Mathematics Notes

Note 70

April 1, 1981

Numerical Evaluation of Jacobian Elliptic Functions, Elliptic Integrals of All Three Kinds and the Jacobi Zeta Function

D.V. Giri
T.L. Brown
LuTech, Inc., Berkeley, CA

and

J.P. Martinez
The Dikewood Corporation
Albuquerque, NM

Abstract

The chief goal of a numerical computation is to arrive at accurate numbers and hence reliable computer routines will always be in demand. In this note, we report a family of computer programs which will be useful in evaluating (1) Jacobian elliptic functions, (2) complete and incomplete elliptic integrals of the first and second kind, (3) Jacobi zeta function, and (4) complete and incomplete elliptic integral of the third kind. Wherever applicable, we will let the apposite arguments of the above functions and integrals take on both real and complex values. Finally, if there exists a belief that calculations involving elliptic functions and integrals are difficult, it is hoped that this note will disprove it.

Acknowledgement

We are thankful to Dr. Carl E. Baum for his suggestions and also for his editorial assistance.

Let us then, be up and doing
With a heart for any fate
Still achieving, still pursuing,
Learn to labour and to wait.

Longfellow

(A Psalm of Life)

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

I. Introduction

This note addresses itself to the problem of numerically evaluating the elliptic functions and integrals. Elliptic functions and integrals arise in several disciplines of applied science, e.g., mutual impedance of two current carrying loops [1, 2], conformal transformation in certain electromagnetic studies [3], an ellipsoid in a gravitational field, ellipsoid in an electromagnetic field [2, 4], and generally speaking, in any problem where ellipsoidal coordinates are suggested. Hence, it is considered useful to prepare and report a package of computer routines which can evaluate all of the elliptic functions and integrals. Although part of this work, namely the evaluation of complete and incomplete elliptic integrals of the first two kinds has already been documented [5] by one of the authors, the listings of the corresponding subroutines have been included in this note. It is noted that some of the subroutines in Table 1.1 may be redundant, e.g., CJEFS which contains JEFS can also be made to do the function of JEFS, but the subroutines are separated in order that each one of the subroutines listed in column 1 of Table 1.1 may be used as a general purpose subroutine in itself. In the following sections, we discuss each of these subroutines and include their listings. An attempt is made to make every section a self-contained unit for the user's convenience. Of course, in using a subroutine of any particular section, it is to be remembered that, whenever applicable, subroutines of earlier section(s) may have to be supplemented.

Table 1.1. Summary of FORTRAN Subroutines Reported in this Note

Subroutine	Quantities That Can Be Computed Along With Restrictions, If Any	Other Subroutines Required
JEFS	The Jacobian elliptic function trio $sn(u\|m)$ and $dn(u\|m)$ of real argument u and parameter m for $0 \le m \le 1$	None
CJEFS	The Jacobian elliptic function trio $sn(w\|m)$, $cn(w\|m)$ and $dn(w\|m)$ of complex argument w and parameter m for $0 \le m \le 1$	JEFS
TEK (Of Ref. 3)	The complete elliptic integrals of the first and second kinds $K(m)$ and $E(m)$ with $0 \le m < 1$	None
TEF (Of Ref. 3)	The incomplete elliptic integrals of the first and second kinds $F(\phi\|m)$, $E(\phi\|m)$ with $0 \le m < 1$ and ϕ real	TEK
CEF	The incomplete elliptic integral of the first and second kinds $F(u + iv\|m)$, $E(u + iv\|m)$ with $0 \le m < 1$	JEFS, CJEFS, TEK and TEF
ZETA	The Jacobi zeta function of a real amplitude $Z(\phi\|m)$ with $0 \le m < 1$	TEK
CZETA	The Jacobi zeta function of a complex argument $Z(u + iv\|m)$ with $0 \le m < 1$	CJEFS, JEFS, TEF and TEK
EI3K	Elliptic integral of the third kind for real ϕ, $\Pi(n; \phi\|m)$ with $0 \le m < 1$	TEF, TEK, E3MLN and E3NLM
CEI3K	Elliptic integral of the third kind for complex w, $\Pi(n; w\|m)$ and real n and m with $0 \le m < 1$	JEFS, CJEFS, TEK, TEF & CEF

II. Jacobian Elliptic Functions

An excellent introduction to all the Jacobian functions given by pq(u/m) where p = s, c, d, n; q = s, c, d, n and m ≡ the parameter; satisfying

$$\left.\begin{array}{ll}(1) & pq(u/m) = 1/[qp(u/m)] \\ (2) & pp(u/m) = 1 \\ (3) & pr(u/m) = \dfrac{pq(u/m)}{rq(u/m)}\ ;\ \text{with } r = s, c, d, n\end{array}\right\} \quad (2.1)$$

may be found in Chapter 16 of Reference [6]. In Eq. (2.1), the vertical stroke separates the argument u from the parameter m. Reference [6] is the basic reference for this entire note and we will have several occasions to refer to it. However, the readers interested in the theoretical aspects of Jacobian functions are referred to the classical work of Neville [7].

Before we proceed to compute the twelve Jacobian elliptic functions, it is noted that nine of them can be related to the trio sn(u), cn(u) and dn(u) according as

$$cd(u) = \frac{cn(u)}{dn(u)}\ ,\ dc(u) = \frac{dn(u)}{cn(u)}\ ,\ ns(u) = \frac{1}{sn(u)}$$

$$sd(u) = \frac{sn(u)}{dn(u)}\ ,\ nc(u) = \frac{1}{cn(u)}\ ,\ ds(u) = \frac{dn(u)}{sn(u)}$$

$$nd(u) = \frac{1}{dn(u)}\ ,\ sc(u) = \frac{sn(u)}{cn(u)}\ ,\ cs(u) = \frac{cn(u)}{sn(u)} \quad (2.2)$$

In Eq. (2.2), the parameter m is implicitly present and we shall write it only when it is required to call specific attention to the parameters. In view of Eq. (2.2), it is sufficient to compute the trio sn, cn and dn of a real argument u and a real parameter m which satisfies the condition

$$0 \le m \le 1 \quad (2.3)$$

If m is outside of the above range, special formulas are available in Sections 16.10 and 16.11 of Reference [6] which are useful in computing the trio. Returning to the situation when Eq. (2.3) is satisfied, the trio has series representations given by

$$
\left.\begin{aligned}
\mathrm{sn}(u) &= u - (1+m)\frac{u^3}{3!} + (1 + 14\,m + m^2)\frac{u^5}{5!} - \cdots \\
\mathrm{cn}(u) &= 1 - \frac{u^2}{2!} + (1 + 4\,m)\frac{u^4}{4!} - (1 + 44\,m + 16\,m^2)\frac{u^6}{6!} + \cdots \\
\mathrm{dn}(u) &= 1 - m\frac{u^2}{2!} + m(m+4)\frac{u^4}{4!} - m(m^2 + 44\,m + 16)\frac{u^6}{6!} + \cdots
\end{aligned}\right\} \quad (2.4)
$$

It is interesting to note that no formulae are known for the general coefficients in all of the above series. Conceivably, one can compute the trio from Eq. (2.4) by using an efficient numerical procedure, e.g., Horner's algorithm which attempts to minimize the round-off errors. However, a far superior method of computing the trio makes use of the Arithmetic-Geometric Mean [5] (A.G.M.) and will be briefly described below.

Starting with a number triple (a_0, b_0, c_0), we compute successively (a_1, b_1, c_1), (a_2, b_2, c_2) ..... (a_N, b_N, c_N) according to the A.G.M. scheme

$$
\begin{array}{lll}
a_0 & b_0 & c_0 = \frac{1}{2}\left(a_0 - b_0\right) \\
a_1 = \frac{1}{2}\left(a_0 + b_0\right) & b_1 = \sqrt{a_0\, b_0} & c_1 = \frac{1}{2}\left(a_0 - b_0\right) \\
a_2 = \frac{1}{2}\left(a_1 + b_1\right) & b_2 = \sqrt{a_1\, b_1} & c_2 = \frac{1}{2}\left(a_1 - b_1\right) \\
\vdots & \vdots & \vdots \\
a_N = \frac{1}{2}\left(a_{N-1} + b_{N-1}\right) & b_N = \sqrt{a_{N-1}\, b_{N-1}} & c_N = \frac{1}{2}\left(a_{N-1} - b_{N-1}\right)
\end{array} \quad (2.5)
$$

The process of determining the two kinds of means stops at the Nth step when $a_N = b_N$ and consequently, $c_N = 0$ to a preassigned degree of accuracy.

To compute $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$ and $\mathrm{dn}(u|m)$ one starts with the triple

$$
a_0 = 1\,,\; b_0 = \sqrt{m_1}\,,\; c_0 = \sqrt{m} \quad (2.6)
$$

where m_1 = the complementary parameter = $(1 - m)$ and proceeds according to the A.G.M. scheme of Eq. (2.5) up to the Nth step. Now, ϕ_N is computed in degrees using

$$\phi_N = 2^N a_N u \, 180°/\pi \tag{2.7}$$

Once ϕ_N is known, then $\phi_{N-1}, \phi_{N-2} \cdots\cdots \phi_0$ are successively computed using the recurrence relation

$$\sin\left(2\phi_{n-1} - \phi_n\right) = \frac{c_n}{a_n} \sin\phi_n \tag{2.8}$$

or

$$\phi_{n-1} = \frac{1}{2}\left[\phi_n + \text{arc} \sin\left\{\frac{c_n}{a_n} \sin\phi_n\right\}\right] \text{ for } n = N, (N - 1), \cdots\cdots 3, 2, 1 \tag{2.9}$$

The trio can now be evaluated using ϕ_1 and ϕ_0 according to

$$sn(u|m) = \sin\phi_0$$

$$cn(u|m) = \cos\phi_0$$

$$dn(u|m) = \frac{\cos\phi_0}{\cos\left(\phi_1 - \phi_0\right)} \tag{2.10}$$

The subroutine JEFS, a listing of which is included at the end of this section computes the functions sn, cn, dn of real argument u and the parameter m. It can be used in conjunction with the familiar Fortran call statement

```
CALL JEFS (U,EM,SN,CN,DN)
```

The input and output variables of the subroutine are self-explanatory except probably for EM which is the parameter m. In this program, the process of computing the means terminates when $|C_N| < 10^{-10}$ or if N = 200 whichever occurs first. If more accuracy is desired, the DIMENSION statement may have to be modified. A sample output of this subroutine is tabulated (Table 2.1) and plotted (Figure 2.1) and the results agree very well with the tables in Reference [8].

Table 2.1. Sample Output of Subroutine JEFS for Three Values of Parameter m and Argument u Ranging from $0 \le u \le 5.00$

m / u	0.3			0.6			0.9		
	sn u	cn u	dn u	sn u	cn u	dn u	sn u	cn u	dn u
0.00	0.00000	1.00000	1.00000	0.00000	1.00000	1.00000	0.00000	1.00000	1.00000
0.25	0.24666	0.96910	0.99083	0.24591	0.96929	0.98169	0.24517	0.96948	0.97258
0.50	0.47422	0.88041	0.96568	0.46902	0.88319	0.93167	0.46384	0.88592	0.89798
0.75	0.66780	0.74434	0.93071	0.65386	0.75662	0.86225	0.63984	0.76851	0.79470
1.00	0.81877	0.57412	0.89380	0.79494	0.60669	0.78794	0.77009	0.63794	0.68284
1.25	0.92408	0.38220	0.86245	0.89448	0.44710	0.72107	0.86051	0.50943	0.57755
1.50	0.98396	0.17840	0.84235	0.95824	0.28597	0.67012	0.92037	0.39104	0.48747
1.75	0.99954	-0.03021	0.83682	0.99198	0.12638	0.63999	0.95847	0.28520	0.41618
2.00	0.97126	-0.23804	0.84676	0.99949	-0.03190	0.63294	0.98162	0.19087	0.36440
2.25	0.89837	-0.43924	0.87056	0.98168	-0.19054	0.64945	0.99445	0.10524	0.33161
2.50	0.77980	-0.62603	0.90420	0.93642	-0.35087	0.68838	0.99969	0.02471	0.31710
2.75	0.61599	-0.78775	0.94136	0.85898	-0.51200	0.74652	0.99851	-0.05458	0.32044
3.00	0.41142	-0.91144	0.97428	0.74327	-0.66899	0.81764	0.99063	-0.12657	0.34174
3.25	0.17657	-0.98429	0.99531	0.58446	-0.81143	0.89165	0.97434	-0.22507	0.38156
3.50	-0.07214	-0.99739	0.99922	0.38296	-0.92376	0.95499	0.94624	-0.32346	0.44064
3.75	-0.31512	-0.94905	0.98499	0.14826	-0.98895	0.99338	0.90090	-0.43403	0.51918
4.00	-0.53413	-0.84540	0.95625	-0.10059	-0.99493	0.99696	0.83068	-0.55676	0.61561
4.25	-0.71601	-0.69809	0.91989	-0.33976	-0.94051	0.96475	0.72632	-0.68735	0.72471
4.50	-0.85391	-0.52042	0.88388	-0.54873	-0.83600	0.90517	0.57925	-0.81515	0.83548
4.75	-0.94599	-0.32419	0.85530	-0.71605	-0.69805	0.83208	0.38621	-0.92241	0.93046
5.00	-0.99297	-0.11837	0.83917	-0.83981	-0.54288	0.75949	0.15499	-0.98792	0.98913

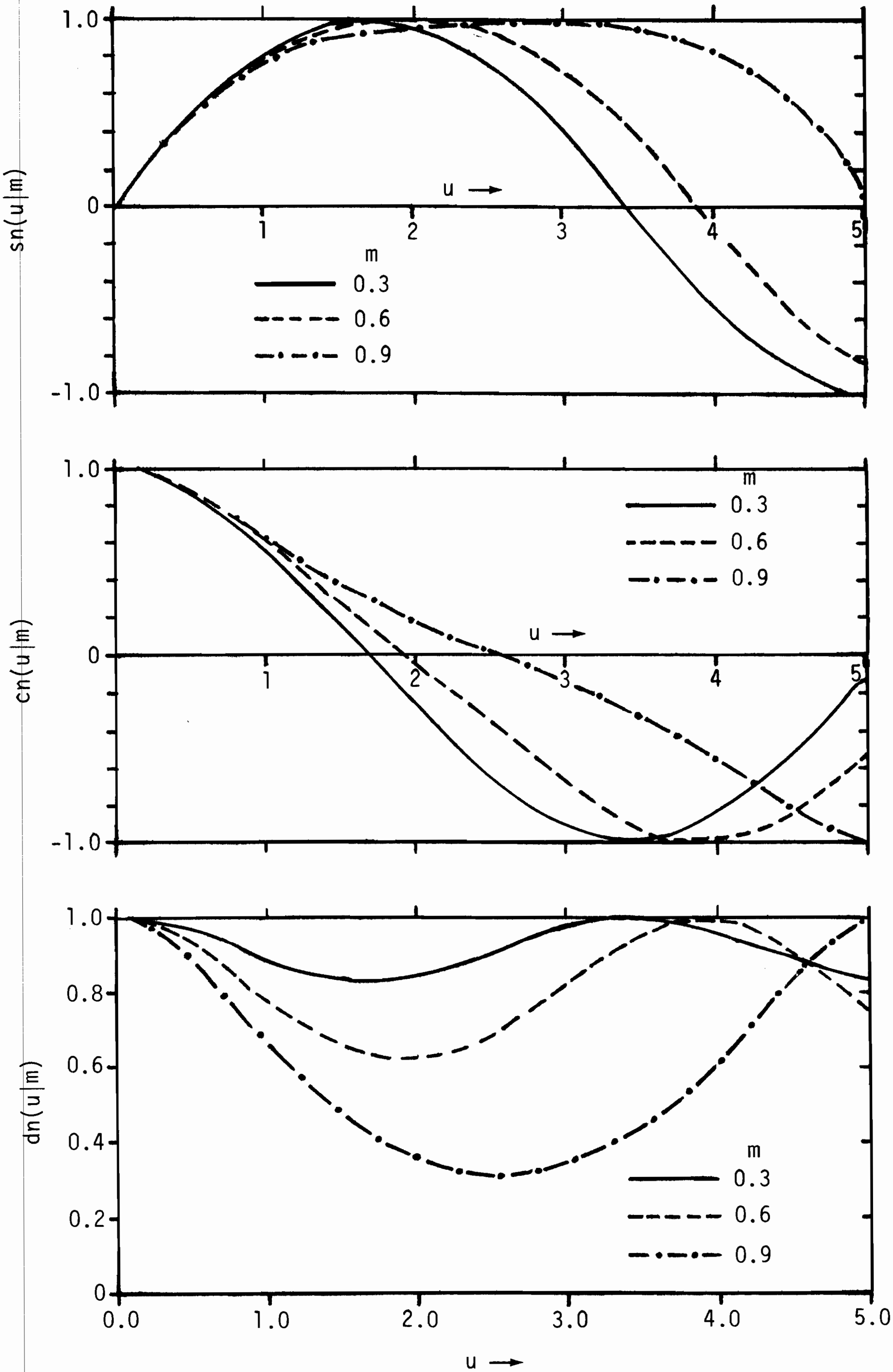

Figure 2.1. Plot of Trio sn(u|m), cn(u|m), and dn(u|m) as a Function of u for Three Values of m (See Table 2.1.)

Thus far in this section, the argument u has been a real variable and we will now proceed to compute the trio sn(w|m), cn(w|m) and dn(w|m) where

$$w = \text{complex variable} = u + iv \tag{2.11}$$

Once again, this is easily accomplished by using Eqs. (16.21) of Reference [4]. Using the short hand notation

$$\begin{aligned} s &= sn(u|m) \;, \quad c = cn(u|m) \;, \quad d = dn(u|m) \\ s_1 &= sn\left(v|m_1\right) \;, \quad c_1 = cn\left(v|m_1\right) \;, \quad d_1 = dn\left(v|m_1\right) \end{aligned} \tag{2.12}$$

one can write down the Jacobian functions as [6],

$$\begin{aligned} sn(u + iv|m) &= \left[s\ d_1 + i\ c\ d\ s_1\ c_1\right] \Big/ \left(c_1^2 + m\ s^2\ s_1^2\right) \\ cn(u + iv|m) &= \left[c\ c_1 - i\ s\ d\ s_1\ d_1\right] \Big/ \left(c_1^2 + m\ s^2\ s_1^2\right) \\ dn(u + iv|m) &= \left[d\ c_1\ d_1 - i\ m\ s\ c\ s_1\right] \Big/ \left(c_1^2 + m\ s^2\ s_1^2\right) \end{aligned} \tag{2.13}$$

Thus the computation for a complex argument reduces to making use of JEFS on the real and imaginary parts separately but with the parameter m and the complementary parameter m_1 respectively. Subroutine CJEFS was written for this purpose and may be called by the Fortran statement

```
CALL CJEFS (C,EM,SN,CN,DN)
```

The input variables C and EM are respectively w = u + iv and m and CJEFS returns the complex numbers SN, CN, DN which respectively are sn(w|m), cn(w|m) and dn(w|m). A sample output of CJEFS and a listing are included. With

$$w = u + iv = Me^{i\theta} \tag{2.14}$$

the complex trio sn, cn and $dn\left(Me^{i\theta}\right)$ are computed for five different values of M = 0.5, 1.0, 1.5, 2.0 and 2.5. For each value of M, θ is varied from 0° to 360° in steps of 20°. Figure 2.2 illustrates the complex w plane, and the dotted grid points in the figure are all of the locations where sn(w|m), cn(w|m) and dn(w|m) are computed and tabulated for three values of m = 0.3, 0.6 and 0.9 (Tables 2.2 through 2.10, respectively).

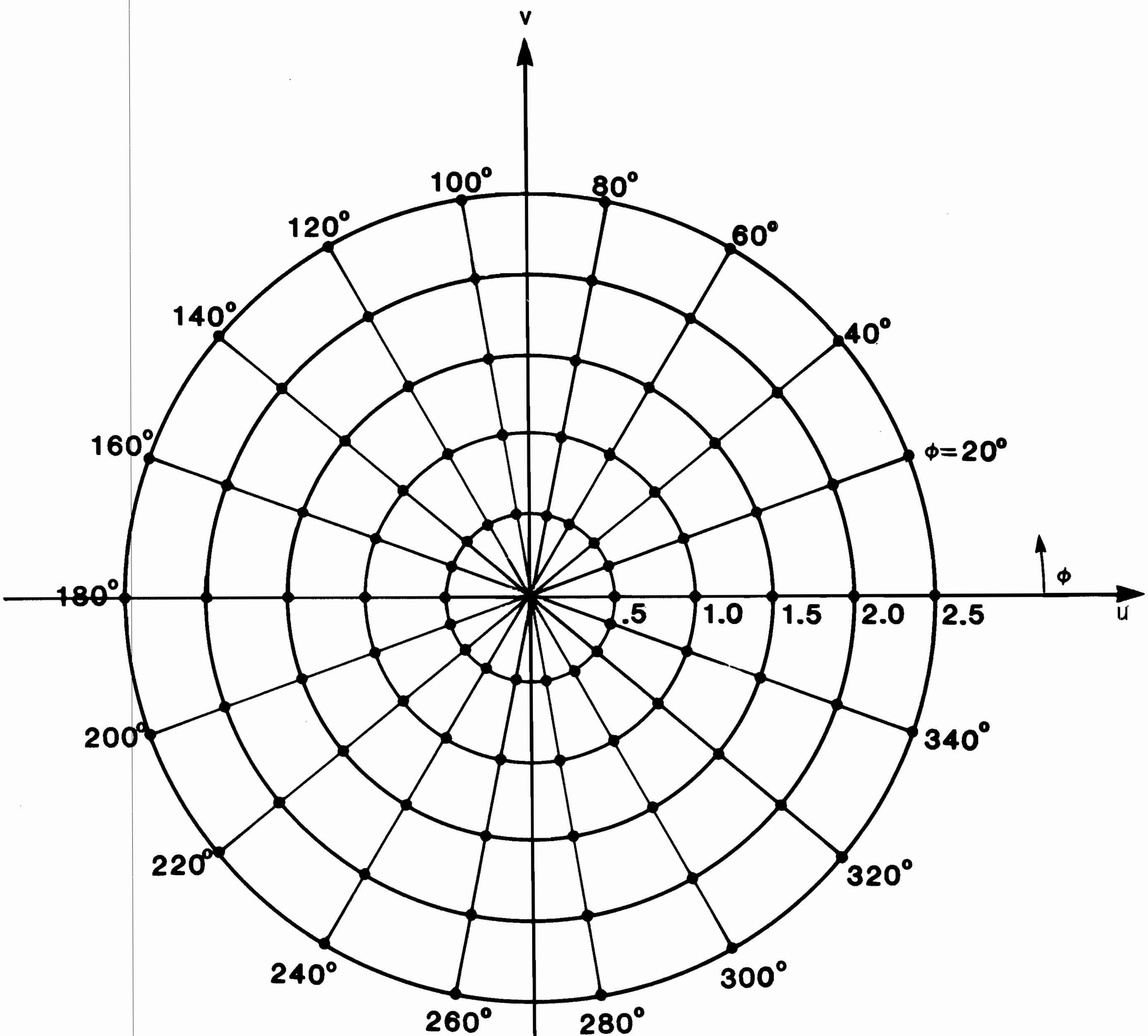

Figure 2.2. Grid Location in the Complex w-Plane at Which the Elliptic Functions and Elliptic Integrals are Calculated.

Table 2.2. $\mathrm{sn}\left(Me^{i\theta}\middle|m\right)$ as Computed from Subroutine CJEFS for m = 0.3

θ°	$\mathrm{sn}\left(0.5\ e^{i\theta}\middle|m\right)$		$\mathrm{sn}\left(1.0\ e^{i\theta}\middle|m\right)$		$\mathrm{sn}\left(1.5\ e^{i\theta}\middle|m\right)$		$\mathrm{sn}\left(2.0\ e^{i\theta}\middle|m\right)$		$\mathrm{sn}\left(2.5\ e^{i\theta}\middle|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.47422	0.00000	0.81877	0.00000	0.98396	0.00000	0.97126	0.00000	0.77980	0.00000
20.00	0.45612	0.14886	0.82980	0.91921	1.06102	0.10777	1.15197	-0.07548	1.13846	-0.37667
40.00	0.39526	0.39755	0.83337	0.45006	1.23973	0.36215	1.50923	0.11747	1.71154	-0.10454
60.00	0.27773	0.43175	0.73073	0.81923	1.46426	0.90860	2.03167	0.41543	1.96750	-0.06274
80.00	0.10150	0.51677	0.32888	1.20411	1.21929	2.32980	4.94852	1.42284	2.51616	-1.93510
100.00	-0.10150	0.51677	-0.32888	1.20411	-1.21929	2.32980	-4.94852	1.42284	-2.51616	-1.93510
120.00	-0.27773	0.43175	-0.73073	0.81923	-1.46426	0.90860	-2.03167	0.41543	-1.96750	-0.06274
140.00	-0.39526	0.29755	-0.83337	0.45006	-1.23973	0.36215	-1.50923	0.11747	-1.71154	-0.10454
160.00	-0.45612	0.14886	-0.82980	0.19121	-1.06102	0.10777	-1.15197	-0.07548	-1.13846	-0.37667
180.00	-0.47422	0.00000	-0.81877	0.00000	-0.98396	0.00000	-0.97126	-0.00000	-0.77980	-0.00000
200.00	-0.45612	-0.14886	-0.82980	-0.19121	-1.06102	-0.10777	-1.15197	-0.07548	-1.13846	0.37667
220.00	-0.39526	-0.29755	-0.83337	-0.45006	-1.23973	-0.36215	-1.50923	-0.11747	-1.71154	0.10454
240.00	-0.27773	-0.43175	-0.73073	-0.81923	-1.46426	-0.90860	-2.03167	-0.41543	-1.96750	0.06274
260.00	-0.10150	-0.51677	-0.32888	-1.20411	-1.21929	-2.32980	-4.94852	-1.42284	-2.51616	1.93510
280.00	0.10150	-0.51677	0.32888	-1.20411	1.21929	-2.32980	4.94852	-1.42284	2.41616	1.93510
300.00	0.27773	-0.43175	0.73073	-0.81923	1.46426	-0.90860	2.03167	-0.41543	1.96750	0.06274
320.00	0.39526	-0.29755	0.83337	-0.45006	1.23973	-0.36215	1.50923	-0.11747	1.71154	0.10454
340.00	0.45612	-0.14886	0.82980	-0.19121	1.06102	-0.10777	1.15197	0.07548	1.13846	0.37667
360.00	0.47422	-0.00000	0.81877	-0.00000	0.98396	-0.00000	0.97126	0.00000	0.77980	0.00000

Table 2.3. $\mathrm{sn}\left(Me^{i\theta}\middle|m\right)$ as Computed from Subroutine CJEFS for m = 0.6

θ°	$\mathrm{sn}\left(0.5\ e^{i\theta}\middle\|m\right)$		$\mathrm{sn}\left(1.0\ e^{i\theta}\middle\|m\right)$		$\mathrm{sn}\left(1.5\ e^{i\theta}\middle\|m\right)$		$\mathrm{sn}\left(2.0\ e^{i\theta}\middle\|m\right)$		$\mathrm{sn}\left(2.5\ e^{i\theta}\middle\|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.46902	0.00000	0.79494	0.00000	0.95824	0.00000	0.99949	0.00000	0.93642	0.00000
20.00	0.45288	0.14452	0.80569	0.17555	1.00453	0.11232	1.08556	0.01611	1.12008	-0.10958
40.00	0.39728	0.29186	0.82559	0.40748	1.14354	0.28425	1.25893	0.09938	1.28385	0.00297
60.00	0.28449	0.43064	0.78555	0.77598	1.48178	0.62464	1.62140	0.04428	1.35478	-0.17450
80.00	0.10565	0.52297	0.40508	1.27515	2.21823	2.35038	2.96593	-1.50910	1.00967	-1.11092
100.00	-0.10565	0.52297	-0.40508	1.27515	-2.21823	2.35038	-2.96593	-1.50910	-1.00967	-1.11092
120.00	-0.28449	0.43064	-0.78555	0.77598	-1.48178	0.62464	-1.62140	0.04428	-1.35478	-0.17450
140.00	-0.39728	0.29186	-0.82559	0.40748	-1.14354	0.28425	-1.25893	0.09938	-1.28385	0.00297
160.00	-0.45288	0.14452	-0.80569	0.17555	-1.00453	0.11232	-1.08556	0.01611	-1.12008	-0.10958
180.00	-0.46902	0.00000	-0.79494	0.00000	-0.95824	0.00000	-0.99949	0.00000	-0.93642	-0.00000
200.00	-0.45288	-0.14452	-0.80569	-0.17555	-1.00453	-0.11232	-1.08556	-0.01611	-1.12008	-0.10958
220.00	-0.39728	-0.29186	-0.82559	-0.40748	-1.14354	-0.28425	-1.25893	-0.09938	-1.28385	-0.00297
240.00	-0.28449	-0.43064	-0.78555	-0.77598	-1.48178	-0.62464	-1.62140	-0.04428	-1.35478	0.17450
260.00	-0.10565	-0.52297	-0.40508	-1.27515	-2.21823	-2.35038	-2.96593	1.50910	-1.00967	1.11092
280.00	0.10565	-0.52297	0.40508	-1.27515	2.21823	-2.35038	2.96593	1.50910	1.00967	1.11092
300.00	0.28449	-0.43064	0.78555	-0.77598	1.48178	-0.62464	1.62140	-0.04428	1.35478	0.17450
320.00	0.39728	-0.29186	0.82559	-0.40748	1.14354	-0.28425	1.25893	-0.99938	1.28385	-0.00297
340.00	0.45288	-0.14452	0.80569	-0.17555	1.00453	-0.11232	1.08556	-0.01611	1.12008	0.10958
360.00	0.46902	-0.00000	0.79494	-0.00000	0.95824	-0.00000	0.99949	-0.00000	0.93642	0.00000

Table 2.4. $\mathrm{sn}\left(Me^{i\theta}\middle|m\right)$ as Computed from Subroutine CJEFS for m = 0.9

θ°	$\mathrm{sn}\left(0.5\ e^{i\theta}\middle\|m\right)$		$\mathrm{sn}\left(1.0\ e^{i\theta}\middle\|m\right)$		$\mathrm{sn}\left(1.5\ e^{i\theta}\middle\|m\right)$		$\mathrm{sn}\left(2.0\ e^{i\theta}\middle\|m\right)$		$\mathrm{sn}\left(2.5\ e^{i\theta}\middle\|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.46384	0.00000	0.77009	0.00000	0.92037	0.00000	0.98162	0.00000	0.99969	0.00000
20.00	0.44965	0.14020	0.78263	0.15890	0.95187	0.10210	0.01242	0.04726	1.02945	0.01244
40.00	0.39926	0.28618	0.81829	0.36768	1.07251	0.21954	1.11316	0.06594	1.08100	0.00051
60.00	0.29124	0.42945	0.83486	0.72981	1.44244	0.40516	1.35460	-0.09082	1.10255	-0.17032
80.00	0.10987	0.52922	0.49399	1.34653	3.27453	1.61032	1.59281	-1.39954	0.66660	-0.69871
100.00	-0.10987	0.52922	-0.49399	1.34653	-3.27453	1.61032	-1.59281	-1.39954	-0.66660	-0.69871
120.00	-0.29124	0.42945	-0.83486	0.72981	-1.44244	0.40516	-1.35460	-0.09082	-1.10255	-0.17032
140.00	-0.39926	0.28618	-0.81829	0.36768	-1.07251	0.21954	-1.11316	0.06594	-1.08100	0.00051
160.00	-0.44965	0.14020	-0.78263	0.15890	-0.95187	0.10210	-1.01242	0.04726	-1.02945	0.01244
180.00	-0.46384	0.00000	-0.77009	0.00000	-0.92037	0.00000	-0.98162	0.00000	-0.99969	0.00000
200.00	-0.44965	-0.14020	-0.78263	-0.15890	-0.95187	-0.10210	-1.01242	-0.04726	-1.02945	-0.01244
220.00	-0.39926	-0.28618	-0.81829	-0.36768	-1.07251	-0.21954	-1.11316	-0.06594	-1.08100	-0.00051
240.00	-0.29124	-0.42945	-0.83486	-0.72981	-1.44244	-0.40516	-1.35460	0.09082	-1.10255	0.17032
260.00	-0.10987	-0.52922	-0.49399	-1.34653	-3.27453	-1.61032	-1.59281	1.39954	-0.66660	0.69871
280.00	0.10987	-0.52922	0.49399	-1.34653	3.27453	-1.61032	1.52981	1.39954	0.66660	0.69871
300.00	0.29124	-0.42945	0.83486	-0.72981	1.44244	-0.40516	1.35460	0.09082	1.10255	0.17032
320.00	0.39926	-0.28618	0.81829	-0.36768	1.07251	-0.21954	1.11316	-0.06594	1.08100	-0.00051
340.00	0.44965	-0.14020	0.78263	-0.15890	0.95187	-0.10210	1.01242	-0.04726	1.02945	-0.01244
360.00	0.46384	-0.00000	0.77009	-0.00000	0.92037	-0.00000	0.98162	-0.00000	0.99969	-0.00000

Table 2.5. $cn\left(Me^{i\theta}\middle|m\right)$ as Computed from Subroutine CJEFS for m = 0.3

θ°	$cn\left(0.5\ e^{i\theta}\middle\|m\right)$		$cn\left(1.0\ e^{i\theta}\middle\|m\right)$		$cn\left(1.5\ e^{i\theta}\middle\|m\right)$		$cn\left(2.0\ e^{i\theta}\middle\|m\right)$		$cn\left(2.5\ e^{i\theta}\middle\|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.88041	0.00000	0.57412	0.00000	0.17840	0.00000	-0.23804	0.00000	-0.62603	0.00000
20.00	0.90539	-0.07499	0.63990	-0.24795	0.26594	-0.42996	-0.14839	-0.58597	-0.59883	-0.71611
40.00	0.97309	-0.12086	0.84085	-0.44606	0.53832	-0.83401	0.15620	-1.13507	-0.12862	-1.39103
60.00	1.05929	-0.11320	1.18076	-0.50699	1.08658	-1.22441	0.47336	-1.78302	-0.07283	-1.69483
80.00	1.12202	-0.04675	1.55141	-0.25526	2.49709	-1.13761	1.45037	-4.85458	-2.03469	-2.39299
100.00	1.12202	0.04675	1.55141	0.25526	2.49709	1.12761	1.45037	4.85458	-2.03469	2.39299
120.00	1.05929	0.11320	1.18076	0.50699	1.08658	1.22441	0.47336	1.78302	-0.07283	1.69483
140.00	0.97309	0.12086	0.84085	0.44606	0.53832	0.83401	0.15620	1.13507	-0.12862	1.39103
160.00	0.90539	0.07499	0.64990	0.24795	0.26594	0.42996	-0.14839	0.58597	-0.59883	0.71611
180.00	0.88041	0.00000	0.57412	0.00000	0.17840	0.00000	-0.23804	0.00000	-0.62603	0.00000
200.00	0.90539	-0.07499	0.63990	-0.24795	0.26594	-0.42996	-0.14839	-0.58597	-0.59883	-0.71611
220.00	0.97309	-0.12086	0.84085	-0.44606	0.53832	-0.83401	0.15620	-1.13507	-0.12862	-1.39103
240.00	1.05929	-0.11320	1.18076	-0.50699	1.08658	-1.22441	0.47336	-1.78302	-0.07283	-1.69483
260.00	1.12202	-0.04675	1.55141	-0.25526	2.49709	-1.13761	1.45037	-4.85458	-2.03569	-2.39299
280.00	1.12202	0.04675	1.55141	0.25526	2.49709	1.13761	1.45037	4.85458	-2.03469	2.39299
300.00	1.05929	0.11320	1.18076	0.50699	1.08658	1.22441	0.47336	1.78302	-0.07283	1.69483
320.00	0.97309	0.12086	0.84085	0.44606	0.53832	0.83401	0.15620	1.13507	-0.12862	1.39103
340.00	0.90539	0.07499	0.63990	0.24795	0.26594	0.42996	-0.14839	-0.58597	0.59883	0.71611
360.00	0.88041	0.00000	0.57412	0.00000	0.17840	0.00000	-0.23804	0.00000	-0.62603	0.00000

Table 2.6. $\mathrm{cn}\left(Me^{i\theta} \middle| m\right)$ as Computed from Subroutine CJEFS for m = 0.6

θ°	$\mathrm{cn}\left(0.5\ e^{i\theta} \middle| m\right)$		$\mathrm{cn}\left(1.0\ e^{i\theta} \middle| m\right)$		$\mathrm{cn}\left(1.5\ e^{i\theta} \middle| m\right)$		$\mathrm{cn}\left(2.0\ e^{i^2} \middle| m\right)$		$\mathrm{cn}\left(2.5\ e^{i\theta} \middle| m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.88319	0.00000	0.60609	0.00000	0.28597	0.00000	-0.03190	0.00000	-0.35087	0.00000
20.00	0.90609	-0.07223	0.65450	-0.21610	0.33855	-0.33328	0.04124	-0.42413	-0.22642	-0.54206
40.00	0.97037	-0.11949	0.81041	-0.41511	0.48046	-0.67655	0.16137	-0.77528	0.00474	-0.80516
60.00	1.05733	-0.11587	1.12969	-0.53959	0.77888	-1.18834	0.05624	-1.27676	-0.25356	-0.93234
80.00	1.12461	-0.04913	1.60184	-0.32246	2.46513	-2.11497	-1.58069	-2.83160	-1.37219	-0.81743
100.00	1.12461	0.04913	1.60184	0.32246	2.46513	2.11497	-1.58069	2.83160	-1.37219	0.81743
120.00	1.05733	0.11587	1.12969	0.53959	0.77888	1.18834	0.05624	1.27676	-0.25356	0.93234
140.00	0.97037	0.11949	0.81041	0.41511	0.48046	0.67655	0.16137	0.77528	0.00474	0.80516
160.00	0.90609	0.07223	0.65450	0.21610	0.33855	0.33328	0.04124	0.42413	-0.22642	0.54206
180.00	0.88319	0.00000	0.60669	0.00000	0.28597	0.00000	-0.03190	0.00000	-0.35087	0.00000
200.00	0.90609	-0.07223	0.65450	-0.21610	0.33855	-0.33328	0.04124	-0.42413	-0.22642	-0.54206
220.00	0.08037	-0.11949	0.81041	-0.41511	0.48046	-0.67655	0.16137	-0.77528	0.00474	-0.80516
240.00	1.05733	-0.11587	1.12969	-0.53959	0.77888	-1.18834	0.05624	-1.27676	-0.25356	-0.93234
260.00	1.12461	-0.04913	1.60184	-0.32246	2.46513	-2.11497	-1.58069	-2.83160	-1.37219	-0.81743
280.00	1.12461	0.04913	1.60184	0.32246	2.46513	2.11497	-1.58069	2.83160	-1.37219	0.81743
300.00	1.05733	0.11587	1.12969	0.53959	0.77888	1.18834	0.05624	1.27676	-0.25356	0.93234
320.00	0.97037	0.11949	0.81041	0.41511	0.48046	0.67655	0.16137	0.77528	0.00474	0.80516
340.00	0.90609	0.07223	0.65450	0.21610	0.33855	0.33328	0.04124	0.42413	-0.22642	0.54206
360.00	0.88319	0.00000	0.60669	0.00000	0.28597	0.00000	-0.03190	0.00000	-0.35087	0.00000

Table 2.7. $cn\left(Me^{i\theta}\middle|m\right)$ as Computed from Subroutine CJEFS for m = 0.9

θ°	$cn\left(0.5\ e^{i\theta}\middle\|m\right)$		$cn\left(1.0\ e^{i\theta}\middle\|m\right)$		$cn\left(1.5\ e^{i\theta}\middle\|m\right)$		$cn\left(2.0\ e^{i\theta}\middle\|m\right)$		$cn\left(2.5\ e^{i\theta}\middle\|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.88592	0.00000	0.63794	0.00000	0.39104	0.00000	0.19087	0.00000	0.02471	0.00000
20.00	0.90681	-0.06952	0.66881	-0.18594	0.40310	-0.24109	0.19441	-0.24610	0.05133	-0.24948
40.00	0.96770	-0.11808	0.78307	-0.38421	0.43576	-0.54034	0.14512	-0.50581	0.00135	-0.41058
60.00	1.05530	-0.11852	1.07545	-0.56654	0.53330	-1.09585	-0.13387	-0.91902	-0.34116	-0.55045
80.00	1.12724	-0.05158	1.65261	-0.40250	1.67338	-3.15113	-1.56524	-1.42419	-1.10519	-0.42143
100.00	1.12724	0.05158	1.65261	0.40250	1.67338	3.15113	-1.56524	1.42419	-1.10519	0.42143
120.00	1.05530	0.11852	1.07545	0.56654	0.53330	1.09585	-0.13387	0.91902	-0.34116	0.55045
140.00	0.96770	0.11808	0.78307	0.38421	0.43576	0.54034	0.14512	0.50581	0.00135	0.41058
160.00	0.90681	0.06952	0.66881	0.18594	0.40310	0.24109	0.19441	0.24610	0.05133	0.24948
180.00	0.88592	0.00000	0.62794	0.00000	0.39104	0.00000	0.19087	0.00000	0.02471	0.00000
200.00	0.90681	-0.06952	0.66881	-0.18594	0.40310	-0.24109	0.19441	-0.24610	0.05133	-0.24948
220.00	0.96770	-0.11808	0.78307	-0.38421	0.43576	-0.54034	0.14512	-0.50581	0.00135	-0.41058
240.00	1.05530	-0.11852	1.07545	-0.56654	0.53330	-1.09585	-0.13387	-0.91902	-0.34116	-0.55045
260.00	1.12724	-0.05158	1.65261	-0.40250	1.67338	-3.15113	-1.56524	-1.42419	-1.10519	-0.42143
280.00	1.12724	0.05158	1.65261	0.40250	1.67338	3.15113	-1.56524	1.42419	-1.10519	0.42143
300.00	1.05530	0.11852	1.07545	0.56654	0.53330	1.09585	-0.13387	0.91902	-0.34116	0.55045
320.00	0.96770	0.11808	0.78307	0.38421	0.43576	0.54034	0.14512	0.50581	0.00135	0.41058
340.00	0.90681	0.06952	0.66881	0.18594	0.40310	0.24109	0.19441	0.24610	0.05133	0.41058
360.00	0.88592	0.00000	0.63794	0.00000	0.39104	0.00000	0.19087	0.00000	0.02471	0.00000

Table 2.8. $dn\left(Me^{i\theta}\middle|m\right)$ as Computed from Subroutine CJEFS for m = 0.3

θ°	$dn\left(0.5\ e^{i\theta}\middle|m\right)$		$dn\left(1.0\ e^{i\theta}\middle|m\right)$		$dn\left(1.5\ e^{i\theta}\middle|m\right)$		$dn\left(2.0\ e^{i\theta}\middle|m\right)$		$dn\left(2.5\ e^{i\theta}\middle|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.96568	0.00000	0.89380	0.00000	0.84235	0.00000	0.84676	0.00000	0.90420	0.00000
20.00	0.97194	-0.02096	0.89845	-0.05298	0.81702	-0.04199	0.77764	0.03354	0.82349	0.15622
40.00	0.99043	-0.03562	0.93114	-0.12084	0.77981	-0.17272	0.57393	-0.09267	0.38003	0.14124
60.00	1.01687	-0.03538	1.03502	-0.17351	0.89603	-0.44544	0.42021	-0.60257	-0.09027	-0.41023
80.00	1.03791	-0.01516	1.18849	-0.09996	1.57345	-0.54162	0.83283	-2.53626	-1.25580	-1.16317
100.00	1.03791	0.01516	1.18849	0.09996	1.57345	0.54162	0.83283	2.53626	-1.25580	1.16317
120.00	1.01687	0.03538	1.03502	0.17351	0.89603	0.44544	0.42021	0.60257	-0.09027	0.41023
140.00	0.99043	0.03562	0.93144	0.12084	0.77981	0.17272	0.57393	0.09267	0.38003	-0.14124
160.00	0.97194	0.02096	0.89845	0.05298	0.81702	0.04199	0.77764	-0.03354	0.82349	-0.15622
180.00	0.96568	0.00000	0.89380	0.00000	0.84235	0.00000	0.84676	-0.00000	0.90420	-0.00000
200.00	0.97194	-0.02096	0.89845	-0.05298	0.81702	-0.04199	0.77764	0.03354	0.82349	0.15622
220.00	0.99043	-0.03562	0.93114	-0.12084	0.77981	-0.17272	0.57393	-0.09267	0.38003	0.14124
240.00	1.01687	-0.03538	1.03502	-0.17351	0.89603	-0.44544	0.42021	-0.60257	-0.09027	-0.41023
260.00	1.03791	-0.01516	1.18849	-0.09996	1.57345	-0.54162	0.83283	-2.53626	-1.25580	-1.16317
280.00	1.03791	0.01516	1.18849	0.09996	1.57345	0.54162	0.83283	2.53626	-1.25580	1.16317
300.00	1.01687	0.03538	1.03502	0.17351	0.89603	0.44544	0.42021	0.60257	-0.09027	0.41023
320.00	0.99043	0.03562	0.93114	0.12084	0.77981	0.17272	0.57393	0.09267	0.38003	-0.14124
340.00	0.97194	0.02096	0.89845	0.05298	0.81702	0.04199	0.77764	-0.03354	0.82349	-0.15622
360.00	0.96568	0.00000	0.89380	0.00000	0.84235	0.00000	0.84676	-0.00000	0.90420	-0.00000

Table 2.9. $dn\left(Me^{i\theta}\middle|m\right)$ as Computed from Subroutine CJEFS for m = 0.6

θ°	$dn\left(0.5\ e^{i\theta}\middle|m\right)$		$dn\left(1.0\ e^{i\theta}\middle|m\right)$		$dn\left(1.5\ e^{i\theta}\middle|m\right)$		$dn\left(2.0\ e^{i\theta}\middle|m\right)$		$dn\left(2.5\ e^{i\theta}\middle|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.93167	0.00000	0.78794	0.00000	0.67012	0.00000	0.63294	0.00000	0.68838	0.00000
20.00	0.94403	-0.04160	0.80016	-0.10606	0.64282	-0.10531	0.54172	-0.01937	0.52367	0.14062
40.00	0.98053	-0.07095	0.86332	-0.23380	0.60614	-0.32176	0.32777	-0.22901	0.10723	-0.02136
60.00	1.03333	-0.07114	1.05423	-0.34693	0.71781	-0.77366	0.05660	-0.76117	-0.32603	-0.43506
80.00	0.07627	-0.03080	1.38816	-0.22326	1.97044	-1.58757	-1.26449	-2.12381	-1.20115	-0.56030
100.00	1.07627	0.03080	1.38816	0.22326	1.97044	1.58757	-1.26449	2.12381	-1.20115	0.56030
120.00	1.03333	0.07114	1.05423	0.34693	0.71781	0.77366	0.05660	0.76117	-0.32603	0.43506
140.00	0.98053	0.07095	0.86332	0.23380	0.60614	0.32176	0.32777	0.22901	0.10723	0.02136
160.00	0.94403	0.04160	0.80016	0.10606	0.64282	0.10531	0.54172	0.01937	0.52367	-0.14062
180.00	0.93167	0.00000	0.78794	0.00000	0.67012	0.00000	0.63294	-0.00000	0.68838	-0.00000
200.00	0.94403	-0.04160	0.80016	-0.10606	0.64282	-0.10531	0.54172	-0.01937	0.52367	0.14062
220.00	0.98053	-0.07095	0.86332	-0.23380	0.60614	-0.32176	0.32777	-0.22901	0.10723	-0.02136
240.00	1.03333	-0.07114	1.05423	-0.34693	0.71781	-0.77366	0.05660	-0.76117	-0.32603	-0.43506
260.00	1.07627	-0.03080	1.38816	-0.22326	1.97044	-1.58757	-1.26449	-2.12381	-1.20115	-0.56030
280.00	1.07627	0.03080	1.38816	0.22326	1.97044	1.58757	-1.26449	2.12381	-1.20115	0.56030
300.00	1.03333	0.07114	1.05423	0.34693	0.71781	0.77366	0.05660	0.76117	-0.32603	0.43506
320.00	0.98053	0.07095	0.86332	0.23380	0.60614	0.32176	0.32777	0.22901	0.10723	0.02136
340.00	0.94403	0.04160	0.80016	0.10606	0.64282	0.10531	0.54172	0.01937	0.52367	-0.14062
360.00	0.93167	0.00000	0.78794	0.00000	0.67012	0.00000	0.63294	-0.00000	0.68838	-0.00000

Table 2.10. $dn\left(Me^{i\theta}\middle|m\right)$ as Computed from Subroutine CJEFS for m = 0.9

θ°	$dn\left(0.5\ e^{i\theta}\middle|m\right)$		$dn\left(1.0\ e^{i\theta}\middle|m\right)$		$dn\left(1.5\ e^{i\theta}\middle|m\right)$		$dn\left(2.0\ e^{i\theta}\middle|m\right)$		$dn\left(2.5\ e^{i\theta}\middle|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.89798	0.00000	0.68284	0.00000	0.48747	0.00000	0.36440	0.00000	0.31710	0.00000
20.00	0.91627	-0.06192	0.70476	-0.15881	0.47702	-0.18336	0.31362	-0.13730	0.22150	-0.05203
40.00	0.97030	-0.10598	0.79660	-0.33992	0.46477	-0.45595	0.17529	-0.37687	0.00219	-0.22742
60.00	1.04936	-0.10727	1.05851	-0.51805	0.52567	-1.00058	-0.13603	-0.81395	-0.37205	-0.45427
80.00	1.11508	-0.04693	1.59769	-0.37470	1.59447	-2.97637	-1.50343	-1.33446	-1.08969	-0.38468
100.00	1.11508	0.04693	1.59769	0.37470	1.59447	2.97637	-1.50343	1.33446	-1.08969	0.38468
120.00	1.04936	0.10727	1.05851	0.51805	0.52567	1.00058	-0.13603	0.81395	-0.37205	0.45427
140.00	0.97030	0.10598	0.79660	0.33992	0.46477	0.45595	0.17529	0.37687	0.00219	0.22742
160.00	0.91627	0.06192	0.70476	0.15881	0.47702	0.18336	0.31362	0.13730	0.22150	0.05203
180.00	0.89798	0.00000	0.68284	0.00000	0.48747	0.00000	0.36440	0.00000	0.31710	0.00000
200.00	0.91627	-0.06192	0.70476	-0.15881	0.47702	-0.18336	0.31362	-0.13730	0.22150	-0.05203
220.00	0.97030	-0.10598	0.79660	-0.33992	0.46477	-0.45595	0.17529	-0.37687	0.00219	-0.22742
240.00	1.04936	-0.10727	1.05851	-0.51805	0.52567	-1.00058	-0.13603	-0.81395	-0.37205	-0.45427
260.00	1.11508	-0.04693	1.59769	-0.37470	1.59447	-2.97637	-1.50343	-1.33446	-1.08969	-0.38468
280.00	1.11508	0.04693	1.59769	0.37470	1.59447	2.97637	-1.50343	1.33446	-1.08969	0.38468
300.00	1.04936	0.10727	1.05851	0.51805	0.52567	1.00058	-0.13603	0.81395	-0.37205	0.45427
320.00	0.97030	0.10598	0.79660	0.33992	0.46477	0.45595	0.17529	0.37687	0.00219	0.22742
340.00	0.91627	0.06192	0.70476	0.15881	0.47702	0.18336	0.31362	0.13730	0.22150	0.05203
360.00	0.89798	0.00000	0.68284	0.00000	0.48747	0.00000	0.36440	0.00000	0.31710	0.00000

In concluding this section, it is emphasized that, after the trio sn, cn and dn is computed for either real or complex argument, the remainder of the Jacobian elliptic functions are easily evaluated using Eq. (2.2).

Listing of the Subroutine JEFS

```
      SUBROUTINE JEFS (U,EM,SN,CN,DN)
C
C     THIS IS A CALCULATION OF THE
C     JACOBIAN ELLIPTIC FUNCTIONS
C     SN , CN , AND DN BY THE METHOD
C     OF ARITHEMETIC/GEOMETRIC MEANS
C     (AMS 55,SECTION 16.4,P.571)
C
      DIMENSION A(200), C(200), PHI(200)
      V=U
      AM=EM
      IF (AM.EQ.0.) GO TO 20
      IF (AM.EQ.1.) GO TO 25
      AM1=1.-AM
      A(1)=1.
      B=SQRT(AM1)
      C(1)=SQRT(AM)
      DO 5 I=2,200
      A(I)=.5*(A(I-1)+B)
      C(I)=.5*(A(I-1)-B)
      CCCC=ABS(C(I))
      IF (CCCC.LT.1.E-10) GO TO 10
      B=SQRT(A(I-1)*B)
    5 CONTINUE
      I=200
   10 CONTINUE
      L=I-1
      PHI(I)=A(I)*V*(2**L)
      DO 15 J=1,L
      K=I+1-J
      ARGU=C(K)*SIN(PHI(K))/A(K)
      T=ASIN(ARGU)
      PHI(K-1)=.5*(T+PHI(K))
   15 CONTINUE
      SN=SIN(PHI(1))
      CN=COS(PHI(1))
      DN=CN/COS(PHI(2)-PHI(1))
      RETURN
   20 SN=SIN(U)
      CN=COS(U)
      DN=1.
      RETURN
   25 SN=TANH(U)
      CN=2./(EXP(U)+EXP(-U))
      DN=CN
      RETURN
      END
```

Listing of the Subroutine CJEFS

Note 1: While using CJEFS, the subroutine JEFS has to be supplemented.

```
      SUBROUTINE CJEFS (C,EM,SN,CN,DN)
C
C     THIS IS A CALCULATION OF THE
C     JACOBIAN ELLIPTIC FUNCTIONS
C     SN , CN , AND DN FOR A COMPLEX
C     ARGUMENT - C
C     (CF. AMS 55 EQNS. 16.21.2,16.21.3,16.21.4)
C
      COMPLEX C,SN,CN,DN
      RU=REAL(C)
      IF (ABS(RU).LT.1.E-8) RU=0.
      CALL JEFS (RU,EM,RSN,RCN,RDN)
      UI=AIMAG(C)
      EM1=1.-EM
      CALL JEFS (UI,EM1,SNI,CNI,DNI)
      DENOM=CNI*CNI+EM*RSN*RSN*SNI*SNI
      SN=CMPLX(RSN*DNI,RCN*RDN*SNI*CNI)/DENOM
      CN=CMPLX(RCN*CNI,-RSN*RDN*SNI*DNI)/DENOM
      DN=CMPLX(RDN*CNI*DNI,-EM*RSN*RCN*SNI)/DENOM
      RETURN
      END
```

III. Elliptic Integrals of the First and Second Kinds

A. Complete Integrals

We shall begin this section by giving certain definitions

$K \equiv$ Complete elliptic integral of the first kind

$E \equiv$ Complete elliptic integral of the second kind

Several representations exist for K and E, e.g.,

$$K = K(m) = F(\pi/2|m) = F(\pi/2\backslash\alpha)$$

$$E = E[K(m)] = E(m) = E(\pi/2\backslash\alpha) \tag{3.1}$$

At this stage, an explanation of the notation in Eq. (3.1) is in order. A vertical or a left-slanted stroke respectively separates the argument from the parameter depending on whether the parameter is represented by an English (m) or a Greek (α) letter. Furthermore, depending on whether the argument itself is an English or Greek symbol, the functions have different, but equivalent representations, e.g.,

$$F(\phi\backslash\alpha) = \int_0^\phi (1 - \sin^2\alpha \sin^2\theta)^{-1/2} d\theta$$

$$F(u|m) = \int_0^u dw = u \tag{3.2}$$

and

$$E(\phi\backslash\alpha) = \int_0^\phi (1 - \sin^2\alpha \sin^2\theta)^{1/2} d\theta$$

$$E(u|m) = \int_0^u \mathrm{dn}^2\, w\, dw \tag{3.3}$$

The equivalence of the two representations above is via

$$\left.\begin{array}{lll} \sin(\phi) & = & \mathrm{sn}(u) \\ \text{or} \quad \phi & = & \arcsin[\mathrm{sn}(u)] \\ & \equiv & \mathrm{am}(u) = \text{the amplitude} \\ \text{and} \quad m & = & \sin^2(\alpha) \end{array}\right\} \qquad (3.4)$$

The elliptic integrals of the first and second kind are said to be complete if the argument $\phi = \pi/2$ and are given by

Complete Elliptic Integral of the First Kind

$$= K(m) \equiv K$$

$$= \int_0^{\pi/2} [1 - m \sin^2(\theta)]^{-\frac{1}{2}} \, d\theta \qquad (3.5)$$

and

Complete Elliptic Integral of the Second Kind

$$= E(m) \equiv E$$

$$= \int_0^{\pi/2} [1 - m \sin^2(\theta)]^{\frac{1}{2}} \, d\theta \qquad (3.6)$$

Subroutine TEK can compute the quantities K(m) and E(m) using their series representations. It may be used by the Fortran call statement

```
CALL TEK (ID,RM,EK,E)
```

The variables ID and RM are inputs to the subroutine and the variables EK and E are returned by the subroutines. The four variables are described below.

ID TYPE INTEGER. This determines which parameter the subroutine expects to receive. If ID = 0, RM $\leftrightarrow$ m. If ID $\neq$ 0, RM $\leftrightarrow m_1$.

RM The real parameter m or its complement m_1 depending on the value of ID.

EK = K(m) irrespective of the value of ID.

E = E(m) irrespective of the value of ID.

f m is close to unity, e.g., $(1 - 10^{-5}) \leq m < 1$, m_1 can be supplied instead of m through the variable RM while setting ID ≠ 0. This improves the accuracy of K(m) and E(m) provided, of course, $m_1 = 1 - m$ be calculated (or otherwise supplied) without a machine subtraction. A test run and a listing of the subroutine TEK follows and for a more detailed discussion of TEK, the reader is referred to an earlier note [5] by one of the authors (Terry L. Brown). For purposes of the test run, the value of the parameter m ranges from 0.00000000 to 0.99999999. As was mentioned earlier, when m is such that $0.99999 \leq m < 1$, the variable ID was set = 1 and hand calculated value of m_1 was fed in place of m as e.g., in order to obtain

K(0.99999900) and E(0.99999900)

The subroutine TEK was called by

```
CALL TEK (1,0.000001,EK,E)
```

and the variables EK and E would contain K(0.99999900) and E(0.99999900).

The results of the test run are tabulated in Table 3.1 and plotted in Figure 3.1.

B. Incomplete Integrals

From the preceding subsection, we have

$$F(\phi|m) = \int_0^{\phi} (1 - m \sin^2\theta)^{-1/2} d\theta$$

$$E(\phi|m) = \int_0^{\phi} (1 - m \sin^2\theta)^{1/2} d\theta \qquad (3.7)$$

which respectively are the incomplete elliptic integrals of the first and the second kind. The subroutine TEF can compute $F(\phi|m)$ and $E(\phi|m)$ and this is achieved by using infinite series representations [9] when $0 \leq m < 0.75$ and by an application of the descending Landen transformation [6] if $0.75 \leq m < 1$. The coefficients arising in the Landen transformation are determined by a process of Arithmetic-Geometric Mean described in an earlier section. We shall not discuss the mechanics of the subroutine TEF here since it is well documented by

Table 3.1. Sample Output of Subroutine TEK

m	K(m)	E(m)
0.00000000	1.57079633	1.57079633
0.05000000	1.59100345	1.55097335
0.10000000	1.61244135	1.53075764
0.15000000	1.63525673	1.51012183
0.20000000	1.65962360	1.48903506
0.25000000	1.68575035	1.46746221
0.30000000	1.71388945	1.44536306
0.35000000	1.74435060	1.42269113
0.40000000	1.77751937	1.39939214
0.45000000	1.81388394	1.37540197
0.50000000	1.85407468	1.35064388
0.55000000	1.89892491	1.32502450
0.60000000	1.94956775	1.29842804
0.65000000	2.00759840	1.27070748
0.70000000	2.07536314	1.24167057
0.75000000	2.15651565	1.21105603
0.80000000	2.25720533	1.17848992
0.85000000	2.38901649	1.14339579
0.90000000	2.57809211	1.10477473
0.95000000	2.90833725	1.06047373
0.96000000	3.01611249	1.05050223
0.97000000	3.15587495	1.03994686
0.98000000	3.35414145	1.02859452
0.99000000	3.69563736	1.01599355
0.99900000	4.84113256	1.00217079
0.99990000	5.99158934	1.00027458
0.99999000	7.14277245	1.00003321
0.99999900	8.29405146	1.00000390
0.99999990	9.44534240	1.00000045
0.99999999	10.59663476	1.00000005

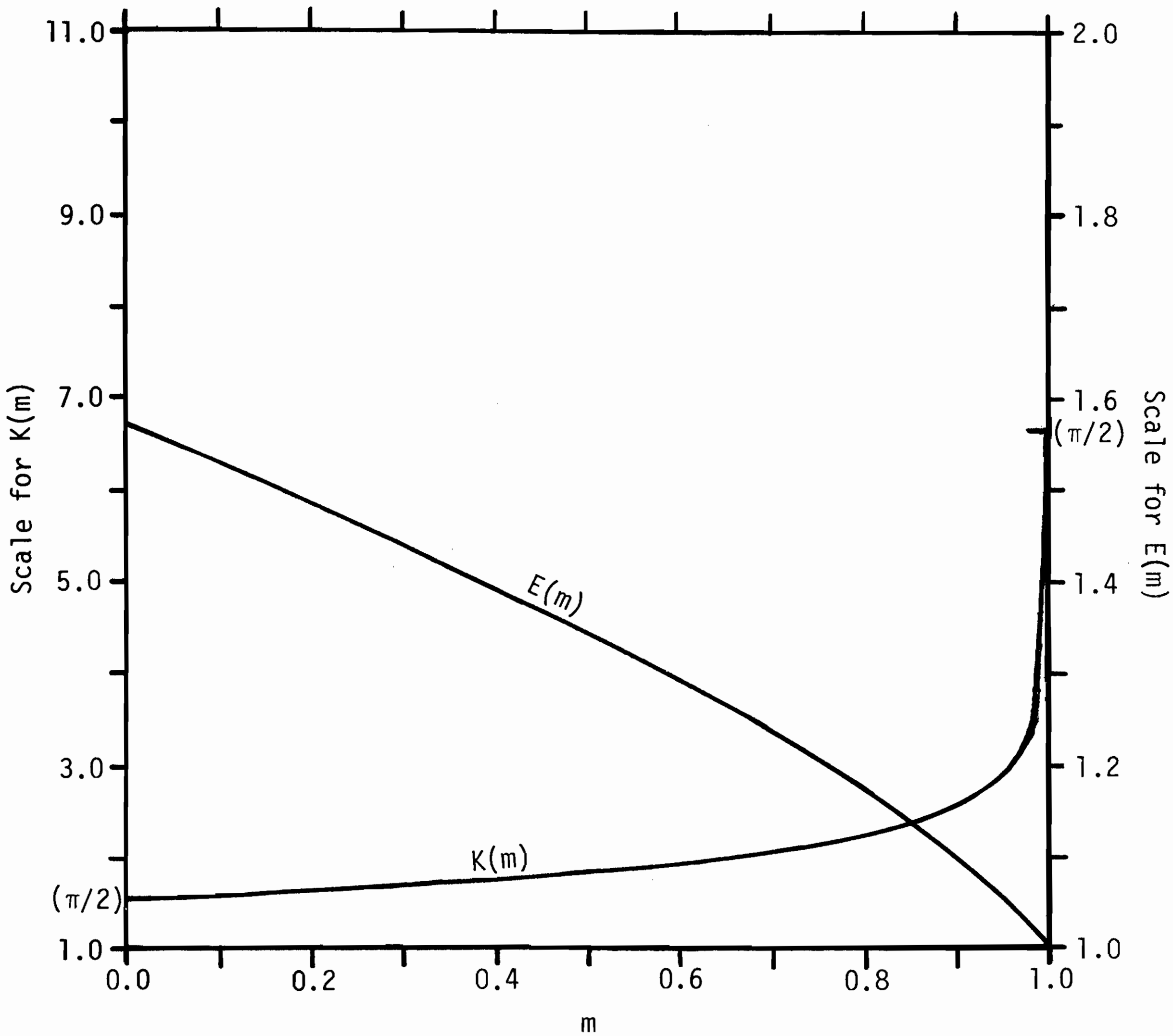

Figure 3.1. Plot of K(m) and E(m) as a Function of m (See Table 3.1.)

Listing of Subroutine TEK

```
      SUBROUTINE TEK (ID,RM,EK,E)
      DIMENSION RKP(60)
      IF (ID) 60,5,60
5     IF (RM-1.) 30,20,10
10    PRINT 15, RM
15    FORMAT (5X,9H*********,3X,13H LOOK OUT M =,F8.3,3X,9H*********)
      RETURN
20    EK=1.E75
      E=1.
25    RETURN
30    EK=1.57079632679489
      E=EK
      IF (RM) 10,25,35
35    IF (RM-.999) 40,40,65
40    RKN=SQRT(RM)
      DO 45 I=1,60
      RKP(I)=SQRT(1.-RKN*RKN)
      RKN=(1.-RKP(I))/(1.+RKP(I))
      IF (I.GE.2.AND.RKN.LT.1.E-20) GO TO 50
45    CONTINUE
      I=60
50    N=I-1
      DO 55 J=1,N
      T1=1.+RKP(I-J)
      EK=2.*EK/T1
55    E=T1*E-EK*RKP(I-J)
      RETURN
60    RPK=SQRT(RM)
      GO TO 70
65    RPK=SQRT(1.-RM)
70    PK2=RPK*RPK
      PKP=PK2
      GOL=ALOG(4./RPK)
      GK=GOL-1.
      FK=.25
      FE=.25
      EK=GOL+FK*GK*PKP
      E=1.+.5*(GOL-.5)*PKP
      GE=GK
      DO 85 I=2,2000
      R=FLOAT(I+I)
      D=R-1.
      PKP=PKP*PK2
      C=D/R
      FK=FK*D*D/(R*R)
      FE=FE*C
      H=1./(D*R)
      GK=GK-1./(D*FLOAT(I))
      GE=GE-H
      T1=FK*GK*PKP
      EK=T1+EK
      T2=FE*GE*PKP
      E=T2+E
      IF (T1-1.E-15) 75,75,80
75    IF (T2-1.E-15) 90,90,80
```

```
80      FE=FE*C
85      GE=GE-H
90      RETURN
        END
```

one of the authors (Terry L. Brown) in Reference [5]. TEF may be used by the Fortran statement

```
CALL TEF (PH1,RM,SIG,TF,TE)
```

The variables in the above CALL statement are described below:

- PH1 - Real amplitude ϕ in radians
- RM - Real parameter m $(0 \leq m < 1)$
- SIG - Real constant used in setting up the error criterion, typically = 10^{-6}.
- TF - The subroutine returns $F(\phi|m)$ in this location.
- TE - The subroutine returns $E(\phi|m)$ in this location.

A sample output from TEF and a listing, which are reproduced from the Reference [3] are included in this section. (See Tables 3.2 and 3.3.)

C. Incomplete Integrals with a Complex Argument

In this subsection, we report a subroutine CEF which can compute the incomplete elliptic integrals of the first and second kind when the argument is complex and the parameter m is restricted in the range $0 \leq m < 1$, i.e., $F(u + iv|m)$ and $E(u + iv|m)$. We shall consider them individually.

$$F(u + iv|m) = \int_0^{(u+iv)} dw = (u + iv)$$

$$= F(\phi + i\psi|m) = \int_0^{(\phi+i\psi)} (1 - m \sin^2\theta)^{-1/2} d\theta \tag{3.8}$$

If the complex argument $(u + iv)$ is given, the value of $F(u + iv|m)$ is simply the complex argument itself, irrespective of the value of the parameter m. On the other hand, if the complex amplitude $(\phi + i\psi)$ is given, the calculation of $F(\phi + i\psi|m)$ is a bit more complicated and one can follow

$$F(\phi + i\psi|m) = u + iv = \text{arc sn}[\sin(\phi + i\psi)] \tag{3.9}$$

Table 3.2. Comparison of $F(\phi\backslash\alpha)$ or $F(\phi|m)$ Returned by Subroutine TEF with the Values Listed in Reference [6] (Reproduced from Reference [5].)

$F(\phi\backslash\alpha)$	Value Listed	Computed Value
F(5°\48°)	0.08732765	0.08732766
F(10°\58°)	0.17517260	0.17517259
F(10°\62°)	0.17522690	0.17522691
F(10°\86°)	0.17542143	0.17542142
F(15°\44°)	0.26324404	0.26324403
F(15°\46°)	0.26335019	0.26335020
F(20°\70°)	0.35547959	0.35547958
F(20°\82°)	0.35622881	0.35622880
F(25°\28°)	0.43932365	0.43932364
F(25°\48°)	0.44404397	0.44404396
F(25°\74°)	0.44967538	0.44967539
F(30°\80°)	0.54842535	0.54842534
F(35°\50°)	0.63363947	0.63363946
F(35°\52°)	0.63511150	0.63511149
F(35°\64°)	0.64351521	0.64351520
F(35°\78°)	0.65067415	0.65067414
F(35°\84°)	0.65228622	0.65228621
F(50°\72°)	0.99163507	0.99163506
F(55°\86°)	1.15261652	1.15261651
F(60°\50°)	1.16431637	0.16431636
F(60°\56°)	1.19275650	1.19275649
F(60°\60°)	1.21259661	1.21259662
F(60°\84°)	1.31117166	1.31117165
F(70°\56°)	1.45726935	1.45726934
F(75°\46°)	1.49668437	1.49668438
F(75°\82°)	1.97316666	1.97316665
F(80°\82°)	2.31643897	2.31643896
F(85°\56°)	1.90143591	1.90143590
F(85°\66°)	2.13070052	2.13070051

Table 3.3. Comparison of $E(\phi\backslash\alpha)$ or $E(\phi|m)$ Returned by Subroutine TEF with the Values Listed in Reference [6] (Reproduced from Reference [5].)

$E(\phi\backslash\alpha)$	Value Listed	Computed Value
E(10°\70°)	0.17375210	0.17375209
E(15°\68°)	0.25924104	0.25924103
E(15°\48°)	0.26016110	0.26016109
E(20°\74°)	0.34256478	0.34256479
E(25°\74°)	0.42368913	0.42368914
E(30°\84°)	0.50026923	0.50026922
E(30°\74°)	0.50186633	0.50186634
E(35°\72°)	0.57733641	0.57733640
E(35°\38°)	0.59723431	0.59723432
E(40°\20°)	0.69206954	0.69206953
E(45°\48°)	0.74409773	0.74409772
E(50°\54°)	0.80601230	0.80601229
E(55°\46°)	0.89246858	0.89246857
E(60°\64°)	0.90689460	0.90689461
E(70°\58°)	1.03614663	1.03614664
E(75°\82°)	0.97598331	0.97598330
E(75°\76°)	0.99517606	0.99517605
E(75°\70°)	1.02171634	1.02171633
E(80°\30°)	1.31605841	1.31605840
E(85°\72°)	1.07377505	1.07377504
E(85°\6°)	1.47970717	1.47970716

Listing of Subroutine TEF

Note 1: While using TEF, the subroutine TEK has to be supplemented.

```
      SUBROUTINE TEF (PH1,RM,SIG,TF,TE)
      DATA PIO4/.785398163397448/,TPI/6.28318530717959/
      DATA PI,PIO2/3.14159265358979323846264 3E0,1.57079632679489661923E0/
      DIMENSION AA(50), BB(50), CC(50), PSAV(50)
      IF (ABS(RM-.5)-.5) 15,15,5
5     PRINT 10, RM
10    FORMAT (5X,9H*********,3X,13HLOOK OUT M = ,F8.3,3X,9H*********)
      RETURN
15    IF (PH1) 20,25,25
20    W=-1.
      PH=-PH1
      GO TO 30
25    W=1.
      PH=PH1
30    RK=SQRT(RM)
      N=PH/TPI
      A=PH-FLOAT(N)*TPI
      B=A/PIO2
      K=B
      NQ=K+1
      GO TO (35,40,45,50), NQ
35    NK=4*N
      SIGNEM=1.
      AP=A
      GO TO 55
40    NK=4*N+2
      SIGNEM=-1.
      AP=PI-A
      GO TO 55
45    NK=4*N+2
      SIGNEM=1.
      AP=A-PI
      GO TO 55
50    NK=4*N+4
      SIGNEM=-1.
      AP=TPI-A
55    CNK=NK
      PHI=AP
      CALL TEK (0,RM,EK,EE)
      PLUS=CNK*EK
      PLUS1=CNK*EE
      IT=0
      IF (ABS(PHI-PIO2)-1.E-10) 60,60,65
60    IT=1
65    IF (ABS(RK-1.E0)-1.E-10) 70,85,85
70    IT=IT+1
      GO TO (75,80), IT
75    TF=W*(PLUS+SIGNEM*ALOG(TAN(PIO4+PHI*.5)))
      TE=W*(PLUS1+SIGNEM*SIN(PHI))
      RETURN
80    TF=W*1.E75
      TE=W*(PLUS1+SIGNEM)
      RETURN
85    IF (ABS(RK)-1.E-15) 90,95,95
90    TF=W*(PLUS+SIGNEM*PHI)
```

```
      TE=W*(PLUS1+SIGNEM*PHI)
      RETURN
95    IT=IT+1
      GO TO (105,100), IT
100   CALL TEK (0,RM,EK,EE)
      TF=W*(PLUS+SIGNEM*EK)
      TE=W*(PLUS1+SIGNEM*EE)
      RETURN
105   IF (ABS(PHI)-1.E-50) 110,115,115
110   TF=W*PLUS
      TE=W*PLUS1
      RETURN
115   IF (RM-.75) 120,140,140
120   CALL TEK (0,RM,EK,EE)
      S=SIN(PHI)
      C=COS(PHI)
      SK=RM
      CE=2.*PHI/PI
      TZ=CE*EK
      T1=CE*EE
      A=.5E0
      T=.5E0*A*SK
      R=T
      SS=S*S
      PS=1.E0
      H=.5
      F=.5E0
      PK=SK
      U1=10.
      DO 130 I=2,20000
      J=I*2
      D=FLOAT(J-1)
      G=FLOAT(J-3)
      E=1./FLOAT(J)
      PS=SS*PS
      A=E*(D*A+PS)
      F=D*E*F
      H=G*E*H
      PK=PK*SK
      U=F*A*PK
      IF (U1*U1/(U1-U)-SIG) 135,135,125
125   U1=U
      T=U+T
130   R=H*A*PK+R
135   TF=W*((TZ-S*C*T)*SIGNEM+PLUS)
      TE=W*((T1+S*C*R)*SIGNEM+PLUS1)
      RETURN
140   ALPHAR=ASIN(RK)
      AA(1)=1.
      BB(1)=COS(ALPHAR)
      DO 145 I=2,50
      II=I-1
      AA(I)=.5*(AA(II)+BB(II))
      BB(I)=SQRT(AA(II)*BB(II))
      CC(I)=.5*(AA(II)-BB(II))
```

```
      IF (ABS(CC(I))-SIG) 150,145,145
145   CONTINUE
      ISTOP=50
      GO TO 155
150   ISTOP=I
155   P=PHI
      P2=1.
      NQ=1
      IOS=1
      M2P=0
      I4=0
      ORELER=1.E25
      OR=1.E25
      DO 215 I=1,ISTOP
      PSAV(I)=P
      P2=P2*2.
      BD=TAN(P)*BB(I)/AA(I)
      BF=ATAN(BD)
160   INS=SIGN(1.,BF)
      IF (IOS*INS) 165,170,170
165   NQ=NQ+1
      IF (NQ.EQ.5) NQ=1
170   GO TO (175,190,190,195), NQ
175   IF (I4) 180,185,180
180   I4=0
      M2P=M2P+1
185   BE=BF+FLOAT(M2P)*TPI
      GO TO 200
190   BE=BF+PI+FLOAT(M2P)*TPI
      GO TO 200
195   BE=BF+TPI+FLOAT(M2P)*TPI
      I4=1
200   IOS=INS
      PR=P/BE
      RELER=ABS(OR-PR)/(PR+OR)
      IF (ORELER-RELER) 205,210,210
205   IOS=-IOS
      GO TO 160
210   P=BE+P
      OR=PR
215   ORELER=RELER
      TF=W*(PLUS+SIGNEM*(P/(P2*AA(ISTOP))))
      CALL TEK (0,RM,EK,EE)
      SUMEM=0.
      DO 220 IK=2,ISTOP
220   SUMEM=SUMEM+CC(IK)*SIN(PSAV(IK))
      TF=W*(PLUS1+SIGNEM*(EE/EK*TF+SUMEM))
      RETURN
      END
```

We have formally written down an inverse Jacobian function in Eq. (3.9) and this subject of inverse Jacobian functions merits a more detailed numerical study. For purposes of this section, however, we are concerned with complex arguments (u + iv) rather than complex amplitudes ($\phi + i\psi$) so that the process of finding $F(u + iv|m)$ is rendered trivial. Moving on to the incomplete elliptic integral of the second kind, we have by making use of the addition theorem

$$E(u + iv|m) = E(u|m) + E(iv|m) - m\ \mathrm{sn}(u|m)\mathrm{sn}(iv|m)\mathrm{sn}(u + iv|m) \qquad (3.10)$$

Once again, using Jacobi's Imaginary Transformation [6] for the second term on the r.h.s. and also the identity [6],

$$\mathrm{sn}(iv|m) = i\ \mathrm{sc}\left(v|m_1\right) \qquad (3.11)$$

we have

$$E(u + iv|m) = E(u|m) + i\left[v + \mathrm{dn}\left(v|m_1\right)\mathrm{sc}\left(v|m_1\right) - E\left(v|m_1\right)\right] - i\left[m\ \mathrm{sn}(u|m)\mathrm{sc}\left(v|m_1\right)\mathrm{sn}(u + iv|m)\right] \qquad (3.12)$$

Expanding the factor $\mathrm{sn}(u + iv|m)$ in terms of its real and imaginary parts [6] we can write $E(u + iv|m)$ in terms of its real and imaginary parts as,

$$E(u + iv|m) \equiv \text{Real Part} + i\ \text{Imaginary Part}$$

with

$$\left.\begin{array}{l}\text{Real}\\ \text{Part}\end{array}\right\} = E(u|m) + \frac{m\ \mathrm{sn}\left(u|m\right)\mathrm{cn}\left(u|m\right)\mathrm{dn}\left(u|m\right)\mathrm{sn}^2\left(v|m_1\right)}{\mathrm{cn}^2\left(v|m_1\right) + m\ \mathrm{sn}^2\left(u|m\right)\mathrm{sn}^2\left(v|m_1\right)}$$

$$\left.\begin{array}{r}\text{Imaginary}\\ \text{Part}\end{array}\right\} = \left[v + \mathrm{dn}\left(v|m_1\right)\mathrm{sc}\left(v|m_1\right) - E\left(v|m_1\right)\right] - \frac{m\ \mathrm{sn}^2\left(u|m\right)\mathrm{sc}\left(v|m_1\right)\mathrm{dn}\left(v|m_1\right)}{\mathrm{cn}^2\left(v|m_1\right) + m\ \mathrm{sn}^2\left(u|m\right)\mathrm{sn}^2\left(v|m_1\right)} \qquad (3.13)$$

where, as before

$$m_1 = \text{Complementary Parameter} = (1 - m)$$

From Eq. (3.13) or otherwise, one can observe the following interesting properties

$$\left.\begin{aligned} &\text{if} \\ &\qquad E\,(u + iv|m) = X + iY \\ &\text{then} \\ &\qquad E(-u + iv|m) = -X + iY \\ &\qquad E(u - iv|m) = X - iY \\ &\qquad E(-u - iv|m) = -X - iY \end{aligned}\right\} \tag{3.14}$$

In arriving at Eq. (3.14), we use the fact that sn, cn and dn are respectively odd, even and even functions of their argument. With the notation $w = u + iv$ and a bar over a quantity signifying the complex conjugate, Eq. (3.14) may be more elegantly represented by

$$\left.\begin{aligned} E(-w|m) &= -E(w|m) \\ E(\overline{w}|m) &= \overline{E(w|m)} \\ E(-\overline{w}|m) = -E(\overline{w}|m) &= -\overline{E(w|m)} \end{aligned}\right\} \tag{3.15}$$

The conjugate properties of $E(w|m)$ also hold if E is replaced by F in Eq. (3.15) for the elliptic integral of the first kind since $F(w|m)$ is simply = w itself. In view of the conjugate properties, it is sufficient to compute $E(w|m)$ in the first quadrant of the complex w-plane. The conjugate properties have been incorporated into the subroutine CEF and it may be used for the argument w anywhere in the complex plane. Subroutine CEF may be called by the standard Fortran statement

```
CALL CE (CW,M,CEM,CFM)
```

where the variables are

CW - Complex Argument $w = u + iv$

M - Real Parameter m $(0 \le m \le 1)$

CEM - Subroutine Returns $E(u + iv|m)$

CFM - Subroutine Returns $F(u + iv|m) = u + iv$
(Although trivial, this is included for completeness and also for possible extension to the case when complex amplitude is given in place of w.)

As is evident from Eq. (3.12), the subroutine CEF in turn makes use of subroutines CJEFS, JEFS, TEK and TEF of earlier sections. With the complex argument

$$u + iv \equiv Me^{i\theta} \tag{3.16}$$

a sample output of $E(u + iv|m)$ or $E(Me^{i\theta}|m)$ as also a listing of the subroutine CEF is included. In the sample output, the values of the magnitude of the argument are set to be 0.5, 1.0, 1.5, 2.0 and 2.5. For each of the five magnitude settings, the phase angle varies from 0 to 360° in steps of 20°. The calculations are tabulated for three values of the parameter m given by 0.3, 0.6 and 0.9 (Tables 3.4, 3.5 and 3.6, respectively). The sample points in the complex w-plane are illustrated in Figure 2.1.

Table 3.4. Sample Output of Subroutine CEF for m = 0.3

θ°	$E\left(0.5\ e^{i\theta}\middle\|m\right)$		$E\left(1.0\ e^{i\theta}\middle\|m\right)$		$E\left(1.5\ e^{i\theta}\middle\|m\right)$		$E\left(2.0\ e^{i\theta}\middle\|m\right)$		$E\left(2.5\ e^{i\theta}\middle\|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.48827	0.00000	0.92130	0.00000	1.29496	0.00000	1.24346	0.00000	0.86294	0.00000
20.00	0.46349	0.16096	0.88849	0.27744	1.24737	0.36000	1.31214	0.45998	0.88503	0.63478
40.00	0.38850	0.31034	0.79184	0.55272	1.15391	0.68343	1.38738	0.74847	1.18915	0.82199
60.00	0.26288	0.43227	0.60866	0.83882	1.08577	1.07365	1.45013	0.96501	1.46715	0.87260
80.00	0.09374	0.50377	0.25071	1.08993	0.76407	1.89840	2.83135	1.57678	1.53256	0.14611
100.00	-0.09374	0.50377	-0.25071	1.08993	-0.76407	1.89840	-2.83135	1.57678	-1.53256	0.14611
120.00	-0.26288	0.43227	-0.60866	0.83882	-1.08577	1.07365	-1.45013	0.96501	-1.46715	0.87260
140.00	-0.38850	0.31034	-0.79184	0.55272	-1.15391	0.68343	-1.38738	0.74847	-1.18915	0.82199
160.00	-0.46349	0.16096	-0.88849	0.27744	-1.24737	0.36000	-1.31214	0.45998	-0.88503	0.63478
180.00	-0.48827	0.00000	-0.92130	0.00000	-1.29496	0.00000	-1.24346	0.00000	-0.86294	0.00000
200.00	-0.46349	-0.16096	-0.88849	-0.27744	-1.24737	-0.36000	-1.31214	-0.45998	-0.88503	-0.63478
220.00	-0.38850	-0.31034	-0.79184	-0.55272	-1.15391	-0.68343	-1.38738	-0.74847	-1.18915	-0.82199
240.00	-0.26288	-0.43227	-0.60866	-0.83882	-1.08577	-1.07365	-1.45013	-0.96501	-1.46714	-0.87260
260.00	-0.09374	-0.50377	-0.25071	-1.08993	-0.76407	-1.89840	-2.83135	-1.57678	-1.53256	-0.14611
280.00	0.09374	-0.50377	0.25071	-1.08993	0.76407	-1.89840	2.83135	-1.57678	1.53256	-0.14611
300.00	0.26288	-0.43227	0.60866	-0.83882	1.08577	-1.07365	1.45013	-0.06501	1.46715	-0.87260
320.00	0.38850	-0.31034	0.79184	-0.55272	1.15391	-0.68343	1.38738	-0.74847	1.18915	-0.82199
340.00	0.46349	-0.16096	0.88849	-0.27744	1.24737	-0.36000	1.31214	-0.45998	0.88503	-0.63478
360.00	0.48827	0.00000	0.92130	0.00000	1.29496	-0.00000	1.24346	-0.00000	0.86294	-0.00000

Table 3.5. Sample Output of Subroutine CEF for m = 0.6

$\theta°$	$E\left(0.5\ e^{i\theta}\middle\|m\right)$		$E\left(1.0\ e^{i\theta}\middle\|m\right)$		$E\left(1.5\ e^{i\theta}\middle\|m\right)$		$E\left(2.0\ e^{i\theta}\middle\|m\right)$		$E\left(2.5\ e^{i\theta}\middle\|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.47685	0.00000	0.84879	0.00000	1.11128	0.00000	1.27824	0.00000	1.06478	0.00000
20.00	0.45711	0.15123	0.83895	0.21978	1.10463	0.23003	1.27790	0.24933	1.06537	0.32240
40.00	0.39363	0.29921	0.80796	0.46579	1.12685	0.45295	1.26661	0.39262	1.29804	0.37755
60.00	0.27591	0.43115	0.71516	0.79547	1.29899	0.77107	1.44817	0.40452	1.29021	0.77845
80.00	0.10102	0.51539	0.35034	1.20464	1.77068	2.13888	2.36685	-0.52117	0.87230	0.52595
100.00	-0.10102	0.51539	-0.35034	1.20464	-1.77068	2.13888	-2.36685	-0.52117	-0.87230	0.52595
120.00	-0.27591	0.43115	-0.71516	0.79547	-1.29899	0.77107	-1.44817	0.40452	-1.29021	0.77845
140.00	-0.39363	0.29921	-0.80796	0.46579	-1.12685	0.45295	-1.26661	0.39262	-1.29804	0.47755
160.00	-0.45711	0.15123	-0.83895	0.21978	-1.10463	0.23003	-1.27790	0.24933	-1.06537	0.32240
180.00	-0.47685	0.00000	-0.84879	0.00000	-1.11128	0.00000	-1.27824	0.00000	-1.06478	0.00000
200.00	-0.45711	-0.15123	-0.83895	-0.21978	-1.10463	-0.23003	-1.27790	-0.24933	-1.06537	-0.32240
220.00	-0.39363	-0.29921	-0.80796	-0.46579	-1.12685	-0.45295	-1.26661	-0.39262	-1.29804	-0.37755
240.00	-0.27591	-0.43115	-0.71516	-0.79547	-1.29899	-0.77107	-1.44817	-0.40452	-1.29021	-0.77845
260.00	-0.10102	-0.51539	-0.35034	-1.20464	-1.77068	-2.13888	-2.36685	0.52117	-0.87230	-0.52595
280.00	0.10102	-0.51539	0.35034	-1.20464	1.77068	-2.13888	2.36685	0.52117	0.87230	-0.52595
300.00	0.27591	-0.43115	0.71516	-0.79547	1.29899	-0.77107	1.44817	-0.40452	1.29021	-0.77845
320.00	0.39363	-0.29921	0.80796	-0.46579	1.12685	-0.45295	1.26661	-0.39262	1.29804	-0.37755
340.00	0.45711	-0.15123	0.83895	-0.21978	1.10463	-0.23003	1.27790	-0.24933	1.06537	-0.32240
360.00	0.47685	0.00000	0.84879	0.00000	1.11128	-0.00000	1.27824	-0.00000	1.06478	-0.00000

Table 3.6. Sample Output of Subroutine CEF for m = 0.9

θ°	$E\left(0.5\ e^{i\theta}\middle\|m\right)$		$E\left(1.0\ e^{i\theta}\middle\|m\right)$		$E\left(1.5\ e^{i\theta}\middle\|m\right)$		$E\left(2.0\ e^{i\theta}\middle\|m\right)$		$E\left(2.5\ e^{i\theta}\middle\|m\right)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.46575	0.00000	0.78239	0.00000	0.95226	0.00000	1.04086	0.00000	1.09695	0.00000
20.00	0.45071	0.14182	0.79088	0.16874	0.97585	0.12507	1.05724	0.08587	1.09414	0.07148
40.00	0.39840	0.28803	0.81533	0.38212	1.07521	0.25865	1.13097	0.12968	1.11549	0.08197
60.00	0.38908	0.42963	0.81730	0.73674	1.40585	0.45062	1.33461	0.21670	1.10839	0.95074
80.00	0.10866	0.52729	0.47740	1.32765	3.11951	1.60262	1.52847	-0.58262	0.65441	1.02330
100.00	-0.10866	0.52729	-0.47740	1.32765	-3.11951	1.60262	-1.52847	-0.58262	-0.65441	1.02330
120.00	-0.28908	0.42963	-0.81730	0.73674	-1.40585	0.45062	-1.33461	0.21670	-1.10839	0.95074
140.00	-0.39840	0.28803	-0.81533	0.38212	-1.07521	0.25865	-1.13097	0.12968	-1.11549	0.08197
160.00	-0.45071	0.14182	-0.79088	0.16874	-0.97585	0.12507	-1.05724	0.08587	-1.09414	0.07148
180.00	-0.46575	0.00000	-0.78239	0.00000	-0.95226	0.00000	-1.04086	0.00000	-1.09695	0.00000
200.00	-0.45071	-0.14182	-0.79088	-0.16874	-0.08595	-0.12507	-1.05724	-0.08587	-1.09414	-0.07148
220.00	-0.39840	-0.28803	-0.81533	-0.38212	-1.07521	-0.25865	-1.13097	-0.12968	-1.11549	-0.08197
240.00	-0.28908	-0.42963	-0.81730	-0.73674	-1.40585	-0.45062	-1.33461	-0.21670	-1.10839	-0.95074
260.00	-0.10866	-0.52729	-0.47740	-1.32765	-3.11951	-1.60262	-1.52847	0.58262	-0.65441	-1.02330
280.00	0.10866	-0.52729	0.47740	-1.32765	3.11951	-1.60262	1.52847	0.58262	0.65441	-1.02330
300.00	0.28908	-0.42963	0.81730	-0.73674	1.40585	-0.45062	1.33461	-0.21670	1.10839	-0.95074
320.00	0.39840	-0.28803	0.81533	-0.38212	1.07521	-0.25865	1.13097	-0.12968	1.11549	-0.08197
340.00	0.45071	-0.14182	0.79088	-0.16874	0.97585	-0.12507	1.05724	-0.08587	1.09414	-0.07148
360.00	0.46575	0.00000	0.78239	0.00000	0.95226	-0.00000	1.04086	-0.00000	1.09695	-0.00000

Listing of Subroutine CEF

Note 1: While using CEF, the subroutines JEFS, CJEFS, TEK and TEF have to be supplemented.

```
      SUBROUTINE CEF(CW,M,CEM,CFM)
      IMPLICIT COMPLEX (C)
      REAL M,M1
C
C     NOTE... IF REAL OR IMAG. PART OF CW LIES BETWEEN -1.E-08
C     AND 1.E-08 , IT WILL BE TREATED AS ZERO.
C
      CI=(0.,1.)
      CFM=CW
      EP=1.E-08
      EM=-EP
      WR=REAL (CW)
      WI=AIMAG(CW)
      S=ABS(WR)
      T=ABS(WI)
      IF (S.LT.EP.AND.T.LT.EP) GO TO 50
C     NOW, WE SET UP THE CONDITION NUMBER...
      IF (WR.GT.EP.AND.WI.GT.EP) IC=1
      IF (WR.LT.EM.AND.WI.GT.EP) IC=2
      IF (WR.LT.EM.AND.WI.LT.EM) IC=3
      IF (WR.GT.EP.AND.WI.LT.EM) IC=4
      IF (T.LE.EP.AND.WR.GE.EP) IC=5
      IF (T.LE.EP.AND.WR.LE.EM) IC=6
      IF (S.LE.EP.AND.WI.GE.EP) IC=7
      IF (S.LE.EP.AND.WI.LE.EM) IC=8
C     COMMENCE COMPUTATION...
      M1=1.-M
      IF (IC-5) 10,20,20
 10   CONTINUE
      CWP=CMPLX(S,T)
      CALL CJEFS(CWP,M,CSN,CCN,CDN)
      CALL JEFS(S,M,USN,UCN,UDN)
      CALL JEFS(T,M1,V1SN,V1CN,V1DN)
      UPHI=ASIN(USN)
      VPHI=ASIN(V1SN)
      CALL TEF(UPHI,M,1.E-06,FUM,EUM)
      CALL TEF(VPHI,M1,1.E-06,FVM1,EVM1)
      SCV1=V1SN/V1CN
      CEIV=CI*(T+(V1DN*SCV1)-EVM1)
      CEM=EUM+CEIV-(CI*M*USN*SCV1*CSN)
      EMR=REAL(CEM)
      EMI=AIMAG(CEM)
      IF (IC.EQ.2) EMR=-EMR
      IF (IC.EQ.4) EMI=-EMI
      CEM=CMPLX(EMR,EMI)
      IF (IC.EQ.3) CEM=-CEM
      RETURN
 20   CONTINUE
```

```
      IF (IC-7) 30,40,40
30    CONTINUE
      CALL JEFS(S,M,USN,UCN,UDN)
      UPHI=ASIN(USN)
      CALL TEF(UPHI,M,1.E-06,FUM,EUM)
      IF (IC.EQ.5) CEM= CMPLX(EUM,0.)
      IF (IC.EQ.6) CEM= CMPLX(-EUM,0.)
      RETURN
40    CONTINUE
      CALL JEFS(T,M1,V1SN,V1CN,V1DN)
      VPHI=ASIN(V1SN)
      CALL TEF(VPHI,M1,1.E-06,FVM1,EVM1)
      SCV1=V1SN/V1CN
      EIV=(T+(V1DN*SCV1)-EVM1)
      IF (IC.EQ.7) CEM=CMPLX(0.,EIV)
      IF (IC.EQ.8) CEM= CMPLX(0.,-EIV)
      RETURN
50    CONTINUE
      CEM=(0.,0.)
      RETURN
      END
```

IV. Jacobi Zeta Function of Real/Complex Argument

For the case of real argument, the Jacobi zeta function is given by

$$Z(\phi|m) = E(\phi|m) - F(\phi|m)\,\frac{E(m)}{K(m)} \tag{4.1}$$

or equivalently

$$Z(u|m) = E(u|m) - u\,\frac{E(m)}{K(m)} \tag{4.2}$$

where, once again

$$\sin\phi = \mathrm{sn}(u|m) \tag{4.3}$$

The subroutine TEF is easily modified and renamed as subroutine ZETA so that it will compute $Z(\phi|m)$ of Eq. (4.1). If it is required to compute $Z(u|m)$ instead of $Z(\phi|m)$, it may be performed using the subroutine CZETA discussed later in this section. Subroutine ZETA may be used in the calling routine by the standard Fortran statement

```
CALL ZETA (PH1,RM,SIG,ZPHI)
```

The variables are described below:

- PH1 - Real amplitude ϕ in radians.
- RM - Real parameter m.
- SIG - Real constant used in setting up the error criterion, typically = 10^{-6}.
- ZPHI - Subroutine returns $Z(\phi|m)$ in this location.

We have included the results of a test run and a listing of the subroutine ZETA. The amplitude ϕ is varied between 0° and 90° in steps of 5° and $Z(\phi|m)$ is tabulated for three values of m = 0.3, 0.6 and 0.9 (Table 4.1), and is also plotted in Figure 4.1.

When the argument is complex, we have

$$Z(w|m) = E(w|m) - w\,\frac{E(m)}{K(m)} \tag{4.4}$$

where

$$w = u + iv$$

Table 4.1. Z(ϕ|m) Computed by Subroutine ZETA for $0 \le \phi \le 90°$ and m = 0.3, m = 0.6 and m = 0.9

θ°	Z(ϕ|0.3)	Z(ϕ|0.6)	Z(ϕ|0.9)
0.00	0.0000000000	0.0000000000	0.0000000000
5.00	0.0136113956	0.0290355734	0.0497282548
10.00	0.0268568814	0.0574078060	0.0986015554
15.00	0.0393769682	0.0844572947	0.1457627027
20.00	0.0508252130	0.1095327905	0.1903495044
25.00	0.0608748582	0.1319960307	0.2314909784
30.00	0.0692263981	0.1512276363	0.2683018733
35.00	0.0756151579	0.1666347986	0.2998747444
40.00	0.0798196342	0.1776612089	0.3252686663
45.00	0.0816702577	0.1838016433	0.3434935634
50.00	0.0810584092	0.1846214224	0.3534892976
55.00	0.0779452734	0.1797838409	0.3540997396
60.00	0.0723698557	0.1690871795	0.3440458136
65.00	0.0644552152	0.1525118009	0.3219126530
70.00	0.0544117214	0.1302764007	0.2861959773
75.00	0.0425360450	0.1028942266	0.2355241794
80.00	0.0292047722	0.0712142984	0.1692995929
85.00	0.0148617652	0.0364253485	0.0890392818
90.00	0.0000000003	0.0000000008	0.0000000019

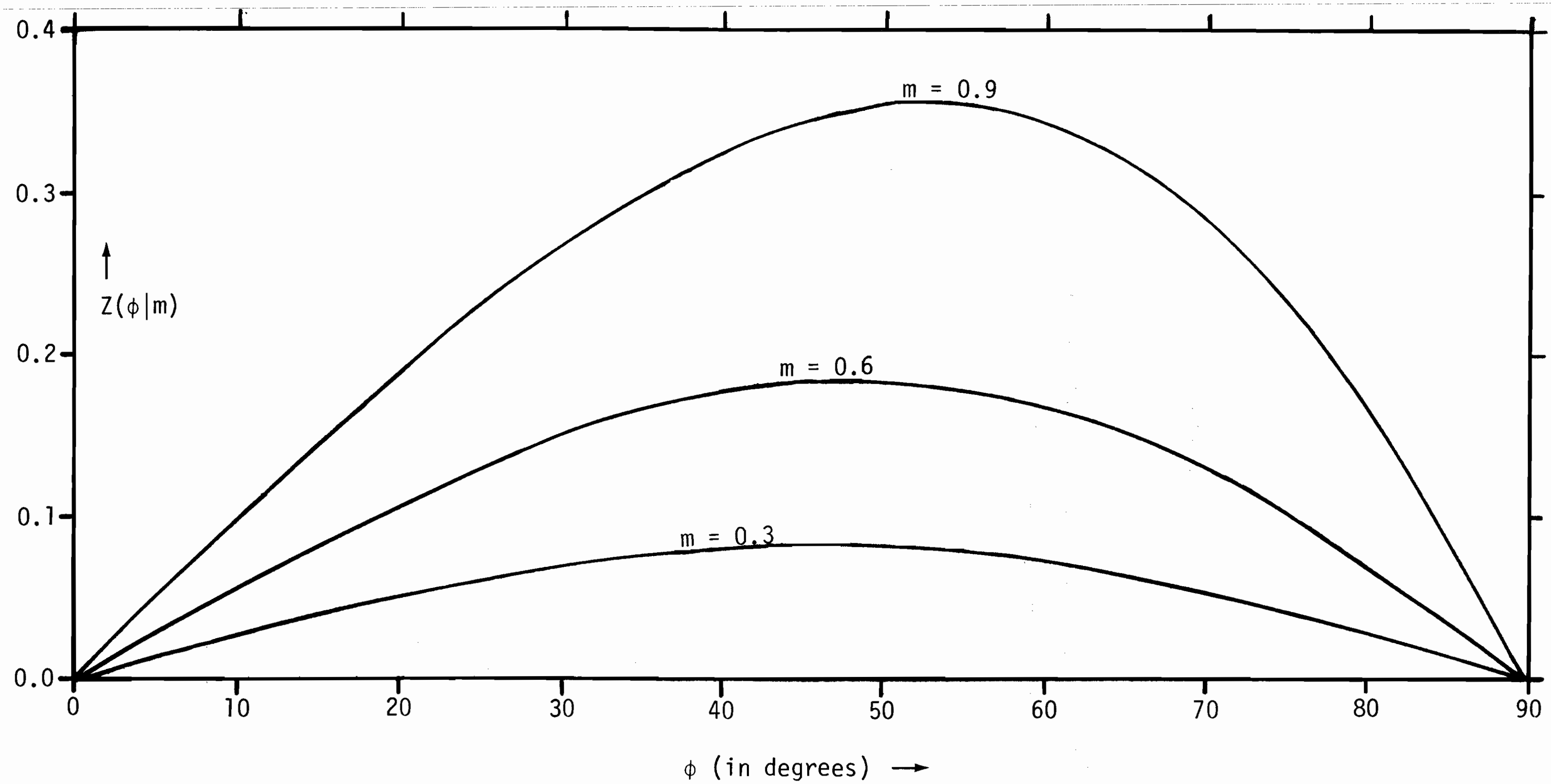

Figure 4.1. Plot of $Z(\phi|m)$ as a Function of ϕ for Three Values of m (See Table 4.1.)

Listing of Subroutine ZETA

Note 1: While using ZETA, the subroutine TEK has to be supplemented.

```
      SUBROUTINE ZETA(PHI,RM,SIG,ZPHI)
      DIMENSION AA(50), BB(50), CC(50), PSAV(50)
      DATA PIO4/.785398163397448/,TPI/6.28318530717959/
      DATA PI,PIO2/3.14159265358979323846264 3E0,1.5707963267948966192E0/
      IF (ABS(RM-.5)-.5) 15,15,5
5     PRINT 10, RM
10    FORMAT (5X,9H*********,3X,13HLOOK OUT M = ,F8.3,3X,9H*********)
      RETURN
15    CONTINUE
      CALL TEK(0,RM,EKM,EM)
      IF (PHI) 20,25,25
20    W=-1.
      PH=-PHI
      GO TO 30
25    W=1.
      PH=PHI
30    RK=SQRT(RM)
      N=PH/TPI
      A=PH-FLOAT(N)*TPI
      B=A/PIO2
      K=B
      NQ=K+1
      GO TO (35,40,45,50), NQ
35    NK=4*N
      SIGNEM=1.
      AP=A
      GO TO 55
40    NK=4*N+2
      SIGNEM=-1.
      AP=PI-A
      GO TO 55
45    NK=4*N+2
      SIGNEM=1.
      AP=A-PI
      GO TO 55
50    NK=4*N+4
      SIGNEM=-1.
      AP=TPI-A
55    CNK=NK
      PHI=AP
      CALL TEK (0,RM,EK,EE)
      PLUS=CNK*EK
      PLUS1=CNK*EE
      IT=0
      IF (ABS(PHI-PIO2)-1.E-10) 60,60,65
60    IT=1
65    IF (ABS(RK-1.E0)-1.E-10) 70,65,85
70    IT=IT+1
      GO TO (75,80), IT
75    TF=W*(PLUS+SIGNEM*ALOG(TAN(PIO4+PHI*.5)))
      TE=W*(PLUS1+SIGNEM*SIN(PHI))
      ZPHI=TE-(TF*(EM/EKM))
      RETURN
```

```
80      TF=W*1.E75
        TE=W*(PLUS1+SIGNEM)
        ZPHI=TE-(TF*(EM/EKM))
        RETURN
85      IF (ABS(RK)-1.E-15) 90,95,95
90      TF=W*(PLUS+SIGNEM*PHI)
        TE=W*(PLUS1+SIGNEM*PHI)
        ZPHI=TE-(TF*(EM/EKM))
        RETURN
95      IT=IT+1
        GO TO (105,100), IT
100     CALL TEK (0,RM,EK,EE)
        TF=W*(PLUS+SIGNEM*EK)
        TE=W*(PLUS1+SIGNEM*EE)
        ZPHI=TE-(TF*(EM/EKM))
        RETURN
105     IF (ABS(PHI)-1.E-50) 110,115,115
110     TF=W*PLUS
        TE=W*PLUS1
        ZPHI=TE-(TF*(EM/EKM))
        RETURN
115     IF (RM-.75) 120,140,140
120     CALL TEK (0,RM,EK,EE)
        S=SIN(PHI)
        C=COS(PHI)
        SK=RM
        CE=2.*PHI/PI
        TZ=CE*EK
        T1=CE*EE
        A=.5E0
        T=.5E0*A*SK
        R=T
        SS=S*S
        PS=1.E0
        H=.5
        F=.5E0
        PK=SK
        U1=10.
        DO 130 I=2,20000
        J=I*2
        D=FLOAT(J-1)
        G=FLOAT(J-3)
        E=1./FLOAT(J)
        PS=SS*PS
        A=E*(D*A+PS)
        F=D*E*F
        H=G*E*H
        PK=PK*SK
        U=F*A*PK
        IF (U1*U1/(U1-U)-SIG) 135,135,125
125     U1=U
        T=U+T
130     R=H*A*PK+R
135     TF=W*((TZ-S*C*T)*SIGNEM+PLUS)
        TE=W*((T1+S*C*R)*SIGNEM+PLUS1)
        ZPHI=TE-(TF*(EM/EKM))
        RETURN
140     ALPHAR=ASIN(RK)
        AA(1)=1.
        BB(1)=COS(ALPHAR)
```

```
      DO 145 I=2,50
      II=I-1
      AA(I)=.5*(AA(II)+BB(II))
      BB(I)=SQRT(AA(II)*BB(II))
      CC(I)=.5*(AA(II)-BB(II))
      IF (ABS(CC(I))-SIG) 150,145,145
145   CONTINUE
      ISTOP=50
      GO TO 155
150   ISTOP=I
155   P=PHI
      P2=1.
      NQ=1
      IOS=1
      M2P=0
      I4=0
      ORELEP=1.E25
      OR=1.E25
      DO 215 I=1,ISTOP
      PSAV(I)=P
      P2=P2*2.
      BD=TAN(P)*BB(I)/AA(I)
      BF=ATAN(BD)
160   INS=SIGN(1.,BF)
      IF (IOS*INS) 165,170,170
165   NQ=NQ+1
      IF (NQ.EQ.5) NQ=1
170   GO TO (175,190,190,195), NQ
175   IF (I4) 180,185,180
180   I4=0
      M2P=M2P+1
185   BE=BF+FLOAT(M2P)*TPI
      GO TO 200
190   BE=BF+PI+FLOAT(M2P)*TPI
      GO TO 200
195   BE=BF+TPI+FLOAT(M2P)*TPI
      I4=1
200   IOS=INS
      PR=P/BE
      RELER=ABS(OR-PR)/(PR+OR)
      IF (ORELER-RELER) 205,210,210
205   IOS=-IOS
      GO TO 160
210   P=BE+P
      OR=PR
215   ORELER=RELER
      TF=W*(PLUS+SIGNEM*(P/(P2*AA(ISTOP))))
      CALL TEK (G,RM,EK,EE)
      SUMEM=0.
      DO 220 IK=2,ISTOP
220   SUMEM=SUMEM+CC(IK)*SIN(PSAV(IK))
      TE=W*(PLUS1+SIGNEM*(EE/EK*TF+SUMEM))
      ZPHI=TE-(TF*(EM/EKM))
      RETURN
      END
```

Subroutine CZETA, which is derived from the subroutine CEF of an earlier section, accepts w and m and returns $Z(w|m)$. It can be used by the statement

```
CALL CZETA (CW,M,CZW)
```

where

- CW - Complex argument $w = u + iv$.
- M - Real parameter m.
- CZW - Subroutine returns the complex number $Z(w|m)$ in this location.

In what follows, $Z(w|m)$ is computed at all of the sample points described by Figure 2.1 and tabulated for m = 0.3, 0.6 and 0.9 (Tables 4.2, 4.3 and 4.4, respectively). A listing of the subroutine CZETA is also included.

Table 4.2. Sample Output of Subroutine CZETA for m = 0.3

$\theta°$	$Z(0.5\ e^{i\theta}\|m)$		$Z(1.0\ e^{i\theta}\|m)$		$Z(1.5\ e^{i\theta}\|m)$		$Z(2.0\ e^{i\theta}\|m)$		$Z(2.5\ e^{i\theta}\|m)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.06661	0.00000	0.07798	0.00000	0.02997	0.00000	-0.44319	0.00000	-1.24537	0.00000
20.00	0.06725	0.01674	0.09602	-0.01099	0.05868	-0.07265	-0.27279	-0.11688	-1.09614	-0.08631
40.00	0.06549	0.03930	0.14582	0.01065	0.18487	-0.12968	0.09534	-0.35569	-0.42591	-0.53320
60.00	0.05205	0.06710	0.18699	0.10848	0.45328	-0.02185	0.60681	-0.49567	0.41300	-0.95325
80.00	0.02052	0.08851	0.10427	0.25942	0.54441	0.65264	2.53847	-0.08425	1.16645	-1.93017
100.00	-0.02052	0.08851	-0.10427	0.25942	-0.54441	0.65264	-2.53847	-0.08425	-1.16645	-1.93017
120.00	-0.05205	0.06710	-0.18699	0.10848	-0.45328	-0.02185	-0.60681	-0.49567	-0.41300	-0.95325
140.00	-0.06549	0.03930	-0.14582	0.01065	-0.18487	-0.12968	-0.09534	-0.33569	0.42591	-0.53320
160.00	-0.06725	0.01674	-0.09602	-0.01099	-0.05868	-0.07265	0.27279	-0.11688	1.09614	-0.08631
180.00	-0.06661	-0.00000	-0.07798	-0.00000	-0.02997	-0.00000	0.44319	-0.00000	1.24537	-0.00000
200.00	-0.06725	-0.01674	-0.09602	0.01099	-0.05868	0.07265	0.27279	0.11688	1.09614	0.08631
220.00	-0.06549	-0.03930	-0.14582	-0.01065	-0.18487	0.12968	-0.09534	0.35569	0.42591	0.53320
240.00	-0.05205	-0.06710	-0.18699	-0.10848	-0.45328	0.02185	-0.60681	0.49567	-0.41300	0.95325
260.00	-0.02052	-0.08851	-0.10427	-0.25942	-0.54441	-0.65264	-2.53847	0.08425	-1.16645	1.93017
280.00	0.02052	-0.08851	0.10427	-0.25942	0.54441	-0.65264	2.53847	0.08425	1.16645	1.93017
300.00	0.05205	-0.06710	0.08699	-0.10848	0.45328	0.02185	0.60681	0.49567	0.41300	0.95325
320.00	0.06549	-0.03930	0.14582	-0.01065	0.18487	0.12968	0.09534	0.33569	-0.42591	0.53320
340.00	0.06725	-0.01674	0.09602	0.01099	0.05868	0.07265	-0.27279	0.11688	-1.09614	0.08631
360.00	0.06661	0.00000	0.07798	0.00000	0.02997	0.00000	-0.44319	0.00000	-1.24537	0.00000

Table 4.3. Sample Output of Subroutine CZETA for m = 0.6

θ°	$Z(0.5\ e^{i\theta}\|m)$		$Z(1.0\ e^{i\theta}\|m)$		$Z(1.5\ e^{i\theta}\|m)$		$Z(2.0\ e^{i\theta}\|m)$		$Z(2.5\ e^{i\theta}\|m)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.14385	0.00000	0.18278	0.00000	0.11227	0.00000	-0.05377	0.00000	-0.60024	0.00000
20.00	0.14419	0.03733	0.21311	-0.00801	0.16587	-0.11165	0.02621	-0.20624	-0.49924	-0.24707
40.00	0.13853	0.08516	0.29777	0.03769	0.36156	-0.18920	0.24623	-0.46358	0.02256	-0.69270
60.00	0.10941	0.14276	0.38215	0.21869	0.79949	-0.09410	0.78216	-0.74904	0.45770	-0.66350
80.00	0.04319	0.18745	0.23469	0.54875	1.59721	1.15504	2.13554	-1.83295	0.58318	-1.11378
100.00	-0.04319	0.18745	-0.23469	0.54875	-1.59721	1.15504	-2.13554	-1.83295	-0.58318	-1.11378
120.00	-0.10941	0.14276	-0.38215	0.21869	-0.79949	-0.09410	-0.78216	-0.74904	-0.45770	-0.66350
140.00	-0.13853	0.08516	-0.29777	0.03769	-0.36156	-0.18920	-0.24623	-0.46358	-0.02256	-0.69270
160.00	-0.14419	0.03733	-0.21311	-0.00801	-0.16587	-0.11165	-0.02621	-0.20624	0.49924	-0.24707
180.00	-0.14385	-0.00000	-0.18278	-0.00000	-0.11227	-0.00000	0.05377	-0.00000	0.60024	-0.00000
200.00	-0.14419	-0.03733	-0.21311	0.00801	-0.16587	0.11165	-0.02621	0.20624	0.49924	0.24707
220.00	-0.12853	-0.08516	-0.29777	-0.03769	-0.36156	0.18920	-0.24623	0.46358	-0.02256	0.69270
240.00	-0.10941	-0.14276	-0.38215	-0.21869	-0.79949	0.09410	-0.78216	0.74904	-0.45770	0.66350
260.00	-0.04319	-0.18745	-0.23469	-0.54875	-1.59721	-1.15504	-2.13554	1.83295	-0.58318	1.11378
280.00	0.04319	-0.18745	0.23469	-0.54875	1.59721	-1.15504	2.13554	1.83295	0.58318	1.11378
300.00	0.10941	-0.14276	0.38215	-0.21869	0.79949	0.09410	0.78216	0.74904	0.45770	0.66350
320.00	0.13853	-0.08516	0.29777	-0.03769	0.36156	0.18920	0.24623	0.46358	0.02256	0.69270
340.00	0.14419	-0.03733	0.21311	0.00801	0.16587	0.11165	0.02621	0.20624	-0.49924	0.24707
360.00	0.14385	0.00000	0.18278	0.00000	0.11227	0.00000	-0.05377	0.00000	-0.60024	0.00000

Table 4.4. Sample Output of Subroutine CZETA for m = 0.9

θ°	$Z(0.5\,e^{i\theta}\vert m)$		$Z(1.0\,e^{i\theta}\vert m)$		$Z(1.5\,e^{i\theta}\vert m)$		$Z(2.0\,e^{i\theta}\vert m)$		$Z(2.5\,e^{i\theta}\vert m)$	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
0.00	0.25149	0.00000	0.35386	0.00000	0.30947	0.00000	0.18381	0.00000	0.02564	0.00000
20.00	0.24937	0.06854	0.38820	0.02218	0.37183	-0.09478	0.25187	-0.20725	0.08743	-0.29493
40.00	0.23426	0.15031	0.48706	0.10667	0.58281	-0.15453	0.47443	-0.42122	0.29482	-0.60666
60.00	0.18194	0.24408	0.60304	0.36562	1.08445	-0.10605	0.90609	-0.52552	0.57273	0.02296
80.00	0.07145	0.31628	0.40299	0.90564	3.00789	0.96960	1.37965	-1.42665	0.46838	-0.03174
100.00	-0.07145	0.31628	-0.40299	0.90564	-3.00789	0.96960	-1.37965	-1.42665	-0.46838	-0.03174
120.00	-0.18194	0.24408	-0.60304	0.36562	-1.08445	-0.10605	-0.90609	-0.52552	-0.57273	0.02296
140.00	-0.23426	0.15031	-0.48706	0.10667	-0.58281	-0.15453	-0.47443	-0.42122	-0.29482	-0.60666
160.00	-0.24937	0.06854	-0.38820	0.02218	-0.37183	-0.09478	-0.25187	-0.20725	-0.09843	-0.29493
180.00	-0.25149	-0.00000	-0.35386	-0.00000	-0.30947	-0.00000	-0.18381	-0.00000	-0.02564	-0.00000
200.00	-0.24937	-0.06854	-0.38820	-0.02218	-0.37183	0.09478	-0.25187	0.20725	0.09843	0.29493
220.00	-0.23426	-0.15031	-0.48706	-0.10667	-0.58281	01.5453	-0.47443	0.42122	-0.29482	0.60666
240.00	-0.18194	-0.24408	-0.60304	-0.36562	-1.08445	0.10605	-0.90609	0.52552	-0.57273	-0.02296
260.00	-0.07145	-0.31628	-0.40299	-0.90564	-3.00789	-0.96960	-1.37965	1.42665	-0.46838	0.03174
280.00	0.07145	-0.31628	0.40299	-0.90564	3.00789	-0.96960	1.37965	1.42665	0.46838	0.03174
300.00	0.18194	-0.24408	0.60304	-0.36562	1.08445	0.10605	0.90609	0.52552	0.57273	-0.02296
320.00	0.23426	-0.15031	0.48706	-0.10667	0.58281	0.15453	0.47443	0.42122	0.29482	0.60666
340.00	0.24937	-0.06854	0.38820	-0.02218	0.37183	0.09478	0.25187	0.20725	0.08743	0.29493
360.00	0.25149	0.00000	0.35386	0.00000	0.30947	0.00000	0.18381	0.00000	0.02564	0.00000

Listing of Subroutine CZETA

Note 1: While using CZETA, subroutines CJEFS, JEFS, TEF and TEK have to be supplemented.

```
      SUBROUTINE CZETA(CW,M,CZW)
      IMPLICIT COMPLEX (C)
      REAL M,M1
C
C     NOTE... IF REAL OR IMAG. PART OF CW LIES BETWEEN -1.E-08
C     AND 1.E-08 , IT WILL BE TREATED AS ZERO.
C
      CI=(0.,1.)
      CALL TEK(0.M,EK,E)
      EP=1.E-08
      EM=-EP
      WR=REAL (CW)
      WI=AIMAG(CW)
      S=ABS(WR)
      T=ABS(WI)
      IF (S.LT.EP.AND.T.LT.EP) GO TO 50
C     NOW, WE SET UP THE CONDITION NUMBER...
      IF (WR.GT.EP.AND.WI.GT.EP) IC=1
      IF (WR.LT.EM.AND.WI.GT.EP) IC=2
      IF (WR.LT.EM.AND.WI.LT.EM) IC=3
      IF (WR.GT.EP.AND.WI.LT.EM) IC=4
      IF (T.LE.EP.AND.WR.GE.EP) IC=5
      IF (T.LE.EP.AND.WR.LE.EM) IC=6
      IF (S.LE.EP.AND.WI.GE.EP) IC=7
      IF (S.LE.EP.AND.WI.LE.EM) IC=8
C     COMMENCE COMPUTATION...
      M1=1.-M
      IF (IC-5) 10,20,20
 10   CONTINUE
      CWP=CMPLX(S,T)
      CALL CJEFS(CWP,M,CSN,CCN,CDN)
      CALL JEFS(S,M,USN,UCN,UDN)
      CALL JEFS(T,M1,V1SN,V1CN,V1DN)
      UPHI=ASIN(USN)
      VPHI=ASIN(V1SN)
      CALL TEF(UPHI,M,1.E-06,FUM,EUM)
      CALL TEF(VPHI,M1,1.E-06,FVM1,EVM1)
      SCV1=V1SN/V1CN
      CEIV=CI*(T+(V1DN*SCV1)-EVM1)
      CEM=EUM+CEIV-(CI*M*USN*SCV1*CSN)
      EMR=REAL(CEM)
      EMI=AIMAG(CEM)
      IF (IC.EQ.2) EMR=-EMR
      IF (IC.EQ.4) EMI=-EMI
      CEM=CMPLX(EMR,EMI)
      IF (IC.EQ.3) CEM=-CEM
      CZW=CEM-(CW*(E/EK))
      RETURN
 20   CONTINUE
```

```
      IF (IC-7) 30,40,40
30    CONTINUE
      CALL JEFS(S,M,USN,UCN,UDN)
      UPHI=ASIN(USN)
      CALL TEF(UPHI,M,1.E-06,FUM,EUM)
      IF (IC.EQ.5) CEM= CMPLX(EUM,0.)
      IF (IC.EQ.6) CEM= CMPLX(-EUM,0.)
      CZW=CEM-(CW*(E/EK))
      RETURN
40    CONTINUE
      CALL JEFS(T,M1,V1SN,V1CN,V1DN)
      VPHI=ASIN(V1SN)
      CALL TEF(VPHI,M1,1.E-06,FVM1,EVM1)
      SCV1=V1SN/V1CN
      EIV=(T+(V1DN*SCV1)-EVM1)
      IF (IC.EQ.7) CEM=CMPLX(0.,EIV)
      IF (IC.EQ.8) CEM= CMPLX(0.,-EIV)
      CZW=CEM-(CW*(E/EK))
      RETURN
50    CONTINUE
      CZW=(0.,0.)
      RETURN
      END
```

V. Complete and Incomplete Elliptic Integral of the Third Kind

A. Real Argument

The elliptic integral of the third kind appears to be less common in physical problems than the first two kinds. It has an integral representation given by

$$\Pi(n;\ \phi\backslash\alpha) = \int_0^{\phi} \frac{1}{\left(1 - n\sin^2\theta\right)\left(1 - \sin^2\alpha\ \sin^2\theta\right)^{1/2}}\, d\theta \tag{5.1}$$

or, with $\sin^2\alpha = m$

$$\Pi(n;\ \phi|m) = \int_0^{\phi} \frac{1}{\left(1 - n\sin^2\theta\right)\left(1 - m\sin^2\theta\right)^{1/2}}\, d\theta \tag{5.2}$$

The integral is said to be complete if $\phi = \pi/2$ and is denoted by $\Pi(n;\ \pi/2|m) \equiv \Pi(n\backslash\alpha) = \Pi(n|m)$. Of course, a lot depends on how the real parameters n and m compare numerically and this leads to 4 cases and 10 special cases as discussed in Section 17.7 of Reference [6]. We shall outline all the different cases here and then proceed with their numerical evaluation.

Case (1): Hyperbolic Case $0 < n < m$

$\Pi(n;\ \phi|m)$ is computed via the following steps:

$$\varepsilon = \arcsin(n/m)^{1/2}\quad,\qquad 0 \le \varepsilon \le (\pi/2)$$

$$\beta = (\pi/2)F(\varepsilon|m)\big/K(m)$$

$$q = q(m) = \exp\left[-\pi\ K(m_1)\big/K(m)\right] = \text{Nome}$$

$$v = (\pi/2)F(\phi|m)\big/K(m)$$

$$\delta_1 = \left[\frac{n}{(1 - n)(m - n)}\right]^{1/2}$$

$$\lambda = 2\sum_{s=1}^{\infty} \frac{q^s \sin(2sv)\sin(2s\beta)}{s\left(1 - q^{2s}\right)}$$

(list of steps continued)

(list of steps concluded)

$$\mu = \cot\beta + 4 \sum_{s=1}^{\infty} \frac{q^{2s}\sin 2\beta}{\left(1 - 2q^{2s}\cos(2\beta) + q^{4s}\right)}$$

$$\Pi(n;\ \phi|m) = \delta_1(-\lambda + \nu\mu)$$

$$\Pi(n|m) = K(m) + \delta_1\ K(m)Z(\varepsilon|m) \tag{5.3}$$

Subroutine E3NLM was written to compute the elliptic integral of the third kind for this hyperbolic case with $(0 < n < m)$ and $(m < 1)$. In this context, it is noted that in evaluating λ and μ, the series are summed up to the M^{th} term so that the following convergence criterion is met:

$$\left|\frac{M^{th}\ \text{term} - (M - 1)^{th}\ \text{term}}{M^{th}\ \text{term}}\right| \le 10^{-4} \tag{5.4}$$

or if $\left|M^{th}\ \text{term}\right| \le 10^{-35}$.

The condition on the absolute value of the M^{th} term is useful and adequate because the successive terms in all of the series in this section decrease rapidly owing to the nome $q(m)$ being < 1.

Case (2): Hyperbolic Case $n > 1$

This case $(n > 1)$ can be reduced to the case $0 < N < m$ by defining

$$N = (m/n)$$

$$p_1 = \left[(n - 1)\left\{1 - (m/n)\right\}\right]^{1/2}$$

from which

$$\Pi(n;\ \phi|m) = -\Pi(N;\ \phi|m) + F(\phi|m) + \frac{1}{2p_1}\ \ell n\left[\frac{\Delta(\phi) + p_1\tan\phi}{\Delta(\phi) - p_1\tan\phi}\right]$$

$$\Pi(n|m) = K(m) - \Pi(N|m) \tag{5.5}$$

In Eq. (5.5), $\Pi(N;\ \phi|m)$ and $\Pi(N|m)$ are computable by the subroutine E3NLM and $\Delta(\phi)$ is given by [4]

$$\Delta(\phi) = \text{the delta amplitude}$$

$$= (1 - m\sin^2\phi)^{1/2} \tag{5.6}$$

Case (3): Circular Case $m < n < 1$

For this case, $\Pi(n; \phi|m)$ is evaluated by using the following steps:

$$\varepsilon = \arcsin[(1 - n)/m]^{1/2} \quad , \quad 0 \le \varepsilon \le (\pi/2)$$

$$\beta = (\pi/2)F(\varepsilon|m_1)/K(m)$$

$$m_1 = (1 - m)$$

$$q = q(m) = \exp\left[-\pi\left(K\ m_1\right)/K(m)\right]$$

$$v = (\pi/2)F(\phi|m)/K(m)$$

$$\delta_2 = \left[\frac{n}{(1 - n)(n - m)}\right]^{1/2}$$

$$\lambda = \arctan(\tanh\beta\ \tan v) + 2\sum_{s=1}^{\infty} \frac{(-1)^{s-1}q^{2s}\sin(2sv)\sinh(2s\beta)}{s\left(1 - s^{2s}\right)}$$

$$\mu = \frac{\left[\sum_{s=1}^{\infty} sq^{s^2}\sinh(2s\beta)\right]}{\left[1 + 2\sum_{s=1}^{\infty} q^{s^2}\cosh(2s\beta)\right]}$$

finally

$$\Pi(n; \phi|m) = \delta_2(\lambda - 4\mu v)$$

$$\Pi(n|m) = K(m) + 0.5\ \pi\delta_2\left[1 - \Lambda_0(\varepsilon|m)\right] \qquad (5.7)$$

where Λ_0 is Heuman's Lambda function [6] given by

$$\Lambda_0(\phi|m) = \frac{F\left(\phi|m_1\right)}{K\left(m_1\right)} + \frac{2}{\pi}K(m)Z\left(\phi|m_1\right) \qquad (5.8)$$

Subroutine E3MLN was written to compute the circular case ($m < n < 1$) and the series were terminated with a similar criterion as expressed by Eq. (5.4).

Case (4): <u>Circular Case $n < 0$</u>

This case can be reduced to the case $m < N < 1$ by defining

$$N = (m - n)/(1 - n)$$

$$p_2 = [-n(m - n)/(1 - n)]^{1/2}$$

then $\Pi(n; \phi|m)$ is computed from

$$\Pi(n; \phi|m) = \left(T_1 + T_2 + T_3\right)/T_4 \tag{5.9}$$

where

$$T_1 = \left[(1 - N)\left\{1 - (m/N)\right\}\right]^{1/2} \Pi(N; \phi|m) \tag{5.10}$$

$$T_2 = \left(m/p_2\right)F\left(\phi|m\right) \tag{5.11}$$

$$T_3 = \arctan\left[0.5p_2 \sin(2\phi)/\Delta(\phi)\right] \tag{5.12}$$

$$T_4 = \left[(1 - n)\left\{1 - (m/n)\right\}\right]^{1/2} \tag{5.13}$$

with

$$\Delta(\phi) = \left[1 - m \sin^2\phi\right]^{1/2} \tag{5.14}$$

$\Pi(N; \phi|m)$ appearing in T_1 may be computed by calling the subroutine E3MLN. For this circular case ($n < 0$), if the integral is complete (i.e., $\phi = \pi/2$), then we use [6]

$$\Pi(n|m) = T_5 + T_6 \tag{5.15}$$

where

$$T_5 = \left\{\frac{-n\, m_1\, \Pi(N|m)}{(1 - n)(m - n)}\right\} \tag{5.16}$$

$$T_5 = m\,K(m)/(m - n) \tag{5.17}$$

$$m_1 = (1 - m) \tag{5.18}$$

We now proceed to the special cases (Eqs. 17.7.18 through 17.7.25 of Reference [6]) which are ten in number.

Special Case (1): <u>n = 0</u>

$$\Pi(0;\ \phi|m) = F(\phi|m) \tag{5.19}$$

Special Case (2): <u>n = 0, m = 0</u>

$$\Pi(0;\ \phi|m) = \phi \tag{5.20}$$

Special Case (3): <u>m = 0</u>

$$\Pi(n;\ \phi|0) = (1 - n)^{-1/2} \arctan\left[(1 - n)^{1/2}\tan\phi\right];\ n < 1 \tag{5.21}$$

$$= (n - 1)^{-1/2} \arctan\left[(n - 1)^{1/2}\tan\phi\right];\ n > 1 \tag{5.22}$$

$$= \tan\phi \qquad ;\ n = 1 \tag{5.23}$$

Special Case (4): <u>m = 1</u>

$$\Pi(n;\ \phi|1) = (1 - n)^{-1}\left[\ell n(\tan\phi + \sec\phi) - \frac{1}{2}\sqrt{n}\,\ell n\,\frac{1 + \sqrt{n}\sin\phi}{1 - \sqrt{n}\sin\phi}\right];\ \begin{matrix} n \neq 1 \\ \phi \neq (\pi/2) \end{matrix} \tag{5.24}$$

Special Cases (5) and (6): <u>$n = \pm\sqrt{m}$</u>

$$\Pi(\pm\sqrt{m};\ \phi|m) = \frac{1}{2}\left[F(\phi|m) + \frac{T_1}{T_2}\right] \tag{5.25}$$

with

$$T_1 = \arctan\left[(1 \mp \sqrt{m})\tan\phi/\Delta(\phi)\right] \tag{5.26}$$

$$T_2 = (1 \mp \sqrt{m}) \tag{5.27}$$

$$\Delta(\phi) = [1 - m\sin^2\phi]^{1/2} \tag{5.28}$$

Special cases (5) and (6) respectively correspond to the top and bottom signs in the previous equation.

Special Cases (7) and (8): $n = 1 \pm \sqrt{(1 - m)}$

$$\Pi\left(1 \pm \sqrt{(1 - m)}\ ;\ \phi|m\right) = \left(T_1 + T_2 + T_3\right) \Big/ T_4 \tag{5.29}$$

where

$$T_1 = \pm \frac{1}{2} \ell n \left[\frac{1 + \tan\phi\ \Delta(\phi)}{1 - \tan\phi\ \Delta(\phi)}\right] \tag{5.30}$$

$$T_2 = \frac{1}{2} \ell n \left[\frac{\Delta(\phi) + \sqrt{(1 - m)}\tan\phi}{\Delta(\phi) - \sqrt{(1 - m)}\tan\phi}\right] \tag{5.31}$$

$$T_3 = \mp \left[1 \mp \sqrt{(1 - m)}\right] F(\phi|m) \tag{5.32}$$

$$T_4 = 2\sqrt{(1 - m)} \tag{5.33}$$

and

$$\Delta(\phi) = [1 - m \sin^2\phi]^{1/2} \tag{5.34}$$

As before, special cases (7) and (8) correspond respectively to the top and bottom signs in the above equation.

Special Case (9): $n = m$

$$\Pi(m;\ \phi|m) = T_1 + T_2 \tag{5.35}$$

where

$$T_1 = E(\phi|m)/(1 - m) \tag{5.36}$$

$$T_2 = -\frac{m \sin(2\phi)}{(1 - m)2\ \Delta(\phi)} \tag{5.37}$$

with

$$\Delta(\phi) = \text{delta amplitude} = [1 - m \sin^2\phi]^{1/2} \tag{5.38}$$

Special Case (10): <u>n = 1</u>

$$\Pi(1;\ \phi|m) = T_1 + T_2 + T_3 \tag{5.39}$$

where

$$T_1 = F(\phi|m) \tag{5.40}$$

$$T_2 = -\ E(\phi|m)/(1 - m) \tag{5.41}$$

$$T_3 = \tan\phi\ \Delta(\phi)/(1 - m) \tag{5.42}$$

with

$$\Delta(\phi) = [1 - m \sin^2\phi]^{1/2} \tag{5.43}$$

Subroutine EI3K combines all of the cases and the special cases into a program package. It is to be supplemented by the subroutines E3NLM, E3MLN, TEF and TEK. EI3K may be called by the standard Fortran statement

CALL EI3K (RN,PHI,RM,SIG,PYE)

where

RN - Real parameter n.

PHI - Real amplitude ϕ in radians.

RM - Real parameter m ($0 < m < 1$).

SIG - Real constant used in certain convergence criterion in TEF, typically = 10^{-6}.

PYE - Subroutine returns $\Pi(n;\ \phi|m)$ in this location.

A test run of the subroutine EI3K was conducted and the results are tabulated in Table 5.1 and plotted in Figure 5.1. The values of n and ϕ were so chosen so that Figure 17.11 of Reference [6] could be reproduced using EI3K, for purposes of comparison. The agreement is found to be very good. A listing of the subroutine is also included in this section.

Table 5.1. Elliptic Integral of the Third Kind as Computed by Subroutine EI3K

m	α°	Π(0.1;15°|m)	Π(0.0;45°|m)	Π(1.0;45°|m)	Π(0.0;90°|m)	Π(0.7;90°|m)	Π(0.8;90°|m)
0.000000	0.00	0.26239175	0.78539816	1.00000000	1.57079633	2.86786860	3.51240736
0.007596	5.00	0.26241425	0.78594111	1.00081663	1.57379213	2.87493944	3.52166996
0.030154	10.00	0.26248114	0.78756494	1.00326027	1.58284280	2.89633714	3.54971480
0.066987	15.00	0.26259056	0.79025417	1.00731143	1.59814200	2.93262999	3.59733145
0.116978	20.00	0.26273944	0.79398144	1.01293509	1.62002590	2.98480954	3.66590018
0.178606	25.00	0.26292362	0.79870518	1.02007685	1.64899522	3.05436380	3.75749812
0.250000	30.00	0.26313789	0.80436613	1.02865728	1.68575035	3.14339453	3.87507018
0.328990	35.00	0.26337598	0.81088316	1.03856470	1.73124518	3.25479833	4.02269379
0.413176	40.00	0.26363106	0.81814777	1.04964597	1.78676913	3.39254403	4.20598638
0.500000	45.00	0.26389552	0.82601809	1.06169590	1.85407468	3.56210765	4.43274857
0.586824	50.00	0.26416142	0.83431268	1.07444622	1.93558109	3.77117175	4.71400082
0.671010	55.00	0.26442064	0.84280571	1.08755595	2.03471531	4.03079796	5.06573801
0.750000	60.00	0.26466511	0.85122407	1.10060512	2.15651565	4.35751349	5.51206094
0.821394	65.00	0.26488715	0.85924936	1.11309636	2.30878680	4.77731081	6.09119578
0.883022	70.00	0.26507961	0.86652996	1.12447185	2.50455008	5.33408433	6.86828536
0.933013	75.00	0.26523628	0.87269924	1.13414359	2.76806314	6.11030683	7.96670645
0.969846	80.00	0.26535205	0.87740833	1.14154642	3.15338524	7.29026749	9.66390669
0.992404	85.00	0.26542308	0.88036502	1.14620341	3.83174198	9.45493373	12.83689900
0.995134	86.00	0.26543168	0.88072675	1.14677365	4.05275815	10.17521466	13.90376041
0.997261	87.00	0.26543838	0.88100915	1.14721889	4.33865394	11.11349771	15.29870780
0.998782	88.00	0.26544318	0.88121142	1.14753785	4.74271722	12.44799748	17.28961750
0.999695	89.00	0.26544606	0.88133302	1.14772959	5.43490973	14.74615364	20.72854123

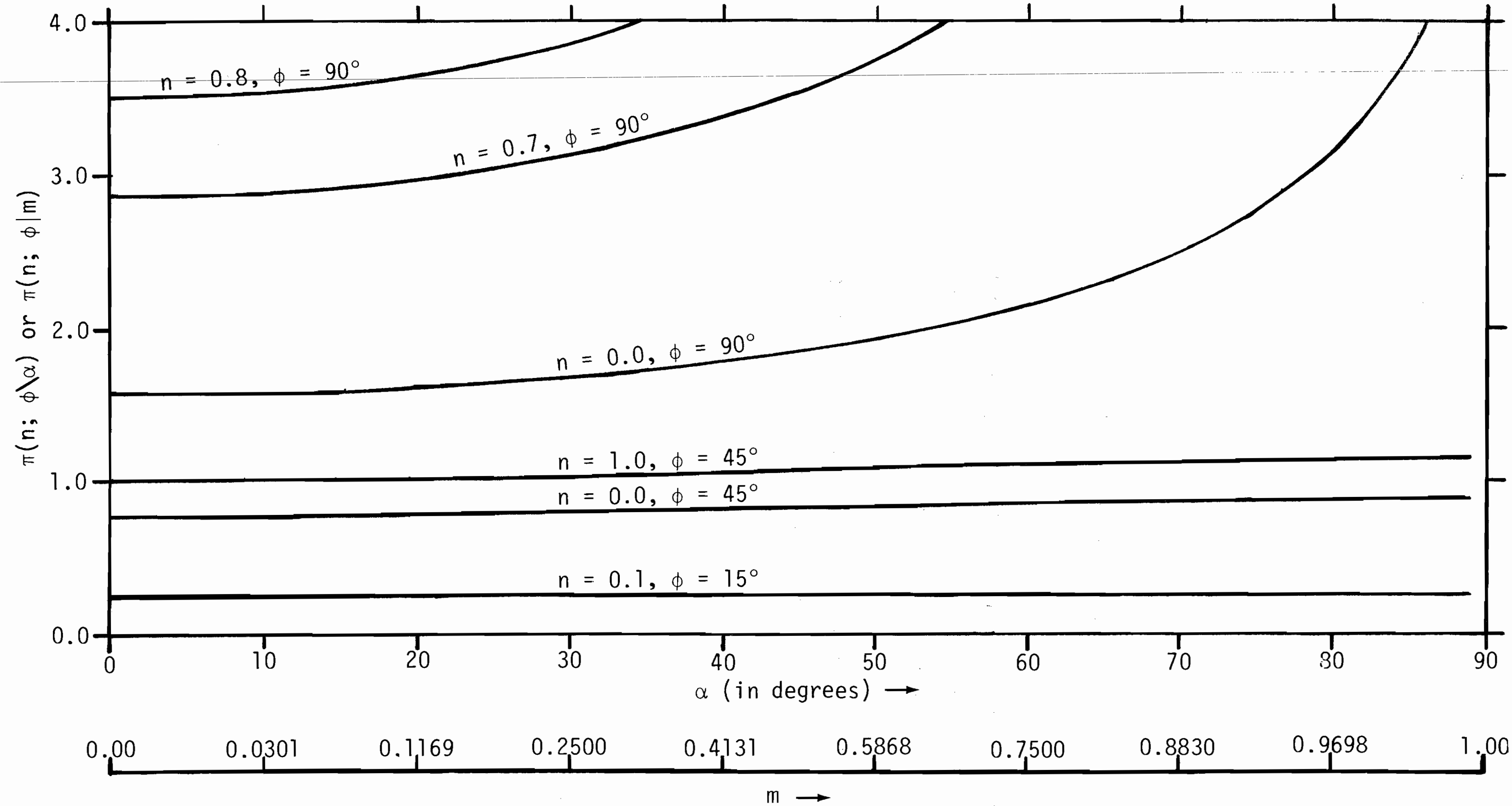

Figure 5.1. Plot of Elliptic Integral of the Third Kind (See Table 5.1 and also Figure 17.11 of Reference [6] for comparison.)

Listing of Subroutine EI3K

Note 1: While using EI3K, subroutines E3NLM, E3MLN, TEF and TEK have to be supplemented.

Note 2: Listings of subroutines E3NLM and E3MLN follow the listing of EI3K.

```
      SUBROUTINE EI3K(RN,PHI,RM,SIG,PYE)
C
C     THIS SUBROUTINE COMPUTES ELLIPTIC INTEGRAL OF THE THIRD KIND
C     PYE(N,PHI,M) GIVEN N,PHI AND M.  IT MAKES USE OF EQUATIONS IN
C     SECTION 17.7 OF HANDBOOK OF MATHEMATICAL FUNCTIONS BY ABRAMOWITZ
C     AND STEGUN A.M.S. 55)
C
      PI=3.14159265
C     NOW, WE FIND THE CASE NUMBER IC. THERE ARE 4 CASES AND 10
C     SPECIAL CASES.
      IF (RN.GT.0..AND.RN.LT.RM) IC=1
      IF (RN.GT.1.0) IC=2
      IF (RN.GT.RM.AND.RN.LT.1.0) IC=3
      IF (RN.LT.0.) IC=4
C     NOW, THE SPECIAL CASE NUMBER.
      EPS=1.E-08
      IF (ABS(RN).LE.EPS.AND.RM.GT.EPS) IC=5
      IF (ABS(RN).LE.EPS.AND.RM.LE.EPS) IC=6
      IF (RM.LE.EPS.AND.ABS(RN).GT.EPS) IC=7
      IF (RM.EQ.1.0.AND.RN.NE.1.0) IC=8
      RMS=SQRT(RM)
      IF (ABS(RN-RMS).LE.EPS) IC=9
      IF (ABS(RN+RMS).LE.EPS) IC=10
      RC=SQRT(1.-RM)
      IF (ABS(RN-1.-RC).LE.EPS) IC=11
      IF (ABS(RN-1.+RC).LE.EPS) IC=12
      IF (ABS(RN-RM).LE.EPS) IC=13
      IF (ABS(RN-1.).LE.EPS.AND.RM.GT.EPS) IC=14
      GO TO (10,20,30,40,50,60,70,80,90,90,110,110,130,140),IC
C     CASE NUMBER =1
   10 CONTINUE
      CALL E3NLM(RN,PHI,RM,SIG,PYE)
      RETURN
C     CASE NUMBER=2
   20 CONTINUE
      RNEW=RM/RN
      P1=SQRT((RN-1.)*(1.-RNEW))
      CALL TEF (PHI,RM,SIG,TFP,TEP)
      DPHI=SQRT(1.-RM*(SIN(PHI)**2))
      DNR=DPHI+(P1*TAN(PHI))
      DDR=DPHI-(P1*TAN(PHI))
      T3=(0.5/P1)*ALOG(DNR/DDR)
      CALL E3NLM(RNEW,PHI,RM,SIG,PYNEW)
      PYE=-PYNEW+TFP+T3
      RETURN
C     CASE NUMBER=3
   30 CONTINUE
      CALL E3MLN (RN,PHI,RM,SIG,PYE)
      RETURN
C     CASE NUMBER=4
   40 CONTINUE
```

```
      RNEW=(RM-RN)/(1.-RN)
      P2=SQRT(-RN*(RM-RN)/(1.-RN))
      CALL E3MLN (RNEW,PHI,RM,SIG,PYNEW)
      T1=SQRT( (1.-RNEW) * (1.-(RM/RNEW) ) ) * PYNEW
      CALL TEF (PHI,RM,SIG,TFP,TEP)
      T2=(RM/P2)*TFP
      DPHI=SQRT(1.-RM*(SIN(PHI)**2))
      T3=ATAN(0.5*P2*SIN(2.*PHI)/DPHI)
      PYE=(T1+T2+T3) / SQRT((1.-RN)*(1.-(RM/RN)))
      RETURN
C     CASE NUMBER=5
   50 CONTINUE
      CALL TEF(PHI,RM,SIG,PYE,TEP)
      RETURN
C     CASE NUMBER=6
   60 CONTINUE
      PYE=PHI
      RETURN
C     CASE NUMBER=7
   70 CONTINUE
      IF (RN-1.) 74,72,75
   72 PYE=TAN(PHI)
      RETURN
   74 SQP=SQRT(1.-RN)
      PYE=(1./SQP)*ATAN(SQP*TAN(PHI))
      RETURN
   75 SQP=SQRT(RN-1.)
      PYE=(1./SQP)*ATAN(SQP*TAN(PHI))
      RETURN
C     CASE NUMBER=8
   80 CONTINUE
      TANP=TAN(PHI)
      SECP=1./COS(PHI)
      SINP=SIN(PHI)
      T1=ALOG(TANP+SECP)
      SQN=SQRT(RN)
      SNP=1.+(SQN*SINP)
      SNM=1.-(SQN*SINP)
      T2=0.5*SQN*ALOG(SNP/SNM)
      PYE=(T1-T2)/(1.-RN)
      RETURN
C     CASE NUMBER 9 AND 10
   90 CONTINUE
      TP=TAN(PHI)
      DP=SQRT(1.-RM*(SIN(PHI)**2))
      SQM=SQRT(RM)
      IF (IC.EQ.9) SGN=-1.
      IF (IC.EQ.10) SGN=1.
      RHS=ATAN((1.+SGN*SQM)*TP/DP)
      PY=RHS/(1.+SGN*SQM)
      CALL TEF(PHI,RM,SIG,TFP,TEP)
      PY2=PY+TFP
      PYE=PY2*0.5
      RETURN
```

```
C     CASE NUMBER = 11 AND 12
 110  CONTINUE
      TP=TAN(PHI)
      CA=SQRT(1.-RM)
      DP=SQRT(1.-RM*(SIN(PHI)**2))
      TPN=1.+TP*DP
      TPD=1.-TP*DP
      IF (IC.EQ.11) SGN=1.
      IF (IC.EQ.12) SGN=-1.
      DPN=DP+CA*TP
      DPD=DP-CA*TP
      CALL TEF (PHI,RM,SIG,TFP,TEP)
      T1=SGN*0.5*ALOG(TPN/TPD)
      T2=0.5*ALOG(DPN/DPD)
      T3=-SGN*(1.-SGN*CA)*TFP
      PYE=(T1+T2+T3)/(2.*CA)
      RETURN
C     CASE NUMBER=13
 130  CONTINUE
      CALL TEF (PHI,RM,SIG,TFP,TEP)
      SE2A=1./(1.-RM)
      T2A=RM*SE2A
      S2P=SIN(2.*PHI)
      DP=SQRT(1.-RM*(SIN(PHI)**2))
      T1=SE2A*TEP
      T2=(T2A*S2P)/(2.*DP)
      PYE=T1-T2
      RETURN
C     CASE NUMBER=14
 140  CONTINUE
      CALL TEF (PHI,RM,SIG,TFP,TEP)
      TP=TAN(PHI)
      SE2A=1./(1.-RM)
      DP=SQRT(1.-RM*(SIN(PHI)**2))
      PYE=TFP-(SE2A*TEP)+(SE2A*TP*DP)
      RETURN
      END
```

Listing of Subroutine E3NLM

```
      SUBROUTINE E3NLM (RN,PHI,RM,SIG,PYE)
C
C     THIS SUBROUTINE COMPUTES ELLIPTIC INTERGRAL OF THE THIRD
C     KIND WHEN (0.LT.RN.LT.RM.LT.1)
C     USING EQNS 17.7.2 TO 17.7.5 OF A.M.S. 55
      DIMENSION T(200)
      PI=3.14159265
      E=ASIN(SQRT(RN/RM))
      CALL TEF (E,RM,SIG,TFE,TEE)
      CALL TEK (0,RM,EKM,EM)
      B=(PI/2.)*(TFE/EKM)
      COTH=1./TAN(B)
      RM1=1.-RM
      CALL TEK (0,RM1,EKM1,EM1)
      Q=EXP(-PI*EKM1/EKM)
      CALL TEF (PHI,RM,SIG,TFP,TEP)
      V=(PI/2.)*(TFP/EKM)
      D1=SQRT( RN / ( (1.-RN)*(RM-RN) ) )
      DO 11 IS=1,200
      IS2=2*IS
      RIS=FLOAT(IS)
      T(IS)=2.*((Q**IS)*SIN(2.*RIS*V)*SIN(2.*RIS*B)/(RIS*(1.-Q**IS2)))
      IF (IS.EQ.1) GO TO 11
      IF (ABS(T(IS)).LE.1.E-35) GO TO 12
      ER=ABS((T(IS)-T(IS-1))/T(IS))
      IF (ER.LT.SIG) GO TO 12
11    CONTINUE
      ISTOP=200
      GO TO 13
12    ISTOP=IS
13    SUM1=0.
      DO 14 ISUM=1,ISTOP
      SUM1=SUM1+T(ISUM)
14    CONTINUE
      DO 15 IS=1,200
      IS2=2*IS
      IS4=4*IS
      T(IS)=4.*(Q**IS2)*SIN(2.*B)/(1.-2.*(Q**IS2)*COS(2.*B)+(Q**IS4))
      IF (IS.LT.2) GO TO 15
      IF (ABS(T(IS)).LE.1.E-35) GO TO 16
      ER=ABS((T(IS)-T(IS-1))/T(IS))
      IF (ER.LT.SIG) GO TO 16
15    CONTINUE
      ISTOP=200
      GO TO 17
16    ISTOP=IS
17    SUM2=0.
      DO 18 ISUM=1,ISTOP
      SUM2=SUM2+T(ISUM)
18    CONTINUE
      PYE=D1*(-SUM1+V*(COTB+SUM2))
      RETURN
      END
```

Listing of Subroutine E3MLN

```
      SUBROUTINE E3MLN (RN,PHI,RM,SIG,PYE)
C
C     THIS SUBROUTINE COMPUTES ELLIPTIC INTEGRALS OF THE THIRD
C     KIND WHEN (RM.LT.RN.LT.1) USING EQNS. 17.7.9 THRU 17.7.15
C     OF A.M.S. 55.
C
      DIMENSION T(200)
      PI=3.14159265
      E=ASIN(SQRT((1.-RN)/(1.-RM)))
      RM1=1.-RM
      CALL TEF (E,RM1,SIG,TFE1,TE1)
      CALL TEK (0,RM,EKM,EM)
      CALL TEF (PHI,RM,SIG,TFP,TEP)
      CALL TEK (0,RM1,EKM1,EM1)
      B=0.5*PI*TFE1/EKM
      Q=EXP(-PI*EKM1/EKM)
      V=0.5*PI*TFP/EKM
      D2=SQRT(RN/((1.-RN)*(RN-RM)))
      DO 31 IS=1,200
      IS2=2*IS
      RIS=FLOAT(IS)
      IS1=IS-1
      SGN=((-1.)**IS1)
      Q2S=(Q**IS2)
      S2SV=SIN(2.*RIS*V)
      SB2=2.*RIS*B
      SH=0.5*(EXP(SB2)-EXP(-SB2))
      T(IS)=2.*SGN*Q2S*S2SV*SH/(RIS*(1.-Q2S))
      IF (IS.EQ.1) GO TO 31
      IF (ABS(T(IS)).LE.1.E-35) GO TO 32
      ER=ABS((T(IS)-T(IS-1))/T(IS))
      IF (ER.LT.SIG) GO TO 32
   31 CONTINUE
      ISTOP=200
      GO TO 33
   32 ISTOP=IS
   33 SUM1=0.
      DO 34 ISUM=1,ISTOP
      SUM1=SUM1+T(ISUM)
   34 CONTINUE
      DO 35 IS=1,200
      IS2=2*IS
      RIS=FLOAT(IS)
      ISS=IS**2
      SB2=2.*RIS*B
      T(IS)=RIS*(Q**ISS)*0.5*(EXP(SB2)-EXP(-SB2))
      IF (IS.LT.2) GO TO 35
      IF (ABS(T(IS)).LE.1.E-35) GO TO 36
      ER=ABS((T(IS)-T(IS-1))/T(IS))
      IF (ER.LT.SIG) GO TO 36
   35 CONTINUE
```

```
      ISTOP=200
      GO TO 37
36    ISTOP=IS
37    SUM2=0
      DO 38 ISUM=1,ISTOP
      SUM2=SUM2+T(ISUM)
38    CONTINUE
      DO 45 IS=1,200
      IS2=2*IS
      RIS=FLOAT(IS)
      ISS=IS**2
      SB2=2.*RIS*B
      T(IS)=(Q**ISS)*(EXP(SB2)+EXP(-SB2))
      IF (IS.LT.2) GO TO 45
      IF (ABS(T(IS)).LE.1.E-35) GO TO 46
      ER=ABS((T(IS)-T(IS-1))/T(IS))
      IF (ER.LT.SIG) GO TO 46
45    CONTINUE
      ISTOP=200
      GO TO 47
46    ISTOP=IS
47    SUM3=0.
      DO 48 ISUM=1,ISTOP
      SUM3=SUM3+T(ISUM)
48    CONTINUE
      SHB=EXP(SB2)-EXP(-SB2)
      CHB=EXP(SB2)+EXP(-SB2)
      THB=SHB/CHB
      TV=TAN(V)
      RLM=ATAN(THB*TV)+SUM1
      RMU=SUM2/(1.+SUM3)
      PYE=D2*(RLM-4.*RMU*V)
      RETURN
      END
```

B. Complex Argument

In this section, we are concerned with computing $\Pi(n;\, w|m)$ where $w = u + iv$ is a complex argument. The integral representation of Eq. (5.2) may be converted into an indefinite integral form as

$$\Pi(n;\, \phi|m) = \int \frac{d\phi}{(1 - n \sin^2\phi)(1 - m \sin^2\phi)^{1/2}} \tag{5.44}$$

Using the following in above,

$$\sin\phi = \mathrm{sn}(w|m) \tag{5.45}$$

$$\cos\phi \, d\phi = \mathrm{cn}(w|m)\mathrm{dn}(w|m)dw \tag{5.46}$$

or

$$d\phi = \mathrm{dn}(w|m)dw \tag{5.47}$$

leads to

$$\Pi(n;\, w|m) = \int \frac{\mathrm{dn}(w|m)}{\left[1 - n\, \mathrm{sn}^2(w|m) \quad 1 - m\, \mathrm{sn}^2(w|m)\right]^{1/2}} dw$$

or

$$\Pi(n;\, w|m) = \int_0^w \frac{d\xi}{\left[1 - n\, \mathrm{sn}^2(\xi|m)\right]} \tag{5.48}$$

with the use of the identity

$$\left[1 - m\, \mathrm{sn}^2(w|m)\right] = \mathrm{dn}^2(w|m) \tag{5.49}$$

One can use Eq. (5.48) to compute the elliptic integral of the third kind $\Pi(n;\, w|m)$ of a complex argument, but this procedure will not be numerically efficient. The procedure of Section V-A was converted to the complex case and subroutine CEI3K was written.

VI. Summary

This note documents a family of computer programs in the form of subroutines which are useful in evaluating the following:

(1) Jacobian elliptic function trio for real arguments, i.e., $sn(u|m)$, $cn(u|m)$ and $dn(u|m)$ with $0 \le m \le 1$.

(2) Jacobian elliptic function trio for complex arguments, i.e., $sn(u + iv|m)$, $cn(u + iv|m)$ and $dn(u + iv|m)$ with $0 \le m \le 1$.

(3) The complete elliptic integrals of the first kind and second kind, i.e., $K(m)$ and $E(m)$ with $0 \le m < 1$.

(4) The incomplete elliptic integrals of the first and second kind, i.e., $F(\phi|m)$ and $E(\phi|m)$ with $0 \le m < 1$ and ϕ real.

(5) The incomplete elliptic integrals of the first and second kind, i.e., $F(u + iv|m)$ and $E(u + iv|m)$ with $0 \le m < 1$.

(6) The Jacobi zeta function of real amplitude, i.e., $Z(\phi|m)$ with $0 \le m < 1$.

(7) The Jacobi zeta function of complex argument, i.e., $Z(u + iv|m)$ with $0 \le m < 1$.

(8) The elliptic integral of the third kind for real argument, i.e., $\Pi(n; \phi|m)$ with $0 \le m < 1$.

(9) The elliptic integral of the third kind for complex argument, i.e., $\Pi(n; w|m)$ with $0 \le m < 1$.

For the names of the computer subroutines that perform each of the above eight functions, the reader is referred to Table 1.1 of this note. In writing the programs, extensive use is made of the equations in Reference [6]. Listings of all of the subroutines as also the results of test runs performed on the CDC 7600 computing system at the Air Force Weapons Laboratory, have been included.

REFERENCES

[1] King, R.W.P., *Fundamental Electromagnetic Theory*, Section VI-13, Second Edition, Dover 1963.

[2] Smythe, W.R., *Static and Dynamic Electricity*, Third Edition, McGraw-Hill 1968.

[3] Baum, Carl E., Giri, D.V., and Gonzalez, Raymond D., "Electromagnetic Field Distribution of the TEM Mode in a Symmetrical Two-Parallel-Plate Transmission Line," Sensor and Simulation Note 219, 1 April 1976.

[4] Lee, K.S.H., "Electrically Small Ellipsoidal Antennas," Sensor and Simulation Note 193, February 1974.

[5] Brown, Terry L., "TEF: A Subroutine for the Calculation of the Incomplete Elliptic Integrals of the First and Second Kind," Mathematics Note 14, August 1970.

[6] Abramowitz, M., and Stegun, I.A., *Handbook of Mathematical Functions*, Dover 1965.

[7] Neville, E.H., *Jacobian Elliptic Functions*, Second Edition, Oxford University Press, London, England, 1951.

[8] Milne-Thomson, L.M., *Jacobian Elliptic Function Tables*, Dover 1950.

[9] Dwight, H.B., *Tables of Integrals and Other Mathematical Data*, The McaMillan Company, 1965, p. 172, Eqs. 775 and 777.

ABBREVIATIONS, ACRONYMS AND SYMBOLS

$\mathrm{sn}(u\|m)$, $\mathrm{cn}(u\|m)$ and $\mathrm{dn}(u\|m)$	Jacobian function trio for real argument u and parameter m.
$\mathrm{sn}(w\|m)$, $\mathrm{cn}(w\|m)$ and $\mathrm{dn}(w\|m)$	Same as above for complex argument $w = u + iv$, also $= Me^{i\theta}$.
m_1	The complementary parameter = 1 - m.
$K(m)$	$\equiv K$ = Complete elliptic integral of the first kind.
$E(m)$	$\equiv E$ = Complete elliptic integral of the second kind.
$F(u\|m)$	Incomplete elliptic integral of the first kind of amplitude ϕ with $\phi \neq (\pi/2)$. Also equivalent to $F(\phi\backslash\alpha)$ via $\sin\phi = \mathrm{sn}(u\|m)$ and $m = \sin^2\alpha$.
$E(u\|m)$	Incomplete elliptic integral of the second kind of amplitude ϕ with $\phi = (\pi/2)$. Also equivalent to $E(\phi\backslash\alpha)$ via the same equations as in above.
$F(w\|m)$, $E(w\|m)$	Incomplete elliptic integrals of the first and second kind for complex argument $w = u + iv$ and real parameter m.
$Z(\phi\|m)$	Jacobi zeta function of real amplitude ϕ and parameter m. Like F and E, this also has an equivalence with $Z(u\|m)$.
$Z(w\|m)$	Jacobi zeta function of a complex argument $w = u + iv = Me^{i\theta}$ and real parameter m.
$\Pi(n; \phi\backslash\alpha)$ or $\Pi(n; \phi\|m)$	Elliptic integral of the third kind of real amplitude ϕ and real parameters n and m with $0 \leq (m = \sin^2\alpha) < 1$.
$\Pi(n; \pi/2\backslash\alpha)$ or $\Pi(n; \pi/2\|m)$ $\equiv \Pi(n\backslash\alpha)$ or $\Pi(n\|m)$	Complete elliptic integral of the third kind.
$\Pi(n; w\|m)$	Elliptic integral of the third kind of complex argument $w = u + iv = Me^{i\theta}$ and real parameters n and m where $0 \leq (m = \sin^2\alpha) < 1$.