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Effect of Record Length on the Correlation of Complex Exponentials

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ABSTRACT: Resolution of two sinusoidal signals of frequencies f_1 and f_2 requires a time record of length at least $1/|f_1-f_2|$. Resolution of two decaying exponentials of complex frequencies $s_1=\sigma_1+j\omega_1$ and $s_2=\sigma_2+j\omega_2$, with $\sigma_2 \geq \sigma_1$, by correlation coefficient is dependent only on the ratios σ_2/σ_1, and $(\omega_1-\omega_2)/\sigma_1$ for a time record of length greater than $2/|\sigma_1|$. This is also the condition for near-orthogonalization of a set of complex exponentials, with small error.

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1. Introduction

With the advent of the Singularity Expansion Method (SEM) there has been a great interest on the identification of a linear time-invariant system by a sum of complex exponentials. In this paper the suitability of the exponential functions for modelling a finite time domain impulse response is examined. More specifically, we address the question of how long a data set does one need so that the record length behaves as if it were infinite. In other words, what is the minimum length of record required to resolve the various components of decaying exponentials. The answer to this question may yield data for a meaningful analysis.

If two signals $\phi(t)$ and $\Psi(t)$ are to be distinguishable the waveforms must have the property of being as different from its shifted self as possible. In mathematical terms, the mean squared departure of $\Psi(t)$ from $\phi(t+\tau)$

$$\int_{-\infty}^{\infty} |\Psi(t)-\phi(t+\tau)|^2 dt \qquad (1)$$

must be as large as possible over the range of τ. By expanding the above integral and noting the independence of the squared terms of τ, we see that minimization of (1) implies

$$\int_{-\infty}^{\infty} \Psi(t)\phi^*(t+\tau)dt \qquad (2)$$

shall be as small as possible. Here * denotes complex conjugate.

The above integral in (2) is defined as the correlation between the two functions $\Psi(t)$ and $\phi(t)$. We now introduce a normalized version of (2) which we define as the correlation coefficient $\rho(\tau)$ between the two signals $\phi(t)$ and $\Psi(t)$. The correlation coefficient $\rho_\infty(\tau)$ is defined as

$$\rho_\infty(\tau) = \frac{\int_{-\infty}^{\infty}\Psi(t)\phi^*(t+\tau)dt}{\{\int_{-\infty}^{\infty}\Psi(t)\Psi^*(t)dt\}^{\frac{1}{2}}\{\int_{-\infty}^{\infty}\phi(t)\phi^*(t)dt\}^{\frac{1}{2}}} \quad . \tag{3}$$

So for perfect correlation, i.e. when the waveforms $\Psi(t)$ and $\phi(t)$ are identical then $\rho_\infty(\tau)=1$. Under this circumstance it is impossible to resolve $\Psi(t)$ from $\phi(t)$. However, when $\rho_\infty(\tau)=0$, we have perfect resolvability, i.e. the two signals $\Psi(t)$ and $\phi(t)$ are as different as possible.

Observe that the limits in the integrals of (3) are from $-\infty$ to $+\infty$. In general when we are performing an experiment it is not possible to have infinitely long data records. If we have finite length data records then the correlation coefficient is defined as

$$\rho_\Delta(\tau) = \frac{\int_{T}^{T+\Delta}\Psi(t)\phi^*(t+\tau)dt}{\{\int_{T}^{T+\Delta}\Psi(t)\Psi^*(t)dt\}^{\frac{1}{2}}\{\int_{T}^{T+\Delta}\phi(t)\phi^*(t)dt\}^{\frac{1}{2}}} \quad . \tag{4}$$

It is clear that because of finite record lengths,

$$|\rho_\Delta(\tau)| \geq |\rho_\infty(\tau)| . \tag{5}$$

In this paper we investigate the value of Δ for which

$$|\rho_\Delta(\tau)| \simeq |\rho_\infty(\tau)| . \tag{6}$$

This value of Δ will then dictate the length of record necessary to reduce the correlation coefficient between the functions $\Psi(t)$ and $\phi(t)$ and thereby increase the chances of resolvability.

2. Correlation Considerations Involving Complex Exponentials

In general, identification of the complex exponential components of a signal involves solution of a set of simultaneous equations (e.g. Prony's method). As the correlation between the components is increased, the equations become more ill-conditioned. Consequently, the correlation coefficient between the two complex exponentials is a measure of the difficulty by which the two components may be resolved. By way of illustration, two signals with unit correlation result in a singular set of equations whereas two signals with zero correlation yield an uncoupled set.

Consider two simple complex exponentials given by

$$\{e^{s_i t}\}; \quad i = 1, 2, \qquad 0 \leq t \leq \infty \tag{7}$$

where $s_i = \sigma_i + j\omega_i$, with $\sigma_i < 0$ and $j = \sqrt{-1}$. The correlation coefficient between exp $[s_1 t]$ and exp $[s_2 t]$ over the time interval $[T, T + \Delta]$ is defined as

$$\rho_\Delta(\tau) = \frac{\int_T^{T+\Delta} e^{s_1 t}\, e^{s_2^* t}\, dt}{\{\int^{\Delta} e^{s_1 t}\, e^{s_1^* t}\, dt\}^{1/2}\, \{\int_T^{T+\Delta} e^{s_2 t}\, e^{s_2^* t}\, dt\}^{1/2}} \tag{8}$$

Performing the integration and rearranging terms, we get

$$\rho_\Delta(\tau) = \frac{\sqrt{4\sigma_1\sigma_2}\cdot e^{(\sigma_2 - j\omega_2)\tau}\cdot e^{j(\omega_1-\omega_2)T}\cdot \left\{e^{[(\sigma_1+\sigma_2)+j(\omega_1-\omega_2)]\Delta} - 1\right\}}{[\sigma_1+\sigma_2+j(\omega_1-\omega_2)]\sqrt{(1-e^{2\sigma_1\Delta})(1-e^{2\sigma_2\Delta})}} \tag{9}$$

In general the correlation coefficient is a complex quantity. The initial time instant T simply adds a constant to the phase. Since we are primarily interested in the magnitude there is no loss in generality by assuming T=0. Also the correlation coefficient varies as $\exp\{[\sigma_2 - j\omega_2]\tau\}$ with τ, which does not enter into our discussions. Hence we define a new coefficient $\rho_{12}(\Delta)$ as

$$\rho_\Delta(\tau) = \rho_{12}(\Delta)\exp\{[\sigma_2 - j\omega_2]\tau\} \tag{10}$$

We want to study the properties of $\rho_{12}(\Delta)$ as the exponential function in (10) does not provide any additional insight.

Of particular interest is the sinusoidal case for which $\sigma_i=0$ for $i=1,2$. In this case

$$\rho_{12}(\Delta)\Big|_{\sigma_{1,2}=0} = \frac{\sin\left[\frac{(\omega_1-\omega_2)\Delta}{2}\right]}{\frac{(\omega_1-\omega_2)\Delta}{2}} \angle \frac{(\omega_1-\omega_2)\Delta}{2} \tag{11}$$

The phase and the magnitude of the correlation coefficient is plotted in figure 1 for $\omega_1 \geq \omega_2$. Observe that the correlation coefficient is zero for

$$\frac{(\omega_1-\omega_2)\Delta}{2} = \frac{2\pi(f_1-f_2)\Delta}{2} = n\pi; \; n=1,2,... \tag{12}$$

Hence, for an observation interval of length Δ, the two components are uncorrelated for

$$f_1-f_2 = \frac{n}{\Delta}; \quad n = 1,2,... \quad . \tag{13}$$

With n = 1, we obtain the conventional condition for resolution of two frequency components which is given by

$$f_1-f_2 = \frac{1}{\Delta} . \tag{14}$$

for other values of f_1-f_2, the magnitude of the correlation coefficient is bounded by

$$|\rho_{12}(\Delta)|_{\sigma_1,\sigma_2=0} \leq \frac{1}{\pi(f_1-f_2)\Delta} \tag{15}$$

This bound is plotted in Fig. 1. When $\omega_1=\omega_2$, the correlation coefficient is unity. This is to be expected since the two components are then identical.

The situation is more complicated when $\sigma_i < 0$; $i=1,2$. This is evident by consideration of the spectra involved. For the infinite interval, $\sigma_i=0$ yields a line spectrum whereas $\sigma_i<0$ results in a

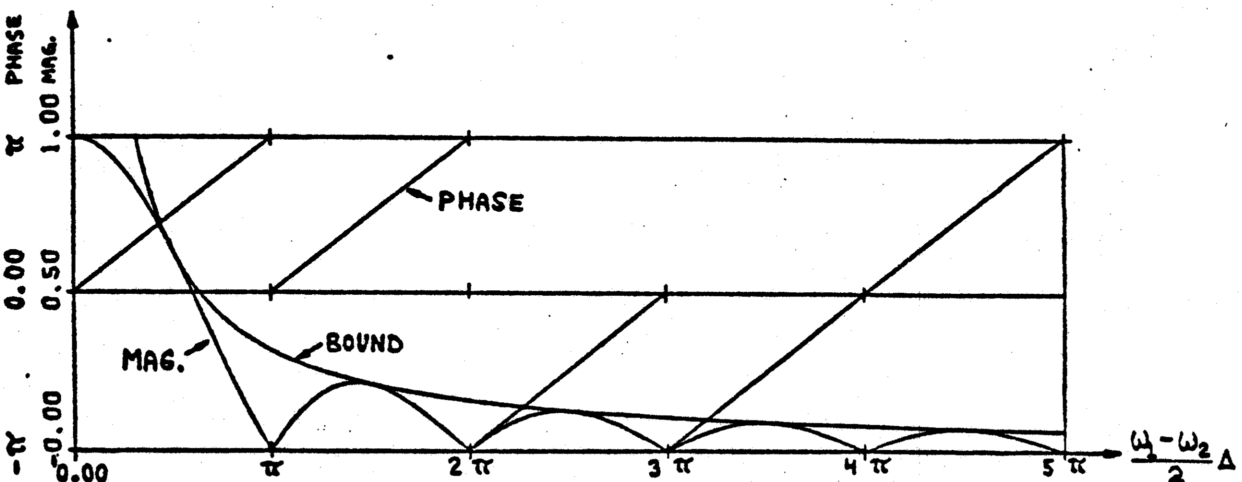

Fig. 1. Phase and magnitude of $\rho_{12}(\Delta)|_{\sigma_1 \sigma_2=0}$

continuous spectrum. Define

$$\alpha = \frac{\sigma_2}{\sigma_1} \; ; \; \beta = \frac{\omega_1 - \omega_2}{\sigma_1} \; ; \quad \gamma = \sigma_1 \Delta \tag{16}$$

where, for convenience, it is assumed $\sigma_2 \leq \sigma_1 \leq 0$. Substitution of (16) into (8), with T=0, the correlation coefficient is expressed as

$$\rho_{12}(\Delta) = \frac{\sqrt{4\alpha}}{1+\alpha+j\beta} \; \frac{e^{(1+\alpha)\gamma} e^{j\beta\gamma} - 1}{(1-e^{2\gamma})(1-e^{2\alpha\gamma})} \tag{17}$$

We define

$$|\rho_{12}(\Delta)| = |\rho_{12}(\infty)| \cdot W(\alpha,\beta,\gamma) \tag{18}$$

where $\rho_{12}(\infty)$ is the correlation coefficient due to the infinite observation interval and is defined as

$$|\rho_{12}(\infty)| = \sqrt{\frac{4\alpha}{(1+\alpha)^2+\beta^2}} \tag{19}$$

The second factor in (18) can then be interpreted as a "windowing" factor due to finite observation interval. The window factor is defined as

$$W(\alpha,\beta,\gamma) = \sqrt{\frac{1-2e^{(1+\alpha)\gamma}\cos\beta\gamma + e^{2(1+\alpha)\gamma}}{1-(e^{2\gamma}+e^{2\alpha\gamma})+e^{2(1+\alpha)\gamma}}} \tag{20}$$

For a fixed value of β, the maximum value of $|\rho_{12}(\infty)|$ occurs for

$$\alpha_M = \sqrt{1+\beta^2} \tag{21}$$

The peak value of $|\rho_{12}(\infty)|$ is then given by

$$|\rho_{12}(\infty)|_{\alpha=\alpha_M} = \sqrt{\frac{2}{1+\sqrt{1+\beta^2}}} \tag{22}$$

It is interesting to note that (17) is unchanged by interchanging the subscripts 1 and 2 in the definitions of (10). For this reason, it is necessary to consider only values of α greater than or equal to unity. The value of $\alpha = 1$ is, therefore, of special interest because it serves as the "origin" in our plots. Interestingly enough, for both large and small values of $|\beta|$, the peak value in (22) is approximately related to the value of the correlation coefficient for $\alpha = 1$. Specifically,

$$|\rho_{12}(\infty)|_{\alpha=1} = \sqrt{\frac{4}{4+\beta^2}} \tag{23}$$

for $|\beta| >> 1$,

$$|\rho_{12}(\infty)|_{\alpha=1} \simeq \frac{2}{|\beta|} \simeq [|\rho_{12}(\infty)|_{\alpha=\alpha_M}]^2 \tag{24}$$

On the other hand, for $|\beta| << 1$,

$$|\rho_{12}(\infty)|_{\alpha=1} \simeq 1 \simeq |\rho_{12}(\infty)|_{\alpha=\alpha_M} \tag{25}$$

Also, for $\alpha >> 1$ and $\alpha >> |\beta|$, observe from (19) that

$$|\rho_{12}(\infty)| \simeq \frac{2}{\sqrt{\alpha}} \tag{26}$$

Hence, for a fixed value of $|\beta|$, the correlation coefficient approaches zero in the limit as α approaches infinity.

Equation (19) is now investigated as a function of β. For a fixed value of α, the maximum value of $|\rho_{12}(\infty)|$ occurs for

$$\beta_M = 0. \tag{27}$$

It is interesting to note that the correlation coefficient peaks when $\omega_1=\omega_2$ but does not necessarily peak when $\sigma_1=\sigma_2$ (i.e., $\alpha=1$). For $\beta_M=0$, the peak value of the correlation coefficient is

$$|\rho_{12}(\infty)|_{\beta=\beta_M=0} = \frac{2\sqrt{\alpha}}{1+\alpha} \tag{28}$$

For $\alpha >> 1$, the peak becomes

$$|\rho_{12}(\infty)|_{\beta=\beta_M=0} \simeq \frac{2}{\sqrt{\alpha}} \tag{29}$$

Asymptotically, for $|\beta| >> \alpha$, (19) reduces to

$$|\rho_{12}(\infty)| \simeq \frac{2\sqrt{\alpha}}{|\beta|} \tag{30}$$

Therefore, for a fixed value of α, the correlation coefficient approaches zero in the limit as $|\beta|$ approaches infinity. By comparison of (26) with (30), it is seen that the correlation coefficient asymptotically approaches zero at a faster rate with respect to $|\beta|$ as opposed to α.

A plot of $|\rho_{12}(\infty)|$ versus $|\beta|$, with α as a parameter is shown in Fig. 2. Recall that the larger is the correlation between signal components, the more ill-conditioned are the equations which arise in the identification problem. Figure 2 points out that the problem of resolution is eased under situations of both large α's and large β's. Recognizing that α is constrained to be greater than unity, small α implies $\sigma_1 \approx \sigma_2$. Then a large value of $|\beta|$ is desirable so that the difference in ω_1 and ω_2 will aid in discriminating between the two components. On the other hand, if $|\beta|$ is small, a large value of α is desirable. It is interesting to note that, for large α, the correlation coefficient is relatively insensitive to $|\beta|$ (e.g., see curve with $\alpha = 100$). This is reasonable since a large value of α implies that one component decays much more rapidly than the other. Hence, the correlation coefficient is more influenced by the relative decays as opposed to the relative oscillations. In the identification problem α and β are specified and the observation interval is finite. Since an infinite observation interval was assumed in obtaining the curves in Fig. 2, they serve as a lower bound on the correlation coefficient for the finite interval.

A second way of viewing $|\rho_{12}(\infty)|$ is presented in Fig. 3 where the magnitude is plotted as a function of α with $|\beta|$ as a parameter. The conclusions arrived at from the previous figure are still valid.

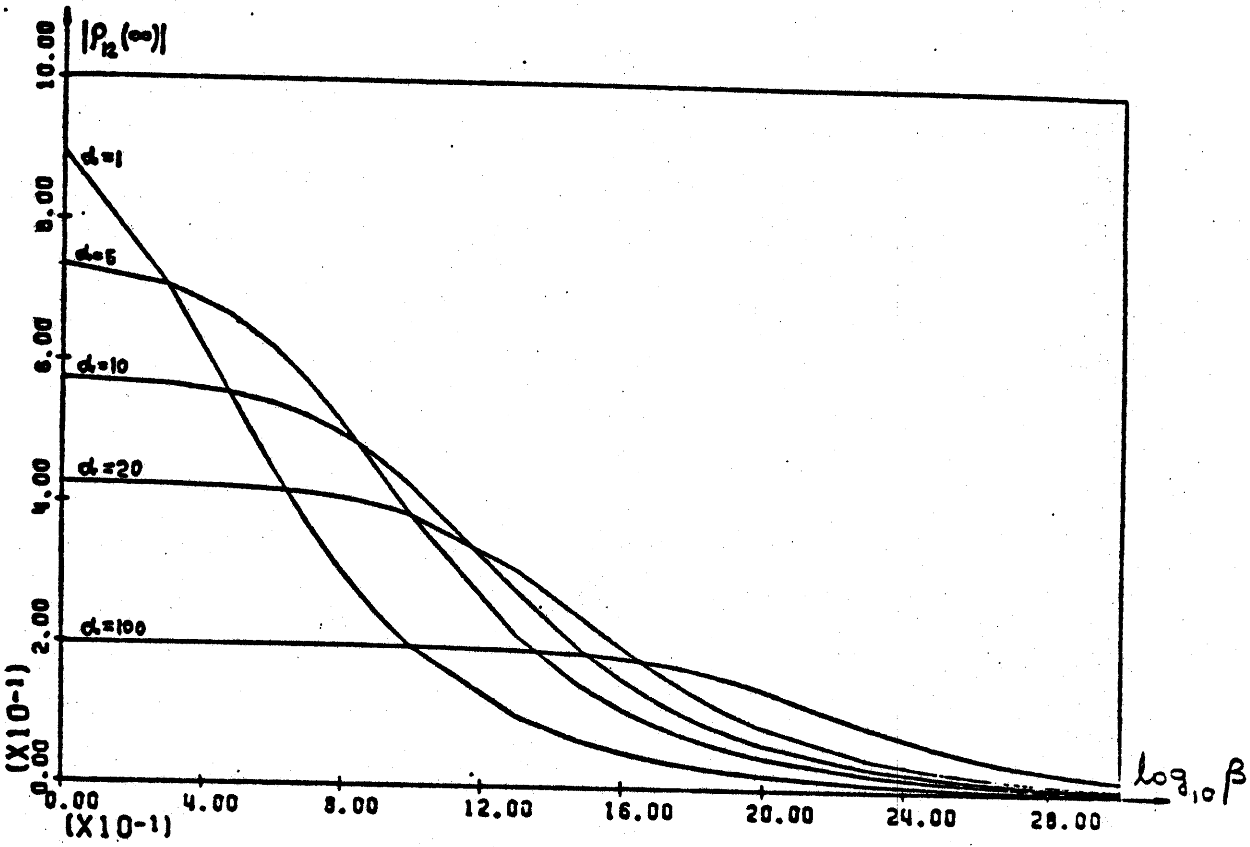

Fig. 2. $|\rho_{12}(\infty)|$ as a function of $|\beta|$, α as a parameter

However, Fig. 3 clearly shows that large values for both α and $|\beta|$ are preferable. Also, note the sequence of peaks at values of α_M predicted by (21). For large values of $|\beta|$, $\alpha_M \simeq |\beta|$. The maximum value of this correlation coefficient then arises when $|\sigma_2| \simeq |\omega_1 - \omega_2|$ where it is assumed $|\sigma_2| > |\sigma_1|$.

Still another way of presenting the results is to construct equal correlation magnitude contours as a function of α and $|\beta|$. Solution of (19) for $|\beta|$ results in

$$|\beta| = \sqrt{2\alpha \left(\frac{2}{|\rho_{12}(\infty)|^2} - 1\right) - (1+\alpha^2)}. \tag{31}$$

The contour plot is shown in Fig. 4 where the parameter is $|\rho_{12}(\infty)|$. This plot enables the user to determine the various combinations of α and $|\beta|$ which result in a given value of the correlation coefficient. The plot also allows one to determine the sensitivity of the signal parameters to small changes in the correlation coefficients. For example, assume $\sigma_1=3$ and $|\beta|=4$. For $|\rho_{12}(\infty)|=0.4$, the allowable value of is approximately 23, as read from Fig. 4. The corresponding value of σ_2 is -69. When $|\rho_{12}(\infty)|=0.5$, the allowable values for α are 1.3 and 13. This yields values for σ_2 of 3.9 and -39, respectively. In this case, the value of σ_2 is seen to be highly sensitive to changes in the correlation coefficient.

Our discussion thus far has dealt with the infinite interval. We now consider the effect of a finite observation. The windowing effect is accounted for in (18) by the second factor which is $W(\alpha,\beta,\gamma)$, and can be shown to approach unity for all possible choices of α and β, as $|\gamma| \to \infty$. Obviously, the effect of windowing is negligible when $W(\alpha,\beta,\gamma) \simeq 1$. Since $\alpha \geq 1$ and $\gamma<0$, $W(\alpha,\beta,\gamma)$ is bounded by

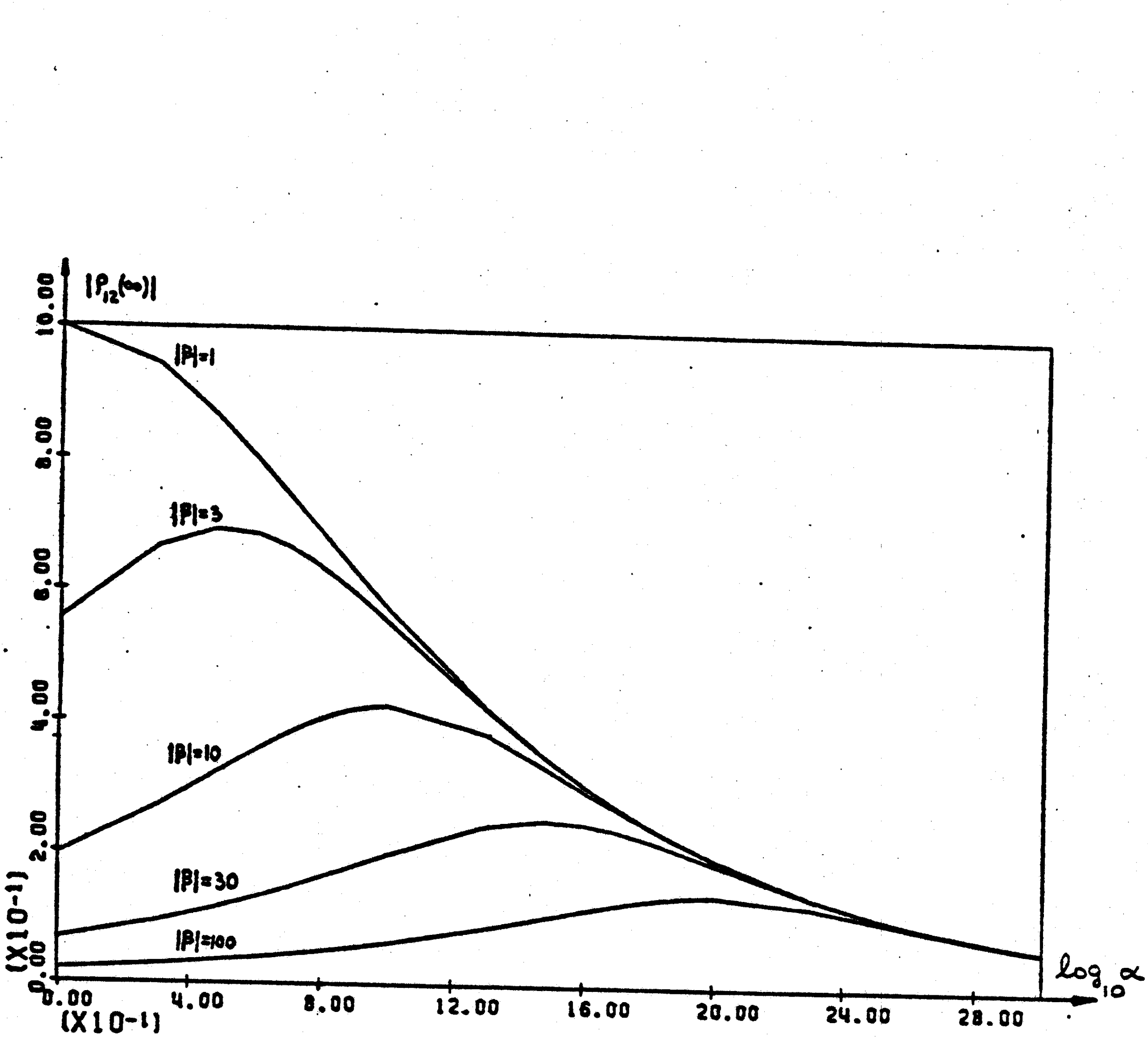

Fig. 3. $|\rho_{12}(\infty)|$ as a function of α, with $|\beta|$ as a parameter.

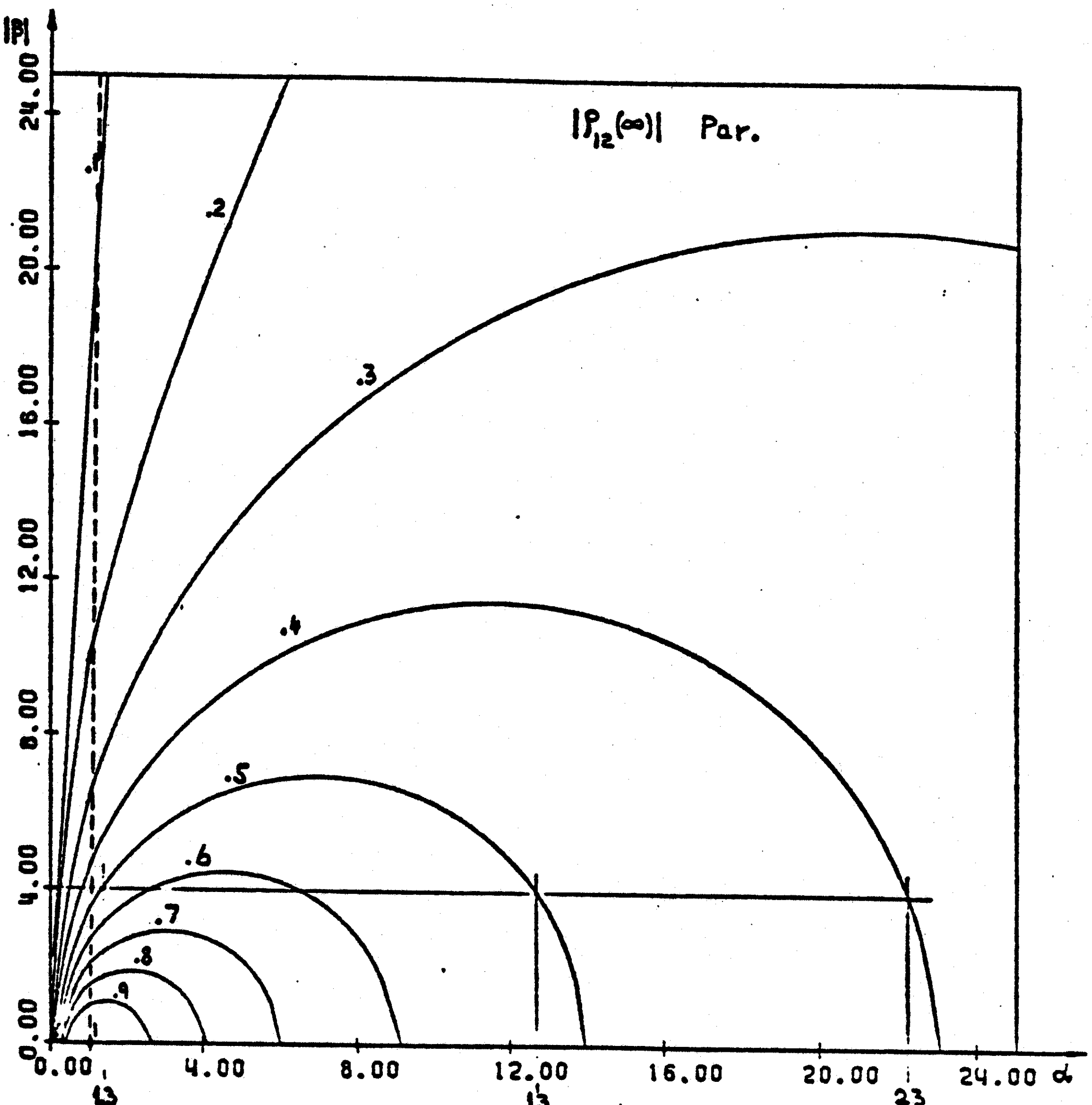

Fig. 4. Equal correlation magnitude contours in the α, |β| plane. α > 1 is assumed in the text.

$$\sqrt{1-e^{2\gamma}} < W(\alpha,\beta,\gamma) < \frac{1+e^{2\gamma}}{1-e^{2\gamma}} \qquad (32)$$

Observe that the bounds are independent of α and β. The inequality in (32) can be used to obtain an estimate for the minimum length of the observation interval in order that the correlation between the two components be approximately the same as for the infinite interval. (In general, windowing tends to increase the correlation). From (32) the following table is obtained.

Table 1. Lower and Upper Bounds on $W(\alpha,\beta,\gamma)$.

γ	$\sqrt{1-e^{2\gamma}}$	$\frac{1+e^{2\gamma}}{1-e^{2\gamma}}$
-1	.930	1.313
-2	.991	1.037
-3	.999	1.005

It is concluded from Table 1 that the record length can be assumed to be infinite, as far as $W(\alpha,\beta,\gamma)$ is concerned, provided $\gamma < -2$. In other words, when the length of the observation interval is such that

$$\Delta \geq \frac{-2}{\sigma_1} \quad \text{where } \sigma_i < \sigma_1 < 0;\ i = 2,3,\ldots, \qquad (33)$$

then $|\rho_{12}(\Delta)| \simeq |\rho_{12}(\infty)|$. Since a finite interval tends to increase the correlation coefficient (see Figs. 5, 6), it is desirable that the inequality in (33) be satisfied.

The behavior of $|\rho_{12}(\Delta)|$ as a function of $|\gamma|$ is illustrated in Figures 5 and 6. In Figure 5, α is constrained to be unity as the parameter $|\beta|$ is varied from 1 to 100. Note that the curves have settled down to their asymptotic behavior for $|\gamma| > 2$. It is interesting

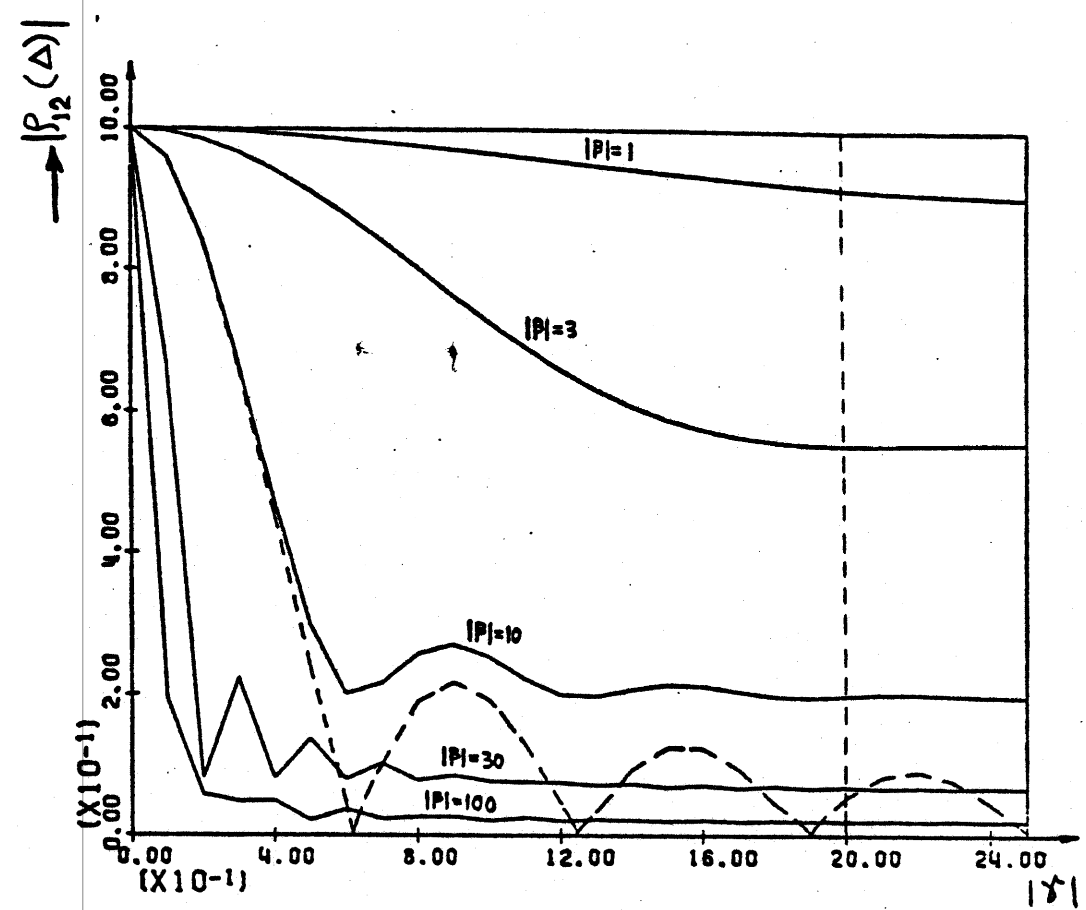

Fig. 5. $|\rho_{12}(\Delta)|$ vs. $|\gamma|$ with $\alpha = 1$ and $|\beta|$ as a parameter.

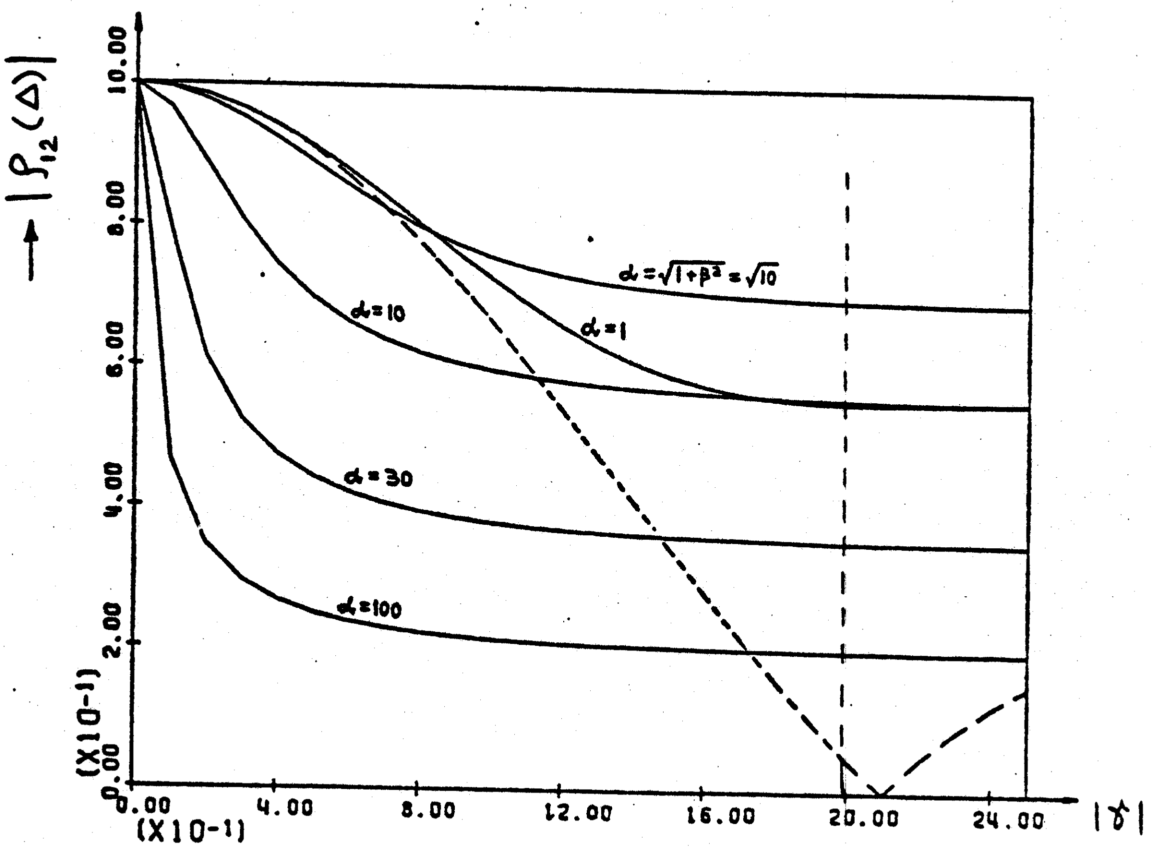

Fig.6. $|\rho_{12}(\Delta)|$ vs. $|\gamma|$ with $|\beta|$ = 3 and α as a parameter.

to compare the damped case (i.e., $\sigma_1 \neq 0$) to the sinusoidal case (i.e., $\sigma_1 = 0$). For $\sigma_1 = -1$, $|\beta| = |\omega_1 - \omega_2|$. The dashed line in Fig. 5 corresponds to the sinusoidal case where $|\omega_1 - \omega_2| = 10$. This serves as a reference for the damped case where $|\beta| = 10$ and $\sigma_1 = -1$. In Figure 6, $|\beta|$ is constrained to equal the value 3 as the parameter α is varied from 1 to 100. Again asymptotic values have been reached for $|\gamma| > 2$. The dashed curve in Fig. 6 represents the sinusoidal case with $|\omega_1 - \omega_2| = 3$. Since $|\beta| = 3$, each curve in the figure may be compared with the sinusoidal case provided σ_1 is assumed equal to -1. for $\alpha > 1$, it is interesting to note that there exist values of $|\gamma|$ where the correlation coefficient is smaller than that of the sinusoidal case. The only exception is for $\alpha = 1$. Notice that the largest asymptote occurs for $\alpha = \sqrt{1+\beta^2} = \sqrt{10}$, as predicted by (21).

3. Orthogonality Property of Complex Exponentials

In general, two complex exponentials as given by (1) are not orthogonal over any interval within $[0,\infty)$. In this section, we would like to investigate under what conditions two complex exponentials (which are orthogonal in the infinite interval) remain almost orthogonal in the finite time interval.

The Gram Schmidt orthogonalization procedure can generally be used to orthoganalize a set of functions. However, a simpler procedure for exponential functions was developed by Kautz [1] and applied in [2-3]. The orthonormalization is carried out over the semi-infinite interval and is based upon the Parseval relation for an inner product between two time functions. Specifically, the Parseval relation is

$$\int_{-\infty}^{+\infty} fg^* \, dt = \int_C F(s) \, G^*(-s) \, \frac{ds}{2\pi j} \tag{34}$$

It follows that the orthogonality in the time domain is equivalent to orthogonality in the frequency domain. If f and g are sums of complex exponentials, they will be orthogonal provided $F(s) \bullet G^*(-s)$ is a rational function which is analytic (i.e. has no poles) either in the left half or in the right half of the s-plane. Consider the set of exponentials

$$\{e^{s_i t}\}, \; i=1,2,\ldots,n,\ldots,m \quad t \geq 0, \; \sigma_i < 0 \tag{35}$$

and construct an orthonormal set where the n^{th} orthonormal basis function is given by

$$\Psi_n(s) = \frac{\sqrt{-2\sigma_n}}{s-s_n} \prod_{i=1}^{n-1} \frac{s+s_i^*}{s-s_i} \tag{36}$$

Eq. (36) can be interpreted in terms of passing the n^{th} exponential time function through an all-pass filter structured from the previous n-1 exponents. The all-pass filter interpretation points out that it is the phase which is responsible for orthogonality.

Thus far, orthogonality has been considered over a semi-infinite interval. Provided a finite interval is suitably long, orthogonality can still be approximated by this procedure. From (36), note that

$$\Psi_n(t) = L^{-1}\{\Psi_n(s)\} = \sum_{i=1}^{n} A_i e^{s_i t} \tag{37}$$

where L^{-1} is the inverse Laplace transform.

In the time domain orthonormality requires

$$\int_0^\infty \Psi_\ell(t)\Psi_q^*(t)\, dt = \begin{cases} 1; & \ell=q \\ 0; & \ell\neq q \end{cases} \tag{38}$$

Substitution of (37) into (38) yields

$$\int_0^\infty \Psi_\ell(t)\Psi_q^*(t)\, dt = \sum_{i=1}^{\ell} \sum_{k=1}^{q} A_i A_k^* \int_0^\infty e^{(s_i+s_k^*)t} dt$$

$$= \sum_{i=1}^{\ell} \sum_{k=1}^{q} -\frac{A_i A_k^*}{s_i+s_k^*} = \begin{cases} 1; & \ell=q \\ 0; & \ell\neq q \end{cases} \tag{39}$$

For a finite interval of length Δ, consider

$$\int_0^\Delta \Psi_\ell(t)\Psi_q^*(t)\, dt = \sum_{i=1}^{\ell} \sum_{k=1}^{q} A_i A_k^* \frac{e^{(s_i+s_k^*)\Delta} - 1}{s_i+s_k^*} \tag{40}$$

Clearly, (40) reduces to (39) provided

$$\left| e^{(s_i+s_k^*)\Delta} \right| << 1 \tag{41}$$

Assume $|\sigma_1| < |\sigma_2|$ for $i = 2,3,\ldots,n$. Then (41) can be replaced by the familiar inequality

$$\left| e^{(s_i+s_k^*)\Delta} \right| \leq e^{2\sigma_1\Delta} = e^{-2\gamma} << 1 \tag{42}$$

Provided the interval length is chosen such that (42) is satisfied, the orthonormal basis generated by (36) is very close to being orthonormal over the finite interval. Interestingly enough, $|\gamma| \leq 2$, which was the condition for $|\rho_{12}(\Delta)| \simeq |\rho_{12}(\infty)|$, also satisfies the orthogonality condition of (42). Thus

$$|\gamma| \geq 2 \tag{43}$$

is the condition to be satisfied if a finite interval is to be considered as though it were an infinite interval.

4. Conclusion

When $|\gamma| \geq 2$ or the record length is greater than $\frac{-2}{\sigma_1}$ (where σ_1 is the real part of the most slowly decaying exponential) then the finite record length may be considered as though it were infinite and, in addition, a set of nearly orthogonal basis can be generated over the finite interval.

5. Acknowledgement

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6. References

[1] W.H. Kautz, "Approximation over a Semi-Infinite Interval," M.S. Thesis, M.I.T., 1949

[2] ____, "Transient Synthesis in the Time Domain," IRE Trans. on Circuit Theory," CT-1, 3, Sept., 1954, pp. 29-39.

[3] W.H. Huggins, "Representation and Analysis of Signals - Part I. The Use of Orthogonalized Exponentials," Dept. of Electrical Engineering, The Johns Hopkins University (AFCRC Report TR-57-357), Sept. 30, 1957.