p. 19

Mathematics Notes

Note 96

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Construction of 2 x 2 Product Integrals from Solutions of Second Order Linear Ordinary Differential Equations

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Abstract

From the study of analytic solutions for special cases of the telegrapher equations for nonuniform single-conductor (plus reference) transmission lines, we construct some 2 x 2 product integrals. This is extended to various forms of the solutions of second order linear ordinary differential equations. This is a general procedure which can be used to construct a *table of product integrals* which can be extended to the N x N case for N > 2.

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1. Introduction

In the study of nonuniform multiconductor transmission lines (NMTL) one is presented with equations of the general form

$$\frac{d}{dz}(f_n(z)) = \left(a_{n,m}(z)\right) \cdot \left(f_n(z)\right) + \left(f_n^{(s)}(z)\right)$$

$$\left(f_n^{(s)}(z)\right) \equiv \text{distributed sources}$$

$$\left(f_n(z)\right) \equiv \text{unknowns (voltages, currents)} \tag{1.1}$$

$$\left(a_{n,m}(z)\right) \equiv \text{N x N matrix (involving line parameters)}$$

This is solved via the matrizant equation

$$\frac{d}{dz}\left(\Phi_{n,m}(z,z_0)\right) = \left(a_{n,m}(z)\right) \cdot \left(\Phi_{n,m}(z,z_0)\right)$$

$$\left(\Phi_{n,m}(z_0,z_0)\right) = \left(1_{n,m}\right) \text{ (boundary condition)} \tag{1.2}$$

In terms of this N x N matrix, (1.1) is solved as

$$\left(f_n(z)\right) = \left(\Phi_{n,m}(z,z_0)\right) \cdot \left(f_n(z_0)\right) + \int_{z_0}^{z} \left(\Phi_{n,m}(z,z')\right) \cdot \left(f_n^{(s)}(z')\right) dz' \tag{1.3}$$

The matrizant can in turn be calculated via the product integral [4, 11]

$$\left(\Phi_{n,m}(z,z_0)\right) = \prod_{z_0}^{z} e^{\left(a_{n,m}(z')\right)dz'}$$

$$= \lim_{\substack{L\to\infty \\ (\Delta z \to 0)}} e^{\left(a_{n,m}(z_L)\right)\Delta z} \cdots e^{\left(a_{n,m}(z_2)\right)\Delta z} \cdot e^{\left(a_{n,m}(z_1)\right)\Delta z} \tag{1.4}$$

$$\Delta z = \frac{z - z_0}{L} \quad , \quad z_p = z_0 + p\Delta z$$

$$\left(\Phi_{n,m}(z,z_0)\right) = \lim_{L\to\infty} \bigodot_{p=1}^{L} e^{\left(a_{n,m}(z_L)\right)\Delta z}$$

with continued dot multiplication taken to the let. This form as a product of matrices gives a way to make a numerical calculation (if L is sufficiently large). This also shows why it is called a product integral by comparison with the usual sum (or Riemann) integral.

Our concern here is with the analytic properties of the product integral. Summarizing some general properties

$$
\begin{aligned}
\left(\Phi(z_2,z_0)\right) &= \prod_{z_0}^{z_2} e^{(a_{n,m}(z'))dz'} = \left[\prod_{z_1}^{z_2} e^{(a_{n,m}(z'))dz'}\right] \cdot \left[\prod_{z_0}^{z_1} e^{(a_{n,m}(z'))dz'}\right] \\
&= \left(\Phi(z_2,z_1)\right) \cdot \left(\Phi(z_1,z_0)\right) \\
\left(\Phi_{n,m}(z,z_0)\right)^{-1} &= \left(\Phi_{n,m}(z_0,z)\right) \\
\det\left(\left(\Phi_{n,m}(z,z_0)\right)\right) &= e^{\int_{z_0}^{z} tr\left(\left(a_{n,m}(z')\right)\right)dz'} \\
tr\left(\left(a_{n,m}(z')\right)\right) &= \sum_{n=1}^{N} a_{n,n}(z) \\
&= \sum_{\beta=1}^{N} \lambda_\beta\left(\left(a_{n,n}(z)\right)\right) \\
\lambda_\beta &\equiv \text{eigenvalue of } \left(a_{n,m}(z)\right)
\end{aligned}
\tag{1.5}
$$

There is also what one can call the matrizant series

$$
\begin{aligned}
\left(\Phi_{n,m}(z,z_0)\right) &= \sum_{\ell=0}^{\infty} \left(\Phi_{n,m}(z,z_0)\right)_\ell \\
\left(\Phi_{n,m}(z,z_0)\right)_0 &= \left(1_{n,m}\right) \\
\left(\Phi_{n,m}(z,z_0)\right)_{\ell+1} &= \int_{z_0}^{z} \left(a_{n,m}(z')\right) \cdot \left(\Phi_{n,m}(z',z_0)\right)_\ell dz'
\end{aligned}
\tag{1.6}
$$

Special cases have analytic solutions. (By analytic we mean here that the solutions are expressible by finite combinations of known special functions.) There is the simple case of

$$
\begin{aligned}
\left(a_{n,m}(z)\right) &\equiv b(z)\left(a_{n,m}\right) \quad , \quad \left(a_{n,m}\right) = \text{constant matrix} \\
\left(\Phi_{n,m}(z,z_0)\right) &= e^{\left(a_{n,m}\right)\int_{z_0}^{z} b(z')dz'}
\end{aligned}
\tag{1.7}
$$

The exponential of a matrix, of course, is given by the usual series. Depending on the exact form of $(a_{n,m})$, terms can be collected into simple trigonometric or hyperbolic functions [2]. The above is extended as [1 (Appendix C)]

$$
\begin{aligned}
\left(a_{n,m}(z)\right) &\equiv \sum_{\ell=1}^{M} b_\ell(z)\left(a_{n,m}^{(\ell)}\right) \\
\left(a_{n,m}^{(\ell)}\right)\cdot\left(a_{n,m}^{(\ell')}\right) &= \left(a_{n,m}^{(\ell')}\right)\cdot\left(a_{n,m}^{(\ell)}\right) \quad \text{(pairwise commutation for all } \ell,\ell'\text{)} \\
\left(\Phi_{n,m}(z,z_0)\right) &= e^{\sum_{\ell=1}^{M}\left(a^{\ell}_{n,m}\right)\int_{z_0}^{z} b_\ell(z')dz'} = \bigodot_{\ell=1}^{M} e^{\left(a^{\ell}_{n,m}\right)\int_{z_0}^{z} b_\ell(z')dz'}
\end{aligned}
\tag{1.8}
$$

More general transformations can also be used including the sum and similarity rules [11], and special matrices involving the direct sum and direct product decomposition [4]. A special simplification rule observes

$$
\begin{aligned}
\left(a_{n,m}(z)\right) &= \frac{1}{N}tr\left(\left(a_{n,m}(z)\right)\right)\left(1_{n,m}\right) + \left(b_{n,m}(z)\right) \\
b_{n,n}(z) &= a_{n,n}(z) - \frac{1}{N}tr\left(a_{n,m}(z)\right) \\
b_{n,m}(z) &= a_{n,m}(z) \text{ for } n \neq m \\
tr\left(b_{n,m}(z)\right) &= 0 \\
\left(\Phi_{n,m}(z,z_0)\right) &= e^{\int_{z_0}^{z}\frac{1}{N}tr\left(\left(a_{n,m}(z')\right)\right)dz'}\; e^{\int_{z_0}^{z}\left(b_{n,m}(z')\right)dz'}
\end{aligned}
\tag{1.9}
$$

The first factor is a scalar involving only the sum integral. The second factor has the property

$$
\det\left(e^{\int_{z_0}^{z}\left(b_{n,m}(z')\right)dz'} \right) = 1
\tag{1.10}
$$

For 2 x 2 matrizants this can reduce the number of "independent" matrix elements from four to three.

Another useful transformation scales the independent variable as

$$\beta z = \zeta \ , \ \beta z_0 = \zeta_0 \ , \ \beta dz = d\zeta$$
$$\prod_{z_0}^{z} e^{\beta(a_{n,m}(z'))dz'} = \prod_{\zeta_0}^{\zeta} e^{\left(a_{n,m}\left(\frac{\zeta'}{\beta}\right)\right)d\zeta'} \tag{1.11}$$

This can be used to scale one of the matrix elements (nonzero) to a desirable value.

Another general approach (generalizable to N x N constant matrices times scalar functions) uses the following theorem

$$e^{b_0(b_{n,m}) + c_0(c_{n,m})} = e^{b_0}\left(b_{n,m}\right) + e^{c_0}\left(c_{n,m}\right) - \left(1_{n,m}\right) \tag{1.12}$$

provided

$$\left(b_{n,m}\right)^{\ell} = \left(b_{n,m}\right) , \ \left(c_{n,m}\right)^{\ell} = \left(c_{n,m}\right) , \ \ell = \text{positive integer}$$
$$\left(b_{n,m}\right) \cdot \left(c_{n,m}\right) = \left(0_{n,m}\right) = \left(c_{n,m}\right) \cdot \left(b_{n,m}\right) \tag{1.13}$$

This can be verified from the series expansion for the exponential. Such is the case if

$$\left(a_{n,m}\right) = b_0\left(b_{n,m}\right) + c_0\left(c_{n,m}\right) \tag{1.14}$$

where b_0 and c_0 are eigenvalues of $(a_{n,m})$ and $(b_{n,m})$ and $(c_{n,m})$ are the corresponding eigendyads (singular idempotent matrices).

2. General Approach

Our approach to finding closed-form 2 x 2 product integrals begins with the first-order homogeneous form of (1.1) as

$$\frac{d}{dz}\left(f_n(z)\right) = \left(a_{n,m}(z)\right) \cdot \left(f_n(z)\right) \tag{2.1}$$

By choosing $(f_n(z))$ in special ways this specifies $(a_{n,m}(z))$. If we have two independent forms of $(f_n(z))$ corresponding to the two independent solutions of a second order differential linear ordinary differential equation, we can use them as the two columns for constructing $(\Phi_{n,m})$. Imposing boundary conditions completes the construction.

Consider the telegrapher equations for a single conductor plus reference

$$\begin{aligned} \frac{d\tilde{V}(z,s)}{dz} &= -sL'(z)\tilde{I}(z,s), \quad L' = \mu f_g(z) \\ \frac{d\tilde{I}(z,s)}{dz} &= -sC'(z)\tilde{V}(z,s), \quad C' = \frac{\varepsilon}{f_g(z)} \end{aligned} \tag{2.2}$$

This can also be transformed to more general forms if the permeability, μ, and permittivity, ε, are allowed to vary as a function of the spatial coordinate, z. The above system is equivalent to a second order linear differential equation. This takes the forms for voltage and current as

$$\begin{aligned} &\frac{d^2\tilde{V}(z,s)}{dz^2} - \left[\frac{d}{dz}\ell n\left(f_g(z)\right)\right]\frac{d\tilde{V}(z,s)}{dz} - \gamma^2\tilde{V}(z,s) = 0 \\ &\frac{d^2\tilde{I}(z,s)}{dz^2} + \left[\frac{d}{dz}\ell n\left(f_g(z)\right)\right]\frac{d\tilde{V}(z,s)}{dz} - \gamma^2\tilde{I}(z,s) = 0 \\ &\gamma = \frac{s}{\mathrm{v}} \ , \quad \mathrm{v} = [\mu\,\varepsilon]^{-1/2} \end{aligned} \tag{2.3}$$

The complex frequency $s = \Omega + j\omega$ can be changed to the regular radian frequency by

$$s = j\omega\,, \quad \gamma = jk\,, \quad \gamma^2 = -k^2 \tag{2.4}$$

thereby giving another form to (2.3).

The reader can note that while these equations are cast in terms of voltages and currents which are functions of s, this identification can be ignored and given arbitrary form of variables in the differential equations.

The general form of the differential equations in (2.3) is

$$b_0(z)\frac{d^2 f(z)}{dz^2} + b_1(z)\frac{df(z)}{dz} + b_2(z)f(z) = 0 \tag{2.5}$$

This is cast in the form of (2.1) as

$$\begin{aligned} \left(f_n(z)\right) &= \begin{pmatrix} f(z) \\ \frac{d}{dz}f(z) \end{pmatrix} \\ \left(a_{n,m}(z)\right) &= \begin{pmatrix} 0 & 1 \\ -\frac{b_2(z)}{b_0(z)} & -\frac{b_1(z)}{b_0(z)} \end{pmatrix} \end{aligned} \tag{2.6}$$

where $f(z)$ can be interpreted as $\tilde{V}(z,s)$ or $\tilde{I}(z,s)$ in (2.3). This gives

$$\begin{aligned} \frac{d}{dz}\begin{pmatrix} \tilde{V}(z,s) \\ \frac{d}{dz}\tilde{V}(z,s) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \gamma^2 & \frac{d}{dz}\ell n\left(f_g(z)\right) \end{pmatrix} \cdot \begin{pmatrix} \tilde{V}(z,s) \\ \frac{d}{dz}\tilde{V}(z,s) \end{pmatrix} \\ \frac{d}{dz}\begin{pmatrix} \tilde{I}(z,s) \\ \frac{d}{dz}\tilde{I}(z,s) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \gamma^2 & -\frac{d}{dz}\ell n\left(f_g(z)\right) \end{pmatrix} \cdot \begin{pmatrix} \tilde{I}(z,s) \\ \frac{d}{dz}\tilde{I}(z,s) \end{pmatrix} \end{aligned} \tag{2.7}$$

with the additional forms from the substitution in (2.4)

Let us next assume that we have two independent solutions for such equations. Call them $f^{(1)}(z)$ and $f^{(2)}(z)$. Then construct

$$\left(A_{n,m}(z)\right) = \begin{pmatrix} f^{(1)}(z) & f^{(2)}(z) \\ \frac{df^{(1)}(z)}{dz} & \frac{df^{(2)}(z)}{dz} \end{pmatrix} \tag{2.8}$$

Of course arbitrary constants can multiply $f^{(1)}$ and $f^{(2)}$. So now construct

$$\begin{aligned}\left(\Phi(z,z_0)\right) &= \left(A_{n,m}(z)\right) \cdot \left(A_{n,m}(z_0)\right)^{-1} \\ &= \prod_{z_0}^{z} e^{\left(a_{n,m}(z')\right)dz'}\end{aligned} \tag{2.9}$$

and verify that the matrizant equation and boundary condition in (1.2) are satisfied. Other product integrals can also be constructed by a similarity transformation with a constant non-singular matrix as

$$\begin{aligned}\left(\Lambda_{n,m}(z,z_0)\right) &= \left(C_{n,m}\right) \cdot \left(\Phi_{n,m}(z,z_0)\right) = \left(C_{n,m}\right)^{-1} \\ \frac{d}{dz}\left(\Lambda_{n,m}(z,z_0)\right) &= \left(C_{n,m}\right) \cdot \left(a_{n,m}(z)\right) = \left(C_{n,m}\right)^{-1} \cdot \left(\Lambda_{n,m}(z,z_0)\right) \\ \left(\Lambda_{n,m}(z,z_0)\right) &= \prod_{z_0}^{z} e^{\left(C_{n,m}\right) \cdot \left(a_{n,m}(z')\right) \cdot \left(C_{n,m}\right)^{-1} dz'} \\ \left(\Lambda_{n,m}(z,z_0)\right) &= \left(C_{n,m}\right) \cdot \left(A_{n,m}(z)\right) \cdot \left(A_{n,m}(z_0)\right)^{-1} \cdot \left(C_{n,m}\right)^{-1} \\ &= \left[\left(C_{n,m}\right) \cdot \left(A_{n,m}(z)\right) \cdot \left(C_{n,m}\right)^{-1}\right] \cdot \left[\left(C_{n,m}\right) \cdot \left(A_{n,m}(z_0)\right) \cdot \left(C_{n,m}\right)^{-1}\right]^{-1}\end{aligned} \tag{2.10}$$

Having two independent solutions of (2.3) we also have via (2.2) relations between $\tilde{V}$ and $\tilde{I}$ for each of these solutions, leading to yet additional forms of $(A_{n,m})$. From the telegrapher equations we have

$$\begin{aligned}\frac{d}{dz}\begin{pmatrix}\tilde{V}(z,s) \\ \tilde{I}(z,s)\end{pmatrix} &= \begin{pmatrix}0 & -s\mu f_g(z) \\ -s\varepsilon f_g^{-1}(z) & 0\end{pmatrix} \cdot \begin{pmatrix}\tilde{V}(z,s) \\ \tilde{I}(z,s)\end{pmatrix} \\ &= -\gamma\begin{pmatrix}0 & Z_c(z) \\ Z_c^{-1}(z) & 0\end{pmatrix} \cdot \begin{pmatrix}\tilde{V}(z,s) \\ \tilde{I}(z,s)\end{pmatrix}\end{aligned} \tag{2.11}$$

$$Z_c(z) = Z_w f_g(z) = \left[\frac{\mu}{\varepsilon}\right]^{1/2} f_g(z) \equiv \text{local characteristic impedance of transmission line}$$

From (2.2) we can convert two independent solutions of (2.3) for $\tilde{V}$ to $\tilde{I}$ and conversely to construct $(A_{n,m}(z))$ and proceed as previously.

Yet other forms can be obtained by transforming the telegrapher equations. It has been convenient on occasion to multiply $\tilde{I}$ by some normalizing impedance [2, 3] to give

$$\frac{d}{dz}\begin{pmatrix} \tilde{V}(z,s) \\ Z(z)\tilde{I}(z,s) \end{pmatrix} = \begin{pmatrix} 0 & -\gamma Z^{-1}(z)Z_c(z) \\ -\gamma Z(z)Z_c^{-1}(z) & \frac{d}{dz}\ell n(Z(z)) \end{pmatrix} \bullet \begin{pmatrix} \tilde{V}(z,s) \\ Z(z)\tilde{I}(z,s) \end{pmatrix} \tag{2.12}$$

$Z(z)$ ≡ normalizing impedance.

If Z is chosen as a constant, this reduces to the same form as in (2.11). If it is chosen as the characteristic impedance, then the coefficient matrix reduces to

$$\left(a_{n,m}(z)\right) = \begin{pmatrix} 0 & -\gamma \\ -\gamma & \frac{d}{dz}\ell n\left(f_g(z)\right) \end{pmatrix} \tag{2.13}$$

for yet another form.

Another form [5] uses a symmetric renormalization with combined voltages as

$$\begin{gathered} \frac{d}{dz}\left(\tilde{\mathrm{v}}_\nu(z,s)\right) = \begin{pmatrix} \gamma & -\frac{d}{dz}\ell n\left(f_g(z)\right) \\ -\frac{d}{dz}\ell n\left(f_g(z)\right) & -\gamma \end{pmatrix} \bullet \left(\mathrm{v}_\nu(z,s)\right) \\ \left(\tilde{\mathrm{v}}_\nu(z,s)\right) = \begin{pmatrix} Z_c^{-1/2}(z)\tilde{V}(z,s) + Z_c^{1/2}(z)\tilde{I}(z,s) \\ Z_c^{-1/2}(z)\tilde{V}(z,s) - Z_c^{1/2}(z)\tilde{I}(z,s) \end{pmatrix} \end{gathered} \tag{2.14}$$

3. Exponential Variation of f_g

Now let us choose [2 (Appendix B), 7]

$$f_g(z) = f_0 \, e^{2\alpha z} = e^{2\alpha z + \ell n(f_0)} \tag{3.1}$$

By appropriate choice of f_0 the characteristic impedance can be fixed at any z of interest. The voltage takes the form

$$\tilde{V}(z,s) = \begin{cases} V_1 \, e^{\left[\alpha - \left[a^2 + \gamma^2\right]^{1/2}\right] z} & (+z \text{ propagation}) \\ V_2 \, e^{\left[\alpha + \left[a^2 + \gamma^2\right]^{1/2}\right] z} & (-z \text{ propagation}) \end{cases} \tag{3.2}$$

The corresponding current takes the form

$$\begin{aligned} \tilde{I}(z,s) &= -\frac{1}{s\mu f_g} \frac{d\tilde{V}(z,s)}{dz} \\ &= -\frac{1}{\gamma Z_w f_0} \begin{cases} \alpha - \left[a^2 + \gamma^2\right]^{1/2} e^{\left[-\alpha - \left[a^2 + \gamma^2\right]^{1/2}\right] z} \\ \alpha + \left[a^2 + \gamma^2\right]^{1/2} e^{\left[-\alpha + \left[a^2 + \gamma^2\right]^{1/2}\right] z} \end{cases} \end{aligned} \tag{3.3}$$

Now (2.3) takes the form

$$\begin{aligned} \frac{d^2\tilde{V}(z,s)}{dz^2} - 2\alpha \frac{d\tilde{V}(z,s)}{dz} - \gamma^2 \tilde{V}(z,s) &= 0 \\ \frac{d^2\tilde{I}(z,s)}{dz^2} + 2\alpha \frac{d\tilde{I}(z,s)}{dz} - \gamma^2 \tilde{I}(z,s) &= 0 \end{aligned} \tag{3.4}$$

This gives, for the voltage equation, the pair,

$$
\begin{aligned}
&\left(a_{n,m}(z)\right) = \begin{pmatrix} 0 & 1 \\ \gamma^2 & 2\alpha \end{pmatrix} \\
&\left(A_{n,m}(z)\right) = \\
&\begin{pmatrix} e^{\left[\alpha-\left[\alpha^2+\gamma^2\right]^{1/2}\right]z} & e^{\left[\alpha+\left[\alpha^2+\gamma^2\right]^{1/2}\right]z} \\ \left[\alpha-\left[\alpha^2+\gamma^2\right]^{1/2}\right] e^{\left[\alpha-\left[\alpha^2+\gamma^2\right]^{1/2}\right]z} & \left[\alpha+\left[\alpha^2+\gamma^2\right]^{1/2}\right] e^{\left[\alpha+\left[\alpha^2+\gamma^2\right]^{1/2}\right]z} \end{pmatrix}
\end{aligned}
\tag{3.5}
$$

The constants (V_1 and V_2) factor out when combined with $(A_{n,m}(z))$ to give the product integral as in (2.9). While we have approached this via transmission-line theory the constants (α, γ, etc.) can be reinterpreted at the reader's choice.

Note that

$$
\begin{aligned}
\det\left(\left(A_{n,m}(z)\right)\right) &= \left[\alpha+\left[\alpha^2+\gamma^2\right]^{1/2}\right] e^{2\alpha z} \\
&\quad - \left[\alpha-\left[\alpha^2+\gamma^2\right]^{1/2}\right] e^{2\alpha z} \\
&= 2\left[\alpha^2+\gamma^2\right]^{1/2} e^{2\alpha z} \\
\det\left(\left(\Phi_{n,m}(z,z_0)\right)\right) &= \det\left(\left(A_{n,m}(z)\right)\right) \det^{-1}\left(\left(A_{n,m}(z_0)\right)\right) \\
&= e^{2\alpha[z-z_0]} \\
&= e^{tr\left(\left(a_{n,m}\right)\right)[z-z_0]}
\end{aligned}
\tag{3.6}
$$

consistent with (1.9). This also gives a Wronskian relationship of the two independent solutions. Of course, since $(a_{n,m})$ is actually independent of z we have

$$
\left(\Phi_{n,m}(z,z_0)\right) = \left(A_{n,m}(z)\right) \cdot \left(A_{n,m}(z_0)\right) = e^{\left(a_{n,m}\right)[z-z_0]} \tag{3.7}
$$

as an alternate representation. Then (3.5) gives an explicit representation of this exponential of a matrix.

Another form comes from (2.11)

$$
\left(a_{n,m}(z)\right) = -\gamma \begin{pmatrix} 0 & Z_w f_0\, e^{2\alpha z} \\ Z_w^{-1} f_0^{-1}\, e^{-2\alpha z} & 0 \end{pmatrix}
$$

$$
\left(A_{n,m}(z)\right) = \begin{pmatrix} e^{\left[\alpha - \left[\alpha^2+\gamma^2\right]^{1/2}\right]z} & e^{\left[\alpha + \left[\alpha^2+\gamma^2\right]^{1/2}\right]z} \\ -\frac{1}{\gamma Z_w f_0}\left[\alpha - \left[\alpha^2+\gamma^2\right]^{1/2}\right] e^{\left[-\alpha - \left[\alpha^2+\gamma^2\right]^{1/2}\right]z} & -\frac{1}{\gamma Z_w f_0}\left[\alpha + \left[\alpha^2+\gamma^2\right]^{1/2}\right] e^{\left[-\alpha + \left[\alpha^2+\gamma^2\right]^{1/2}\right]z} \end{pmatrix}
$$

$$
\begin{aligned}
\det\left(\left(A_{n,m}(z)\right)\right) &= -\frac{1}{\gamma Z_w f_0}\left[\alpha + \left[\alpha^2+\gamma^2\right]^{1/2}\right] \\
&\quad + \frac{1}{\gamma Z_w f_0}\left[\alpha - \left[\alpha^2+\gamma^2\right]^{1/2}\right] \\
&= -\frac{2}{\gamma Z_w f_0}\left[\alpha^2+\gamma^2\right]^{1/2}
\end{aligned} \tag{3.8}
$$

$$
\det\left(\left(\Phi_{n,m}(z)\right)\right) = \det\left(\left(A_{n,m}(z)\right)\right) \cdot \det^{-1}\left(\left(A_{n,m}(z_0)\right)\right)
$$

noting that $(a_{n,m}(z))$ has a zero trace. Also the combination $Z_w f_0$ is just an arbitrary constant.

For the special form in (2.13) we have

$$
\left(a_{n,m}(z)\right) = \begin{pmatrix} 0 & -\gamma \\ -\gamma & 2\alpha \end{pmatrix}
$$

$$
\left(A_{n,m}(z)\right) = \begin{pmatrix} e^{\left[\alpha - \left[\alpha^2+\gamma^2\right]^{1/2}\right]z} & e^{\left[\alpha + \left[\alpha^2+\gamma^2\right]^{1/2}\right]z} \\ -\frac{1}{\gamma}\left[\alpha - \left[\alpha^2+\gamma^2\right]^{1/2}\right] e^{\left[\alpha - \left[\alpha^2+\gamma^2\right]^{1/2}\right]z} & -\frac{1}{\gamma}\left[\alpha + \left[\alpha^2+\gamma^2\right]^{1/2}\right] e^{\left[\alpha + \left[\alpha^2+\gamma^2\right]^{1/2}\right]z} \end{pmatrix}
$$

$$
\begin{aligned}
\det\left(\left(A_{n,m}(z)\right)\right) &= -\frac{1}{\gamma}\left[\alpha + \left[\alpha^2+\gamma^2\right]^{1/2}\right] e^{2\alpha z} \\
&\quad + \frac{1}{\gamma}\left[\alpha - \left[\alpha^2+\gamma^2\right]^{1/2}\right] e^{2\alpha z} \\
&= -\frac{2}{\gamma}\left[\alpha^2+\gamma^2\right]^{1/2} e^{2\alpha z}
\end{aligned} \tag{3.9}
$$

$$
\det\left(\left(\Phi_{n,m}(z, z_0)\right)\right) = e^{2\alpha[z - z_0]}
$$

From (2.14) we also have

$$\boxed{\begin{array}{l}\left(a_{n,m}(z)\right) = \begin{pmatrix} \gamma & -2\alpha \\ -2\alpha & -\gamma \end{pmatrix} \\ \left(A_{n,m}(z)\right) = \\ Z_w^{-1/2} f_0^{-1/2} \begin{pmatrix} e^{-\left[\alpha^2+\gamma^2\right]^{1/2} z}\left[1-\frac{1}{\gamma}\left[\alpha-\left[\alpha^2+\gamma^2\right]^{1/2}\right]\right] & e^{\left[\alpha^2+\gamma^2\right]^{1/2} z}\left[1-\frac{1}{\gamma}\left[\alpha-\left[\alpha^2+\gamma^2\right]^{1/2}\right]\right] \\ e^{-\left[\alpha^2+\gamma^2\right]^{1/2} z}\left[1-\frac{1}{\gamma}\left[\alpha+\left[\alpha^2+\gamma^2\right]^{1/2}\right]\right] & e^{\left[\alpha^2+\gamma^2\right]^{1/2} z}\left[1+\frac{1}{\gamma}\left[\alpha+\left[\alpha^2+\gamma^2\right]^{1/2}\right]\right] \end{pmatrix}\end{array}}$$

$$\begin{aligned} \det\left(\left(A_{n,m}(z)\right)\right) &= Z_w^{-1} f_0^{-1}\left[1+\frac{2}{\gamma}\left[\alpha^2+\gamma^2\right]^{1/2} + 1\right. \\ &\quad \left. -1 + \frac{2}{\gamma}\left[\alpha^2+\gamma^2\right]^{1/2} - 1\right] \\ &= \frac{4}{\gamma Z_w f_0}\left[\alpha^2+\gamma^2\right]^{1/2} \\ \det\left(\left(\Phi_{n,m}(z,z_0)\right)\right) &= 1 \end{aligned} \tag{3.10}$$

Note that the constant $Z_w^{-1/2} f_0^{-1/2}$ can be neglected in the above for a simpler form due to its cancellation by $(A_{n,m}(z_0))^{-1}$.

4. General Form of the Product Integral of a Constant 2 x 2 Matrix Times a Scalar Function

Consider the problem

$$\left(\Phi_{n,m}(z,z_0)\right) = \prod_{z_0}^{z} e^{\beta(z')(a_{n,m})dz'} = e^{(a_{n,m})\int_{z_0}^{z}\beta(z')dz'} \tag{4.1}$$

Transform this as

$$\beta(z)dz = d\zeta \quad , \quad \zeta = \int_{z_0}^{z}\beta(z')dz' \tag{4.2}$$

By this scaling, the product integral is reduced to a product integral of $(a_{n,m})$. So without loss of generality consider

$$\left(\Phi_{n,m}(z,z_0)\right) = \prod_{z_0}^{z} e^{(a_{n,m})dz'} = e^{(a_{n,m})[z-z_0]} \tag{4.3}$$

Also from (4.2) one of the $a_{n,m}$ can be scaled to unity.

One approach would scale $a_{1,2}$ to unity so that we would evaluate

$$\begin{aligned}
&\prod_{a_{1,2}z_0}^{a_{1,2}z} \exp\left(\begin{pmatrix} \frac{a_{1,1}}{a_{1,2}} & 1 \\ \frac{a_{2,1}}{a_{1,2}} & \frac{a_{2,2}}{a_{1,2}} \end{pmatrix} d\left[a_{1,2}z'\right]\right) \\
&= e^{a_{1,1}[z-z_0]} \prod_{a_{1,2}z_0}^{a_{1,2}z} \exp\left(\begin{pmatrix} 0 & 1 \\ \frac{a_{2,1}}{a_{1,2}} & \frac{a_{2,2}}{a_{1,2}} - \frac{a_{1,1}}{a_{1,2}} \end{pmatrix} d\left[a_{1,2}z'\right]\right)
\end{aligned} \tag{4.4}$$

where we have pulled out $a_{1,1}(1_{n,m})$ and closed-form product integrated this. We now have a form as in (2.5) which is one form of our solution.

A general approach begins from (1.9) as

$$
\begin{aligned}
\prod_{z_0}^{z} e^{(a_{n,m})dz'} &= e^{(a_{n,m})[z-z_0]} \\
&= e^{\frac{a_{1,1}+a_{2,2}}{2}[z-z_0]} = e^{(b_{n,m})[z-z_0]} \\
(b_{n,m}) &= \begin{pmatrix} b & a_{1,2} \\ a_{2,1} & -b \end{pmatrix} \quad , \quad tr(b_{n,m}) = 0 \\
b &= \frac{a_{1,1}-a_{2,2}}{2} \\
\det\left(e^{(b_{n,m})[z-z_0]}\right) &= 1
\end{aligned}
\tag{4.5}
$$

Then we diagonalize $(b_{n,m})$ as

$$
\begin{aligned}
(b_{n,m}) &= \sum_{\ell=1}^{2} \lambda_\ell (r_n)_\ell (\ell n)_\ell \\
\det\begin{pmatrix} b-\lambda_{\frac{1}{2}} & a_{1,2} \\ a_{2,1} & -b-\lambda_{\frac{1}{2}} \end{pmatrix} &= 0 \\
\lambda_{\frac{1}{2}} &= \pm\left[b^2 + a_{1,2}a_{2,1}\right] \\
(b_{n,m}) \cdot (r_n)_\ell &= \lambda_\ell (r_n)_\ell \quad , \quad (\ell_n)_\ell \cdot (b_{n,m}) = \lambda_\ell (\ell_n)_\ell \\
(r_n)_{\frac{1}{2}} &= C_{\frac{1}{2}} \begin{pmatrix} a_{1,2} \\ \lambda_{\frac{1}{2}} - b \end{pmatrix} \\
(\ell_n)_{\frac{1}{2}} &= D_{\frac{1}{2}} \begin{pmatrix} \lambda_{\frac{1}{2}} + b \\ a_{1,2} \end{pmatrix} \\
(r_n)_{\ell_1} \cdot (\ell_n)_{\ell_2} &= 1_{\ell_1,\ell_2} \quad \text{(biorthonormal)} \\
C_{\frac{1}{2}} D_{\frac{1}{2}} &= \left[2a_{1,2}\,\lambda_{\frac{1}{2}}\right]^{-1}
\end{aligned}
\tag{4.6}
$$

Note that the eigendyads only use the CD product. This of course assumes that

$$\lambda_1 \neq \lambda_2 \quad , \quad b \neq \pm j\left[a_{1,2}\, a_{2,1}\right]^{1/2}, \det\left(\left(b_{n,m}\right)\right) \neq 0 \tag{4.7}$$

In such a case

$$\left(b_{n,m}\right)^2 = \left(0_{n,m}\right) \tag{4.8}$$

and the exponential series truncates after two terms, giving a yet simpler result.

Then applying (1.12) we have

$$e^{\left(a_{n,m}\right)[z-z_0]} = e^{\frac{a_{1,1}+a_{2,2}}{2}[z-z_0]}\left[\left[\sum_{\ell=1}^{2} e^{\frac{\lambda_1}{2}[z-z_0]}\left(r_n\right)_\ell\left(\ell_n\right)_\ell\right] - \left(1_{n,m}\right)\right] \tag{4.9}$$

as the general form for the product integral of a constant 2 x 2 matrix. A similar diagonalization procedure can extend this to larger N. See an alternate approach in [6].

5. Power-Law Variation of f_g

Choose now (see also [2(Appendix A)])

$$f_g(z) = f_0 z^p \tag{5.1}$$

Now (2.3) takes the form

$$\begin{aligned} &\frac{d^2\tilde{V}(z,s)}{dz^2} - \frac{p}{z}\frac{d\tilde{V}(z,s)}{dz} - \gamma^2\tilde{V}(z,s) = 0 \\ &\frac{d^2\tilde{I}(z,s)}{dz^2} + \frac{p}{z}\frac{d\tilde{I}(z,s)}{dz} - \gamma^2\tilde{I}(z,s) = 0 \end{aligned} \tag{5.2}$$

which transforms to

$$\begin{aligned} &[\gamma z]^2 \frac{d^2\tilde{V}(z,s)}{d[\gamma z]^2} - p\gamma z\frac{d\tilde{V}(z,s)}{d[\gamma z]} - [\gamma z]^2\tilde{V}(z,s) = 0 \\ &[\gamma z]^2 \frac{d^2\tilde{I}(z,s)}{d[\gamma z]^2} + p\gamma z\frac{d\tilde{V}(z,s)}{d[\gamma z]} - [\gamma z]^2\tilde{I}(z,s) = 0 \end{aligned} \tag{5.3}$$

This starts to look like a Bessel equation. Following [8], which also calls this case a Bessel transmission line, we have the general form of the current differential equation which transforms to a Bessel equation as

$$\begin{aligned} &u^2\frac{d^2 f(u)}{du^2} + p\,u\frac{df(u)}{du} - u^2 f(u) = 0 \\ &f(u) = u^q g(u) \\ &\quad q = \frac{1-p}{2} \quad , \quad p = 1-2q \\ &u^2\frac{d^2 g(u)}{du^2} + u\frac{dg(u)}{du} - \left[u^2 + q^2\right]g(u) = 0 \end{aligned} \tag{5.4}$$

which has solutions $I_q(u)$ and $I_{-q}(u)$ for q not an integer, and $I_q(u)$ and $K_q(u)$ for q an integer. These are called modified Bessel functions and have some properties as [10]

$I_{-m}(u) = I_m(u)$ for m an integer

$$
\begin{aligned}
K_{-q}(u) &= K_q(u) \\
I_q(u) &= [-j]^q J_q(-ju) \quad , \quad K_q(u) = \frac{\pi}{2}[-j]^q H_q^{(2)}(-ju) \quad \text{(for imaginary } u\text{)}
\end{aligned}
\tag{5.5}
$$

$$
\frac{d}{du}\begin{bmatrix} u^q I_q(u) \\ u^q K_q(u) \end{bmatrix} = u^q \begin{bmatrix} I_{q-1}(u) \\ K_{q-1}(u) \end{bmatrix}
$$

$$
W\{I_q(u),\, I_{-q}(u)\} = I_q(u) I_{-q-1}(u) = I_{\nu+1}(u) I_{-\nu}(u) = -2\sin(q\pi)/[\pi z]
$$

$$
W\{K_q(u),\, I_q(u)\} = I_q(u) K_{q+1}(u) = I_{q+1}(u) K_q(u) = 1/u
$$

This leads to several forms of product integrals. For the modified Bessel equation we have

$$
\begin{aligned}
(a_{n,m}(u)) &= \begin{pmatrix} 0 & 1 \\ \dfrac{u^2+q^2}{u^2} & -u^{-1} \end{pmatrix} \\
(A_{n,m}(u)) &= \begin{pmatrix} I_q(u) & I_{-q}(u) \\ \dfrac{d}{du} I_q(u) & \dfrac{d}{du} I_{-q}(u) \end{pmatrix}
\end{aligned}
\tag{5.6}
$$

with the substitution of $K_q(u)$ for integer q. The determinant of $(A_{n,m})$ is just the Wronskians in (5.5). For imaginary u (corresponding to propagation in frequency domain, $\gamma = jk,\ s = j\omega$) one merely substitutes the J_q and $H_q^{(2)}$ functions.

Going back to the current form of the equation we have

$$
\begin{aligned}
(a_{n,m}(u)) &= \begin{pmatrix} 0 & 1 \\ 1 & p/u \end{pmatrix} \\
(A_{n,m}(u)) &= \begin{pmatrix} u^q I_q(u) & u^q I_{-q}(u) \\ u^q I_{q-1}(u) & u^q I_{-q-1}(u) \end{pmatrix}
\end{aligned}
\tag{5.7}
$$

with the same substitution of functions as before. For the voltage form of the equation replace p by $-p$, including the change in q per (5.4).

Beginning with the current form we note

$$
\tilde{V}(z,s) = -\frac{Z_c(z)}{\gamma}\frac{d\tilde{I}(z,s)}{dz} = -Z_w f_0 z^p \frac{d\tilde{I}(z,s)}{du} = -Z_w \gamma^p f_0 u^p \frac{d\tilde{I}(z,s)}{du}
\tag{5.8}
$$

In terms of the independent variable u we then have (per (2.11))

$$\begin{aligned}
\left(a_{n,m}(u)\right) &= -\begin{pmatrix} 0 & Z_w f_0 \gamma^p u^p \\ Z_w^{-1} f_0^{-1} \gamma^{-p} u^{-p} & 0 \end{pmatrix} \\
\left(A_{n,m}(u)\right) &= -\begin{pmatrix} -Z_w \gamma^{-p} f_0 u^p u^q I_{q-1}(u) & -Z_w \gamma^{-p} f_0 u^p u^q I_{-q-1}(u) \\ u^q I_q(u) & u^q I_{-q-1}(u) \end{pmatrix} \\
&= \begin{pmatrix} -Z_w \gamma^{-p} f_0 u^{1-q} I_{q-1}(u) & -Z_w \gamma^{-p} u^{1-q} I_{-q-1}(u) \\ u^q I_q(u) & u^q I_{-q-1}(u) \end{pmatrix}
\end{aligned} \tag{5.9}$$

Note that the combination $Z_w f_0 \gamma^{-p}$ is just a constant to be chosen at our convenience.

For the form in (2.13) we have the same result as (5.7). The form in (2.14) gives

$$\begin{aligned}
\left(a_{n,m}(u)\right) &= \begin{pmatrix} 1 & -\frac{p}{u} \\ -\frac{p}{u} & -1 \end{pmatrix} \\
\left(A_{n,m}(u)\right) &= \\
Z_w^{1/2} f_0^{1/2} \gamma^{-p/2} u^{p/2} & \begin{pmatrix} u^q\left[I_{q-1}(u) + I_q(u)\right] & u^q\left[I_{-q-1}(u) + I_{-q}(u)\right] \\ u^q\left[I_{q-1}(u) - I_q(u)\right] & u^q\left[I_{-q-1}(u) - I_{-q}(u)\right] \end{pmatrix} \\
&= \left[Z_w f_0 \gamma^{-p}\right]^{1/2} u^{1/2} \begin{pmatrix} I_{q-1}(u) + I_q(u) & I_{-q-1}(u) - I_{-q}(u) \\ I_{q-1}(u) + I_q(u) & I_{-q-1}(u) - I_{-q}(u) \end{pmatrix}
\end{aligned} \tag{5.10}$$

Note that the constant in front can be dropped due to the dot maultiplication by $(A_{n,m}(u_0))^{-1}$ on the right, giving a simpler-looking result.

6. Concluding Remarks

What we have here is a procedure for constructing product integrals that can be used as a beginning of a *table of product integrals.* The emphasis here has been on the 2 x 2 case based on the solution of second order linear differential equations. While our starting point has been the telegrapher equations, the results are more general. The results presented here are based on two general forms of transmission lines: exponential and power law. Various others have been studied which can give yet more 2 x 2 product integrals.

More generally the 2 x 2 case can be extended from the large literature concerning second order linear ordinary differential equations. Such solutions are tabulated in [9]. The N x N case for N > 2 can be treated by the usual way of converting them to a first order linear N x N differential equation. Solutions to such *N*th order equations are also found in [9]. By various product-integral transformations yet other forms can be found.

References

1. C. E. Baum, "High-Frequency Propagation on Nonuniform Multiconductor Transmission Lines in Uniform Media", Interaction Note 463, March 1988; Int'l J. Numerical Modelling: Electronic Networks, Devices, and Fields, 1988, pp. 175-188.

2. C. E. Baum, "Approximation of Nonuniform Multiconductor Transmission Lines by Analytically Solvable Sections", Interaction Note 490, October 1992.

3. C. E. Baum, "Nonuniform Multiconductor Transmission Lines", Interaction Note 516, February 1996.

4. C. E. Baum, "Symmetric Renormalization of the Nonuniform Multiconductor-Transmission-Line Equations with a Single Modal Speed for Analytically Solvable Sections", Interaction Note 537, January 1998.

5. C. E. Baum, "High-Frequency Propagation on Nonuniform Multiconductor Transmission Lines", Interaction Note 589, October 2003.

6. S. Steinberg and C. E. Baum, "Lie-Algebraic Representations of Product Integrals of Variable Matrices", Mathematics Note 92, September 1998.

7. C. E. Baum and J. M. Lehr, "Nonuniform-Transmission-Line Transformers for Fast High-Voltage Transients", Circuit and Electromagnetic System Design Note 44, February 2000; Tapered Transmission-Line Transformers for Fast High-Voltage Transients", IEEE Trans. Plasma Science, 2002, pp. 1712-1721.

8. S. Ballantine, "Non-Uniform Lumped Electric Lines: I. The Conical Line", J. Franklin Inst., vol. 203, 1927, pp. 561-582.

9. G. M. Murphy, Ordinary *Differential Equations and Their Solutions*, D. Van Nostrand, 1960.

10. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, U. S. Gov't Printing Office, 1964.

11. J. D. Dollard and C. N. Friedman, *Product Integration*, Addison-Wesley, 1979.