Mathematics Notes

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Product-Integral Interpolation

Carl E. Baum

Air Force Research Laboratory
Directed Energy Directorate

Abstract

This paper generalizes some previous results for approximating the product integrand by two matrices with variable coefficients. In this case the coefficient functions of both matrices are allowed to vary throughout the interval of integration. This reduces the error in the approximation.

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1. Introduction

This paper is motivated by a recent paper [2] concerning nonuniform multiconductor transmission lines (NMTLs). In that paper a technique was developed for approximating the integrand of a product integral representing the NMTL over the ℓth section of the line, say for $z_\ell \leq z \leq z_{\ell+1}$. This technique assured continuity of the integrand from one section of the line to the next, with at most a slope discontinuity (discontinuity in d/dz), thereby significantly reducing the high-frequency reflections at the section ends.

The present paper is concerned with a generalization of the technique to achieve lower error by more closely approximating the integrand with the interpolated form. This still allows the same general form of the splitting of the interpolated integrand into two constant matrices times scalar functions of z.

2. General Problem

The solution for NMTLs is via a general matrizant differential equation of the form involving N x N matrices as

$$\frac{d}{dz}\left(U_{n,m}(z,z_0)\right) = \left(h_{n,m}(z)\right) \cdot \left(U_{n,m}(z,z_0)\right)$$

$$\left(U_{n,m}(z_0,z_0)\right) = \left(1_{n,m}\right) \qquad \text{boundary condition}$$

$$\left(h_{n,m}(z)\right) \equiv \text{known (specified) coefficient matrix} \tag{2.1}$$

$$\left(U_{n,m}(z,z_0)\right) \equiv \text{matrizant}$$

The general solution for the matrizant takes the form of a product integral

$$\left(U_{n,m}(z,z_0)\right) = \prod_{z_0}^{z} e^{\left(h_{n,m}(z')\right)dz'} \tag{2.2}$$

There is much known about product integrals [4, 5], which will be used liberally here.

In [2] the integrand was approximated on the ℓth section as

$$\left(h_{n,m}(z)\right) = \left(h_{n,m}^{(0)}\right) + \left(h_{n,m}^{(1)}(z)\right)$$

$$\left(h_{n,m}^{(0)}\right) = \frac{1}{2}\left[\left(h_{n,m}(z_\ell)\right) + \left(h_{n,m}(z_{\ell+1})\right)\right]$$

$$= \text{constant matrix}$$

$$\left(h_{n,m}^{(1)}(z)\right) = f^{(\ell)}(z)\left(C_{n,m}\right)$$

$$= \text{constant matrix times scalar function of } z \tag{2.3}$$

$$\left(C_{n,m}\right) = \frac{1}{2}\left[\left(h_{n,m}(z_{\ell+1})\right) - \left(h_{n,m}(z_\ell)\right)\right]$$

$$f^{(\ell)}(z_\ell) = -1 \quad , \quad f^{(\ell)}(z_{\ell+1}) = +1$$

$$f^{(\ell)}(z) = \text{monotone nondecreasing function in } z_\ell \le z \le z_{\ell+1}$$

with an example case as

$$f^{(\ell)}(z) = \frac{2z - z_{\ell+1} - z_\ell}{z_{\ell+1} - z_\ell} \tag{2.4}$$

Note that the above reproduces $(h_{n,m}(z_\ell))$ and $(h_{n,m}(z_{\ell+1}))$ exactly, and as long as $d(h_{n,m}(z))/dz$ and $df^{(\ell)}(z)/dz$ are bounded gives no step discontinuities, including at the end points of the interval. The approximation appears by approximating $(h_{n,m}^{(1)}(z))$ by a scalar function times a constant matrix. Ideally this term is small so as to introduce only a small error. This implies that the interval of length $z_{\ell+1} - z_\ell$ be small enough that $(h_{n,m}(z))$ does not vary much over the interval.

Applying the sum rule for the product integral gives

$$\begin{aligned}
\left(U_{n,m}(z,z_\ell)\right) &= \left(U_{n,m}^{(0)}(z,z_\ell)\right) \cdot \left(U_{n,m}^{(1)}(z,z_\ell)\right) \\
\left(U_{n,m}^{(0)}(z,z_\ell)\right) &= \prod_{z_\ell}^{z} e^{\left(h_{n,m}^{(0)}\right)dz'} \\
&= e^{[z-z_\ell]\left(h_{n,m}^{(0)}\right)} \\
\left(U_{n,m}^{(1)}(z,z_\ell)\right) &= \prod_{z_\ell}^{z} e^{\left(U_{n,m}^{(0)}(z',z_\ell)\right)^{-1} \cdot \left(h_{n,m}^{(1)}(z')\right) \cdot \left(U_{n,m}^{(0)}(z',z_\ell)\right)dz'} \\
&\simeq \prod_{z_\ell}^{z} e^{f^{(\ell)}(z')\left(U_{n,m}^{(0)}(z_\ell,z')\right) \cdot \left(C_{n,m}\right) \cdot \left(U_{n,m}^{(0)}(z',z_\ell)\right)dz'}
\end{aligned} \tag{2.5}$$

Note how the approximation enters via the second term (superscript 1). The first term has a simple analytic form which can be computed using the eigenvectors of $(h_{n,m}^{(0)})$ which *do not vary* in the interval.

Assuming that $(C_{n,m})$ is sufficiently small (by some appropriate norm), then we can use the matrizant series to write

$$\begin{aligned}
\left(U_{n,m}^{(1)}(z,z_\ell)\right) &\simeq \left(1_{n,m}\right) + \int_{z_\ell}^{z} f^{(\ell)}(z')\left(U_{n,m}^{(0)}(z_\ell,z')\right) \cdot \left(C_{n,m}\right) \cdot \left(U^{(0)}(z',z_\ell)\right)dz' \\
&\quad + \mathrm{O}\left(\left(\chi_{\max}|z-z_\ell|\right)^2\right) \text{ as } z-z_\ell \to 0 \\
\chi_{\max}^2 &= \text{maximum eigenvalue of } \left(C_{n,m}\right)^\dagger \cdot \left(C_{n,m}\right)
\end{aligned} \tag{2.6}$$

where the norm is here taken in the associated 2-norm sense. (Note that eigenvalues are unchanged by a similarity transformation.) So if $(h^{(1)}_{n,m})$ is sufficiently small this should give a good approximate solution.

If, however, $(h_{n,m}(z_{\ell+1}))$ is significantly different from $(h_{n,m}(z_\ell))$, the above approach in general breaks down. Of course, one may redefine z_ℓ and/or $z_{\ell+1}$ such that $z_{\ell+1} - z_\ell$ is smaller, so that the two matrices are more closely approximating each other.

Another approach comes from the recognition that if $(h_{n,m}(z))$ has a special form as

$$\left(h_{n,m}(z)\right) = h(z)\left(h^{(0)}_{n,m}\right) \tag{2.7}$$

then the product integral also has a simple solution as

$$\left(U_{n,m}(z,z_0)\right) = \prod_{z_\ell}^{z} e^{h(z')\left(h^{(0)}_{n,m}\right)dz'} = e^{\left(h^{(0)}_{n,m}\right)\int_{z_\ell}^{z} h(z')dz'} \tag{2.8}$$

So even with $h(z_{\ell+1})$ quite different from $h(z_\ell)$ we still have a solution in terms of the usual sum integral. This clue leads us to a more general solution procedure.

3. Interpolation of the Integrand

To generalize the technique let us try

$$\left(h_{n,m}(z)\right) \approx g_0(z)\left(h_{n,m}(z_\ell)\right) + g_1(z)\left(h_{n,m}(z_{\ell+1})\right) \equiv \left(h_{n,m}^{(\alpha)}(z)\right)$$

$$g_0(z) = \begin{cases} 1 \ \text{ for } \ z = z_\ell \\ \text{monotone non-increasing} \ \text{ for } \ z_\ell < z < z_{\ell+1} \\ 1 \ \text{ for } \ z = z_{\ell+1} \end{cases} \tag{3.1}$$

$$g_1(z) = \begin{cases} 0 \ \text{ for } \ z = z_\ell \\ \text{monotone non-decreasing} \ \text{ for } \ z_\ell < z < z_{\ell+1} \\ 1 \ \text{ for } \ z = z_{\ell+1} \end{cases}$$

We could constrain

$$g_0(z) + g_1(z) \equiv 1 \tag{3.2}$$

with a possible choice of linear interpolation as

$$g_0(z) = \frac{z_{\ell+1} - z}{z_{\ell+1} - z_\ell} \quad , \quad g_1(z) = \frac{z - z_\ell}{z_{\ell+1} - z_\ell} \tag{3.3}$$

but this is just one special choice.

The fundamental choice in (3.1) assures

$$\left(h_{n,m}^{(a)}(z_\ell)\right) = \left(h_{n,m}(z_\ell)\right) \quad , \quad \left(h_{n,m}^{(a)}(z_{\ell+1})\right) = \left(h_{n,m}(z_{\ell+1})\right) \tag{3.4}$$

if we constrain $g_0(z)$ and $g_1(z)$ to have bounded derivatives on the interval, and in particular as $z \to z_\ell$, $z_{\ell+1}$, then there are at most ramp discontinuities, including as one goes from one interval to the next.

Next, manipulate (3.1) ino the form

$$\left(h_{n,m}^{(a)}(z)\right) = f_0(z)\left(a_{n,m}^{(0)}\right) + f_1(z)\left(a_{n,m}^{(1)}\right) \tag{3.5}$$

where we want $f_0(z)\left(a_{n,m}^{(0)}\right)$ large in the interval and $f_1(z)\left(a_{n,m}^{(1)}\right)$ small in the interval. Clearly $\left(a_{n,m}^{(0)}\right)$ and $\left(a_{n,m}^{(1)}\right)$ are linear combinations of $(h_{n,m}(z_0))$ and $(h_{n,m}(z_1))$

If, for example, we can find h_0, h_1 such that (implying that one matrix is the scalar multiple of the other)

$$h_0\left(h_{n,m}(z_\ell)\right) + h_1\left(h_{n,m}(z_{\ell+1})\right) = \left(0_{n,m}\right) \quad , \quad h_0, h_1 \neq 0 \tag{3.6}$$

then we can choose

$$\begin{aligned} \left(h_{n,m}^{(a)}(z)\right) &\equiv g_2(z)\left(h_{n,m}(z_\ell)\right) \\ \prod_{z_\ell}^{z} e^{\left(h_{n,m}^{(a)}(z')\right)dz'} &= e^{\left(h_{n,m}(z_\ell)\right)\int_{z_\ell}^{z} g_2(z')dz'} \end{aligned} \tag{3.7}$$

for an analytic solution using the approximate interpolating matrix. Here $g_2(z)$ needs to be chosen such that

$$\left\|\left(h_{n,m}(z)\right) - \left(h_{n,m}^{(a)}(z)\right)\right\| \text{ (for some appropriate norm)} \tag{3.8}$$

is minimized over z and the matrix elements. If $\left(h_{n,m}^{(a)}(z)\right)$ is *exact* for all $z_\ell \leq z \leq z_{\ell+1}$, then (3.7) is *exact* (as in (2.8)).

How small can $f_1(z)\left(a_{n,m}^{(1)}\right)$ be on the interval? If desired we can normalize the problem by setting

$$\sup_{z_\ell \leq z \leq z_1} \left|f_1(z)\right| = 1 \tag{3.9}$$

So form an "error" matrix

$$\left(h_{n,m}^{(b)}(z_\ell)\right) = \left(h_{n,m}(z)\right) - \left(h_{n,m}^{(a)}(z)\right) = \left(h_{n,m}(z)\right) - f_0(z)\left(a_{n,m}^{(0)}\right) - f_1(z)\left(a_{n,m}^{(1)}\right) \tag{3.10}$$

Impose boundary conditions

$$\begin{aligned}
\left(h_{n,m}^{(b)}(z_\ell)\right) &= \left(0_{n,m}\right) = \left(h_{n,m}(z_\ell)\right) - f_0(z_\ell)\left(a_{n,m}^{(0)}\right) - f_1(z_\ell)\left(a_{n,m}^{(1)}\right) \\
\left(h_{n,m}^{(b)}(z_{\ell+1})\right) &= \left(0_{n,m}\right) = \left(h_{n,m}(z_{\ell+1})\right) - f_0(z_{\ell+1})\left(a_{n,m}^{(0)}\right) - f_1(z_{\ell+1})\left(a_{n,m}^{(1)}\right)
\end{aligned} \tag{3.11}$$

Solving for $\left(a_{n,m}^{(0)}\right)$ and $\left(a_{n,m}^{(1)}\right)$ gives

$$\begin{aligned}
\left(a_{n,m}^{(0)}\right) &= D^{-1}\left[f_1(z_\ell)\left(h_{n,m}(z_{\ell+1}) - f_1(z_{\ell+1})\right)\left(h_{n,m}(z_\ell)\right)\right] \\
\left(a_{n,m}^{(1)}\right) &= -D^{-1}\left[f_0(z_\ell)\left(h_{n,m}(z_{\ell+1}) - f_0(z_{\ell+1})\right)\left(h_{n,m}(z_\ell)\right)\right] \\
D &\equiv f_0(z_{\ell+1}) f_1(z_\ell) - f_0(z_\ell) f_1(z_{\ell+1})
\end{aligned} \tag{3.12}$$

Consider first the minimization of $f_1(z)\left(a_{n,m}^{(1)}\right)$. In (3.12) we have the boundary values at z_ℓ and $z_{\ell+1}$. Consistent with the normalization in (3.9), let us take

$$f_1(z_\ell) = -1\,,\quad f_1(z_{\ell+1}) = 1 \tag{3.13}$$

appropriate to real matrices. Since we expect $(h_{n,m}(z_\ell))$ and $(h_{n,m}(z_{\ell+1}))$ to be approximately the same, and hence $f_0(z_{\ell+1})$ and $f_0(z_\ell)$ as well, this makes D large and we have

$$\begin{aligned}
\left(a_{n,m}^{(1)}\right) &= \left[f_0(z_\ell) + f_0(z_{\ell+1})\right]^{-1}\left[f_0(z_\ell)\left(h_{n,m}(z_{\ell+1})\right) - f_0(z_{\ell+1})\left(h_{n,m}(z_\ell)\right)\right] \\
&= \left[\alpha + \frac{1}{2}\right]\left(h_{n,m}(z_{\ell+1})\right) + \left[\alpha - \frac{1}{2}\right]\left(h_{n,m}(z_\ell)\right) \\
\alpha &\equiv \frac{1}{2}\,\frac{f_0(z_\ell) - f_0(z_\ell)}{f_0(z_\ell) + f_0(z_\ell)}
\end{aligned} \tag{3.14}$$

With f_1 normalized we can take a norm of $\left(a_{n,m}^{(1)}\right)$ as

$$\begin{aligned}
a^{(1)}(\alpha) &\equiv \left\|\left(a_{n,m}^{(1)}\right)\right\| = \left\|\left[a + \frac{1}{2}\right]\left(h_{n,m}(z_{\ell+1})\right) + \left[a - \frac{1}{2}\right]\left(h_{n,m}(z_\ell)\right)\right\| \\
&= \left\|a\left[\left(h_{n,m}(z_{\ell+1})\right) + \left(h_{n,m}(z_\ell)\right)\right] + \frac{1}{2}\left[\left(h_{n,m}(z_{\ell+1})\right) - \left(h_{n,m}(z_\ell)\right)\right]\right\|
\end{aligned} \tag{3.15}$$

This can be used to choose α so that

$$a_0^{(1)} \equiv \inf_{\alpha} a^{(1)}(\alpha) \equiv a^{(1)}(\alpha_0) \tag{3.16}$$

Note that for $(h_{n,m}(z_{\ell+1}))$ near $(h_{n,m}(z_\ell))$ we have α near zero, $a_0^{(1)}$ near zero, and $\left(a_{n,m}^{(1)}(\alpha_0)\right)$ near $(0_{n,m})$.

For the case that $(h_{n,m}(z_\ell))$ and $(h_{n,m}(z_{\ell+1}))$ are scalar multiples of each other as in (3.6) we have

$$\begin{aligned} a_0^{(1)} &= \left\| \left[\left[\alpha_0 + \frac{1}{2}\right]\left[-\frac{h_1}{h_0}\right] + \left[\alpha_0 - \frac{1}{2}\right] \right] (h_{n,m}(z_\ell)) \right\| \\ &= \left| -\left[\alpha_0 + \frac{1}{2}\right]\frac{h_1}{h_0} + \left[\alpha_0 - \frac{1}{2}\right] \right| \left\|(h_{n,m}(z_\ell))\right\| \end{aligned} \tag{3.17}$$

which is zero provided

$$\begin{aligned} -\left[\alpha_0 + \frac{1}{2}\right]\frac{h_1}{h_0} + \alpha_0 - \frac{1}{2} &= 0 \\ \alpha_0 = \frac{1}{2}\,\frac{1+\dfrac{h_1}{h_0}}{1-\dfrac{h_1}{h_0}} &= \frac{1}{2}\,\frac{h_0+h_1}{h_0-h_1} \end{aligned} \tag{3.18}$$

Now we are in a position to evaluate $\left(a_{n,m}^{(0)}\right)$. From (3.11) we now have (adding the two equations)

$$\left(a_{n,m}^{(0)}\right) = \left[f_0(z_\ell) + f_0(z_{\ell+1})\right]^{-1}\left[\left(h_{n,m}(z_\ell)\right) + \left(h_{n,m}(z_{\ell+1})\right)\right] \tag{3.19}$$

Choosing an obvious normalization we have

$$f_0(z_\ell) + f_0(z_{\ell+1}) \equiv 2 \;\;,\;\; \left(a_{n,m}^{(0)}\right) = \frac{1}{2}\left[\left(h_{n,m}(z_\ell)\right) + \left(h_{n,m}(z_{\ell+1})\right)\right] \tag{3.20}$$

From (314) for $\alpha = \alpha_0$ we now have

$$\begin{aligned} 4\alpha_0 &= f_0(z_\ell) - f_0(z_{\ell+1}) \\ f_0(z_\ell) &= 1+2\alpha_0 \\ f_0(z_{\ell+1}) &= 1-2\alpha_0 \end{aligned} \tag{3.21}$$

Combining the results we have

$$\begin{aligned} \left(a_{n,m}^{(1)}(\alpha_0)\right) &= \left[\alpha_0+\frac{1}{2}\right]\left(h_{n,m}(z_{\ell+1})\right) + \left[\alpha_0-\frac{1}{2}\right]\left(h_{n,m}(z_\ell)\right) \\ \left(a_{n,m}^{(0)}\right) &= \frac{1}{2}\left[\left(h_{n,m}(z_{\ell+1})\right) + \left(h_{n,m}(z_\ell)\right)\right] \\ f_0(z_\ell) &= 1 + 2\alpha_0 \quad , \quad f_0(z_{\ell+1}) = 1 - 2\alpha_0 \\ f_1(z_\ell) &= 1 \quad , \quad f_1(z_{\ell+1}) = -1 \end{aligned} \tag{3.22}$$

with α_0 the solution of the minimum norm in (3.15).

Having satisfied the boundary conditions with a small $\left(a_{n,m}^{(1)}(\alpha_0)\right)$, we now need to consider $f_0(z)$ and $f_1(z)$ in the interval to minimize the error norm throughout the interval. From (3.8) and (3.10) we need to minimize

$$h^{(b)} \equiv \left\| \left(h_{n,m}(z)\right) - f_0(z)\left(a_{n,m}^{(0)}\right) - f_1(z)\left(a_{n,m}^{(1)}(\alpha_0)\right) \right\| \tag{3.23}$$

subject to the previous constraints. Here the norm is not only over the N x N matrix, but over z on the interval as well. We might use the same norm as in the previous equations for the matrix at each z and the norm (2-norm, or whatever) over z, or one might norm each of the matrix elements over z and then performs a norm of the resulting matrix.

Following the first approach, let us define

$$b(z) = \left\| \left(h_{n,m}(z)\right) - f_0(z)\left(a_{n,m}^{(0)}\right) - f_1(z)\left(a_{n,m}^{(1)}\right) \right\| \tag{3.24}$$

as a matrix norm only. Then for each z on the interval we can solve for $f_0(z)$ and $f_1(z)$ to minimize this as

$$b_0(z) = \inf_{f_0,f_1} \left\| \left(h_{n,m}(z)\right) - f_0(z)\left(a_{n,m}^{(0)}\right) - f_1(z)\left(a_{n,m}^{(1)}\right) \right\| \tag{3.25}$$

This defines $f_0(z)$ and $f_1(z)$. Note that by our previous construction

$$b_0(z_\ell) = b_0(z_{\ell+1}) = 0 \tag{3.26}$$

with the boundary values of f_0 and f_1 already determined. Then we can form

$$h_0^{(b)} = \|b_0(z)\| \tag{3.27}$$

As a p norm this is

$$h_0^{(b)} = \left[\int_{z_\ell}^{z_{\ell+1}} b_0^p(z)\,dz \right]^{1/p} \tag{3.28}$$

as a typical norm one might use.

4. Application of Norms

In the previous section the parameters of concern are solved via norms of the various matrices. The general properties are discussed in [1, 3]. The p-norm of a vector (N components) is

$$\left\| (\chi_n) \right\|_p = \left[\sum_{n=1}^{N} |x_n|^p \right]^{1/p} , \quad \left\| (x_n) \right\|_\infty = \max_r |\chi_n| \tag{4.1}$$

For a function of z (for $z_\ell \le z \le z_{\ell+1}$) which can be thought of as vector of an infinite number of components, this becomes

$$\left\| f(z) \right\|_p = \left[\int_{z_\ell}^{z_{\ell+1}} |f(z)|^p \, dz \right]^{1/p} , \quad \left\| f(z) \right\|_\infty = \sup_{z_\ell \le z \le z_{\ell+1}} |f(z)| \tag{4.2}$$

Generalizing to matrice (N x N) the *associated matrix norm* is defined from the vector norm by

$$\left\| (X_{n,m}) \right\| \equiv \sup_{(x_n) \ne (0_n)} \frac{\left\| (X_{n,m}) \cdot (x_n) \right\|}{\left\| (x_n) \right\|} \tag{4.3}$$

For example, the 2-norm of a matrix is known to be

$$\begin{gathered} \left\| (X_{n,m}) \right\|_2 \equiv \left[\text{max eigenvalue of } (X_{n,m})^\dagger \cdot (X_{n,m}) \right]^{1/2} \\ \dagger \equiv \text{adjoint} \equiv \mathrm{T}^* \equiv \text{transpose conjugate} \end{gathered} \tag{4.4}$$

This is different from the *Euclidean norm* of a matrix

$$\left\| (X_{n,m}) \right\|_e \equiv \left[\sum_{n,m} |X_{n,m}|^2 \right]^{1/2} \tag{4.5}$$

where the summation is over all values of n and m. While this is a simple norm to calculate it does not in general have the property in (4.3). One way to view the Euclidean norm is as the 2-norm of a vector with N^2 components.

Applying the Euclidean norm to $\left(a_{n,m}^{(1)}\right)$ in (3.15) we have

$$a^{(1)^2} = \left\|\left(a_{n,m}^{(1)}\right)\right\|_e^2 = \sum_{n,m}\left|\alpha\left[h_{n,m}(z_{\ell+1}) + h_{n,m}(z_\ell)\right] + \frac{1}{2}\left[h_{n,m}(z_{\ell+1}) - h_{n,m}(z_\ell)\right]\right|^2 \qquad (4.6)$$

For real matrix elements and real α this reduces to

$$\begin{aligned} a^{(1)^2} &= p_1\alpha^2 + p_2\alpha + p_3 \\ p_1 &= \sum_{n,m}\left[h_{n,m}(z_{\ell+1}) + h_{n,m}(z_\ell)\right]^2 \\ p_2 &= \sum_{n,m}\left[h_{n,m}^2(z_{\ell+1}) - h_{n,m}^2(z_\ell)\right] \\ p_3 &= \frac{1}{4}\sum_{n,m}\left[h_{n,m}(z_{\ell+1}) - h_{n,m}(z_\ell)\right]^2 \end{aligned} \qquad (4.7)$$

This allows us to find α_0 by setting

$$\begin{aligned} \frac{d}{d\alpha}\left[a^{(1)^2}\right]_{\alpha=\alpha_0} &= 2p_1\alpha_0 + p_2 = 0 \\ \alpha_0 &= -\frac{p_2}{2p_1} \\ a_0^{(1)^2} &= \frac{p_2^2}{4p_1} - \frac{p_2^2}{2p_1} + p_3 = -\frac{p_2^2}{4p_1} + p_3 \end{aligned} \qquad (4.8)$$

This is a convenient explicit solution. For complex matrix elements (4.6) in general involves α and α^*. However, if the matrix elements involve a common complex factor (such as $j\omega$) then this comes through the equation as a magnitude squared, leaving real elements in the equation for α, yielding a real α_0. From (3.22) we now have explicit expressions for the interpolating matrices and their coefficient functions at the section boundaries.

Continuing, let us consider the minimization of the error norm in the interval via the 2-norm. From (3.23) we have

$$h^{(b)} = \left\| \left\| \left(h_{n,m}(z)\right) - f_0(z)\left(a_{n,m}^{(0)}\right) - f_1(z)\left(a_{n,m}^{(1)}\right) \right\|_e \right\|_2 \tag{4.9}$$

with the e-norm over the matrices, followed by the 2-norm over z. Let us first minimize the e-norm for each z.

$$h^2(z) = \sum_{n,m} \left| h_{n,m}(z) - f_0(z)\,a_{n,m}^{(0)} - f_1(z)\,a_{n,m}^{(1)} \right|^2 \tag{4.10}$$

For real matrix elements we have

$$\begin{aligned}
h^2(z) &= p_1 f_0^2(z) + p_2(z) f_0(z) + p_3 f_1^2(z) + p_4(z) f_1(z) + p_5 f_0(z) f_1(z) + p_6(z) \\
p_1 &= \sum_{n,m} a_{n,m}^{(0)^2} \quad , \quad p_2(z) = 2\sum_{n,m} h_{n,m}(z)\,a_{n,m}^{(0)} \\
p_3 &= \sum_{n,m} a_{n,m}^{(1)^2} \quad , \quad p_4(z) = 2\sum_{n,m} h_{n,m}(z)\,a_{n,m}^{(1)} \\
p_5 &= 2\sum_{n,m} a_{n,m}^{(0)} a(1)_{n,m} \quad , \quad p_6(z) = \sum_{n,m} h_{n,m}^2(z)
\end{aligned} \tag{4.11}$$

Again this also applies if there is a common complex factor. Minimize for each z by taking partial derivatives with respect to f_1 and f_2 giving

$$\begin{aligned}
\frac{\partial\left(h^2(z)\right)}{\partial f_0} &= 2p_1 f_0(z) + p_2(z) + p_5 f_1(z) = 0 \\
\frac{\partial\left(h^2(z)\right)}{\partial f_1} &= 2p_3 f_1(z) + p_4(z) + p_5 f_0(z) = 0 \\
\begin{pmatrix} 2p_1 & p_5 \\ p_5 & 2p_3 \end{pmatrix} \cdot \begin{pmatrix} f_0(z) \\ f_1(z) \end{pmatrix} &= -\begin{pmatrix} p_2(z) \\ p_4(z) \end{pmatrix} \\
\begin{pmatrix} f_0(z) \\ f_1(z) \end{pmatrix} &= -\left[4p_1p_3 - p_5^2\right]^{-1} \begin{pmatrix} 2p_3 & -p_5 \\ -p_5 & 2p_1 \end{pmatrix} \cdot \begin{pmatrix} p_2(z) \\ p_4(z) \end{pmatrix} \\
&= -\left[4p_1p_3 - p_5^2\right]^{-1} \begin{pmatrix} 2p_3p_2(z) & -p_5p_4(z) \\ 2p_1p_4(z) & 2p_5p_2(z) \end{pmatrix}
\end{aligned} \tag{4.12}$$

This solves for $f_0(z)$ and $f_1(z)$ in terms of two functions of z and three constant terms.

Then we have the remaining error

$$h^{(b)} = \left[\int_{z_\ell}^{z_{\ell+1}} h^2(z)\,dz\right]^{1/2}$$

$$= \left[\int_{z_\ell}^{z_{\ell+1}} \left[p_1 f_0^2(z) + p_2 f_0(z) + p_3 f_1^2(z) + p_4 f_1(z) + p_5 f_0(z) f_1(z) + p_6(z)\right] dz\right]^{1/2} \tag{4.13}$$

For an rms error on the interval we can also form

$$h^{(b)}\left[z_{\lambda+1} - z_\ell\right]^{-1/2} = \left[\left[z_{\ell+1} - z_\ell\right]^{-1} \int_{z_\ell}^{z_{\ell+1}} h^2(z)\,dz\right]^{1/2} \tag{4.14}$$

One can then decide if the error is small enough, or if the interval should be made smaller.

5. Application of Sum Rule

Having $\left(a_{n,m}^{(0)}\right)$ and $\left(a_{n,m}^{(1)}\right)$ as well as $f_0(z)$ and $f_1(z)$, then we form the product integral of $\left(h_{n,m}^{(a)}(z)\right)$ in (3.5) as

$$\left(U_{n,m}^{(a)}(z,z_\ell)\right) \equiv \prod_{z_\ell}^{z} e^{\left[f_0(z')\left(a_{n,m}^{(0)}\right)+f_1(z')\left(a_{n,m}^{(1)}\right)\right]dz'}$$

$$= \left(U_{n,m}^{(a,0)}(z,z_\ell)\right) \bullet \left(U_{n,m}^{(a,1)}(z,z_\ell)\right)$$

$$\left(U_{n,m}^{(a,0)}(z,z_\ell)\right) = \prod_{z_\ell}^{z} e^{f_0(z')\left(a_{n,m}^{(0)}\right)dz'} \tag{5.1}$$

$$= e^{\left(a_{n,m}^{(0)}\right)\int_{z_\ell}^{z} f_0(z')dz'}$$

$$\left(U_{n,m}^{(a,0)}(z,z_\ell)\right) = \prod_{z_\ell}^{z} e^{f_1(z')\left(U_{n,m}^{(a,0)}(z',z_\ell)\right)^{-1} \bullet \left(a_{n,m}^{(1)}\right) \bullet \left(U_{n,m}^{(a,0)}(z',z_\ell)\right)dz'}$$

Diagonalizing $\left(a_{n,m}^{(0)}\right)$ gives a set of eigenvectors which are invariant to z for ready calculation of the first term. With sufficiently small $\left(a_{n,m}^{(1)}\right)$ we have

$$\left(U_{n,m}^{(a,1)}(z,z_\ell)\right) = \left(1_{n,m}\right) + \int_{z_\ell}^{z} f_1(z')\left(U_{n,m}^{(a,0)}(z_\ell,z')\right) \bullet \left(a_{n,m}^{(1)}\right) \bullet \left(U_{n,m}^{(a,0)}(z',z_\ell)\right)dz'$$

$$+ \mathrm{O}\left(\left[\chi_{\max}\left|z-z_\ell\right|\right]^2\right) \quad \text{as } z-z_\ell \to 0 \tag{5.2}$$

$$\chi_{\max} = \left[\text{maximum eigenvalue magnetidue of } \left(a_{n,m}^{(1)}\right)^{\dagger} \bullet \left(a_{n,m'}^{(1)}\right)\right]^{1/2}$$

So one can also check to see if this expansion can be truncated as indicated.

6. Concluding Remarks

This interpolation technique is more general than that in [2]. In this case the first matrix (the large one) in the approximation of the integrand is allowed to have a variable coefficient function. This extra degree of freedom allows the integrand to be more closely approximated in general.

There is, however, an additional degree of complexity introduced in the computation associated with the calculation of the optimal choice of the various parameters (Sections 3 and 4).

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