Mathematics Notes

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A Note on an Initial Value Problem Associated with a Distributed Switch for Launching Spherical Waves

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Abstract

In this note a nonlinear differential equation which arose in an investigation of a distributed switch is studied. A numerical solution, power series solution, and approximate solutions are obtained and the results are compared.

A Note on an Initial Value Problem Associated With a Distributed Switch for Launching Spherical Waves

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1 Introduction

In this paper we study a nonlinear differential equation which arose in an investigation (see [1]) of a distributed switch for launching spherical waves. The distributed switch, which can be incorporated in typical Marx/peaker geometries, generalizes the concept of a distributed source. This source provides one way of launching a fast transient pulse in the TEM mode of a biconic (or monoconic) antenna geometry. In any case, the differential equation which arose in [1] was solved by a numerical method and no exact solution was actually obtained. Since the design of an array of switches which close at appropriate times is a worthy objective, more precise information on the solution to the initial value problem studied in [1] is desirable.

The initial value problem investigated in this paper consists of a first order nonlinear differential equation together with an initial value of the unknown function. This problem was solved in [1] by a numerical procedure. In Section 2 of this paper a power series solution is specified on a suitable interval and the results are seen to be in excellent agreement with the numerical solution. An approximate solution, which would be of value in a design application, is obtained in Section 3. Over much of the range of interest, the approximate solution is within 1/2% of the series solution. The techniques which were used to obtain these solutions should moreover be of independent mathematical interest. The results are briefly summarized in Section 4.

2 Series Solution of an Initial Value Problem

The initial value problem which arose in [1] is given on an interval $0 \le \psi < \pi/2$ by

$$\begin{aligned} \left(\frac{dR}{d\psi}\right)^2 + R^2 &= \sec^2(\psi) \\ R(0) &= 1 \end{aligned} \tag{2.1}$$

Here $R = R(\psi)$ represents a radial distance and ψ is an angle. A brief investigation into the functional form of $R(\psi)$ will establish the result, through use of series expansions of $R(\psi)$, that

$$R(\psi) = 1 + \left[\frac{-1+\sqrt{5}}{4}\right]\psi^2 + O(\psi^4)\,. \tag{2.2}$$

In this section we obtain more detailed information on the form of $R(\psi)$.

We start our analysis by recalling certain elementary results (see [2]) concerning the series expansions of $\sec(\psi)$ and $\sec^2(\psi)$. First we note that

$$\sec(\psi) = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!}\psi^{2n}\,,\ |\psi| < \frac{\pi}{2}\,. \tag{2.3}$$

The numbers E_n are the Euler numbers which are tabulated in the literature (see [3], for example). While there is no simple formula for E_n, these numbers may be obtained from a recursion relation

$$\sum_{R=0}^{n} \binom{2n}{2k} E_{2n-2k} = 0 \tag{2.4}$$

where $E_0 = 1$. The equation (2.4) is obtained from the Cauchy product of the series expansions of $\cos(\psi)$ and $\sec(\psi)$, noting the simple fact that $\cos(\psi)\sec(\psi) = 1$. The first eight nonzero Euler numbers are

$$\begin{aligned} E_0 &= 1 & E_8 &= 1,385 \\ E_2 &= -1 & E_{10} &= -50,521 \\ E_4 &= 5 & E_{12} &= 2,702,765 \\ E_6 &= -61 & E_{14} &= -199,360,981\,. \end{aligned} \tag{2.5}$$

A power series expansion for $\sec^2(\psi)$, which appears in the differential equation (2.1), is obtained by taking the Cauchy product of the series expansion of $\sec(\psi)$ with itself. The result, valid on $|\psi| < \frac{\pi}{2}$, is

$$\sec^2(\psi) = 1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} E_{2n-2k} \right) \frac{(-1)^n}{(2n)!} \psi^{2n}\,. \tag{2.6}$$

Hence we obtain

$$\left. \frac{d^{2n} \sec^2(\psi)}{d\psi^{2n}} \right|_{\psi=0} = (-1)^n \sum_{k=0}^{n} \binom{2n}{2k} E_{2k} E_{2n-2k}\,. \tag{2.7}$$

The differential equation which appears in (2.1), namely,

$$\left(\frac{dR}{d\psi} \right)^2 + R^2 = \sec^2(\psi) \tag{2.8}$$

can now be investigated. Instead of making a power series substitution for R into (2.8) we take a slightly different, though equivalent, approach to obtaining a series expansion for R. The procedure will be to repeatedly differentiate both sides of (2.8) and evaluate all results at $\psi = 0$. This process will yield the coefficients of ψ^{2n} in the power series

$$R(\psi) = \sum_{n=0}^{\infty} \left(\frac{d^{2n} R}{d\psi^{2n}} \right) \bigg|_{\psi=0} \frac{1}{(2n)!} \psi^{2n} \tag{2.9}$$

which represents the solution R on the interval $0 \le \psi < \pi/2$. In order to simplify the notation in the calculation, we define for $p = 0, 1, 2, 3, \ldots$

$$\left. \frac{d^p R}{d\psi^p} \right|_{\psi=0} = R_0^{(p)}\,. \tag{2.10}$$

Thus, if we differentiate both sides of (2.8) and take $\psi = 0$, we obtain

$$\begin{aligned} R_0^{(1)} R_0^{(2)} + R_0^{(0)} R_0^{(1)} &= \sec^2(\psi)\tan(\psi)|_{\psi=0} \\ R_0^{(1)}[R_0^{(2)} + R_0^{(0)}] &= \left[\frac{\sec^2(\psi)}{2}\right]^{(1)}\Bigg|_{\psi=0} = 0 \end{aligned} \tag{2.11}$$

Since $R_0^{(1)} = 0$, equation (2.11) results in no additional information. If this procedure is repeated on (2.11), the result is

$$R_0^{(2)}]R_0^{(2)} + R_0^{(0)}] = \left[\frac{\sec^2(\psi)}{2}\right]^{(2)}\Bigg|_{\psi=0} \tag{2.12}$$

and hence, from (2.7),

$$R_0^{(2)}[R_0^{(2)} + R_0^{(0)}] = -\frac{1}{2}\left[\binom{2}{0} E_0 E_2 + \binom{2}{2} E_2 E_0\right] = 1\,. \tag{2.13}$$

Thus from $[R_0^{(2)}]^2 + R_0^{(0)} R_0^{(2)} - 1 = 0$ we find

$$R_0^{(2)} = \frac{-1 \pm \sqrt{5}}{2}\,. \tag{2.14}$$

If we choose the positive root in (2.14), we are selecting the positive root of $\frac{dR}{d\psi}$ in the differential equation

$$\frac{dR}{d\psi} = \pm\sqrt{\sec^2(\psi) - R^2}\,. \tag{2.15}$$

Thus selection of the positive root will result in a solution $R(\psi)$ which is increasing on $0 \le \psi < \pi/2$. Hence, let us take

$$R_0^{(2)} = \frac{-1 + \sqrt{5}}{2} \equiv \alpha \tag{2.16}$$

in which case the numbers $R_0^{(p)}$, $p = 4, 6, 8, \ldots$ will turn out to be rational functions of α. Note that α satisfies the quadratic equation

$$\alpha^2 + \alpha - 1 = 0\,. \tag{2.17}$$

If we now continue this process of repeated differentiations of (2.8) followed by evaluations at $\psi = 0$, and if we make use of the fact that $R_0^{(p)} = 0$ for all odd positive integers p, we arrive at the identity

$$\sum_{k=0}^{m-1} \binom{2m-1}{2k} R_0^{(2m-2k)}[R_0^{(2k+2)} + R_0^{(2k)}] = \left[\frac{\sec^2(\psi)}{2}\right]^{(2m)}\Bigg|_{\psi=0} \tag{2.18}$$

and hence for $m = 1, 2, 3, \ldots$ we must have

$$\sum_{k=0}^{m-1} \binom{2m-1}{2k} R_0^{(2m-2k)}[R_0^{(2k+2)} + R_0^{(2k)}] = \frac{(-1)^m}{2} \sum_{k=0}^{m} \binom{2m}{2k} E_{2k}E_{2m-2k}\,. \tag{2.19}$$

From the equation (2.19) we can now obtain all of the coefficients in our power series expansion for $R(\psi)$. Thus, if $m = 2$, we obtain

$$\binom{3}{0} R_0^{(4)}[R_0^{(2)} + R_0^{(0)}] + \binom{3}{2} R_0^{(2)}[R_0^{(4)} + R_0^{(2)}] = \frac{1}{2}\left[\binom{4}{0} E_0E_4 + \binom{4}{2} E_2^2 + \binom{4}{4} E_4E_0\right] \tag{2.20}$$

and hence we find

$$R_0^{(4)} = \frac{8 - 3\alpha^2}{(1 + 4\alpha)} \tag{2.21}$$

and so $R_0^{(4)}$ is a rational function of α. Similarly, if $m = 3$,

$$R_0^{(6)} = \frac{90\alpha^4 + 45\alpha^3 + 2176\alpha^2 + 968\alpha - 504}{(1 + 4\alpha)^2(1 + 6\alpha)} \tag{2.22}$$

which again is a rational function of α. The series expansion for $R(\psi)$, which is

$$\begin{aligned} R(\psi) &= 1 + \sum_{n=1}^{\infty} \frac{R_0^{(2n)}}{(2n)!}\psi^{2n} \\ &= 1 + \sum_{n=1}^{\infty} a_{2n}\psi^{2n} \end{aligned} \tag{2.23}$$

is thus determined. Since $\alpha = (-1+\sqrt{5})/2$, the coefficients a_{2n} are also known. The values of a_{2n}, for $n = 1, 2, 3, \ldots, 8$ are given explicitly, through (2.18), as

$$\begin{aligned} a_2 &= 0.30902 & a_{10} &= 0.00228 \\ a_4 &= 0.08225 & a_{12} &= 0.00076 \\ a_6 &= 0.02322 & a_{14} &= 0.00026 \\ a_8 &= 0.00707 & a_{16} &= 0.00009 \end{aligned} \tag{2.24}$$

and the corresponding values of $R(\psi)$, for $0 \leq \psi < \pi/2$ may then be computed. Table 2.1 gives a comparison of these values with those obtained by a numerical solution of (2.1). For values of $\psi \leq 1$ radian the agreement between these values of R is nearly perfect.

3 An Approximate Solution

It would indeed be satisfying if the initial value problem

$$\begin{aligned} \left(\frac{dR}{d\psi}\right)^2 + R^2 &= \sec^2(\psi) \\ R(0) &= 1 \end{aligned} \tag{3.1}$$

studied in section had a reasonably simple exact solution for $R(\psi)$ expressible in closed form in terms of known functions. Unfortunately this does not appear to be the case. However, an approximate solution is possible and obtainable as follows.

We first observe that the initial value problem

$$\begin{aligned} (y')^2 + y^2 &= \sec^4(\psi) \\ y(0) &= 1 \end{aligned} \tag{3.2}$$

does have an exact solution $y(\psi) = \sec(\psi)$ and so, since $\sec^2(\psi) \leq \sec^4(\psi)$ for $0 \leq \psi < \pi/2$, we must have $1 \leq R(\psi) \leq \sec(\psi)$ if the positive root in (2.15) is chosen. Alternatively, the substitution $R(\psi) = \lambda(\psi) + \sec(\psi)$ into (3.1) leads to a differential equation for $\lambda(\psi)$ from which it is clear that $R(\psi) \leq \sec(\psi)$. Note also that the differential equation (3.1) suggests a possible substitution given by

$$R = \frac{\sec(\psi)}{\sec(u)} \,. \tag{3.3}$$

The change of dependent variables leads to a new differential equation

$$u' = \frac{\tan(\psi) - \tan(u)}{\tan(u)} \,. \tag{3.4}$$

Note that the right-hand side of (3.4) is not defined when $\psi = 0$, since (3.3) implies $u = 0$. However,

$$\lim_{\psi \to 0} \frac{\tan(\psi) - \tan(u)}{\tan(u)} = \frac{1 - u'(0)}{u'(0)} = u'(0) \tag{3.5}$$

and hence $u'(0) = (-1 \pm \sqrt{5})/2$. The choice of the positive root then yields $u'(0) = \alpha$.

If we now assume that ψ is near 0, so that u also is near 0, then (3.4) assumes the form

$$u' = \frac{\psi - u}{u} . \tag{3.6}$$

Note that the right-hand side of (3.6) is a homogeneous function. Such equations are always transformable to a separable form through a substitution $u = \psi v$. In the case of (3.6) we have an even simpler situation since one solution of this equation is obviously

$$u = \alpha\psi . \tag{3.7}$$

(Recall that α is one of the roots of equation (2.17).)

Thus an approximate solution to (3.1) is

$$\tilde{R}(\psi) = \frac{\sec(\psi)}{\sec(\alpha\psi)} . \tag{3.8}$$

Comparisons of the values of $\tilde{R}(\psi)$ with those obtained for $R(\psi)$ appear in Table 3.1. For a value of $\psi = 0.4$ radians, the error is approximately 1/10% while if $0.4 \le \psi \le 0.7$ the error is still less than 1%. In the range $0.7 \le \psi \le 1.0$ the error increases to almost 6%. Thus the approximate solution given by (3.8) appears to be adequate for engineering purposes.

The approximate solution, (3.8), to the initial value problem, (3.1), can be used to generate yet another approximate solution. If (3.8) is substituted into the right-hand side of

$$\frac{dR}{d\psi} = \sqrt{\sec^2(\psi) - R^2} \tag{3.9}$$

we obtain

$$\frac{dR}{d\psi} = \sec(\psi)\sin(\alpha\psi) . \tag{3.10}$$

If $\sec(\psi)$ and $\sin(\alpha\psi)$ are replaced by their series representations on $0 \le \psi < \pi/2$, then one obtains after integration of a Cauchy product, the result that

$$R^*(\psi) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} \left(\sum_{k=0}^{n} \binom{2n+1}{2k} E_{2k} \alpha^{2n-2k+1} \psi^{2n+2} \right) \tag{3.11}$$

where $R^*(\psi)$ is a second approximation of $R(\psi)$. Thus

$$\begin{aligned} R^*(\psi) &= 1 + \frac{1}{2!}\alpha\psi^2 - \frac{1}{4!}(\alpha^3 - 3\alpha)\psi^4 + \frac{1}{6!}(\alpha^5 - 10\alpha^3 + 25\alpha)\psi^6 \\ &\quad - \frac{1}{8!}(\alpha^7 - 21\alpha^5 + 175\alpha^3 - 427\alpha)\psi^8 + \frac{1}{10!}(\alpha^9 - 36\alpha^7 + 630\alpha^5 - 5124\alpha^3 + 12465\alpha)\psi^{10} \\ &\quad - \cdots \end{aligned} \tag{3.12}$$

and if the value $\alpha = (-1 + \sqrt{5})/2$ is substituted into (3.12) we obtain

$$\begin{aligned} R^*(\psi) &= 1 + (0.30902)\psi^2 + (0.06742)\psi^4 + (0.01831)\psi^6 \\ &\quad + (0.00557)\psi^8 + (0.00180)\psi^{10} + \cdots . \end{aligned}$$

Table 3.2 compares the values of $R^*(\psi)$ with the numerical solution and the series solution obtained in (2.23). At the value of $\psi = 1$ radian the error that results from comparing $R^*(1)$ with the value of R from the numerical solution is approximately 1.6%.

4 Summary

In this paper a study has been made of a certain initial value problem, given in (2.1). While efforts to produce an exact solution for $R(\psi)$ in a closed form were not successful, a series solution (2.23) and an approximate solution (3.8) were obtained and the agreement between these solutions is quite good. It is possible to introduce further transformations of the dependent variable u in (3.4) which lead to Abel differential equations of either the first or second kind. However, further efforts to solve these equations, which have the general form

$$v' = a(\psi) + b(\psi)v + c(\psi)v^2 + d(\psi)v^3 \tag{4.1}$$

failed. Occasionally equations of the for (4.1) can be put into a form in which the variables separate, but such was not the case here. The interested reader should consult [4] or [5] for further details concerning Abel equations.

ψ (Radians)	R(ψ) (Numerical Solution)	R(ψ) (Series Solution)	sec(ψ)	% Error
0.00	1.00000	1.00000	1.00000	0.00000
0.01	1.00003	1.00003	1.00005	0.00000
0.02	1.00012	1.00012	1.00020	0.00000
0.03	1.00028	1.00028	1.00045	0.00000
0.04	1.00049	1.00049	1.00085	0.00000
0.05	1.00077	1.00077	1.00125	0.00000
0.06	1.00111	1.00111	1.00180	0.00000
0.07	1.00152	1.00152	1.00245	0.00000
0.08	1.00198	1.00198	1.00321	0.00000
0.09	1.00251	1.00251	1.00406	0.00000
0.10	1.00310	1.00310	1.00502	0.00000
0.11	1.00375	1.00375	1.00608	0.00000
0.12	1.00447	1.00447	1.00724	0.00000
0.13	1.00525	1.00525	1.00851	0.00000
0.14	1.00609	1.00609	1.00988	0.00000
0.15	1.00699	1.00699	1.01136	0.00000
0.16	1.00797	1.00797	1.01294	0.00000
0.17	1.00900	1.00900	1.01463	0.00000
0.18	1.01010	1.01010	1.01642	0.00000
0.19	1.01126	1.01126	1.01832	0.00000
0.20	1.01249	1.01249	1.02034	0.00000
0.21	1.01379	1.01379	1.02246	0.00000
0.22	1.01515	1.01515	1.02470	0.00000
0.23	1.01658	1.01658	1.02705	0.00000
0.24	1.01808	1.01808	1.02951	0.00000
0.25	1.01964	1.01964	1.03208	0.00000
0.26	1.02127	1.02127	1.03476	0.00000
0.27	1.02297	1.02297	1.03759	0.00000
0.28	1.02474	1.02474	1.04052	0.00000
0.29	1.02658	1.02658	1.04358	0.00000
0.30	1.02850	1.02850	1.04675	0.00000
0.31	1.03048	1.03048	1.05005	0.00000
0.32	1.03253	1.03253	1.05348	0.00000
0.33	1.03466	1.03466	1.05703	0.00000
0.34	1.03686	1.03686	1.06072	0.00000
0.35	1.03913	1.03913	1.06454	0.00000
0.36	1.04148	1.04148	1.06849	0.00000
0.37	1.04391	1.04391	1.07258	0.00000
0.38	1.04641	1.04641	1.07681	0.00000
0.39	1.04899	1.04899	1.08119	0.00000
0.40	1.05165	1.05165	1.08570	0.00000
0.41	1.05439	1.05439	1.09037	0.00000
0.42	1.05720	1.05720	1.09518	0.00000
0.43	1.06010	1.06010	1.10015	0.00000
0.44	1.06309	1.06309	1.10528	0.00000
0.45	1.06615	1.06615	1.11056	0.00000
0.46	1.06931	1.06931	1.11601	0.00000
0.47	1.07254	1.07254	1.12162	0.00000
0.48	1.07587	1.07587	1.12740	0.00000
0.49	1.07928	1.07928	1.13336	0.00000
0.50	1.08279	1.08279	1.13949	0.00000

Table 2.1. Comparison of Numerical Values of R(ψ) with the Series (Exact) Values of R(ψ): Solution is bounded by sec(ψ).

ψ (Radians)	R(ψ) (Numerical Solution)	R(ψ) (Series Solution)	sec(ψ)	% Error
0.51	1.08638	1.08638	1.14581	0.00000
0.52	1.09007	1.09007	1.15231	0.00000
0.53	1.09386	1.09386	1.15901	0.00000
0.54	1.09774	1.09774	1.16590	0.00000
0.55	1.10171	1.10171	1.17299	0.00000
0.56	1.10579	1.10579	1.18028	0.00000
0.57	1.10997	1.10997	1.18779	0.00000
0.58	1.11425	1.11425	1.19551	0.00000
0.59	1.11863	1.11863	1.20345	0.00000
0.60	1.12312	1.12312	1.21163	0.00000
0.61	1.12772	1.12772	1.22004	0.00000
0.62	1.13244	1.13244	1.22868	0.00000
0.63	1.13726	1.13726	1.23758	0.00000
0.64	1.14220	1.14220	1.24673	0.00000
0.65	1.14725	1.14725	1.25615	0.00000
0.66	1.15243	1.15243	1.26583	0.00000
0.67	1.15773	1.15773	1.27580	0.00000
0.68	1.16315	1.16315	1.28605	0.00000
0.69	1.16870	1.16870	1.29660	0.00000
0.70	1.17438	1.17438	1.30746	0.00000
0.71	1.18020	1.18020	1.31863	0.00000
0.72	1.18615	1.18615	1.33013	0.00000
0.73	1.19224	1.19224	1.34197	0.00000
0.74	1.19847	1.19847	1.35415	0.00000
0.75	1.20485	1.20485	1.36670	0.00000
0.76	1.21137	1.21137	1.37962	0.00000
0.77	1.21805	1.21805	1.39293	0.00000
0.78	1.22489	1.22489	1.40664	0.00000
0.79	1.23189	1.23188	1.42077	0.00081
0.80	1.23905	1.23905	1.43532	0.00000
0.81	1.24638	1.24638	1.45033	0.00000
0.82	1.25388	1.25388	1.46580	0.00000
0.83	1.26156	1.26156	1.48175	0.00000
0.84	1.26943	1.26942	1.49821	0.00079
0.85	1.27748	1.27748	1.51519	0.00000
0.86	1.28572	1.28572	1.53271	0.00000
0.87	1.29417	1.29416	1.55080	0.00077
0.88	1.30282	1.30281	1.56949	0.00077
0.89	1.31167	1.31167	1.58878	0.00000
0.90	1.32075	1.32074	1.60873	0.00076
0.91	1.33005	1.33004	1.62934	0.00075
0.92	1.33957	1.33956	1.65065	0.00075
0.93	1.34934	1.34932	1.67270	0.00074
0.94	1.35935	1.35933	1.69552	0.00148
0.95	1.36961	1.36959	1.71915	0.00146
0.96	1.38013	1.38011	1.74362	0.00144
0.97	1.39092	1.39089	1.76897	0.00216
0.98	1.40199	1.40196	1.79526	0.00213
0.99	1.41335	1.41331	1.82252	0.00284
1.00	1.42501	1.42496	1.85081	0.00350

Table 2.1. Continuation

ψ (Radians)	R(ψ) (Numerical Solution)	R(ψ) (Series Solution)	$\tilde{R}(\psi)$ (Approximate Solution)	sec(ψ)	% Error
0.00	1.00000	1.00000	1.00000	1.00000	0.00000
0.01	1.00003	1.00003	1.00003	1.00005	0.00000
0.02	1.00012	1.00012	1.00012	1.00020	0.00000
0.03	1.00028	1.00028	1.00028	1.00045	0.00000
0.04	1.00049	1.00049	1.00049	1.00080	-0.00001
0.05	1.00077	1.00077	1.00077	1.00125	-0.00002
0.06	1.00111	1.00111	1.00111	1.00180	-0.00005
0.07	1.00152	1.00152	1.00152	1.00245	-0.00009
0.08	1.00198	1.00198	1.00198	1.00321	-0.00015
0.09	1.00251	1.00251	1.00251	1.00406	-0.00024
0.10	1.00310	1.00310	1.00310	1.00502	-0.00037
0.11	1.00375	1.00375	1.00376	1.00608	-0.00054
0.12	1.00447	1.00447	1.00447	1.00724	-0.00076
0.13	1.00525	1.00525	1.00526	1.00851	-0.00105
0.14	1.00609	1.00609	1.00610	1.00988	-0.00142
0.15	1.00699	1.00699	1.00701	1.01136	-0.00187
0.16	1.00797	1.00797	1.00799	1.01294	-0.00243
0.17	1.00900	1.00900	1.00903	1.01463	-0.00310
0.18	1.01010	1.01010	1.01014	1.01642	-0.00390
0.19	1.01126	1.01126	1.01131	1.01832	-0.00484
0.20	1.01249	1.01249	1.01255	1.02034	-0.00595
0.21	1.01379	1.01379	1.01386	1.02246	-0.00725
0.22	1.01515	1.01515	1.01524	1.02470	-0.00874
0.23	1.01658	1.01658	1.01669	1.02705	-0.01046
0.24	1.01808	1.01808	1.01820	1.02951	-0.01243
0.25	1.01964	1.01964	1.01979	1.03208	-0.01466
0.26	1.02127	1.02127	1.02145	1.03476	-0.01718
0.27	1.02297	1.02297	1.02318	1.03759	-0.02002
0.28	1.02474	1.02474	1.02498	1.04052	-0.02320
0.29	1.02658	1.02658	1.02686	1.04358	-0.02676
0.30	1.02850	1.02850	1.02881	1.04675	-0.03071
0.31	1.03048	1.03048	1.03084	1.05005	-0.03510
0.32	1.03253	1.03253	1.03294	1.05348	-0.03994
0.33	1.03466	1.03466	1.03513	1.05703	-0.04529
0.34	1.03686	1.03686	1.03739	1.06072	-0.05116
0.35	1.03913	1.03913	1.03973	1.06454	-0.05760
0.36	1.04148	1.04148	1.04216	1.06849	-0.06465
0.37	1.04391	1.04391	1.04466	1.07258	-0.07234
0.38	1.04641	1.04641	1.04725	1.07681	-0.08072
0.39	1.04899	1.04899	1.04993	1.08119	-0.08982
0.40	1.05165	1.05165	1.05270	1.08570	-0.09970
0.41	1.05439	1.05439	1.05555	1.09037	-0.11040
0.42	1.05720	1.05720	1.05849	1.09518	-0.12197
0.43	1.06010	1.06010	1.06153	1.10015	-0.13445
0.44	1.06309	1.06309	1.06466	1.10528	-0.14791
0.45	1.06615	1.06615	1.06789	1.11056	-0.16239
0.46	1.06931	1.06931	1.07121	1.11601	-0.17796
0.47	1.07254	1.07254	1.07463	1.12162	-0.19467
0.48	1.07587	1.07587	1.07816	1.12740	-0.21258
0.49	1.07928	1.07928	1.08178	1.13336	-0.23176
0.50	1.08279	1.08279	1.08552	1.13949	-0.25228

Table 3.1. Comparison of Series Solution for R(ψ) with an Approximate Solution, $\tilde{R}(\psi)=\sec(\psi)/\sec(\alpha\psi)$, $\alpha=(-1+\sqrt{5})/2$.

ψ (Radians)	R(ψ) (Numerical Solution)	R(ψ) (Series Solution)	$\tilde{R}(\psi)$ (Approximate Solution)	sec(ψ)	% Error
0.51	1.08638	1.08638	1.08936	1.14581	-0.27420
0.52	1.09007	1.09007	1.09332	1.15231	-0.29759
0.53	1.09386	1.09386	1.09738	1.15901	-0.32254
0.54	1.09774	1.09774	1.10157	1.16590	-0.34912
0.55	1.10171	1.10171	1.10587	1.17299	-0.37741
0.56	1.10579	1.10579	1.11030	1.18028	-0.40749
0.57	1.10997	1.10997	1.11484	1.18779	-0.43946
0.58	1.11425	1.11425	1.11952	1.19551	-0.47341
0.59	1.11863	1.11863	1.12433	1.20345	-0.50943
0.60	1.12312	1.12312	1.12927	1.21163	-0.54763
0.61	1.12772	1.12772	1.13436	1.22004	-0.58810
0.62	1.13244	1.13244	1.13958	1.22868	-0.63096
0.63	1.13726	1.13726	1.14495	1.23758	-0.67632
0.64	1.14220	1.14220	1.15047	1.24673	-0.72429
0.65	1.14725	1.14725	1.15615	1.25615	-0.77501
0.66	1.15243	1.15243	1.16198	1.26583	-0.82860
0.67	1.15773	1.15773	1.16798	1.27580	-0.88519
0.68	1.16315	1.16315	1.17414	1.28605	-0.94492
0.69	1.16870	1.16870	1.18048	1.29660	-1.00795
0.70	1.17438	1.17438	1.18700	1.30746	-1.07443
0.71	1.18020	1.18020	1.19371	1.31863	-1.14451
0.72	1.18615	1.18615	1.20060	1.33013	-1.21836
0.73	1.19224	1.19224	1.20769	1.34197	-1.29616
0.74	1.19847	1.19847	1.21498	1.35415	-1.37810
0.75	1.20485	1.20485	1.22249	1.36670	-1.46436
0.76	1.21137	1.21137	1.23021	1.37962	-1.55516
0.77	1.21805	1.21805	1.23816	1.39293	-1.65069
0.78	1.22489	1.22489	1.24634	1.40664	-1.75119
0.79	1.23189	1.23188	1.25476	1.42077	-1.85688
0.80	1.23905	1.23905	1.26343	1.43532	-1.96802
0.81	1.24638	1.24638	1.27236	1.45033	-2.08486
0.82	1.25388	1.25388	1.28156	1.46580	-2.20767
0.83	1.26156	1.26156	1.29104	1.48175	-2.33673
0.84	1.26943	1.26942	1.30081	1.49821	-2.47236
0.85	1.27748	1.27748	1.31088	1.51519	-2.61486
0.86	1.28572	1.28572	1.32127	1.53271	-2.76458
0.87	1.29417	1.29416	1.33198	1.55080	-2.92185
0.88	1.30282	1.30281	1.34303	1.56949	-3.08707
0.89	1.31167	1.31167	1.35444	1.58878	-3.26062
0.90	1.32075	1.32074	1.36621	1.60873	-3.44291
0.91	1.33005	1.33004	1.37838	1.62934	-3.63440
0.92	1.33957	1.33956	1.39094	1.65065	-3.83554
0.93	1.34934	1.34932	1.40393	1.67270	-4.04685
0.94	1.35935	1.35933	1.41736	1.69552	-4.26884
0.95	1.36961	1.36959	1.43125	1.71915	-4.50208
0.96	1.38013	1.38011	1.44562	1.74362	-4.74716
0.97	1.39092	1.39089	1.46050	1.76897	-5.00472
0.98	1.40199	1.40196	1.47592	1.79526	-5.27545
0.99	1.41335	1.41331	1.49189	1.82252	-5.56007
1.00	1.42501	1.42496	1.50845	1.85081	-5.85935

Table 3.1. Continuation

ψ (Radians)	R(ψ) (Numerical Solution)	R*(ψ) (Second Approximation)	sec(ψ)	% Error
0.00	1.00000	1.00000	1.00000	0.00000
0.01	1.00003	1.00003	1.00005	0.00000
0.02	1.00012	1.00012	1.00020	0.00000
0.03	1.00028	1.00028	1.00045	0.00000
0.04	1.00049	1.00049	1.00080	0.00000
0.05	1.00077	1.00077	1.00125	0.00000
0.06	1.00111	1.00111	1.00180	0.00000
0.07	1.00152	1.00152	1.00245	0.00000
0.08	1.00198	1.00198	1.00321	0.00000
0.09	1.00251	1.00251	1.00406	0.00000
0.10	1.00310	1.00310	1.00502	0.00000
0.11	1.00375	1.00375	1.00608	0.00000
0.12	1.00447	1.00446	1.00724	0.00031
0.13	1.00525	1.00524	1.00851	0.00042
0.14	1.00609	1.00608	1.00988	0.00057
0.15	1.00699	1.00699	1.01136	0.00000
0.16	1.00797	1.00796	1.01294	0.00097
0.17	1.00900	1.00899	1.01463	0.00123
0.18	1.01010	1.01008	1.01642	0.00155
0.19	1.01126	1.01124	1.01832	0.00193
0.20	1.01249	1.01247	1.02034	0.00236
0.21	1.01379	1.01376	1.02246	0.00288
0.22	1.01515	1.01512	1.02470	0.00347
0.23	1.01658	1.01654	1.02705	0.00414
0.24	1.01808	1.01803	1.02951	0.00491
0.25	1.01964	1.01958	1.03208	0.00578
0.26	1.02127	1.02120	1.03478	0.00677
0.27	1.02297	1.02289	1.03759	0.00787
0.28	1.02474	1.02465	1.04052	0.00911
0.29	1.02658	1.02648	1.04358	0.01049
0.30	1.02850	1.02837	1.04675	0.01201
0.31	1.03048	1.03034	1.05005	0.01370
0.32	1.03253	1.03237	1.05348	0.01556
0.33	1.03466	1.03448	1.05703	0.01760
0.34	1.03686	1.03665	1.06072	0.01984
0.35	1.03913	1.03890	1.06454	0.02229
0.36	1.04148	1.04122	1.06849	0.02495
0.37	1.04391	1.04362	1.07258	0.02785
0.38	1.04641	1.04609	1.07681	0.03099
0.39	1.04899	1.04863	1.08119	0.03440
0.40	1.05165	1.05125	1.08570	0.03807
0.41	1.05439	1.05394	1.09037	0.04204
0.42	1.05720	1.05672	1.09518	0.04630
0.43	1.06010	1.05957	1.10015	0.05089
0.44	1.06309	1.06249	1.10528	0.05580
0.45	1.06615	1.066550	1.11056	0.06107
0.46	1.06931	1.06859	1.11601	0.06670
0.47	1.07254	1.07176	1.12162	0.07272
0.48	1.07587	1.07502	1.12740	0.07913
0.49	1.07928	1.07836	1.13336	0.08597
0.50	1.08279	1.08178	1.13949	0.09323

Table 3.2. Comparison of Numerical Solution for R(ψ) with a Second Approximation, R*(ψ).

ψ (Radians)	R(ψ) (Numerical Solution)	R*(ψ) (Second Approximation)	sec(ψ)	% Error
0.51	1.08638	1.08529	1.14581	0.10095
0.52	1.09007	1.08888	1.15231	0.10915
0.53	1.09386	1.09257	1.15901	0.11783
0.54	1.09774	1.09634	1.16590	0.12703
0.55	1.10171	1.10021	1.17299	0.13676
0.56	1.10579	1.10416	1.18028	0.14704
0.57	1.10997	1.10821	1.18779	0.15790
0.58	1.11425	1.11236	1.19551	0.16935
0.59	1.11863	1.11660	1.20345	0.18142
0.60	1.12312	1.12094	1.21163	0.19413
0.61	1.12772	1.12538	1.22004	0.20750
0.62	1.13244	1.12993	1.22868	0.22156
0.63	1.13726	1.13457	1.23758	0.23634
0.64	1.14220	1.13932	1.24673	0.25185
0.65	1.14725	1.14418	1.25615	0.26813
0.66	1.15243	1.14914	1.26583	0.28519
0.67	1.15773	1.15422	1.27580	0.30308
0.68	1.16315	1.15941	1.28605	0.32181
0.69	1.16870	1.16471	1.29660	0.34141
0.70	1.17438	1.17013	1.30746	0.36191
0.71	1.18020	1.17567	1.31863	0.38336
0.72	1.18615	1.18133	1.33013	0.40576
0.73	1.19224	1.18712	1.34197	0.42917
0.74	1.19847	1.19303	1.35415	0.45361
0.75	1.20485	1.19907	1.36670	0.47911
0.76	1.21137	1.20525	1.37962	0.50572
0.77	1.21805	1.21155	1.39293	0.53347
0.78	1.22489	1.21800	1.40664	0.56240
0.79	1.23189	1.22459	1.42077	0.59255
0.80	1.23905	1.23132	1.43532	0.62396
0.81	1.24638	1.23819	1.45033	0.65669
0.82	1.25388	1.24522	1.46580	0.69076
0.83	1.26156	1.25240	1.48175	0.72624
0.84	1.26943	1.25974	1.49821	0.76317
0.85	1.27748	1.26724	1.51519	0.80160
0.86	1.28572	1.27490	1.53271	0.84160
0.87	1.29417	1.28274	1.55080	0.88322
0.88	1.30282	1.29074	1.56949	0.92653
0.89	1.31167	1.29893	1.58878	0.97159
0.90	1.32075	1.30730	1.60873	1.01846
0.91	1.33005	1.31585	1.62934	1.06723
0.92	1.33957	1.32460	1.65065	1.11797
0.93	1.34934	1.33354	1.67270	1.17077
0.94	1.35935	1.34268	1.69552	1.22572
0.95	1.36961	1.35204	1.71915	1.28291
0.96	1.38013	1.36160	1.74362	1.34244
0.97	1.39092	1.37139	1.76897	1.40442
0.98	1.40199	1.38140	1.79526	1.46898
0.99	1.41335	1.39164	1.82252	1.53622
1.00	1.42501	1.40212	1.85081	1.60629

Table 3.2. Continuation

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