Mathematics Notes

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Norms of Time-Domain Functions and Convolution Operators

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Abstract

This note develops various norms of time-domain functions and convolution operators to obtain bounds for transient system response. Besides the usual p-norm one can define another norm, the residue norm (or r-norm), based on the singularities in the complex-frequency (or Laplace-transform) plane.

Contents

I. Introduction

In describing the interaction of electromagnetic fields with complex systems electromagnetic topology is used to organize the system into parts which can be separately analyzed and the results subsequently combined [1, 7, 9]. Due to the complexity of the problem it is often more practical to look for bounds on the electromagnetic response instead of "exact" answers. In this context the use of norms has been developed to obtain rigorous bounds which are in some sense also "tight" bounds. For this purpose several papers discuss the use of vector norms and associated matrix norms of wave variables and associated scattering matrices, primarily in frequency domain [1, 2, 3, 5, 7].

More recently these norm considerations have been extended into time domain [4, 6]. In this form the transient waveforms are characterized by positive scalars related to appropriate system response parameters. This note considers the p-norms of transient waveforms and the associated operator norms for time-domain convolution operators (what transfer functions become in time domain). Another type of norm called the residue norm is defined based on the singularities in the complex-frequency (or Laplace-transform) plane, and related to the p-norm.

II. Norms of Time-Domain Functions and Operators

Norms of vectors have been defined by the properties [8]

$$
\|(x_n)\| \begin{cases} = 0 & \text{iff } (x_n) = (0_n) \\ > 0 & \text{otherwise} \end{cases}
$$

$$
\|\alpha(x_n)\| = |\alpha| \; \|(x_n)\| \quad , \quad \alpha \equiv \text{a complex scalar} \tag{2.1}
$$

$$
\|(x_n) + (y_n)\| \le \|(x_n)\| + \|(y_n)\| \quad \text{(triangle inequality)}
$$

$$
\|(x_n)\| \text{ depends continuously on } (x_n)
$$

Associated matrix norms are defined by

$$
\|(A_{n,m})\| \equiv \sup_{(x_n)\neq(0_n)} \frac{\|(A_{n,m}) \cdot (x_n)\|}{\|(x_n)\|} \tag{2.2}
$$

where the matrices are allowed to be rectangular as long as there is a compatibility of numbers of rows and columns to allow dot multiplication. These matrix norms have the properties

$$
\|(A_{n,m})\| \begin{cases} = 0 & \text{iff } (A_{n,m}) = (0_{n,m}) \\ > 0 & \text{otherwise} \end{cases}
$$

$$
\|\alpha(A_{n,m})\| = |\alpha| \; \|(A_{n,m})\|
$$

$$
\|(A_{n,m}) + (B_{n,m})\| \le \|(A_{n,m})\| + \|(B_{n,m})\| \quad \text{(triangle inequality)} \tag{2.3}
$$

$$
\|(A_{n,m}) \cdot (B_{n,m})\| \le \|(A_{n,m})\| \; \|(B_{n,m})\| \quad \text{(Schwarz inequality)}
$$

$$
\|(A_{n,m})\| \text{ depends continuously on } (A_{n,m})
$$

Here $(A_{n,m})$ and $(B_{n,m})$ are general rectangular matrices as long as they are compatible for addition and/or dot multiplication as required.

The vector norms are generalized to functional norms in the sense that a vector of infinitely many components (an infinite dimensioned vector space) can be considered as a function of a real variable (taken as t (or time) in this case, but could represent other kinds of parameters). These have the properties [4, 14]

$$\|f(t)\| \begin{cases} = 0 & \text{iff } f(t) \equiv 0 \text{ or has zero "measure" per the particular norm} \\ > 0 & \text{otherwise} \end{cases}$$

$$\|\alpha f(t)\| = |\alpha| \, \|f(t)\| \quad , \quad \alpha \equiv \text{a complex scalar} \tag{2.4}$$

$$\|f(t) + g(t)\| \le \|f(t)\| + \|g(t)\|$$

In a manner similar to associated matrix norms we can define associated operator norms via

$$\|\Lambda(\ )\| = \sup_{f(t) \not\equiv 0} \frac{\|\Lambda(f(t))\|}{\|f(t)\|} \tag{2.5}$$

where now Λ is an operator which operates on a function to produce another function, say as

$$F(t) = \Lambda(f(t)) \tag{2.6}$$

Here Λ can include integration, differentiation, or any kind of linear operation that results in the following properties of an operator norm

$$\|\Lambda(\ )\| \begin{cases} = 0 & \text{iff } \Lambda(\ ) \equiv 0 \text{ or has zero "measure" per the particular norm} \\ > 0 & \text{otherwise} \end{cases}$$

$$\|\alpha\Lambda(\ )\| = |\alpha| \, \|\Lambda(\ )\|$$

$$\|\Lambda(\ ) + \Upsilon(\ )\| \le \|\Lambda(\ )\| + \|\Upsilon(\ )\| \tag{2.7}$$

$$\|\Lambda(\Upsilon(\ ))\| \le \|\Lambda(\ )\| \, \|\Upsilon(\ )\|$$

Note that Λ and Υ are required to operate over the same range of t on which $f(t)$ is defined

One should be careful in considering the norms of functions and operators. There are considerations of continuity, continuous derivatives, integrability, etc., which may need to be considered, depending on the particular norm under consideration. Note here that while we are defining function and operator norms as a natural extension of vector and matrix norms, this concept can be further generalized to vectors of functions, matrices of operators, functions of several variables and associated operators, etc.

There are various kinds of operators of physical interest. A very important one is the convolution operator (with respect to time) that characterizes linear, time-invariant systems. We symbolize this special operator by $g(t)\circ$ where

$$\begin{aligned} F(t) &\equiv g(t) \circ f(t) \\ &= \int_{-\infty}^{\infty} g(t - t')\, f(t')\, dt' \\ &= \int_{-\infty}^{\infty} g(t')\, f(t - t')\, dt' \end{aligned} \tag{2.8}$$

If (as we normally do) we assume that $g(t)\circ$ is causal (no response before an excitation) then

$$g(t) = 0 \quad \text{for} \quad t < 0 \tag{2.9}$$

and

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} g(t - t')\, f(t')\, dt' \\ &= \int_{0}^{\infty} g(t')\, f(t - t')\, dt' \end{aligned} \tag{2.10}$$

The concept of convolution is closely related to the Laplace transform (two sided) defined by

$$\begin{aligned} \tilde{f}(s) &\equiv \int_{-\infty}^{\infty} f(t)\, e^{-st} dt \\ f(t) &= \frac{1}{2\pi j} \int_{Br} \tilde{f}(s)\, e^{st} ds \end{aligned} \tag{2.11}$$

Br ≡ Bromwich contour in strip of convergence of Laplace transform (parallel to $j\omega$ axis)

$s = \Omega + j\omega$

In terms of the Laplace transform (2.8) (or (2.10)) becomes

$$\tilde{F}(s) = \tilde{g}(s)\, \tilde{f}(s) \tag{2.12}$$

so that convolution in time domain becomes multiplication in complex-frequency (or Laplace or Fourier) domain.

Still in time domain convolution is an operation for which we consider a general convolution operator $g(t)\circ$ which is to be distinguished from the function $g(t)$. Following (2.5) we define the associated norm of a convolution operator as

$$||g(t)\circ|| \equiv \sup_{f(t)\not\equiv 0} \frac{||g(t) \circ f(t)||}{||f(t)||} \tag{2.13}$$

which depends on the particular function norm chosen (and which requires that $f(t)$ be limited to functions for which such a norm exists). Note that in general the norm of the operator $g(t)\circ$ is not the same as the norm of the function $g(t)$.

III. p-Norm of Time-Domain Convolution Operators

The p-norm of time-domain waveforms is [4]

$$||f(t)||_p \equiv \{\int_{-\infty}^{\infty} |f(t)|^p dt\}^{\frac{1}{p}} \tag{3.1}$$

$$1 \leq p < \infty$$

with a special case of the ∞-norm as

$$||f(t)||_\infty = \sup_{-\infty<t<\infty} |f(t)| \tag{3.2}$$

where isolated values of f(t) are excluded by considering limits from both sides of values of f(t) of concern.

Considering a convolution operator $g(t)\circ$ as in (2.2) we have from (2.13) and (2.10) for p-norms

$$||g(t)\circ||_p = \sup_{f(t)\not\equiv 0} \frac{||g(t) \circ f(t)||_p}{||f(t)||_p} \tag{3.3}$$

$$= \frac{\{\int_{-\infty}^{\infty} |\int_0^{\infty} g(t') f(t - t') dt'|^p dt\}^{\frac{1}{p}}}{||f(t)||_p}$$

where $g(t)\circ$ has been assumed to be causal. Apply the Hölder inequality (Appendix A) with

$$\begin{aligned} f_1(t') &= |g(t')|^{\frac{p-1}{p}} \\ f_2(t') &= |g(t')|^{\frac{1}{p}} |f(t - t')| \\ p_1 &= [1 - \frac{1}{p}]^{-1} = \frac{p}{p - 1} \\ p_2 &= p \end{aligned} \tag{3.4}$$

Then

$$|\int_0^{\infty} g(t') f(t - t') dt'|$$
$$\leq \{\int_0^{\infty} |g(t')| dt'\}^{\frac{p-1}{p}} \{\int_0^{\infty} |g(t')| |f(t - t')|^p dt'\}^{\frac{1}{p}} \tag{3.5}$$

Integrating over t of the pth power of the above (as in (3.3)) gives

$$\int_{-\infty}^{\infty} \left|\int_0^{\infty} g(t')\ f(t - t')\ dt'\right|^p dt$$

$$< \int_{-\infty}^{\infty} \{\int_0^{\infty} |g(t')|\ dt'\}^{p-1}\ \{\int_0^{\infty} |g(t')|\ |f(t - t')|^p dt'\}\ dt$$

$$= \int_{-\infty}^{\infty} ||g(t')||_1^{p-1}\ \{\int_0^{\infty} |g(t')|\ |f(t - t')|^p dt'\}\ dt$$

$$= ||g(t)||_1^{p-1} \int_0^{\infty} |g(t')|\ \{\int_{-\infty} |f(t - t')|^p dt\}\ dt'$$

$$= ||g(t)||_1^{p-1} \int_0^{\infty} |g(t')|\ ||f(t)||_p^p dt$$

$$= ||g(t)||_1^{p-1}\ ||f(t)||_p^p \int_0^{\infty} |g(t')|\ dt'$$

$$= ||g(t)||_1^{p-1}\ ||f(t)||_p^p\ ||g(t)||_1$$

$$= ||g(t)||_1^p\ ||f(t)||_p^p \qquad (3.6)$$

Substituting this result in (3.3) we have

$$||g(t)\circ||_p < ||g(t)||_1$$
$$1 < p < \infty \qquad (3.7)$$

This is a remarkably compact result saying that the p-norm of $g(t)\circ$ (the convolution operator) is bounded by the 1-norm of g(t) (the function).

This result is related to what in linear system theory [15] is called bounded-input-bounded-output stability. Here the important point is that a convolution-operator bound is given by the 1-norm of the convolution function.

IV. 1-Norm of Time-Domain Convolution Operators

In the case of the 1-norm the results of the previous section also apply as can be seen from

$$||g(t)\circ||_1 \equiv \sup_{f(t)\not\equiv 0} \frac{\int_{-\infty}^{\infty} \left|\int_0^{\infty} g(t')\, f(t-t')\, dt'\right| dt}{||f(t)||_1} \tag{4.1}$$

$$\begin{aligned}\int_{-\infty}^{\infty} \left|\int_0^{\infty} g(t')\, f(t-t')\, dt'\right| dt &\le \int_{-\infty}^{\infty} \left[\int_0^{\infty} |g(t')|\, |f(t-t')|\, dt'\right] dt \\ &= \int_0^{\infty} |g(t')| \left[\int_{-\infty}^{\infty} |f(t-t')|\, dt\right] dt' \\ &= ||f(t)||_1 \int_0^{\infty} |g(t')|\, dt' \\ &= ||f(t)||_1\, ||g(t)||_1\end{aligned}$$

giving

$$||g(t)\circ||_1 \le ||g(t)||_1 \,. \tag{4.2}$$

For a lower bound consider the definition in (4.1) and substitute in a special f(t) as

$$f(t) = \delta(t) \quad \text{(delta function)}$$

$$||\delta(t)||_1 = \int_{-\infty}^{\infty} |\delta(t)|\, dt = \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{4.3}$$

$$\int_0^{\infty} g(t')\, \delta(t-t')\, dt' = g(t)$$

giving

$$||g(t)\circ||_1 \ge \int_{-\infty}^{\infty} |g(t)|\, dt = \int_0^{\infty} |g(t)|\, dt = ||g(t)||_1 \tag{4.4}$$

Note that it is important that the δ function have a 1 norm (in effect, be integrable) for this result.

Combining (4.2) and (4.4) we have

$$||g(t)\circ||_1 = ||g(t)||_1 \tag{4.5}$$

V. 2-Norm of Time-Domain Convolution Operators

For the 2-norm it is convenient to use Laplace- (or Fourier-) transform concepts. The two-sided Laplace transform is

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)\, e^{-st} dt$$

$$s = \Omega + j\omega = \text{complex frequency} \qquad (5.1)$$

$$f(t) = \frac{1}{2\pi j} \int_{B_r} \tilde{f}(s)\, e^{st} dt$$

Br ≡ Bromwich contour in strip of convergence (parallel to $j\omega$ axis)

In terms of the Laplace transform our convolution problem is just multiplication, i.e.

$$F(t) = g(t) \circ f(t)$$

$$\tilde{F}(s) = \tilde{g}(s)\, \tilde{f}(s) \qquad (5.2)$$

As discussed in another paper [6] the 2-norm with respect to time is related to the 2-norm with respect to frequency ω. As

$$||f(t)||_2 \equiv \{\int_{-\infty}^{\infty} |f(t)|^2 dt\}^{\frac{1}{2}}$$

$$||\tilde{f}(j\omega)||_2 \equiv \{\int_{-\infty}^{\infty} |\tilde{f}(j\omega)|^2 d\omega\}^{\frac{1}{2}} \qquad (5.3)$$

$$||f(t)||_2 = \frac{1}{\sqrt{2\pi}} ||\tilde{f}(j\omega)||_2$$

which is one way to state the Parseval theorem.

Now the 2-norm is expressible in both time and frequency domains as

$$||g(t)\circ||_2 = \sup_{f(t) \neq 0} \frac{||g(t) \circ f(t)||_2}{||f(t)||_2}$$

$$= \sup_{\tilde{f}(j\omega) \neq 0} \frac{||\tilde{g}(j\omega)\, \tilde{f}(j\omega)||_2}{||\tilde{f}(j\omega)||_2}$$

$$= \sup_{\tilde{f}(j\omega) \neq 0} \frac{\{\int_{-\infty}^{\infty} |\tilde{g}(j\omega)\, \tilde{f}(j\omega)|^2 d\omega\}^{\frac{1}{2}}}{\{\int_{-\infty}^{\infty} |\tilde{f}(j\omega)|^2 d\omega\}^{\frac{1}{2}}} \qquad (5.4)$$

We have the inequality

$$\int_{-\infty}^{\infty} |\tilde{g}(j\omega)\ \tilde{f}(j\omega)|^2\, d\omega \leq \int_{-\infty}^{\infty} |\tilde{g}(j\omega)|^2_{max}\ |\tilde{f}(j\omega)|^2 d\omega$$

$$= |\tilde{g}(j\omega)|^2_{max} \int_{-\infty}^{\infty} |\tilde{f}(j\omega)|^2 d\omega \tag{5.5}$$

Here the maximum of $|\tilde{g}(j\omega)|$ is over all real ω. This can be used to define an ω_{max} via

$$|\tilde{g}(j\omega_{max})| = |\tilde{g}(j\omega)|_{max} \tag{5.6}$$

Of course there may be more than one ω_{max} meeting this definition. This gives

$$||g(t)\circ||_2 \leq |\tilde{g}(j\omega)|_{max} = |\tilde{g}(j\omega_{max})| \tag{5.7}$$

For a special case consider

$$|\tilde{f}(j\omega)|^2 = \delta(\omega - \omega_{max}) \tag{5.8}$$

which gives

$$||g(t)\circ||_2 \geq |\tilde{g}(j\omega_{max})| \tag{5.9}$$

assuming that $|\tilde{g}(j\omega)|$ is sufficiently smooth near $\omega = \omega_{max}$.

Combining (5.7) and (5.9) gives

$$||g(t)\circ||_2 = |\tilde{g}(j\omega)|_{max} = |\tilde{g}(j\omega_{max})| \tag{5.10}$$

Combining this with (3.7) gives

$$|\tilde{g}(j\omega)|_{max} \leq ||g(t)||_1 \tag{5.11}$$

Interpreting (5.8) physically this means that the spectrum of f(t) is concentrated in some region near $\omega = \omega_{max}$ leading to (5.9). This should be considered as some sort of limit process.

VI. ∞-Norm of Time-Domain Convolution Operators

For the ∞-norm the results of section 3 also apply as can be seen from

$$||g(t)\circ||_\infty \equiv \sup_{f(t)\not\equiv 0} \frac{\left|\int_0^\infty g(t')\, f(t - t')\, dt'\right|_{\substack{\sup \\ t}}}{|f(t)|_{\sup}} \tag{6.1}$$

$$\begin{aligned}\left|\int_0^\infty g(t')\, f(t - t')\, dt'\right| & \\ &< \int_0^\infty |g(t')|\; |f(t - t')|\; dt' \\ &< |f(t)|_{\sup} \int_0^\infty |g(t')|\; dt' \\ &= |f(t)|_{\sup}\; ||g(t)||_1\end{aligned}$$

giving

$$||g(t)\circ||_\infty < ||g(t)||_1 \tag{6.2}$$

For a lower bound choose f(t) in a special way. Think of fixing t and choosing

$$f(t - t') \equiv \begin{cases} +1 & \text{if} \quad g(t') > 0 \\ 0 & \text{if} \quad g(t') = 0 \\ -1 & \text{if} \quad g(t') < 0 \end{cases} \tag{6.3}$$

This gives

$$|f(t)|_{\sup} = 1$$

$$\begin{aligned}\left|\int_0^\infty g(t')\, f(t - t')\, dt'\right| & \\ &= \left|\int_0^\infty |g(t')|\; dt'\right| \\ &= ||g(t)||_1\end{aligned} \tag{6.4}$$

which when substituted in (5.1) gives

$$||g(t)\circ||_\infty > ||g(t)||_1 \tag{6.5}$$

Combining (5.2) and (5.5) we have

$$||g(t)\circ||_\infty = ||g(t)||_1 \tag{6.6}$$

VII. Residue Norm of Time-Domain Functions

Let us assume that our time-domain functions are of the form

$$f(t) = \sum_n R_n \, e^{s_n t} u(t) \tag{7.1}$$

with conjugate symmetry as

$$\begin{aligned} s_{-n} &= s_n^* \\ R_{-n} &= R_n^* \end{aligned} \tag{7.2}$$

except that for n = 0 another index is needed to allow more than one pole on the negative real axis and so that in time domain the function is real.

In complex-frequency domain the above is

$$\tilde{f}(s) = \sum_n R_n [s - s_\alpha]^{-1} \tag{7.3}$$

which is a sum of first order poles. So that f(t) may be bounded let us require

$$\mathrm{Re}[s_n] < \text{ for all } n \tag{7.4}$$

Let us now define another norm as the residue norm or r-norm as

$$||f(t)||_r \equiv \sum_n |R_n| \tag{7.5}$$

with the restriction this sum converge. Here the subscript r is purely symbolic and does not assume numerical values. One can verify that this is a norm by application of the required properties in (2.4). Let us assume that all the s_n are distinct so that no terms in (7.1) cancel (in particular so that $f(t) \not\equiv 0$ unless all the R_n are zero). Then

$$||f(t||_r \begin{cases} = 0 & \text{iff all } R_\alpha = 0, \text{ or equivalently } f(t) \equiv 0 \\ > 0 & \text{otherwise} \end{cases} \tag{7.6}$$

Also we have

$$||\alpha f(t)||_r = \sum_n |\alpha R_n| = |\alpha| \sum_n |R_n|$$

$$= |\alpha| \; ||f(t)||_r \tag{7.7}$$

The triangle inequality is verified by considering two separate functions, distinguished by superscripts, as

$$f^{(1)}(t) \equiv \sum_n R_n^{(1)} \; e^{s_n^{(1)} t} u(t)$$

$$f^{(2)}(t) \equiv \sum_n R_n^{(2)} \; e^{s_n^{(2)} t} u(t) \tag{7.8}$$

Then if the two sets of natural frequencies $\{s_n^{(1)}\}$ and $\{s_n^{(2)}\}$ are distinct (have no common elements)

$$||f^{(1)}(t) + f^{(2)}(t)||_r = ||[\sum_n R_n^{(1)} \; e^{s_n^{(1)} t} u(t)] + [\sum_n R_n^{(2)} \; e^{s_n^{(2)} t} u(t)]||_r$$

$$= \sum_n |R_n^{(1)}| + \sum_n |R_n^{(2)}|$$

$$= ||f^{(1)}(t)||_r + ||f_n^{(2)}(t)||_r \tag{7.9}$$

However, if some $s_n^{(1)} = s_{n'}^{(2)}$ then the associated residue is $R_n^{(1)} + R_{n'}^{(2)}$ and the term in the residue norm is

$$|R_n^{(1)} + R_{n'}^{(2)}| \leq |R_n^{(1)}| + |R_{n'}^{(2)}| \tag{7.10}$$

This being true for all such pairs of $s_n^{(1)}$ and $s_{n'}^{(2)}$, then

$$||f^{(1)}(t) + f^{(2)}(t)||_r \leq ||f^{(1)}(t)||_r + ||f^{(2)}(t)||_r \tag{7.11}$$

and the r-norm has all the properties of a norm.

While (7.1) can describe many interesting waveforms, a general time-domain waveform can contain other types of terms as well. In the general theory of the singularity expansion method (SEM) there can be branch singularities which take the form [13]

$$f_{n'}(t) = \int_{C_{n'}} \tilde{R}_{n'}(s') \, e^{s't} u(t) \, ds'$$

$$\tilde{f}_{n'}(s) = \int_{C_{n'}} \tilde{R}_{n'}(s')[s - s']^{-1} ds' \tag{7.12}$$

$$C_{n'} \equiv \text{n'th contour in left half s' plane}$$

Near the branch point(s) one may need to be careful in defining the branch contribution. Here for contours not on the negative real axis we take the contours in pairs

$$C_{-n'} = C^*_{n'} \quad \text{(symbolic)}$$

$$\tilde{R}_{-n'}(s') = \tilde{R}_{n'}(s'^*) = R^*_{n'}(s') \tag{7.13}$$

so that the resulting time-domain function is real valued.

Comparing (7.12) with (7.1) and (7.3) note that the form is very similar. In particular one can think of approximating an integral as in (7.12) by a sum as in (7.1) and (7.3). So let us define the r-norm of $f_{n'}(t)$ by

$$||f_{n'}(t)||_r \equiv \int_{C'_n} |\tilde{R}_{n'}(s')| \, |ds'| \tag{7.14}$$

in agreement with (7.5), provided of course that this integral exist. Note that if C'_n is moved in the s plane (even with fixed branch points) a different result may be obtained since the integrand is not analytic. Hence the definition of C'_n must in general be fixed for the problem at hand. The reader will note that a sum of such terms $f_{n'}(t)$ with those in (7.1) to give a more general f(t) and the r-norm as defined by a sum of terms as in (7.5) and (7.14) is a legitimate norm satisfying (7.6), (7.7), and (7.11).

Another type of singularity (at ∞) is referred to as an entire function. As discussed in [13] this can be represented by a contour integral at ∞ of a form similar to that in (7.12). In this case one needs to be careful of convergence of the integrals particularly in the norm as in (7.14).

The problem of higher order poles can be addressed by noting that the r-norm of a single term in (7.1) is the same as the ∞-norm, i.e.

$$||R_n e^{s_n t} u(t)||_\infty = |R_n| = ||R_n e^{s_n t} u(t)||_r \qquad (7.15)$$

Then consider a multiple order pole of the form

$$f_n^{(m)}(t) = R_n^{(m)} \frac{t^{n-1}}{(m-1)!} e^{s_n t} u(t)$$

$$\tilde{f}_n^{(m)}(s) \equiv R_n^{(m)} [s - s_n]^{-m} \qquad (7.16)$$

$$m = 1,2,3, \cdots$$

The peak magnitude is found from

$$|f_n^{(m)}(t)| = |R_n^{(m)}| \frac{t^{m-1}}{(m-1)!} e^{Re[s_n]t} u(t) \qquad (7.17)$$

where the time of the peak satisfies the equation

$$\frac{d}{dt} |f_n^{(m)}(t)| \Bigg|_{t=t_p} = 0 = \frac{|R_n^{(m)}|}{(m-1)!} [(m-1) t_p^{m-2} + t_p^{m-1} Re[s_n]] e^{Re[s_n]t_p}$$

$$t_p = \frac{m-1}{-Re[s_n]} \qquad (7.18)$$

giving

$$||f_n^{(m)}(t)||_r \equiv \max_t |f_n^{(m)}(t)| \equiv ||f_n^{(m)}(t)||_\infty$$

$$= \frac{|R_n^{(m)}|}{(m-1)!} \left[\frac{m-1}{-Re[s_n]}\right]^{m-1} e^{1-m} \qquad (7.19)$$

$$m = 2,3,4, \cdots$$

$$||f_n^{(1)}(t)||_r = |R_n^{(1)}|$$

Note that for the higher order poles we restrict

$$Re[s_n] < 0 \quad \text{for all } n \qquad (7.20)$$

Then our general form for the r-norm is

$$
\begin{aligned}
f(t) &= \sum_{n,m} f_n^{(m)}(t) + \sum_{n'} f_{n'}(t) \\
||f(t)||_r &= \sum_{n,m} ||f_n^{(m)}(t)||_r + \sum_{n'} ||f_{n'}(t)||_r
\end{aligned}
\qquad (7.21)
$$

where the individual terms are defined in (7.14) and (7.19).

VIII. Relation Between the r-Norm and p-Norm

Consider first the case of simple poles as in (7.1). If the p-norm of f(t) is to exist we require

$$\mathrm{Re}[s_n] \begin{cases} \le 0 & \text{for } p = \infty \\ < 0 & \text{for } 1 \le p < \infty \end{cases} \quad \text{for all } n \tag{8.1}$$

Then the p-norm of (7.1) can be bounded as

$$\begin{aligned} \|f(t)\|_p &= \|\sum_n R_n e^{s_n t} u(t)\|_p \\ &\le \sum_n \|R_n e^{s_n t} u(t)\|_p \\ &= \sum_n |R_n| \, \|e^{s_n t} u(t)\|_p \end{aligned} \tag{8.2}$$

using the fundamental properties of norms in (2.4). Considering the individual terms

$$\begin{aligned} \|e^{s_n t} u(t)\|_p &= \{\int_0^\infty |e^{s_n t}|^p dt\}^{\frac{1}{p}} \\ &= \{\int_0^\infty e^{p\mathrm{Re}[s_n]} dt\}^{\frac{1}{p}} \\ &= [\frac{-1}{p\mathrm{Re}[s_n]}]^{\frac{1}{p}} = [\frac{1}{-\mathrm{Re}[s_n]}]^{\frac{1}{p}} p^{-\frac{1}{p}} \end{aligned} \tag{8.3}$$

$$\text{for } 1 \le p < \infty$$

For $p = \infty$ we have

$$\|e^{s_n t} u(t)\|_\infty = |e^{s_n t} u(t)|_{sup} = 1 \tag{8.4}$$

Then (8.2) becomes

$$\|f(t)\|_p \le \begin{cases} p^{-\frac{1}{p}} \sum_n |R_n| \, [\frac{1}{-\mathrm{Re}[s_n]}]^{\frac{1}{p}} & \text{for } 1 \le p < \infty \\ \sum_n |R_n| = \|f(t)\|_r & \text{for } p = \infty \end{cases} \tag{8.5}$$

Note for these results to apply, not only must (8.1) apply, but also the series in (8.5) must converge. An interesting term in (8.5) is

$$p^{-\frac{1}{p}} = \begin{cases} 1 & \text{for } p = 1 \\ \frac{1}{\sqrt{2}} \simeq .707 & \text{for } p = 2 \\ \lim\limits_{p\to\infty} p^{-\frac{1}{p}} = \lim\limits_{p\to\infty} e^{-\frac{1}{p}\ln(p)} = 1 & \text{for } p = \infty \end{cases} \tag{8.6}$$

$$\min_{1<p<\infty} p^{-\frac{1}{p}} = e^{-\frac{1}{e}} \simeq .692 \quad \text{at } p = e \simeq 2.718$$

$$\max_{1<p<\infty} p^{-\frac{1}{p}} = 1 \qquad \text{at } p = 1,\infty$$

If we define

$$\Omega_{max} = \sup_n \mathrm{Re}[s_n] < 0 \tag{8.7}$$

Then from (8.5) we have the looser (but simpler) bound

$$||f(t)||_p < p^{-\frac{1}{p}}\left[\frac{1}{-\Omega_{max}}\right]^{\frac{1}{p}} ||f(t)||_r \tag{8.8}$$

Thus the p-norm can be bounded in terms of the r-norm.

A contour integral contribution as in (7.12) can be bounded provided the contour $C_{n'}$ has its location in the left half s' plane bounded to the left of the $j\omega'$ axis as

$$\sup_{s'\varepsilon C_{n'}} \mathrm{Re}[s'] = \Omega_{n'} < 0 \tag{8.9}$$

Then we have

$$||f_{n'}(t)||_p = \left\{\int_0^\infty \left|\int_{C_n'} \tilde{R}_{n'}(s')\, e^{s't}u(t)\, ds'\right|^p dt\right\}^{\frac{1}{p}}$$

$$< \left\{\int_0^\infty \left[\int_{C_{n'}} |\tilde{R}_{n'}(s')|\, e^{\mathrm{Re}[s']t}u(t)\, |ds'|\right]^p dt\right\}^{\frac{1}{p}} \tag{8.10}$$

This can be bounded by regarding an integral as the limit of a summation, i.e., generalize (8.3) as

$$||f_{n'}(t)||_p = \left|\left|\int_{C_{n'}} \tilde{R}_{n'}(s')\, e^{s't}u(t)\, ds'\right|\right|_p$$

$$< \int_{C_{n'}} ||\tilde{R}_{n'}(s')\, e^{s't}u(t)||_p\, |ds'|$$

$$= \int_{C_{n'}} |\tilde{R}_{n'}(s')|\; ||e^{s't}u(t)||_p\, |ds'| \tag{8.11}$$

Then applying (8.3) and (8.4) a result analogous to (8.5) is

$$||f_{n'}(t)|| < \begin{cases} p^{-\frac{1}{p}} \int_{C_{n'}} |\tilde{R}_{n'}(s')| \left[\frac{1}{-\mathrm{Re}[s']}\right]^{\frac{1}{p}} |ds'| & \text{for } 1 < p < \infty \\ \int_{C_{n'}} |\tilde{R}_{n'}(s')|\ |ds'| = ||f_{n'}(t)||_r & \text{for } p = \infty \end{cases} \tag{8.12}$$

Again we need that $|\tilde{R}_n(s')|$ be integrable on $C_{n'}$ with special attention paid to $\mathrm{Re}[s'] \to 0$ and $\mathrm{Re}[s'] \to -\infty$ (if such cases occur). For the bound on the general p-norm it may be possible to allow $\mathrm{Re}[s'] \to 0$ and/or $-\infty$ provided the behavior of $|\tilde{R}_n(s')|$ is such as to allow integrability there (thereby loosening (8.9)).

A looser (but simpler) bound is found from (8.12) with the restriction of (8.9) as

$$||f_{n'}(t)||_p < p^{-\frac{1}{p}} \left[\frac{1}{-\Omega_{n'}}\right]^{\frac{1}{p}} ||f_{n'}(t)||_r \tag{8.13}$$

Thus for a branch contribution as well the p-norm can be bounded in terms of the r-norm.

One can also consider entire-function contributions which also have the form of a contour integral (at ∞) with bounds as above.

In the case of higher order poles with the restriction (for $m > 1$) of

$$\mathrm{Re}[s_n] < 0 \quad \text{for } 1 < p < \infty \tag{8.14}$$

we have the extension of (8.2) using (7.16) as

$$||f_n^{(m)}(t)||_p = |R_n^{(m)}|\ ||\frac{t^{m-1}}{(m-1)!} e^{s_n t} u(t)||_p \tag{8.15}$$

The individual terms as

$$||\frac{t^{m-1}}{(m-1)!} e^{s_n t} u(t)||_p = \left\{\int_0^\infty \left|\frac{t^{m-1}}{(m-1)!} e^{s_n t}\right|^p dt\right\}^{\frac{1}{p}}$$

$$= \frac{1}{(m-1)!} \left\{\int_0^\infty t^{p(m-1)} e^{p\mathrm{Re}[s_n]t} dt\right\}^{\frac{1}{p}} \tag{8.16}$$

are solved via a common integral as [12(6.11)]

$$\Gamma(z) = k^2 \int_0^\infty t^{2-1} e^{-kt} dt \quad \text{for } Re[2] > 0 \;, \quad Re[k] > 0$$

$$k = -pRe[s_n]$$

$$z = 1 + p(m - 1) \tag{8.17}$$

$$\Gamma(z) \equiv \text{gamma function}$$

$$\Gamma(z) = (z - 1)!$$

This gives

$$\int_0^\infty t^{p(m-1)} e^{pRe[s_n]t} dt = [-pRe[s_n]]^{-1-p(m-1)} \Gamma(1 + p(m - 1)) \tag{8.18}$$

and

$$||\frac{t^{m-1}}{(m - 1)!} e^{s_n t} u(t)||_p = \frac{1}{(m - 1)!} [-pRe[s_n]]^{-\frac{1}{p} -m+1} \Gamma^{\frac{1}{p}}(1 + p(m - 1))$$

$$= p^{-\frac{1}{p} -m+1} [\frac{1}{-Re[s_n]}]^{\frac{1}{p} +m-1} \frac{\Gamma^{\frac{1}{p}}(1 + p(m - 1))}{\Gamma(m)} \tag{8.19}$$

which is a direct extension of (8.3). Then for higher order poles we have

$$||f_n^{(m)}(t)||_p = |R_n^{(m)}| \, p^{-\frac{1}{p} -m+1} [\frac{1}{-Re[s_n]}]^{\frac{1}{p} +m-1} \frac{\Gamma^{\frac{1}{p}}(1 + p(m - 1))}{\Gamma(m)} \tag{8.20}$$

With the r-norm of a higher order pole as in (7.19) (defined via the ∞-norm) we can write the p-norm as

$$||f_n^{(m)}(t)||_p = ||f_n^{(m)}(t)||_r \, p^{-\frac{1}{p} -m+1} [\frac{1}{-Re[s_n]}]^{\frac{1}{p}} [\frac{e}{m - 1}]^{m-1} \Gamma^{\frac{1}{p}}(1 + p(m - 1))$$

$$= ||f_n^{(m)}(t)||_r \, p^{-\frac{1}{p}} [\frac{1}{-Re[s_n]}]^{\frac{1}{p}} A(p,m) \tag{8.21}$$

$$A(p,m) = [\frac{e}{p(m - 1)}]^{m-1} \Gamma^{\frac{1}{p}}(1 + p(m - 1)) \quad \text{for } m = 2,3, \ldots$$

where $p^{-\frac{1}{p}}$ has been considered in (8.6). The additional factor A(p,m) can be considered for special cases. For p = 1 (the 1-norm) we have

$$A(1,m) = \left[\frac{e}{m-1}\right]^{m-1} = e^{(m-1)\ln(\frac{e}{m-1})}$$

$$= e^{(m-1)[1-\ln(m-1)]}$$

$$= \begin{cases} 1 & \text{for } m = 1 \text{ (from 8.5))} \\ e \approx 2.718 & \text{for } m = 2 \\ \infty & \text{for } m = \infty \end{cases} \tag{8.22}$$

For p = 2 (the 2-norm) we have

$$A(2,m) = \left[\frac{e}{2(m-1)}\right]^{m-1}[(2m-2)!]^{\frac{1}{2}} = e^{(m-1)[1-\ln[2(m-1)]]}[(2m-2)!]^{\frac{1}{2}}$$

$$= \begin{cases} 1 & \text{for } m = 1 \text{ (from (8.5))} \\ \frac{e}{\sqrt{2}} \approx 1.922 & \text{for } m = 2 \\ \infty & \text{for } m = \infty \end{cases} \tag{8.23}$$

For p = ∞ (the ∞-norm) use the Stirling approximation [12(6.1.37)] as

$$\Gamma(z) = e^{-z} z^{z-\frac{1}{2}} \sqrt{2\pi}\,[1 + O(z^{-1})] \qquad \text{as } z \to \infty$$

$$\Gamma(1 + p(m-1)) = \sqrt{2\pi}\, e^{-[1+p(m-1)]+[\frac{1}{2}+p(m-1)]\ln[1+p(m-1)]}[1 + O(p^{-1})] \qquad \text{as } p \to \infty$$

$$\Gamma^{\frac{1}{p}}(1 + p(m-1)) = (2\pi)^{\frac{1}{2p}} e^{-[\frac{1}{p}+m-1]+[\frac{1}{2p}+(m-1)]\ln[1+p(m-1)]}[1 + O(p^{-2})] \qquad \text{as } p \to \infty$$

$$= e^{1-m+(m-1)\ln[1+p(m-1)]}[1 + O(p^{-1})] \qquad \text{as } p \to \infty$$

$$= [1 + p(m-1)]^{m-1}\, e^{1-m}[1 + O(p^{-1})] \qquad \text{as } p \to \infty \tag{8.24}$$

giving

$$A(\infty,m) = \lim_{p\to\infty} A(p,m)$$

$$= \lim_{p\to\infty} \left[\frac{e}{p(m-1)}\right]^{m-1}[1 + p(m-1)]^{m-1}\, e^{1-m}$$

$$= \lim_{p\to\infty} \left[\frac{1}{p(m-1)} + 1\right]^{m-1}$$

$$= 1 \tag{8.25}$$

which agrees with our definition of the r-norm via the ∞-norm in (7.15).

Analogous to (8.5) and (8.12) we have the bound for a sum of higher order poles

$$||f(t)||_p < \sum_{n,m} ||f_n^{(m)}(t)||_p$$

$$= \begin{cases} \displaystyle\sum_{n,m} |R_n^{(m)}| \, p^{-\frac{1}{p} - m + 1} \left[\frac{1}{-Re[s_n]}\right]^{\frac{1}{p} + m - 1} \frac{\Gamma^{\frac{1}{p}}(1 + p(m - 1))}{\Gamma(m)} & \text{for } 1 < p < \infty \\ \displaystyle\sum_{n,m} |R_n^{(m)}| \left[\frac{m - 1}{-Re[s_n]}\right]^{m-1} \frac{e^{1-m}}{\Gamma(m)} = ||f(t)||_r & \text{for } p = \infty \end{cases} \qquad (8.26)$$

With the restriction of (8.7) this bound is loosened somewhat by replacing all the $Re[s_n]$ by Ω_{max}.

Then our general form for the p-norm is

$$f(t) = \sum_{n,m} f_n^{(m)}(t) + \sum_{n'} f_{n'}(t)$$

$$||f(t)||_p < \sum_{n,m} ||f_n^{(m)}(t)||_p + \sum_{n'} ||f_{n'}(t)||_p \qquad (8.27)$$

where the individual terms are defined in (8.5), (8.12), and (8.26).

IX. Residue Norm of Time-Domain Convolution Operators

Now consider the r-norm of a time-domain convolution operator $g(t)\circ$ defined by

$$||g(t)\circ||_r \equiv \sup_{f(t)\not\equiv 0} \frac{||g(t) \circ f(t)||_r}{||f(t)||_r} \tag{9.1}$$

Section 7 has considered the r-norm of time domain functions. For later use let us define bounds on the real parts of the singularities of the functions, i.e.

$$\begin{aligned} \tilde{f}(s) \text{ analytic for } Re[s] > \Omega_f \\ \tilde{g}(s) \text{ analytic for } Re[s] > \Omega_g \end{aligned} \tag{9.2}$$

Now for the r-norm of f(t) to exist and for the response $g(t) \circ f(t)$ to be bounded we require

$$\Omega_f < 0 \quad , \quad \Omega_g < 0 \tag{9.3}$$

Furthermore it will be useful to bound one or both of these to the left of the imaginary axis for various applications.

1. First order poles

Let us restrict the consideration at first to first order poles as in (7.1)

$$f(t) = \sum_n R_n e^{s_n t} u(t) \tag{9.4}$$

with conjugate symmetry as in (7.2). Similarly let g(t) be represented by

$$g(t) = \sum_\ell G_\ell e^{s'_\ell t} u(t) \tag{9.5}$$

with conjugate symmetry as

$$\begin{aligned} s'_{-\ell} = s'^*_\ell \\ G_{-\ell} = G^*_\ell \end{aligned} \tag{9.6}$$

except that for $\ell = 0$ another index is needed to allow more than one pole on the negative real axis.

In complex frequency domain we have

$$\tilde{f}(s) = \sum_n R_n[s - s_n]^{-1}$$
$$\tilde{g}(s) = \sum_\ell G_\ell[s - s'_\ell]^{-1} \tag{9.7}$$

which gives a product

$$\tilde{g}(s)\,\tilde{f}(s) = \sum_{\ell,n} G_\ell R_n[s - s'_\ell]^{-1}[s - s_n]^{-1}$$
$$= \sum_{\ell,n} G_\ell R_n\{[s - s'_\ell]^{-1}[s'_\ell - s_n]^{-1} + [s_n - s'_\ell]^{-1}[s - s_n]^{-1}\} \tag{9.8}$$
$$\text{for } s_n \neq s'_\ell \text{ for all } (n,\ell)$$

In time domain this is

$$g(t) \circ f(t) = \sum_{\ell,n} G_\ell R_n\{[s'_\ell - s_n]^{-1}\, e^{s'_\ell t} u(t) + [s_n - s'_\ell]^{-1}\, e^{s_n t} u(t)\} \tag{9.9}$$

From this one can write the r-norm from (7.5) for first order poles as

$$||g(t) \circ f(t)||_r = 2 \sum_{\ell,n} |G_\ell|\ |R_n|\ |s'_\ell - s_n|^{-1}$$
$$= \sum_n |R_n|\{2 \sum_\ell |G_\ell|\ |s'_\ell - s_n|^{-1}\} \tag{9.10}$$

Defining

$$G^{(0)} \equiv \max_n 2 \sum_\ell |G_\ell|\ |s'_\ell - s_n|^{-1} \tag{9.11}$$

(provided sum exists) we have

$$||g(t) \circ f(t)||_r \leq G^{(0)} ||f(t)||_r \tag{9.12}$$

and

$$||g(t)\circ||_r \leq G^{(0)} \tag{9.13}$$

as one way to consider the r-norm of a convolution operator.

However, note that $G^{(0)}$ is a function of the poles s_n of $\tilde{f}(s)$, not just of the G_ℓ and s'_ℓ. This problem can be alleviated if we can give a lower bound to the $|s'_\ell - s_n|$, say

$$\Delta = \inf_{\ell,n} |s'_\ell - s_n| > 0 \tag{9.14}$$

Then we have

$$G^{(0)} < \frac{2}{\Delta} ||g(t)||_r \tag{9.15}$$

giving

$$\begin{aligned} ||g(t) \circ f(t)||_r &< \frac{2}{\Delta} ||g(t)||_r \; ||f(t)||_r \\ ||g(t)\circ||_r &< \frac{2}{\Delta} ||g(t)||_r \end{aligned} \tag{9.16}$$

assuming the sum

$$||g(t)||_r = \sum_\ell |G_\ell| \tag{9.17}$$

converges. Note that the result in (9.16) is consistent with the symmetry of the convolution operation, i.e.

$$g(t) \circ f(t) = f(t) \circ g(t) \tag{9.18}$$

so that either $g(t)\circ$ or $f(t)\circ$ can be considered as the convolution operator. However, the presence of Δ in (9.16) is still undesirable in that it depends on both f(t) and g(t) in the sense of closest approach of corresponding poles.

2. Second order poles appearing from convolution

Even though $\tilde{f}(s)$ and $\tilde{g}(s)$ have each been constrained to have only first order poles, the product can, in principle, have two such first order poles (or pole pairs) coincident giving a second order pole (or pair of second order poles). Say for some (n,ℓ) pair

$$\begin{aligned} s_n &= s'_\ell \\ s_n^* &= s'^*_\ell \quad (\text{or } s_{-n} = s'_{-\ell}) \end{aligned} \tag{9.19}$$

Then considering one such case we have from (7.19)

$$\begin{aligned} ||[G_\ell e^{s'_\ell t} u(t)] \circ [R_n e^{s_n t} u(t)]||_r &= \frac{1}{2\pi j} ||\int_{Br} G_\ell R_n [s - s_n]^{-2} e^{st} ds||_r \\ &= \frac{1}{e} \frac{1}{-\mathrm{Re}[s_n]} |G_\ell| \; |R_n| \end{aligned} \tag{9.20}$$

In this formula $2/\Delta$ in (9.16) has been replaced by $1/(-e \, Re[s_n])$. So it is not the nearness of s_n and s'_ℓ (coincidence in this case) which blows up the norm, but rather the nearness to the $j\omega$ axis. If we require Ω_f and Ω_g in (9.3) to be bounded to the left of the $j\omega$ axis then such a coincidence causes no problem.

3. Close approach of two poles appearing in a convolution

Well, if coincidence of two poles in $\tilde{f}(s)$ and $\tilde{g}(s)$ does not cause the r-norm of the convolution to blow up, then close approach of these two should not either, or rather the definition of the r-norm can be modified to take this into account. Let us say that s_n and s'_ℓ are near to each other, and consider a term of the form

$$A(t) \equiv [G_\ell e^{s'_\ell t} u(t)] \circ [R_n e^{s_n t} u(t)] = \frac{1}{2\pi j} \int_{Br} G_\ell R_n [s - s'_\ell]^{-1} [s - s_n]^{-1} e^{st} ds \tag{9.21}$$

Now back in section 7 when considering the r-norm as a sum of norms of s-plane singularity terms, the ∞-norm (or peak value) was used to define the norm of each term. Then for the case of close approach above let us consider these two poles as a single term and find the ∞-norm and use this to define the r-norm for such a case.

Expanding the product of poles gives

$$\begin{aligned} A(t) &= G_\ell R_n \frac{1}{2\pi j} \int_{Br} \{[s - s'_\ell]^{-1} [s'_\ell - s_n]^{-1} + [s_n - s'_\ell]^{-1} [s - s_n]^{-1}\} e^{st} ds \\ &= G_\ell R_n [s'_\ell - s_n]^{-1} [e^{s'_\ell t} - e^{s_n t}] \, u(t) \end{aligned} \tag{9.22}$$

Defining

$$a \equiv \frac{1}{2} [s'_\ell + s_n]$$

$$b \equiv \frac{1}{2} [s'_\ell - s_n] \tag{9.23}$$

so that

$$s'_\ell = a + b$$

$$s_n = a - b \tag{9.24}$$

we have

$$A(t) = G_\ell R_n \frac{1}{2b} [e^{(a+b)t} - e^{(a-b)t}] u(t)$$

$$= G_\ell R_n \frac{e^{at}}{b} \sinh(bt)\ u(t) \qquad (9.25)$$

In magnitude this is [12 (4.5.49, 4.5.54)]

$$|A(t)| = |G_\ell|\ |R_n|\ \frac{|e^{at}|}{|b|} |\sinh(bt)|\ u(t)$$

$$= |G_\ell|\ |R_n|\ \frac{e^{Re[a]t}}{|b|} \{\sinh^2(Re[b]t) \cos^2[Im[b]t) + \cosh^2(Re[b]t) \sin^2(Im[b]t)\}^{\frac{1}{2}}$$

$$= |G_\ell|\ |R_n|\ \frac{e^{Re[a]t}}{|b|} \{\sinh^2(Re[b]t) + \sin^2(Im[b]t)\}^{\frac{1}{2}} \qquad (9.26)$$

Let us now assume that

$$|b| << |a| \qquad (9.27)$$

Since b represents the difference and a the sum of two complex frequencies that are assumed very close to each other. Noting that

$$Re[a] < \Omega_f + \Omega_g \qquad (9.28)$$

assume that

$$Re[a] < 0 \qquad (9.29)$$

and take the limiting form for small $|b|$ in (9.26). This gives

$$|A(t)| = |G_\ell|\ |R_n|\ t\ e^{Re[a]t} u(t)[1 = O((bt)^2)] \quad \text{as } b \to 0 \qquad (9.30)$$

Note now for small b/a that $|A(t)|$ has the form of a second order pole. From (7.19) we have

$$||A(t)||_\infty = ||A(t)||_r = \frac{1}{e} \frac{1}{-Re[a]} |G_\ell|\ |R_n| \qquad (9.31)$$

which is the result for a second order pole in (9.20) noting in (9.23) that

$$a \to s_n \quad \text{for} \quad s'_\ell \to s_n \tag{9.32}$$

Rewriting (9.31) for s'_ℓ near s_n we have a definition for the r-norm in such a case

$$||A(t)||_r \equiv \frac{1}{e} \frac{2}{-\mathrm{Re}[s'_\ell] - \mathrm{Re}[s_n]} |G_\ell| \, |R_n| \tag{9.33}$$

Thus as long as either (or both) s'_ℓ and s'_n are bounded to the left of the $j\omega$ axis then the $2/\Delta$ in (9.16) can be replaced by $2/(e[-\mathrm{Re}[s'_\ell] - \mathrm{Re}[s_n]])$ for the case of closely approaching poles due to the product of $\tilde{\tilde{g}}(s)$ and $\tilde{f}(s)$. This allows even some $s'_\ell = s_n$ cases without the r-norm blowing up. Note, however, that this does not allow $s'_\ell = s_n$ on the $j\omega$ axis.

Note that this case of closely approaching poles due to a product is quite different from a case of say two s_n in a sum such as represents f(t) in (7.1) appearing close together. In such a case the residues of the two poles may be quite independently choosable. However the product as in (9.22) inherently brings the s'_ℓ and s_n into the effective compound residue or residues.

4. Combination of results

Then in (9.22) let us exclude cases in which s'_ℓ is near s_n and replace $2/|s'_\ell - s_n|$ by $2/(e[-\mathrm{Re}[s'_\ell] - \mathrm{Re}[s_n]])$. Note that nearness is defined by (9.27), i.e. $|s'_\ell - s_n| << |s'_\ell + s_n|$ which for both s'_ℓ and s'_n in the second quadrant of the complex s plane is approximately achieved. Note for such a close approach in the second quadrant there is also another close approach of the conjugate poles in the third quadrant.

With this modification then $G^{(0)}$ in (9.11) can be used to better bound the r-norm of a convolution, especially for the case of closely approaching poles. In (9.16) we can use these results to replace $2/\Delta$ as above to remove cases of closely approaching poles so that $G^{(0)}$ is bounded in (9.15).

X. Summary

This note has developed some of the norm properties of time-domain waveforms and convolution operators. This is done in the context of the usual p-norm and a new norm which we call the residue norm or r-norm.

The r-norm has been related to the p-norm, being defined basically as the ∞-norm on a termwise (singularity by singularity in the s plane) basis. For use in bounding time-domain waveforms and operators, the r-norm has significant potential as it can be applied in the context of the singularity expansion method (SEM).

Appendix A: The Hölder Inequality

For this paper there is an important inequality **known** as the Hölder inequality. This is discussed in various texts such as [10, 11]. For vector p-norms we have

$$||(x_n)||_p = \{\sum_{n=1}^{N} |x_n|^p\}^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$

$$||(x_n)||_\infty = \max_n |x_n| \qquad \text{(A1)}$$

$$1 \leq n \leq N$$

The Hölder inequality is

$$|(x_n) \cdot (y_n) \equiv |\sum_{n=1}^{N} x_n y_n| \leq \{\sum_{n=1}^{N} |x_n|^{p_1}\}^{\frac{1}{p_1}} \{\sum_{n=1}^{N} |y_n|^{p_2}\}^{\frac{1}{p_2}} \qquad \text{(A2)}$$

$$|(x_n) \cdot (y_n)| \leq ||(x_n)||_{p_1} ||(y_n)||_{p_2}$$

$$1 = \frac{1}{p_1} + \frac{1}{p_2}$$

$$p_1 > 1 \quad , \quad p_2 > 1$$

with equality if 2 conditions are met

$$\frac{|x_n|^{p_1}}{||(x_n)||_{p_1}^{p_1}} = \frac{|y_n|^{p_2}}{||(y_n)||_{p_2}^{p_2}}$$

$x_n y_n$ has the same sign (all + or, all -) (A3)

for all $1 \leq n \leq N$

A special case is that for the ∞-norm and 1-norm

$$
\begin{aligned}
|(x_n) \cdot (y_n)| &= \left|\sum_{n=1}^{\infty} x_n y_n\right| \\
&\le \sum_{n=1}^{\infty} |x_n|\,|y_n| \\
&\le \{\max_n |x_n|\}\{\sum_{n=1}^{\infty} |y_n|\} \\
&= ||(x_n)||_{\infty}||(y_n)||_1 \qquad \text{(A4)}
\end{aligned}
$$

$$|(x_n) \cdot (y_n)| \le ||(x_n)||_{\infty}||(y_n||_1$$

Another case concerns the 2-norm which is also known as the Schwarz inequality

$$
\begin{aligned}
|(x_n) \cdot (y_n)| &\le \{\sum_{n=1}^{N}|y_n|^2\}^{\frac{1}{2}}\{\sum_{n=1}^{N}|y_n|^2\}^{\frac{1}{2}} \\
|(x_n) \cdot (y_n)| &\le ||(x_n)||_2||(y_n)||_2
\end{aligned}
\qquad \text{(A5)}
$$

In terms of functions the vector p-norm is generalized (for real t) as

$$
\begin{aligned}
||f(t)||_p &\equiv \{\int_a^b |f(t)|^p dt\}^{\frac{1}{p}} \quad \text{for } 1 \le p < \infty \\
||f(t)||_{\infty} &= \sup_{a \le t \le b} |f(t)|
\end{aligned}
\qquad \text{(A6)}
$$

Here the supremum technically can exclude isolated values of f(t) by considering limits from both sides of values of t of concern. The Hölder inequality is

$$
\begin{aligned}
\left|\int_a^b f_1(t)\, f_2(t)\, dt\right| &\le \{\int_a^b |f_1(t)|^{p_1} dt\}^{\frac{1}{p_1}}\{\int_a^b |f_2(t)|^{p_2} dt\}^{\frac{1}{p_2}} \\
\left|\int_a^b f_1(t)\, f_2(t)\, dt\right| &\le ||f_1(t)||_{p_1}||f_2(t)||_{p_2} \\
1 &= \frac{1}{p_1} + \frac{1}{p_2} \\
p_1 > 1 \quad &, \quad p_2 > 1
\end{aligned}
\qquad \text{(A7)}
$$

with equality if 2 conditions are met

$$\frac{|f_1(t)|^{p_1}}{||f_1(t)||_{p_1}^{p_1}} = \frac{|f_2(t)|^{p_2}}{||f_2(t)||_{p_2}^{p_2}} \tag{A8}$$

$f_1(t)\ f_2(t)$ has the same sign (all + or all -)

"almost everywhere"

A special case is that for the ∞-norm and 1-norm

$$\left|\int_a^b f_1(t)\ f_2(t)\ dt\right| \leq \int_a^b |f_1(t)|\ |f_2(t)|\ dt$$

$$\leq \{\sup_t |f_1(t)|\}\{\int_a^b |f_2(t)|\ dt\}$$

$$= ||f_1(t)||_\infty ||f_2(t)||_1$$

$$\left|\int_a^b f_1(t)\ f_2(t)\ dt\right| \leq ||f_1(t)||_\infty ||f_2(t)||_1 \tag{A9}$$

$$a < b$$

The special case of the 2-norm is also known as the Schwarz inequality as

$$\left|\int_a^b f_1(t)\ f_2(t)\ dt\right| \leq \{\int_a^b |f_1(t)|^2 dt\}^{\frac{1}{2}}\{\int_a^b |f_2(t)|^2 dt\}^{\frac{1}{2}} \tag{A10}$$

$$\left|\int_a^b f_1(t)\ f_2(t)\ dt\right| \leq ||f_1(t)||_2 ||f_2(t)||_2$$

In this paper in dealing with time-domain waveforms the case of interest has

$$a = -\infty$$
$$b = \infty \tag{A11}$$

so that we are dealing with integrals over all times (of interest). In general such times, while of course finite, are much longer than times for which the waveforms of concern are significant so that $-\infty < t < \infty$ is a resonable approximation.

References

1. C.E. Baum, Electromagnetic Topology: A Formal Approach to the Analysis and Design of Complex Electronic Systems, Interaction Note 400, September 1980, and Proc. EMC Symposium, Zürich, March 1981, pp. 209-214.

2. A.K. Agrawal and C.E. Baum, Bounding of Signal Levels of Terminations of a Multiconductor Transmission-Line Network, Interaction Note 419, April 1983.

3. F.C. Yang and C.E. Baum, Use of Matrix Norms of Interaction Supermatrix Blocks for Specifying Electromagnetic Performance of Subshields, Interaction Note 427, April 1983.

4. C.E. Baum, Black Box Bounds, Interaction Note 429, May 1983, and Proc. EMC Symposium, Zürich, March 1985, pp. 381-386.

5. C.E. Baum, Bounds on Norms of Scattering Matrices, Interaction Note 432, June 1983.

6. C.E. Baum, Some Bounds Concerning the Response of Linear Systems with a Nonlinear Element, Interaction Note 438, June 1984.

7. C.E. Baum, On the Use of Electromagnetic Topology for the Decomposition of Scattering Matrices for Complex Physical Structures, Interaction Note 454, July 1985.

8. C.E. Baum, Norms and Eigenvector Norms, Mathematics Note 63, November 1979.

9. F.M. Tesche, Topological Concepts for Internal EMP Interaction, IEEE Trans. Antennas and Propagation, January 1978, pp. 60-64, and IEEE Trans. EMC, February 1978, pp. 60-64.

10. F. Riesz and B. Sz.-Nagy, Functional Analysis, Frederick Ungar Publishing Co., 1955.

11. V.I. Krylov, Approximate Calculation of Integrals, MacMillan, 1962.

12. M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, AMS 55, National Bureau of Standards, 1964.

13. C.E. Baum, Toward an Engineering Theory of Electromagnetic Scattering: The Singularity and Eigenmode Expansion Methods, in P.L.E. Uslenghi (ed.)., Electromagnetic Scattering, Academic Press, 1978.

14. I. Stakgold, Green's Functions and Boundary Value Problems, Wiley, 1979.

15. C.-T. Chen, Linear System Theory and Design, Holt, Rinehart, and Winston, 1984.