta\theta(\omega)}{\omega} \quad \text{because } \Delta\theta(\omega)\ \delta(\omega) = 0$$

$$\therefore a(t) - a_0(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{H_0(\omega)\Delta\theta(\omega)}{\omega} e^{j\omega t} dt$$

$$\begin{aligned} a(t) - a_0(t) &= -\frac{1}{\pi}\int_0^{\infty} A_0\, e^{-j\omega t_0} p_{\omega c}(\omega) \frac{e^{j\omega t}\Delta\theta(\omega)}{\omega} d\omega \\ \Delta\theta(\omega) &= \begin{cases} +\alpha\omega^2 & \omega > 0 \\ -\alpha\omega^2 & \omega < 0. \end{cases} \\ \therefore a(t) - a_0(t) &= -\frac{A_0\alpha}{\pi}\int_0^{\omega c} \omega \cos\omega(t - t_0)\ d\omega. \end{aligned}$$

Integrating by parts,

$$= +\frac{A_0\alpha}{\pi}\left\{-\frac{\omega \sin\omega_t(t - t_0)}{(t - t_0)}\right\}_0^{\omega c} + \int_0^{\omega c} \frac{\sin\omega(t - t_0)}{(t - t_0)} d\omega\Bigg\}$$

$$= +\frac{A_0\alpha}{\pi}\left[-\frac{\omega \sin\omega(t - t_0)}{(t - t_0)} - \frac{\cos\omega(t - t_0)}{(t - t_0)}\right]_0^{\omega_c}$$

$$\therefore a(t) - a_0(t) = \frac{A_0\alpha}{\pi}\left[-\frac{\cos\omega_c(t - t_0)}{(t - t_0)} + 1 - \omega_c \frac{\sin\omega_c(t - t_0)}{(t - t_0)}\right\}$$

$$\simeq \frac{A_0\alpha}{\pi}\left\{\frac{2\sin^2\omega_c(t - t_0)/2}{(t - t_0)} - \frac{\omega_c \sin\omega_c(t - t_0)}{(t - t_0)}\right\}.$$

∴ The Step Response $a(t)$ is given by

$$a(t) \cong a(0) + \frac{A_0\alpha}{\pi}\left\{\frac{2\sin^2\omega_c(t - t_0)/2}{(t - t_0)} - \frac{\omega_c \sin\omega_c(t - t_0)}{(t - t_0)}\right\}$$