Mathematics Notes

Note 83

1 May 1984

Resonance (Natural-Frequency) Calculation and Extraction from Transient Fields

A.G. Ramm
Kansas State University
Department of Mathematics

Abstract

Mathematical formulation and analysis of numerical methods for calculating the natural frequencies (resonances) are given. Stability of these methods towards round-off errors and small perturbations of the obstacles is established. Some formulas for the variations of the natural frequencies due to small perturbations of the surface of the obstacle are given. A simple new method for extraction of resonances from transient fields is given.

PREFACE

The author thanks Dr. C. Baum, AFWL, and Mr. T. Brown, Dikewook, for useful discussions.

CONTENTS

I. INTRODUCTION

Let $\mathcal{D}$ be a finite obstacle with a smooth surface , Ω be the exterior domain. The obstacle (scatterer) is three dimensional. The smoothness of Γ is of the type that ensures the applicability of Green's formulas. Roughly speaking, the cusp-type singular points of the surface are not admissible, but the edges (as in a cube) or conical points are admissible. The scalar wave scattering will be discussed for simplicity, but the results and arguments are valid for electromagnetic wave scattering. The Green's function for a reflecting obstacle satisfies the equations

$$(-\nabla^2 - k^2)G(x,y,k) = \delta(x - y) \quad \text{in} \quad \Omega, \ k > 0, \ x = (x_1,x_2,x_3) \qquad (1)$$

$$G = 0 \quad \text{if} \quad x \in \Gamma \qquad (2)$$

$$r\left(\frac{\partial G}{\partial r} - ikG\right) \to 0 \quad \text{as} \quad r = |x| \to \infty \qquad (3)$$

Here y is the position vector of the source, ∇^2 is the Laplacian. The function G is uniquely determined by Equations 1 through 3 and can be continued analytically on the whole complex plane of k as a meromorphic function of k. Its poles lie in the half-plane $\mathrm{Im}k < 0$ and are called resonances, natural frequencies or complex poles. The meromorphic nature of G as a function of k and the (closely connected with it) behavior of solutions to the time-dependent wave equation as $t \to +\infty$ was studied in the series of papers starting with Reference 1. In

Reference 2 there is a bibliography of the subject. In Reference 3 one can find a collection of papers and an extensive bibliography of the singularity and eigenmode expansion methods. In References 4 and 5 there are reviews of the subject for engineers. The connection of the complex poles asymptotic with the behavior of solutions to the time-dependent wave equation is the foundation of the singularity expansion method (SEM). If

$$\nabla^2 u = u_{tt} \quad \text{in} \quad \Omega, \ t > 0, \ u = 0 \quad \text{on} \quad \Gamma$$

$$u(x,0) = 0, \quad u_t(x,0) = f(x) \tag{4}$$

then the function v defined as

$$v(x,k) = \int_0^\infty \exp(ikt)u(x,t)dt \tag{5}$$

satisfies the equations

$$(\nabla^2 + k^2)v = -f, \ v = 0 \quad \text{on} \quad \Gamma, \ r\left(\frac{\partial v}{\partial r} - ikv\right) \to 0, \ r \to \infty \tag{6}$$

$$v = \int_\Omega G(x,y,k)fdy \tag{7}$$

$$u = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-ikt)v(x,k)dk \tag{8}$$

Assume that f is a smooth function which vanishes outside of a bounded domain (compactly supported). In the engineering literature (e.g., in Ref. 4) the complex variables $s = -ik$ is often used. In the physical mathematical literature k is usually the compex variable. The half-plane $\mathrm{Im}k < 0$ (used in this report) corresponds to the half-plane $\mathrm{Re}\, s < 0$ on the s-plane. If one knows that (Ref. 2):

$$v \text{ is meromorphic (and analytic if } \mathrm{Im}k \geq 0) \tag{9}$$

$$|v| \leq C(b)(1 + |k|)^{-a}, \ a > \frac{1}{2}, \ b = \mathrm{Im}k, \ |\mathrm{Re}k| \to \infty \tag{10}$$

$$|\mathrm{Im}k_j| < |\mathrm{Im}k_{j+1}| \to \infty \quad \text{as} \quad j \to \infty \tag{11}$$

then one can move the contour of integration in Equation 8 in k-plane down and obtain the SEM expansion (Ref. 2)

$$u(x,t) = \sum_{j=1}^{N} c_j(x,t)e^{-ik_j t} + 0(e^{-|\mathrm{Im}k_{N+1}|t}), \quad t \to +\infty \tag{12}$$

Here k_j are the complex poles of $v(x,k)$, $c_j(x,t) = i \operatorname{Res}_{k=k_j} \{e^{-ikt}v(x,k)\}$ and N is the number of the poles in the strip $0 > \mathrm{Im}k \geq \mathrm{Im}k_N$. Usually it is assumed by engineers (Ref. 4) that the poles k_j are simple, in which case $c_j(x,t) = c_j(x)$. If $m_j + 1$ is the multiplicity of the pole k_j then $c_j(x,t) = 0(t^{m_j})$, and one can write Equation 12 as

$$u(x,t) = \sum_{j=1}^{N} \sum_{m=0}^{m_j} c_{jm}(x)t^m e^{-ik_j t} + 0(e^{-|\mathrm{Im}k_{N+1}|t}) \quad \text{as} \quad t \to +\infty \tag{13}$$

Equations 12 and 13 were called in Reference 2 asymptotic SEM expansions. These expansions are proved under the assumption that $\mathcal{D}$ is strictly convex. The expansion

$$u = \sum_{j=1}^{\infty} c_j(x,t)e^{-ik_j t} \tag{14}$$

which one can see in the literature, is not proved and probably is not valid in general. Equation 13 is not valid, in general, for nonconvex obstacles. For example, in Reference 6 it is proved that if the obstacle consists of two strictly convex bodies than $G(x,y,k)$ has countably many complex poles on the line $\mathrm{Im}k = c_0$. These poles asymptotically are equidistant and the distance between the poles depend on the distance between the bodies and the curvatures and principal directions of the surfaces Γ_1 and Γ_2 of the bodies $\mathcal{D}_1$ and $\mathcal{D}_2$ at the two closest points $a_1 \in \Gamma_1$ and $a_2 \in \Gamma_2$. (That is $|a_1 - a_2| = \min_{\substack{s_1 \in \Gamma_1 \\ s_2 \in \Gamma_2}} |s_1 - s_2|$). This result shows that SEM Equation 13 is not valid for two convex bodies. It suggests also that Equation 13 is not valid for a single body with nonconvex boundary which can hold a trapping mode (i.e. a standing wave in the geometrical optics approximation). This, however, is not proved yet.

In principle, one can tell the difference between convex obstacles and nonconvex obstacles, capable of holding a trapping mode, by the behavior of complex poles k_j for large j: for convex obstacles (Eq. 11) holds and $|\mathrm{Im}k_j| \to +\infty$, while in the other case there exist infinitely many poles on a line $\mathrm{Im}k_j = c_0 < 0$.

The significance of the complex poles is manyfold. We mention only two areas important in applications. First, one can tabulate the complex poles and use them for target identification. Practically, it is expected that different scatterers produce different sets of complex poles. Although this is not proved, but there are some supporting arguments (Ref. 2, p. 585-586). Secondly, the systems theory uses representations of impulse responses as sums of exponentials. The problem is to find these exponentials from transient fields.

It is a long-standing open problem to prove that infinitely many complex not purely imaginary poles of G exist for any reflecting obstacle. So far it was proved that infinitely many purely imaginary poles exist (this is a result from Ref. 7, a simple proof of one can find in Ref. 2).

The objectives of this paper include:

(a) Formulation of the mathematical methods for numerical calculating the complex poles.

(b) Analysis of convergence and stability of these methods.

(c) Formulation of a simple technique for extracting resonances (natural frequencies) from transient fields.

An extensive bibliography on the third question can be found in Reference 8. The techniques used in the literature and reviewed in Reference 8, are based mostly on the Prony's method. Some other methods were also used (Refs. 8 and 9). Here we present a very simple numerical technique which seems to be new and does not require solving nonlinear or even linear equations. The most difficult part of this problem is the question of the noisy data. This report is organized as follows: Section II discusses the first two objectives while Section III covers the third objective.

poles under perturbations of the surface of the scatterer. Numeration of formulas is separate in each of the sections.

II. METHODS OF CALCULATING NATURAL FREQUENCIES

1. BASIC EQUATIONS.

From the Green's formula one obtains

$$G(x,y,k) = g(x,y,k) - \int_{\Gamma} g(x,s')h(s',y)ds' \qquad (15)$$

$$g = \frac{e^{ik|x - y|}}{4\pi|x - y|} \qquad (16)$$

$$h = \frac{\partial G}{\partial N_s} \qquad (17)$$

where N_s is the outer normal to Γ at the point s, and the dependence on k is supressed in some of the functions for brevity. Let $x = s \varepsilon \Gamma$ in Equation 15. Then

$$\int_{\Gamma} ghds' = g \qquad (18)$$

If k_j is a pole of G then it is a pole of h, so that $h = \frac{\psi}{(k - k_j)^m}$. Multiply (4) by $(k - k_j)^m$ and let $k = k_j$ to obtain

$$Qh \equiv \int_{\Gamma} g(s,s',k_j)\psi ds' = 0 \qquad (19)$$

Therefore, the complex poles are the points k_j at which Equation 19 has a nontrivial solution.

Let us differentiate Equation 15 in the direction N_s and then take $x \to s \varepsilon \Gamma$ to obtain

$$[I + A(k)]h \equiv h + Ah = 2 \frac{\partial g}{\partial N_s} \tag{20}$$

$$Ah = \int_{\Gamma} 2 \frac{\partial g}{\partial N_s} hds' \tag{21}$$

This gives the second way to characterize the complex poles: they are the points at which the equation

$$B\psi \equiv [I + A(k)]\psi = 0 \tag{22}$$

has a nontrivial solution.

2. PROJECTION METHODS FOR CALCULATING THE POLES.

First, consider Equation 22. Take a complete in $L^2(\Gamma) = H$ set of linearly independent functions $\{\phi_j\}$. The linear span of the first n functions is a linear subspace H_n, $H_n \subset H_{n+1}$. Since the system $\{\phi_j\}$ is complete in H one concludes that the system of subspaces H_n is limit dense in H, that is $\text{dist}(\psi, H_n) \to 0$ as $n \to \infty$ for any $\psi \in H$. This property is crucial for the analysis below. Here dist is the distance between the elements ψ and the subspaces H_n. Let $\psi_n = \sum_{j=1}^{n} c^{(n)} \phi_j$. Consider the projection method for solving Equation 22:

$$(B\psi_n, \phi_m) = 0, \ 1 \leq m \leq n, \ \sum_{j=1}^{n} (B\phi_j, \phi_m) c_j = 0, \ 1 \leq m \leq n \tag{23}$$

The necessary and sufficient condition for Equation 23 to have a nontrivial solution is

$$\det b_n(k) = 0, \quad b_n(k) \equiv [b_{jm}(k)]_{j,m=1\ldots n}, \quad b_{jm} \equiv (B\phi_j, \phi_m) \tag{24}$$

The parentheses denote the inner product in $L^2(\Gamma)$, $(u,v) = \int_\Gamma u\bar{v}ds$. The elements $b_{jm}(k)$ are entire functions of k since the operator A in Equation 21 is an entire analytic operator function of k. Therefore:

(a) it is not obvious that Equation 24 has zeros (e.g., $\exp(k)$ does not have zeros),

(b) if Equation 24 has zeros $k_j^{(n)}$, $j = 1,2,\ldots$ then one should prove convergence of this method, that is one should prove that

$$\lim_{n\to\infty} k_j^{(n)} = k_j \tag{25}$$

where k_j are the complex poles of G and that all of the complex poles can be obtained in this way. This will be done later.

Consider Equation 19. In the same way as in the previous section one can derive the equation

$$\det Q_n(k) = 0, \quad Q_n(k) \equiv [Q_{jm}(k)]_{j,m=1\ldots n}, \quad Q_{jm} = (Q\phi_j, \phi_m) \tag{26}$$

This equation is of the same structure as Equation 24, and the same questions (a) and (b) should be investigated for Equation 26. The difference between operators Q and B is that Q is compact while B is of Fredholm type, so that Equation 19 is of the first kind, while Equation 22 is of the second kind. The element Q_{jm} is easier to compute than b_{jm}.

3. VARIATIONAL METHODS FOR CALCULATING THE POLES.

Consider the problem:

$$|Qf| = \min, \quad |f| = 1 \tag{27}$$

where $|f|$ is the $L^2(\Gamma)$ norm, $|f|_p$ is the Sobolev space $W^{2,p}(\Gamma) = H^p$ norm, $|f| = |f|_0$, $|f|_p = \int_\Gamma \{|u|^2 + |Du|^2 + |D^p u|^2\} ds$, D denotes the first order derivative on Γ. For $p < 0$ the space $W^{2,p}$ is defined as a dual to $W^{2,|p|}$.

Take $f_n = \sum_{j=1}^{n} c^{(n)} \phi_j$, substitute in Equation 13, and obtain the problem

$$\sum_{j=1}^{n} q_{mj}^{(n)} c_j^{(n)} = \lambda c_m^{(n)}, \quad 1 \le m \le n, \quad q_{mj}^{(n)} \equiv (Q\phi_j, Q\phi_m) \tag{28}$$

where λ is an eigenvalue of the matix $q_{mj}^{(n)}$. This matrix is an entire function of k. Its minimal eigenvalue $\lambda_1^{(n)}(k)$ is minimum of the functional $|Qf|$ under the constraint of Equation 27. The points $k_j^{(n)}$ which are zeros of $\lambda_1^{(n)}(k)$:

$$\lambda_1^{(n)}(k_j^{(n)}) = 0 \tag{29}$$

converge to the complex poles of $G(x,y,k)$ and all of the complex poles can be obtained as limits of $k_j^{(n)}$ as $n \to \infty$:

$$k_j = \lim_{n\to\infty} k_j^{(n)} \tag{30}$$

A similar idea was used in Equation 25. Convergence of the methods given in Section II.2 and a study of their stability we give in the next subsection.

4. CONVERGENCE AND STABILITY OF THE METHODS FOR CALCULATING THE POLES.

The basic ideas and methods of the analysis and proofs are taken from Reference 10 (Refs. 2, 11 and 12). The basic results consist in a proof of convergence and stability of the methods given in Section II.2 towards the round-off errors and perturbations of the data.

We start with the method given in Section II.2. Let us assume that there exist a countable discreet set P of points k_j at which Equation 22 has a nontrivial solution. In paragraph II.1 we proved that any complex pole of G belongs to P. Let us show that any point $k_0 \varepsilon P$ is a complex pole of G. Let ψ be a nontrivial solution to Equation 8. Define the simple layer potential $v = \int_\Gamma g\psi ds'$. From the known formula (Ref. 10, p. 240): $\frac{\partial v}{\partial N_i} = \frac{A\psi + \psi}{2}$ (in which $\frac{\partial}{\partial N_i}$ denotes the limit value of the normal derivative on Γ from the interior and A is given in Eq. 21) and Equation 22 it follows that $\frac{\partial v}{\partial N_i} = 0$. We know that $(\nabla^2 + k^2)v = 0$ in $\mathcal{D}$. Since k_0^2 is a complex number and the spectrum of the interior Neumann Laplacian consists of positive numbers only, we conclude that $v = 0$ in $\mathcal{D}$. Therefore $v = 0$ on Γ. If G does not have a pole at $k = k_0$ then the problem $(\nabla^2 + k_0^2)v = 0$ in Ω, $v = 0$ on Γ, $v(x,k_0)$ is the limit value of a function $v(x,k)$ analytic in k in a neighborhood of k_0 and belonging to $L^2(\Omega)$ when $Imk > 0$, has only the trivial solution. Thus, $v = 0$ in Ω if k_0 is not a pole of G. Therefore $\psi = \frac{\partial v}{\partial N_i} - \frac{\partial v}{\partial N_e} = 0$, where $\frac{\partial v}{\partial N_e}$ is the limit value of the normal derivative on Γ from the exterior domain. This contradicts the assumption that $\psi \not\equiv 0$. Therefore k_0 is a pole of G.

Let us prove now that for sufficiently large n: (1) Equation 24 has solutions, (2) Equation 25 holds, (3) all the complex poles can be obtained as limits (Eq. 25) and (4) complex poles k_j are stable towards small perturbations of the data the notion of the small perturbation will be specified.

Equation 23 can be written as an operator equation $P_n B P_n \psi = 0$, or

$$(I + P_n A)\psi_n = 0 \tag{31}$$

where P_n is the orthoprojection onto H_n, $\psi_n = P_n\psi \in H_n$, I is the identity. Since A(k) is compact in H for any k and $P_n \to I$ as $n \to \infty$, where the arrow denotes strong convergence, one has $||B - P_nB|| \to 0$ as $n \to \infty$. Therefore, the operator $I + P_nA = I + A(k) - P^{(n)}A(k)$ is invertible for sufficiently large n in a neighborhood of any point k_0 at which $I + A(k_0)$ is invertible. Here $P^{(n)} \equiv I - P_n$, $P^{(n)} \to 0$ as $n \to \infty$. This argument shows that if k_0 is not a complex pole then there are no roots $k_j^{(n)}$ of Equation 24 in a neighborhood of k_0. It remains to be proved that if k_0 is a pole of G then for sufficiently large n there exists a root $k_j^{(n)}$ of Equation 24 which lies in the circle $C_\delta : |k - k_0| \leq \delta$, where $\delta > 0$ is arbitrary small number. Suppose that for some $\delta > 0$ and all n there are no roots $k_j^{(n)}$ of Equation 24 in the circle C_δ. Then the operator $I + P_n A(k)$ is invertible in C_δ, the operator $(I + P_n A(k))^{-1}$ is analytic in k in C_δ, and therefore $||(I + P_n A(k))^{-1}|| \leq c$ where c is constant which does not depend on n. On the other hand, $(I + A(k))^{-1} = (I + P_n A(k) + P^{(n)} A(k))^{-1} = (I + P_n A(k))^{-1}(I + P^{(n)} A(k)(I + P_n A(k))^{-1})^{-1}$. Since $||P^{(n)} A(k)|| \to 0$ and $||(I + P_n A)^{-1}|| \leq c$ one concludes that $(I + A(k))^{-1}$ is a bounded operator in C_δ. This is a contradiction since k_0 is a pole of the operator $(I + A(k))^{-1}$. The contradiction proves that for any $\delta > 0$ and sufficiently large n there is a root $k_j^{(n)}$ of Equation 24 in the circle C_δ.

The above argument settles also the question about stability of the poles towards small perturbations of the data and round-off errors. Indeed, the small perturbations of the data and the round-off errors are equivalent to small perturbations of the matrix $b_{jm}(k)$.

Let us assume that a small perturbation of the matrix $b_{jm}(k)$ is caused by a small perturbation of the operator $B = I + A$. Let us denote $\tilde{B} = I + \tilde{A} = I + A + T$ the perturbed operator. In this formulation the perturbed matrix $\tilde{b}_{jm}^{(n)}$ is the matrix of the operator $P_n \tilde{B} P_n$. The perturbation T can describe both the perturbation of Γ and the round-off errors in computing matrix $b_{jm}^{(n)}$. Our aim is to prove that in any finite domain on the k-plane the poles $\tilde{k}_j^{(n)}$ of the perturbed operator $(I + P_n \tilde{B}(k) P_n)^{-1}$ differ from the poles $k_j^{(n)}$ of the unperturbed operator $(I + P_n B(k) P_n)^{-1}$ a little: $|\tilde{k}_j^{(n)} - k_j^{(n)}| \leq$

$\varepsilon(n, ||T||)$, $\varepsilon \to 0$ if $||T|| \to 0$ and $n \to \infty$. Since we have already established the convergence property (Eq. 25), it is sufficient to prove that

$$|\tilde{k}_j - k_j| \to 0 \quad \text{if} \quad ||T|| \to 0, \; T = \tilde{B} - B \tag{32}$$

Let k_j be a pole of $(I + A(k))^{-1}$ and there are no other poles of this operator in the circle C_δ. One has $(I + A(k) + T)^{-1} = (I + A(k))^{-1}(I + T(I + A(k))^{-1})^{-1}$. Suppose k_j is a pole of multiplicity ν. Then $||(I + A(k))^{-1}|| \leq \frac{c}{|k - k_j|^\nu}$, $k \in C_\delta$, $c = \text{const}$. Thus $||T(I + A(k))^{-1}|| \leq c||T|| |k - k_j|^{-m}$. If $c||T||\delta^{-m} < 1$ then the perturbed pole $\tilde{k}_j$, corresponding to the unperturbed pole k_j, lies inside the circle C_δ, that is $|\tilde{k}_j - k_j| < \delta$. In other words $|\tilde{k}_j - k_j| = 0(||T||^{1/m})$ where m is the multiplicity of the pole k_j and $0(||T||^{1/m})$ means $\leq$ const $||T||^{1/m}$.

The smallness of the perturbation of the surface is described in terms of the smallness of the norm $||T||$. One can give a relationship between the equation of the perturbed surface and the norm of T. This is cumbersome and is done in Appendix A.

Let us study the method based on Equation 26. The results will be the same: (1) Equation 26 has roots $k^{(n)}$ for sufficiently large n, (2) Equation 25 holds, (3) all the complex poles can be obtained as limits (Eq. 11), and (4) small perturbations of the data lead to small perturbations of the complex poles uniformly on any bounded domain on the complex k-plane.

Analysis of Equations 19 and 26 is more complicated than that of Equations 22 and 24 because Equation 19 is an equation of the first kind. The basic tool in our analysis is the factorization formula

$$Q(k) = Q_0(I + V), \; Q_0 \equiv Q(0), \; V \equiv Q^{-1}(Q(k) - Q_0) \tag{33}$$

Here $Q_0 f = \int \frac{f ds'}{4\pi|s - s'|}$ is a self-adjoint positive definite operator on $H = L^2(\Gamma)$. This operator is an isomorphism between $H = H^0$ and H^1, while the operator V is compact in any space H^p (see [2] and [10] for details). Therefore, the bilinear form $(Q_0 u, v)$ defines an inner product equivalent to the inner product $(u,v)_{-1/2}$ in $H^{-1/2}$. The matrix $Q_{jm} = (Q_0(I + V)\phi_j, \phi_m) = ((I + V)\phi_j, \phi_m)_{-1/2}$. Our previous arguments in paragraph 4 are fully applicable to this matrix because:

(1) V is compact in $H^{-1/2}$ and depends analytically on k,

(2) If a system $\{\phi_j\}$ is complete in H^0 then it is complete in H^p for any $p < 0$.

Compactness of V was already mentioned. To explain the second statement assume that $f_p \in H^p$, $p < 0$. It is well known that $H^p \subset H^q$ if $p > q$, H^p is dense in H^q (that is for any $\varepsilon > 0$ and any $f \in H^q$ there exists an $f_\varepsilon \in H^p$ such that $|f_\varepsilon - f|_q < \varepsilon$, where $|\cdot|_q$ is the norm in H^q), and $|f|_p \leq |f|_q$ if $p < q$. Let $f \in H^p$, $p < 0$, and $\varepsilon > 0$ is fixed. Find $f_\varepsilon \in H^0$ such that $|f - f_\varepsilon|_p < \frac{\varepsilon}{2}$. Use completeness of the system $\{\phi_j\}$ in H^0 to find $h_\varepsilon = \sum_{j=1}^{n(\varepsilon)} c_j(\varepsilon)\phi_j$ such that $|h_\varepsilon - f_\varepsilon|_0 < \frac{\varepsilon}{2}$. Then

$|f - h_\varepsilon|_p \leq |f - f_\varepsilon|_p + |f_\varepsilon - h_\varepsilon|_p \leq \frac{\varepsilon}{2} + f_\varepsilon - h_\varepsilon|_0 < \varepsilon$, if $p < 0$. Therefore the system $\{\phi_j\}$ is complete in $H^{-1/2}$ and the matrix Q_{jm} is a matrix of the operator $I + V(k)$ in $H^{-1/2}$ where $V(k)$ is compact in $H^{-1/2}$ and analytic in k. The rest of the arguments is the same as in subsection II.4 and the conclusions are formulated in the beginning of this section.

In this paragraph the variational method given in Section II.3 is studied. If $\lambda_1^{(n)}(k_j^{(n)}) = 0$ then Equation 28 corresponds to the projection method for the equation $Q^*Qf = 0$. The factorization in paragraph II.4 is sufficient for the arguments of paragraph 4.3 to hold for the operator Q^*Q (the reason is that $Q^*Q = (I + V^*)Q_0^2(I + V) = Q_0^2(I + V_1)$ where V_1 is compact). Here we used self-adjointness of Q_0. Compactness of V_1 follows from simple properties of pseudodifferential operators: ord $Q_0 = -1$, ord $V = -3$ $V_1 \equiv Q_0^{-2}V^*Q_0^2 + V + Q_0^{-2}V^*Q_0^2V$, ord $V_1 \leq$ ord $V < 0$. Here ord Q is the order of pseudodifferential operator Q. Properties of pseudodifferential operators one can find, e.g., in Reference 13. Thus, one concludes that the results (Eqs. 15 through 18) in Section II.4 hold for the variational method described in paragraph 3.

III. EXTRACTING NATURAL FREQUENCIES FROM TRANSIENT FIELDS

1. PRELIMINARIES

Consider the problem

$$u_{tt} = \nabla^2 u, \quad t \geq 0, \quad x \in \Omega \quad \mathbb{R}^3, \quad u = 0$$

$$\text{on } \Gamma, \quad u(x,0) = 0, \quad u_t(x,0) = f(x), \tag{34}$$

where $D = R^3 \backslash \Omega$ is a bounded connected domain with a smooth strictly convex boundary, $f \in C_0^\infty$. In Reference 2 the basic results on the asymptotic behavior of u as $t \to +\infty$ are described. In particular, the following asymptotic SEM (singularity expansion method) formula holds:

$$u = \sum_{j=1}^{N} \sum_{m=0}^{m_j} c_{jm}(x)\, t^m e^{-ik_j t} + 0(e^{-|\mathrm{Im}k_{N+1}|t}), \quad t \to +\infty \tag{35}$$

where $c_{jm}(x)$ do not depend on t, $k_j = a_j - ib_j$, $b_j > 0$, are complex poles of the resolvent kernel G of the Dirichlet Laplacian in Ω:

$$(\nabla^2 + k^2)G(x,y,k) = -\delta(x - y) \quad \text{in } \Omega, \quad G = 0 \quad \text{on } \Gamma,$$

$$|x|\left(\frac{\partial G}{\partial |x|} - ikG\right) \to 0 \quad \text{as} \quad |x| \to \infty,$$

and $m_j + 1$ is the multiplicity of the pole k_j, $\mathrm{Im}k_j < 0$, $|\mathrm{Im}k_j| \to \infty$ as $j \to \infty$. The poles k_j are called resonances or natural frequencies. The signal (Eq. 35) is the transient field that can be observed experimentally. The knowledge of the resonances k_j may serve for target identification: the scatterers of various shapes produce various sets of resonances, and this is one of the reasons to be interested in resonances.

The other reason is that in systems theory one often models impulse responses as a sum of the type given in Equation 35. The important problem of system identification can be formulated as follows: from the observation of the transient field (Eq. 2) find the numbers k_j and m_j. There is an extensive literature on the subject. Many researchers contributed to the field (Prony, Bruns, Dale, Lagrange, Kühnen and quite a few modern researchers). A large bibliography can be found in References 8 and 14. Only the case $m_j = 0$ (simple poles) was treated in the literature.

The purpose of this section is: (1) to give a simple numerical procedure for computing the numbers m_j and k_j, $1 \leq j \leq N$ for any fixed N from the exact transient data, (2) to discuss this problem for the noisy data, and (3) to briefly review the classical methods (Prony, Bruns, Lagrange, Dale).

2. A SIMPLE METHOD FOR EXTRACTING RESONANCES FROM THE TRANSIENT FIELD

Assume first that the scatterer is a strictly convex reflecting body so that Equation 35 holds. By $u(n)$ let us denote the sequence $u(x,nh)$, where $h > 0$ is a fixed number. It follows from Equation 35 that

$$u(n) = c_{1m_1} h^{m_1} n^{m_1} e^{-ia_1 nh} e^{-b_1 nh} (1 + 0(\frac{1}{n})) \quad \text{as} \quad n \to \infty \qquad (36)$$

From Equation 36 one obtains

$$\frac{u(n+1)}{u(n)} = e^{-ia_1 h} e^{-b_1 h} (1 + 0(\frac{1}{n})) \quad \text{as} \quad n \to \infty \tag{37}$$

Thus,

$$b_1 = \frac{1}{h} \ell n \left| \frac{u(n)}{u(n+1)} \right| + 0(\frac{1}{n}), \quad \text{as} \quad n \to \infty \tag{38}$$

$$b_1 + ia_1 = \frac{1}{h} \ell n \frac{u(n)}{u(n+1)} + 0(\frac{1}{n}), \quad \text{as} \quad n \to \infty \tag{39}$$

Suppose that k_j, m_j and c_{jm}, $1 \leq m \leq m_j$, $1 \leq j \leq N$, are computed. Let u_N denote the sum in Equation, $u - u_N \equiv w_N$. Then, as above,

$$b_{N+1} = \frac{1}{n} \ell n \left| \frac{w_N(n)}{w_N(n+1)} \right| + 0(\frac{1}{n}) \quad \text{as} \quad n \to \infty \tag{40}$$

$$b_{N+1} + ia_{N+1} = \frac{1}{n} \ell n \frac{w_N(n)}{w_N(n+1)} + 0(\frac{1}{n}) \quad \text{as} \quad n \to \infty \tag{41}$$

If a_1 and b_1 are found then m_1 can be found by the formula:

$$m_1 = \frac{\ell n\{u(n) e^{ia_1 nh + b_1 nh}\}}{\ell n \; n} + 0(\frac{1}{\ell n \, n}) \quad \text{as} \quad n \to \infty \tag{42}$$

Similarly,

$$m_{j+1} = \frac{\ell n\{w_j(n) e^{ia_{j+1} nh + b_{j+1} nh}\}}{\ell n \; n} + 0(\frac{1}{\ell n \, n})$$

$$\text{as} \quad n \to \infty, \quad j = 1,2,\ldots \tag{43}$$

If a_1, b_1 and m_1 are found then

$$c_{1m_1} = \frac{u(n)e^{ia_1nh+b_1nh}}{(nh)^{m_1}} + O(\frac{1}{nh}), \text{ as } n \to \infty \quad (43a)$$

In the literature (Refs. 8 and 14 through 21) the case of simple resonances $(m_j = 0)$ only was discussed. In this case $O(\frac{1}{n})$ in Equation 44 can be substituted by $O(e^{-(b_2-b_1)nh})$ as $n \to \infty$.

In a similar way all the coefficients c_{1m} can be computed. Practically one takes n large, neglects the remainder in formulas of the type (Eq. 10a), and uses the main term in the right hand side of Equation 10a as the formula for c_{1m_1}, etc.

If k_1, a_1, b_1 and c_{1m}, $1 \leq m \leq m_1$, are found then one works with $w_1 = u - u_1$ and so on. This is a method for computing the coefficients c_{jm} from the transient field.

An alternative method is the usual least squares method. If m_1, a_1, and b_1 are found then c_{1m}, $1 \leq m \leq m_1$ can be found from the requirement $\sum_{n=1}^{\infty} \left| u(n) - \sum_{m=0}^{m_1} c_{1m}(nh)^m e^{(ia_1-b_1)nh} \right|^2 =$ min. This leads to a uniquely solvable linear system for the coefficients c_{1m}, $1 \leq m \leq m_1$. If k_1, a_1, b_1, m_1 and c_{1m}, $1 \leq m \leq m_1$, are found then one works with $u - u_1 = w_1$ and considers w_1 as the transient field. The method based on Equation 43a is quite simple and does not require solving linear systems.

Equations 38-43 give a simple method for extracting resonances and their multiplicities from the exact transient field. The much more complicated case of noisy data is discussed as follows.

In systems theory u often does not depend on x being an impulse response of a system.

Noisy data. Assume that $y(n) = u(n) + \varepsilon(n)$ is measured instead of $u(n)$. Here $\varepsilon(n)$ is noise. Assume that $\varepsilon(n)$ is uniformly distributed on the interval $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$ is a given number. In practice the level of noise is not known exactly since it comes not only from the errors in measurements but also from the unknown background noise in the environment of the scatterer. But without some assumptions about the noise nothing can be derived. One has

$$y(n) = c_{1m_1} h^{m_1} n^{m_1} e^{-ia_1 nh - b_1 nh} (1 + 0(\frac{1}{n})) + \varepsilon(n) =$$

$$= c_{1m_1} h^{m_1} n^{m_1} e^{-ia_1 nh - b_1 nh}$$

$$\{1 + 0(\frac{1}{n}) + \varepsilon(n) c_{1m_1}^{-1} h^{-m_1} e^{ia_1 nh + b_1 nh}\} \qquad (44)$$

From Equation 44 it follows that, regardless of the method used, the extraction of the complex poles k_j from noisy data is highly unstable and depends on the magnitude of $\alpha_n \equiv 0(\frac{1}{n}) + \varepsilon h^{-m_1} e^{b_1 nh} c_{1m_1}^{-1}$. If there exists n such that $\alpha_n \ll 1$ (say $\alpha_n < 0.1$), then the pole $k_1 = a_1 - ib_1$ can be computed by Equations 38 and 39 in which $y(n)$ (α_n) should stay in place of $u(n)$ $(0(\frac{1}{n}))$. Similar considerations hold for other poles. Since $b_j > 0$ the factor $e^{b_j nh}$ is growing as $n \to \infty$. Therefore ε should be small in order that α_n be small and k_j

could be computed. In this case it is not advisable to take n too large because for large n the second term in α_n becomes large. Since the bound on $O(\frac{1}{n})$ is not available it is not worth while to compute the optimal n, but practically n should be taken as a value for which

$$\frac{1}{h} \ln \left| \frac{y(n)}{y(n + 1)} \right|$$

is stationary when one computes b_1, and

$$\frac{1}{h} \ln \left| \frac{z_j(n)}{z_j(n + 1)} \right|$$

is stationary when one computes b_{j+1}.

If the constant c_{1m_1} is small, then the second term in α_n is large unless ε is sufficiently small. Therefore, it is difficult to compute resonance with small Laurent coefficients (coupling coefficients) in front of the singular terms $(k - k_j)^{-m}$. All these arguments are simple but they clearly show the nature of the difficulties for which noise is responsible and the limitations of any method of resonances extraction from noisy data.

We assumed that the scatterer was convex. This assumption implies the basic result: the validity of Equation 35. In the outstanding paper (Ref. 6) it is proved that for the scatterer consisting of two strictly convex reflecting bodies (Eq. 35) does not hold: there exists countably many poles k_j on some line $\mathrm{Im}k = \text{const}$. Therefore, one cannot order the poles by the rule $|\mathrm{Im}k_j| < |\mathrm{Im}k_{j+1}|$ in the case of two disjoint convex reflecting bodies. If the scatterer is just one strictly convex reflecting body then it is known that $|\mathrm{Im}\, k_j| \to +\infty$ as $j \to \infty$ and Equation 35 (Ref. 1).

3. A BRIEF REVIEW OF THE EXISTING METHODS FOR THE RESONANCES EXTRACTION

The most popular is Prony's method (Refs. 8 and 15). One assumes that $u = u(t) = \sum_{j=1}^{N} c_j e^{s_j t}$, $s_j \equiv -ik_j$, $c_j = \text{const}$. One observes $u(t)$ experimentally and wants to compute s_j and c_j. If the data are exact (there is no noise) then the Prony's method consists in the following. Let $f_n \equiv u(nh)$ where $h > 0$ is a fixed number, $e^{s_j h} \equiv z_j$. Then $f_n = \sum_{j=1}^{N} c_j z_j^n$, $n \leq 0$. An obvious linear algebra argument shows that $\det A_{pq}^{(m)} = 0$, $m \geq 0$, $0 \leq p$, $q \leq N$, where $A_{p0}^{(m)} \equiv f_{p+m}$, $A_{pq}^{(m)} = z_q^p$, $0 \leq p \leq N$, $1 \leq q \leq N$. Therefore,

$$0 = \sum_{p=0}^{N} f_{p+m} A_p, \quad m \geq 0 \tag{45}$$

where A_p are the cofactors corresponding to the elements f_{p+m} of the matrix $A_{pq}^{(m)}$. Notice that A_p do not depend on m. Write $N + 1$ Equations 45 taking $m = 0,1,\ldots N$, and find a nontrivial solution $(A_0, A_1, \ldots, A_N)$ to the $N + 1$ simultaneous Equations 45. Consider the equation

$$\sum_{p=0}^{N} A_p z^p = 0 \tag{46}$$

From the structure of the matrix $A_{pq}^{(0)}$ it is clear that Equation 46 has solutions $z_j = e^{s_j h}$. Thus, $s_j = h^{-1} \ell n z_j$. If one does not know the number N (and this is usually the case in practice) then there is a problem of choosing the right N. In Reference 14, p. 140, there is a method (due to Kühnen) for choosing N. If the data are noisy then one faces difficulties explained in Section II and reported in the literature (Ref. 8). If the data are noisy then the matrix f_{p+m}, $0 \leq p, m \leq N$ is nonsingular and system (45) with $0 \leq m \leq N$ has only the trivial solution $A_p = 0$ which can not be used since Equation 46 in this case gives no information. Therefore, in practice one takes as A_p, $0 \leq p \leq N$ the components of an eigenvector corresponding to the minimal eigenvalue of the matrix F*F. Here F* is the adjoint matrix, and F is the matrix of the noisy data, $F_{p+m} = f_{f+m} + \varepsilon_{p+m}$, where ε_{p+m} is noise. If there are several eigenvectors corresponding to the minimal eigenvalue, one has no rule to pick up any particular eigenvector. But this situation is not generic in the sense that a small perturbation of the matrix will split up the multiple eigenvalue into a number of simple ones. However the simple eigenvalues will be close to each other and it will be difficult to find the minimal eigenvalue numerically. (Recall that an eigenvalue is called simple if there is only one linearly independent eigenvector corresponding to this eigenvalue). An extensive bibliography and a discussion of Prony's method one can find in Reference 8. The Bruns method described in Reference 14 is essentially the Prony method for real resonances.

Let us outline another method for extracting the resonances. Let

$$f(t) = \sum_{j=1}^{N} c_j e^{s_j t}$$

and N is assumed known. Then

$$f^{(m)}(t) = \sum_{j=1}^{N} c_j s^m e^{s_j t} .$$

Taking t = 0 and m = 0,...N,... one obtains

$$\begin{vmatrix} f(0) & 1 & \cdots & 1 \\ f'(0) & s_1 & & s_N \\ \cdots & \cdots & & \\ f^{(N)}(0) & s_1^N & & s_N^N \end{vmatrix} = 0, \quad \begin{vmatrix} f'(0) & 1 & 1 \\ \cdots & \cdots & \cdots \\ f^{(N+1)} & s_1^N & s_N^N \end{vmatrix} = 0, \ldots \qquad (47)$$

Therefore

$$\sum_{p=0}^{N} f^{(m+p)}(0) A_p = 0, \quad m = 0,1,2\ldots \qquad (48)$$

Here A_p is the cofactor corresponding to the element $f^{(p)}(0)$ in the first matrix in Equation 47. This argument above is similar to that given previously, and $f^{(m+p)}(0)$ plays the role of f_{m+p}. Taking $m = 0,1\ldots N$ in Equation 48 one obtains a homogeneous system of linear equations. If $A_0,\ldots A_N$ is a solution to this system then the N roots of Equation 46 3.2 are equal to s_j, $1 \leq j \leq N$. This gives a method of extracting the resonances s_j from the exact transient data. It is interesting that only the data near $t = 0$ is used in this procedure, while the procedure in Section III.2 needs the data at large t. On the other hand, the procedure based on Equation 48 is sensitive to the noise in the data because one needs to differentiate the data.

We assumed that the number N of resonances was known. If N is not known, then one can find N as the smallest number for which $\det f_{p+m} = 0$, $0 \le p, m \le N$.

The simple algorithm in Section II requires that $b_1 < b_2 < b_3 \ldots$. In practice the poles $a_j - ib_j$ occur in pairs $\pm a_j - ib_j$ and the measured transient field is a real-valued function. Assuming that the poles are simple, i.e. $m_j = 0$, one has $u(t) = \sum_{j=1}^{N} c_j \exp(-b_j t)\cos(a_j t + \phi_j) + 0(e^{-b_{N+1}t})$ as $t \to +\infty$. Therefore for large t one obtains $u(t) = c_1 \exp(-b_1 t)\cos(a_1 t + \phi_1) + 0(e^{-b_2 t})$, $t \to +\infty$. If the values $u_n \equiv u(nh)$ are measured, then the values c_1, b_1, a_1 and ϕ_1, $0 < \phi_1 < 2\pi$, can be numerically obtained from the requirement

$$F(c_1, b_1, a_1, \phi_1) \equiv \frac{1}{m} \sum_{n=n_0}^{n_0+m} \left| c_1 \exp(-b_1 hn)\cos(a_1 hn + \phi_1) - u_n \right|^2 = \min \qquad (49)$$

Here n_0 is a large number such that $\exp(-b_1 hn_0) \gg \exp(-b_2 hn_0)$, $m > 4$ is a fixed number, and the function $F(c_1, b_1, a_1, \phi_1)$ is to be minimized numerically. If this minimization problem is solved one can consider $w_1 \equiv u(t) - u_1(t)$, where $u_1(t) \equiv$

$c_1 \exp(-b_1 t)\cos(a_1 t + \phi_1)$ and apply the same procedure for finding c_2, b_2, a_2, ϕ_2. Each step requires minimization of a function of four variables only. The basic new idea in this method is to use the asymptotic behavior as $t \to +\infty$ of the transient field. One should have in mind that the basic asymptotic SEM expansion (Eq. 35) is proved only if the scatterer is convex (Ref. 1) (or, more generally, star-shaped). It does not hold, for example, when the scatterer consists of two convex obstacles. In this case there exists infinitely many poles k_j, such that $|k_j - ic_0 - \pi d_j^{-1}| \le c(1 + |j|)^{-1/2}$ for all large j, $j = \pm j_0, \pm(j_0 + 1), \ldots$. Here d is the distance between the two obstacles, c_0 depends on d, on the principal curvatures, and principal directions of the surfaces Γ_1 and Γ_2 of the two obstacles at the points $s_1 \in \Gamma_1$ and $s_2 \in \Gamma_2$, such that $|s_1 - s_2| = d$. This remarkable result was proven recently in Reference 5.

4. BIBLIOGRAPHICAL REMARKS.

Of the older references only Prony's paper (Ref. 15) is often mentioned by the modern authors. There is a translation of this paper in Reference 8. Bruns (Ref. 16) used practically the same idea as Prony. His work is discussed in Reference 14. There are several authors, mostly astronomers, which were interested in detection of hidden periodicities (Refs. 16 through 21). Although only the case $m_j = 0$, $b_j = 0$ was discussed in these papers, the basic questions (extracting the resonances from the transient field, determining the number N of resonances etc.) were actually identical with the questions discussed in a very recent review (Ref. 8) of the state of art in this field. Papers (Refs. 16 through 21) are not cited by modern western authors in the field. One can find review of these papers in Reference 14.

An extensive modern literature exists on the extracting of resonances. One can find a large bibliography and a review of the basic results in Reference 8. We did not discuss here some of the methods mentioned in Reference 8.

There are many reasons for being interested in the extracting of resonances. We mention only two major theories: singularity expansion method (see Refs. 2, 22 and 10) for the mathematical results) and systems identification (see the bibliography in Ref. 8).

REFERENCES

1. Ramm, A. G., On the analytic continuation of the solution of the Schrodinger equation in the spectral parameter and the behavior of the solution to the nonstationary problem as t → ∞, Uspechi Mat. Nauk, 19, pp. 192-194, (1964).

2. Ramm, A. G., Mathematical foundations of the singularity and eigenmode expansion methods, J. Math. Anal. Appl., 86, pp. 562-591, 1982.

3. Ramm, A. G., On the singularity and eigenmode expansion methods, Electromagnetics, 1, pp. 385-394, 1981.

4. Baum, C., Energing technology for transient and broad-band analysis and synthesis of antennas and scatterers, Proc. IEEE, 64, pp. 1598-1616, (1976).

5. Dolph, C., and Scott, R., Recent developments in the use of complex singularities in electromagnetic theory and elastic wave propagation, in Electromagnetic scattering, Acad. Press, New York, 1978. (Editor P. Vslenghi).

6. Ikawa, M., On the poles of the S-matrix for two strictly convex obstacles, J. of Math. Kyoto Univ., 23, pp. 127-194, pp. 795-802, 1983.

7. Lax, P. and Phillips, R., Decaying modes for the wave equation in the exterior of an obstacle, Comm. Pure and Appl. Math., 22, pp. 737-787, 1969.

8. Auton, J. and Van Blaricum, M., Investigation of procedures for automatic resonances extraction from noisy transient electromagnetic data, AFWL Math. Notes, 79, vol. 1-3, 1981.

9. Senior, T. and Pond, J., Pole extraction in the frequency domain, AFWL, Interaction Note 411, Dec. 1981.

10. Ramm, A. G., Theory and applications of some new classes of integral equations, Springer-Verlag, New York, 1980.

11. Ramm, A. G., A variational principle for resonances, I, II, J. Math. Phys, 21, pp. 2052-2054, 1980 23, pp. 1112-1114, 1982.

12. Ramm, A. G., Convergence of the T-matrix approach, J. Math. Phys., 23, 1123-1125, pp. 2408-2410, 1982 24. pp. 2619-2631, 1983, the last paper is joint with G. Kristensson and S. Strom.

13. Kumano-go, H., Pseudodifferential operators, MIT Press, Cambridge, 1981.

14. Serebrennikov, M. and Pervozvanskii, A., <u>Detection of hidden periodicities,</u> Moscow, Nauka, 1965 (in russian).

15. Prony, R., Essai experimentale ......, <u>J. Ecole Polytech.,</u> vol. 1, pp. 24-76, 1795.

16. Bruns, H., Uber die Analyse periodischer Vorgange, <u>Astronom. Nachrichte,</u> vol. 188, 1911.

17. Dale, J., The resolution of a compound periodic functions into simple periodic functions, <u>Monthly Notices Roy. Astron. Soc.,</u> vol. 74, p. 628, 1914.

18. Plumer, H., Notes on Dale's method of periodic analysis, <u>ibid.,</u> vol. 76, pp. 60-63, 1913.

19. Hopfner, F., Uber die praktische Vervendbarkeit einer neuen Methode zur Auffindung der Periode einer periodischen Erscheinung, <u>Wiener Sitzungsberichte,</u> vol. 119, pp. 351-370, 1910.

20. Oppenheim, S., Uber die Bestimmung der Periode einer periodischer Erscheinung, <u>ibid.,</u> vol. 118, 1909.

21. Kuhnen, F., Methode zur Aufsuchung periodischer Erscheinungen in Reien equidistanten Beobachtungen, <u>Astron. Nachrichten,</u> vol. 182, pp. 1-11, 1909.

22. Ramm, A. G., Theoretical and practical aspects of singularity and eigenmode expansion methods, <u>IEEE Trans., AP-28, N6,</u> pp. 897-901, 1980.

23. Vainberg, M. and Trenogin, V., Bifurcation of solutions of nonlinear equations, <u>Noordhoff,</u> Leiden, 1974.

APPENDIX A

PERTURBATION OF RESONANCES

1. ABSTRACT SCHEME

First let us present an abstract scheme. Assume that a compact operator function $A(k)$ on a Hilbert space H is analytic in k in a domain on the complex plane k, and $A(k_0)$ has an eigenvalue -1. Then $(I + A(k))^{-1}$ has a pole at $k = k_0$. Suppose that $A(k,\varepsilon)$ is a compact operator function such that $A(k,0) = A(k)$, which is analytic in k and ε, $|k - k_0| < \delta_0$, $|\varepsilon| < \delta_1$. Assume that k_0 is an isolated pole of $I + A(k)$. (This is the case if $I + A(k)$ is invertible for at least one k in the disk $|k - k_0| < \delta_0$). Then, in a neighborhood of k_0, there exist a finite number m_0 of points $k_j(\varepsilon) = k_0 + \sum_{m=1}^{\infty} a_{mh}\, \varepsilon^{m/p}$, $1 \leq j \leq j_0$, such that the operator $(I + A(k,\varepsilon))^{-1}$ has poles $k(\varepsilon)$. Here j_0 is the multiplicity of the pole k_0, and the meaning of the integer p will be explained in the proof which is based on the idea in Reference 2, p. 582.

Let $\phi_1,\ldots,\phi_n$ be an orthonormal basis of $N(I + A(k_0))$ where $N(A)$ is the null space of an operator A. Let $\psi_1,\ldots,\psi_n$ be an orthonormal basis of $N(I + A^*(k_0))$, where the star denotes the adjoint operator. Let $Th \equiv \sum_{j=1}^{n} (h,\phi_j)\psi_j$. The operator $I + A(k_0) + T$ is invertible in H. Indeed, $(I + A(k_0) + T)h = 0$ implies that $(Th,\psi_j) = 0$, $1 \leq j \leq n$. This leads to $(h,\phi_j) = 0$, $1 \leq j \leq n$, i.e. $Th = 0$, and $(I + A(k_0))h = 0$. Thus $h \in N(I + A(k_0))$ and $h \perp N(I + A(k_0))$. Therefore $h = 0$, and by Fredholm's alternative $(I + A(k_0) + T)^{-1} = \Gamma$ is bounded. Consider $(I + A(k,\varepsilon))^{-1} = (I + A(k_0) + T + A(k,\varepsilon) - A(k_0) - T)^{-1} =$

$= (I + a(k,\varepsilon))^{-1} \cdot \Gamma(k,\varepsilon)$, where $\Gamma(k,\varepsilon) = (I + A(k_0) + T + A(k,\varepsilon) - A(k_0))^{-1}$ is analytic in k and ε in a neighborhood Δ of $(k_0,0)$, and $a(k,\varepsilon) = -\Gamma(k,\varepsilon)T$ is a finite-dimensional operator analytic in k and ε in Δ, $ah = -\sum_{j=1}^{n} (h,\phi_j)\Gamma(k,\varepsilon)\psi_j$. Since $\Gamma(k,\varepsilon)$ is an isomorphism of H onto H for $k,\varepsilon \in \Delta$, the elements $\psi_j(k,\varepsilon) = -\Gamma(k,\varepsilon)\psi_j$ are linearly independent and analytic in $k,\varepsilon \in \Delta$. Therefore the operator $(I + a(k,\varepsilon))^{-1}$ can be constructed explicitly. If $(I + a)h = f$, then $h + \sum_{j=1}^{n} h_j\psi_j(k,\varepsilon) = f$, $h_j \equiv (h,\phi_j)$. Multiply by ϕ_m to obtain $h_m + \sum_{j=1}^{n} c_{mj}h_j = f_m$, where $f_m = (f,\phi_m)$, $c_{mj} = (\psi_j(k,\varepsilon),\phi_m)$, c_{mj} are analytic in $k,\varepsilon \in \Delta$. Thus, $h_m = \frac{d_m(k,\varepsilon)}{d(k,\varepsilon)}$, where d_m and $d = \det(\delta_{mj} + c_{mj})$ are analytic in Δ. One has $(I + a(k,\varepsilon))^{-1}f = f - \frac{1}{d}\sum_{j=1}^{n} d_j(k,\varepsilon)\psi_j(k,\varepsilon)$. From this formula it is clear that the poles of $(I + a(k,\varepsilon))^{-1}$ can occur only at the zeros of $d(k,\varepsilon)$. Thus the equation for the perturbed poles is

$$d(k,\varepsilon) = \det [\delta_{jm} - (\Gamma(k,\varepsilon)\psi_j,\phi_m)] = 0,$$

$$\Gamma(k,\varepsilon) = (I + T + A(k,\varepsilon))^{-1} \tag{A1}$$

For $\varepsilon = 0$ the function $d(k) = d(k,0)$ has, by assumption, a zero of multiplicity j_0. By the Weierstrass' preparation theorem (see e.g. Ref. 2, p. 583) one has $d(k,\varepsilon) = [k^{j_0} + \sum_{j=1}^{j_0-1} c_j(\varepsilon)k^j]g(k,\varepsilon)$ where $g(0,0) \neq 0$, $c_j(0) = 0$, c_j and $g(k,\varepsilon)$

are holomorphic functions. Therefore Equation (A1) is equivalent to

$$k^{j_0} + \sum_{j=1}^{j_0-1} c_j(\varepsilon)k^j = 0 \qquad \text{(A2)}$$

This equation has j_0 roots. These roots can be divided into several groups so that the p roots $(k_1(\varepsilon),\dots k_p(\varepsilon))$ in ν-th group can be expanded in a Puiseux series $k_j(\varepsilon) = k_0 + \sum_{j=1}^{\infty} a_{mj}\,\varepsilon^{m/p_\nu}$, $\sum_\nu p_\nu = j_0$. The number of groups and the integers p_ν can be computed by the algorithm known as Newton's diagram method (Ref. 23). Let us summarize our arguments: a method for computing the poles $k_j(\varepsilon)$ of the perturbed operator $(I + A(k,\varepsilon))^{-1}$ is given. The method is valid under the following assumptions: $A(k,\varepsilon)$ is a compact analytic in k and ε linear operator function on a Hilbert space, the operator $(I + A(k,0))^{-1}$ has an isolated pole at $k = k_0$. To use the method computationally one needs to compute (or to know) the bases in the subspaces $N(I + A(k_0))$ and $N(I + A^*(k_0))$. This is a linear algebra problem.

2. REDUCTION OF A CONCRETE PERTURBATION PROBLEM TO THE ABSTRACT ONE.

Suppose that the surface Γ of the obstacle is perturbed. Let $x_j = x_j(u,v)$, $1 \leq j \leq 3$ be a parametric equation of Γ, and $z_j = x_j(u,v) + \varepsilon y_j(u,v)$ be the equation of the perturbed surface Γ_ε, where ε is a small parameter. Assume that the functions x_j and y_j, $1 \leq j \leq 3$, $u,v \in S \equiv \{u,v: 0 \leq u,\ v \leq 1\}$ are smooth.

Consider equation (8) in section 1. Assume that k_0 is a pole of the operator $(I + A(k))^{-1}$. Suppose that the bases of the subspaces $N(I + A(k_0))$ and $N(I + A^*(k_0))$ are computed. Consider the problem corresponding to the perturbed surface Γ_ε. The operator $A(k,\varepsilon)$, associated with this problem, is of the form $\int_{\Gamma_\varepsilon} 2 \frac{\partial g}{\partial N_s} h ds = \int_S A(u,v,u',v',k,\varepsilon) h du'dv'$. Here $A(u,v,u',v',k,\varepsilon)$ is the kernel of the integral operator in the variables u,v,u',v'. If $\varepsilon = 0$ then $\Gamma_\varepsilon = \Gamma$ and we assume that the sets $\{\phi_j\}$ $(\{\psi_j\})$, $1 \leq j \leq n$ of all linearly independent solutions of the equation $\phi + \int_S A(u,v,u',v',k_0,0)\phi du'dv' = 0$ $(\psi + \int_S \overline{A(u',v',u,v,k_0,0)}\psi du'dv' = 0)$ is known. Then the abstract scheme is applicable.

Since small perturbations of the kernel cause small perturbations of the poles, one can approximate the kernel A by a degenerate kernel and consider the corresponding matrix problem.

As an example consider a simple case when the matrix is 2×2. Let

$$\begin{pmatrix} 1 & e^{\pi k} \\ e^{\pi k} & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} .$$

Then the inverse matrix is $\begin{vmatrix} 1 & -e^{\pi k} \\ -e^{\pi k} & 1 \end{vmatrix} \frac{1}{1 - e^{2\pi k}}$. It has simple poles $k_m = im$, $m = 0, \pm 1, \ldots$. Consider, for example, pole $k_0 = 0$. The set $\{\phi_j\}$ corresponding to this pole is the set of linearly independent solutions to the equation

$\begin{cases} c_1 + c_2 = 0 \\ c_1 + c_2 = 0. \end{cases}$ Thus, there exists one linearly independent solution $\phi_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. In our example the matrix $\begin{pmatrix} 1 & e^{\pi k} \\ e^{\pi k} & 1 \end{pmatrix}$ is selfadjoint for $k = 0$. Thus $\psi_1 = \phi_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Consider the perturbed matrix $\begin{pmatrix} 1 + \varepsilon a_{11} & e^{\pi k} + \varepsilon a_{12} \\ \pi k & \\ e^{\pi k} + \varepsilon a_{21} & 1 + \varepsilon a_{22} \end{pmatrix}$. The poles of this matrix are the roots of the equation $(1 + \varepsilon a_{11})(1 + \varepsilon a_{22}) - (e^{\pi k} + \varepsilon a_{12})(e^{\pi k} + \varepsilon a_{21}) = 0$. We are interested in the root $k(\varepsilon)$ such that $k(\varepsilon) \to 0$ as $\varepsilon \to 0$. One has $1 - e^{2\pi k} + \varepsilon[a_{11} + a_{22} - e^{\pi k}(a_{12} + a_{21})] + \varepsilon^2(a_{11}a_{22} - a_{12}a_{21}) = 0$. Let $e^{2\pi k} = z$, $a_{11} + a_{22} = a$, $a_{12} + a_{21} = b$, $a_{11}a_{22} - a_{12}a_{21} = c$. Then $z^2 + z\varepsilon b - 1 - \varepsilon a - \varepsilon^2 c = 0$, $z(\varepsilon) = -\frac{\varepsilon b}{2} + \sqrt{\frac{\varepsilon^2 b^2}{4} + 1 + \varepsilon a + \varepsilon^2 c} \approx 1 + \frac{\varepsilon}{2}(a - b)$. The plus sign in front of the radical is chosen because $z = 1$ if $k = 0$. Thus, in this example the perturbed pole $k_0 = 0$ can be computed for small ε as $k(\varepsilon) = \frac{1}{\pi} \ell n\, z(\varepsilon) = \frac{(a - b)\varepsilon}{2\pi}$. Depending on the values of a and b the perturbed pole can move in any direction. If $a = b$ then $k(\varepsilon) = O(\varepsilon)$. If $a = b$ then $k(\varepsilon) = O(\varepsilon^2)$. The bifurcation theory and the Newton diagram method solve the following problem. Given an equation $F(k,\varepsilon) = 0$ find its solutions $k(\varepsilon)$ such that $k(\varepsilon \to 0$ as $\varepsilon \to 0$. It is assumed that $F(0,0) = 0$. If $F_k(0,0) = 0$ then the solution is well known and is given by the standard implicit function theorem. If $F_k(0,0) = 0$ then the solution is more complicated.

Methods and algorithms for solving this problem can be found in Reference 23.

APPENDIX B

METHODS OF CALCULATING THE ZEROS FOR ANALYTIC FUNCTIONS

1. A brief review of the known methods of calculating the zeros for smooth functions: Some methods for finding zeros of smooth functions are described in References B1 through B3. They include Newton's method, various modifications of this method, and some methods used in computational practice but not well understood theoretically in the sense that convergence of these methods is not proved and theoretical estimates of the rate of convergence are not obtained. Among these methods we mention Muller's method and Wegstein's method (Ref. B1). Newton's method was discussed in dozens of papers and books. There are some modifications of this method which converge globally and not only in a neighborhood of the zeros (Refs. B2, B4, and B5).

2. In Reference B6 a method for calculating zeros of analytic functions is suggested. The method is based on the formula

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \phi(z)dz = \sum_j \phi(z_j) \qquad \text{(B1)}$$

where z_j are the zeros of $f(z)$ which lie inside the domain Δ on the complex plane z with boundary C, f and ϕ are analytic functions of z in Δ. If one takes $\phi(z) = z^m$, $m = 0,1,2$, then Equation B1 yields

$$J_m \equiv \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} z^m dz = \sum_{j=1}^{N} z_j^m, \quad m = 0,1,2,... \qquad \text{(B2)}$$

provided that there are N zeros (counting their multiplications) of $f(z)$ in Δ. If one computes the numbers J_m then Equation B2 is a system of nonlinear equations for the unknown roots z_j, $1 \leq j \leq r$. Among the roots in Equation B2 some can be equal to each other. Let z_j be a root of multiplicity r_j so that $z_j = z_i$ if $j = i$. Then Equation B2 can be written as

$$J_m = \sum_{j=1}^{r} r_j z_j^m \tag{B3}$$

where r in Equation B3 denotes the number of different roots, $N = \sum_{j=1}^{r} r_j$.

It is now clear that the unknown quantities are the multiplicities r_j, the different roots z_j, and the number r. The difficulty in the numerical solution to Equation B3 is to discriminate between two close roots and one multiple root. Suppose one wishes to find the roots in the domain $\Delta_{-N,N} = \{-N \leq x \leq N, -b \leq y \leq 0\}$, $x + iy = z$. Computing integral J_0 with C_n where C_n is the boundary of $\Delta_{-N,n} = \{-N \leq x \leq nh, -b \leq y \leq 0\}$ $hn = N, N-1, \ldots,$ where $h > 0$ is a discretization parameter, one can locate the rectangle $\{nh \leq x \leq (n+1)h, -b \leq y \leq 0\}$ in which a zero of $f(z)$ lies by the jump of the quantity $J^0(n)$ provided that in this rectangle there is only one root possibly multiple. By this scanning procedure which can be used in the y variable as well, one can isolate the roots in the rectangles $\{n^1 h \leq x \leq (n^1 + 1)h,$

$n_2 \leq y \leq (n_2 + 1)h\}$. This requires many computations, but the computations are of the same type: one computes the integral in the left-hand side of Equation B2 along various rectangles.

3. The general methods for solving equation $f(x) = 0$ do not use analyticity of $f(x)$ (Newton's method, various relaxation methods, e.g. the steepest descend method etc.) and therefore are not discussed here (see Refs. B2 through B5).

REFERENCES FOR APPENDIX B

B1. Lance, G., Numerical methods for high speed computers, London, Iliffe, (1960).

B2. Ortega, J. and Rheinboldt, W., Iterative solution of nonlinear equations in several variables, Acad. Press, New York, (1970).

B3. Ostrowski, A., Solution of equations and systems of equations, Acad. Press, New York, (1960).

B4. Sandberg, I., "Diffeomorphisms and Newton-Direction algorithms," Bell Syst. Tech. J., 59, pp. 1721-1733, (1980).

B5. Hirsch, N. and Smale, S., "On algorithms for solving f(x) = 0," Comm. Pure Appl. Math. 32, pp. 281-312, (1979).

B6. Singaraju, B., Giri, D. and Baum, C., "Further developments in the application of contour integration to the evaluation of the zeros of analytic functions and relevant computer programs," Math. Note 42, March 1976.