Mathematics Notes

Note 63

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Norms and Eigenvector Norms

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Abstract

Norms of vectors and matrices are reviewed; this concept is extended to a norm based on the eigenvectors (left and right) of some square complex matrix of interest. The properties of this norm are discussed and special cases are noted depending on the assumed form of the generating matrix for the norm. The eigenvector norm is related to the euclidean or spectral norm (or magnitude) under certain conditions related to the defining matrix and its eigenvectors. Some applications of the eigenvector norm to physical (electromagnetic) problems are pointed out.

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I. Introduction

In the analysis of distributions of electromagnetic fields, current densities, etc. one may wish to assign some single scalar number to such spatial and frequency or time functions as a means of comparing one such function to another. This scalar number might be something which is to go to zero as the function itself goes to zero in some sense; if the function of interest is some kind of "error" field (say the difference of between the "real" field distribution from an "ideal" one) then such a scalar number is quite appropriate. Another use of such a scalar might be to represent a maximum or minimum of the function in some sense.

One way to reduce such functions to a single scalar is to define what is often referred to as a norm in the mathematical literature. As will be discussed such norms have some useful properties.

For present purposes norms are discussed in the context of vectors (of finite dimension) which can also be considered as functions of a discrete variable (the vector-component index (integer subscript)). This simplifies matters somewhat. The concepts are readily generalizable to the continuous case (functions, including vector functions of any number of variables).

Having considered vector norms the property of matrices in transforming one vector into another vector is used to define appropriate matrix norms. Such matrix norms can also be applied to more general operators encountered in the case of functions of continuous variables.

After reviewing commonly used vector and matrix norms, a special kind of norm, the eigenvector norm (or eigenmode norm) is defined and its properties discussed. This kind of norm has some important applications in electromagnetic interaction problems for which an appropriate set of eigenmodes of the object of interest can be defined (say from some integral equation describing the electromagnetic interaction process).

This note is intended to lay some of the groundwork for the use of norm concepts in EMP problems. Applications include bounds on EMP response (interaction), accuracy of EMP simulation, bounds on response to EMP criteria from EMP test data, accuracy of numerical electromagnetic calculations, and perhaps various other areas to be investigated.

II. Vector Norms

Following a traditional definition [6] let us call something a vector norm if the following properties are satisfied

$$\|(a_n)\| \geq 0 \text{ with } \|(a_n)\| = 0 \text{ iff } (a_n) = (0_n)$$

$$\|\alpha(a_n)\| = |\alpha| \; \|(a_n)\| \qquad (2.1)$$

$$\|(a_n) + (b_n)\| \leq \|(a_n)\| + \|(b_n)\| \text{ (triangle inequality)}$$

$\|(a_n)\|$ depends continuously on (a_n)

where

$(a_n),(b_n)$ are N-component complex vectors
$(n = 1,2,\ldots,N)$

$\|(a_n)\| \equiv$ norm of (a_n) (2.2)

α is a complex scalar

$|\alpha| \equiv$ magnitude of α

For the special case of one-component vectors (scalars) this reduces to

$$\|a + b\| \leq \|a\| + \|b\| \qquad (2.3)$$

$$\|ab\| = |a| \, \|b\| = \|a\| \, |b|$$

One can consider the notion of norm as a generalization of the magnitude concept. For scalars (the one-dimensional case) we have

$$N = 1$$

$$(a_n) \equiv a \;,\; (b_n) \equiv b \tag{2.4}$$

$$|a| = \{[Re[a]]^2 + [Im[a]]^2\}^{\frac{1}{2}}$$

One choice for norm is then

$$||a|| \equiv |a| \tag{2.5}$$

since the magnitude satisfies all the properties (2.1). Another possible choice for norm would be a positive constant times the magnitude. In general one might wish to define a particular norm so that (2.5) is satisfied in the case of one-component vectors.

A common vector norm is the vector magnitude or euclidean norm given by

$$|(a_n)| \equiv \left\{ \sum_{n=1}^{N} |a_n|^2 \right\}^{\frac{1}{2}} \equiv \left\{ (a_n) \cdot (a_n)^* \right\}^{\frac{1}{2}} \tag{2.6}$$

where * indicates complex conjugate. For $N = 1$ this reduces to the usual magnitude of a complex scalar.

Another common type of vector norm is referred to as the p norm defined by

$$||(a_n)||_p \equiv \left\{ \sum_{n=1}^{N} |a_n|^p \right\}^{1/p} \tag{2.7}$$

This has important special cases

$$||(a_n)||_1 \equiv \sum_{n=1}^{N} |a_n|$$

$$||(a_n)||_2 \equiv \left\{ \sum_{n=1}^{N} |a_n|^2 \right\}^{\frac{1}{2}} \equiv \left\{ (a_n) \cdot (a_n)^* \right\}^{\frac{1}{2}} \equiv |(a_n)| \tag{2.8}$$

$$||(a_n)||_\infty \equiv \max_{1 \le n \le N} |a_n|$$

The 2 norm is then the euclidean norm or magnitude which has a common interpretation in physical problems in terms of energy concepts. The ∞ norm or maximum norm is also useful in physical problems; this might represent, for example, the maximum of a set of signals of interest. Note again for the case of one-dimensional vectors we have

$$N = 1$$

$$(a_n) \equiv a \tag{2.9}$$

$$\|(a_n)\|_p \equiv \|a\|_p \equiv |a|$$

so that all p norms reduce to (or are consistent with) the magnitude of a complex scalar.

III. Matrix Norms

Norms can also be defined for matrices (and for more general operators as well). Let us first consider general matrix norms, and then associate them with vector norms.

In a form similar to (2.1) let us call something a matrix norm if the following properties are satisfied [5]

$$\|(d_{n,m})\| \geq 0 \text{ with } \|(d_{n,m})\| = 0 \text{ iff } (d_{n,m}) = (0_{n,m})$$

$$\|\alpha(d_{n,m})\| = |\alpha| \, \|(d_{n,m})\|$$

$$\|(c_{n,m}) + (d_{n,m})\| \leq \|(c_{n,m})\| + \|(d_{n,m})\| \text{ (triangle inequality)} \tag{3.1}$$

$$\|(f_{n,m}) \cdot (g_{n,m})\| \leq \|(f_{n,m})\| \, \|(g_{n,m})\| \text{ (Schwarz inequality)}$$

$$\|(d_{n,m})\| \text{ depends continuously on } (d_{n,m})$$

with the additional multiplication property for the norm of matrix products (as compared to the list in (2.1)).

For the above relationships to be meaningful we must have matrices of <u>compatible order</u> [3]. In general we allow the matrices to be rectangular, say $N \times M$ (N rows, M columns). In (3.1) this requires

$(c_{n,m})$, $(d_{n,m})$ have same number of rows and same number of columns

$$N_c = N_d \; , \; M_c = M_d$$

(3.2)

$(f_{n,m})$ has number of rows equal to number of columns of $(g_{n,m})$

$$M_f = N_g$$

Other terms are as defined in (2.2).

There are some interesting properties which follow from (3.1). If $(d_{n,m})$ is a square matrix we have

$$\|(d_{n,m})^q\| \le \|(d_{n,m})\|^q \qquad (3.3)$$

$$q \equiv \text{positive integer}$$

Considering the special case that one of the matrices reduces to a vector ($N \times 1$ or $1 \times M$ matrix) we have

$$\|(d_{n,m}) \cdot (a_n)\| \le \|(d_{n,m})\| \, \|(a_n)\|$$

$$\|(b_n) \cdot (d_{n,m})\| \le \|(b_n)\| \, \|(d_{n,m})\| \qquad (3.4)$$

with again the requirement that the vectors be of compatible order to the $N \times M$ rectangular matrix. Furthermore if both matrices are reduced to N-component vectors we have

$$\|(a_n) \cdot (b_n)\| \le \|(a_n)\| \, \|(b_n)\| \qquad (3.5)$$

One can also choose the special case of matrices as scalars which gives

$$\|a + b\| \le \|a\| + \|b\|$$

$$\|ab\| = |a| \, \|b\| = \|a\| \, |b| \le \|a\| \, \|b\| \qquad (3.6)$$

which are evidently satisfied for the case of norm being defined as magnitude for complex scalars. In general for scalars the above gives the requirement

$$\|a\| \ge |a| \qquad (3.7)$$

One might wish to choose the definition of a particular norm of interest so that (3.7) gives equality.

There are many ways of defining a matrix norm consistent with (3.1). For our purposes we will only consider matrix norms associated with (or generated from) vector norms.

IV. Associated Matrix Norms

A common way of constructing matrix norms uses the role of matrices in relating vectors via dot multiplication as in

$$
\begin{aligned}
&(b_n) = (d_{n,m}) \cdot (a_n) \\
&(d_{n,m}) \equiv N \times M \text{ complex matrix} \\
&(a_n) \equiv M\text{-component complex vector} \\
&(b_n) \equiv N\text{-component complex vector}
\end{aligned}
\tag{4.1}
$$

Assume that we have defined a vector norm which applies to vectors of arbitrary numbers of complex components; examples are given in section 2. Then we can write

$$\|(b_n)\| = \|(d_{n,m}) \cdot (a_n)\| \tag{4.2}$$

If we define a matrix norm via

$$
\begin{aligned}
&\|(d_{n,m})\| \equiv \sup_{(a_n)\neq(0_n)} \frac{\|(d_{n,m}) \cdot (a_n)\|}{\|(a_n)\|} \\
&\text{sup} \equiv \text{supremum} \equiv \text{least upper bound}
\end{aligned}
\tag{4.3}
$$

then we have

$$\|(b_n)\| \leq \|(d_{n,m})\| \, \|(a_n)\| \tag{4.4}$$

which makes the matrix norm a least upper bound over all (a_n) in this inequality (4.4). Hence the associated matrix norm can be thought of as a minimum norm consistent with the chosen vector norm.

One can take definition (4.3), make various substitutions for $(d_{n,m})$, apply the vector-norm properties in (2.1), and generate the matrix-norm properties in (3.1) showing that the associated matrix norm is a true matrix norm.

For the special case of square matrices (N = M) we can write the identity $(1_{n,m})$ where

$$1_{n,m} = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$
$$\|(1_{n,m})\| = 1 \tag{4.5}$$

which is easily seen from the defining relation (4.3) since

$$(1_{n,m}) \cdot (a_n) = (a_n) \tag{4.6}$$

If we let the $N \times M$ matrix degenerate to a complex scalar (1×1 matrix) we then have from (4.3)

$$\begin{aligned} (d_{n,m}) &\equiv d \\ (a_n) &\equiv a \\ \|(d_{n,m})\| &\equiv \|d\| \equiv \sup_{a\neq 0} \frac{\|da\|}{\|a\|} \\ &= \sup_{a\neq 0} \frac{|d|\,\|a\|}{\|a\|} \\ &= |d| \end{aligned} \tag{4.7}$$

Hence for all vector norms the associated matrix norm reduces to magnitude in the scalar case. Compare this result to (3.7) which has the magnitude as the lower bound for the matrix norm of a scalar. This shows that the associated matrix norm is a generalization of the magnitude of a scalar.

V. Eigenvector Expansions

For use later let us review some concepts concerning the eigenvector expansion of matrices. For a square complex matrix $(N \times N)$ we define

$$
\begin{aligned}
&(d_{n,m}) \cdot (r_n)_\beta \equiv \lambda_\beta (r_n)_\beta \\
&(\ell_n)_\beta \cdot (d_{n,m}) \equiv \lambda_\beta (\ell_n)_\beta \\
&\det((d_{n,m}) - \lambda_\beta (1_{n,m})) = 0 \quad \text{for } \beta = 1,2,\ldots,N \\
&(r_n)_\beta \equiv \text{right eigenvectors} \\
&(\ell_n)_\beta \equiv \text{left eigenvectors} \\
&\lambda_\beta \equiv \text{eigenvalues}
\end{aligned}
\tag{5.1}
$$

For the case of distinct eigenvalues we have the usual result

$$(r_n)_{\beta_1} \cdot (\ell_n)_{\beta_2} = 0 \; , \; \lambda_{\beta_1} \neq \lambda_{\beta_2} \tag{5.2}$$

which is the biorthogonalization property. Note that for our purposes in electromagnetics which uses in general non-hermitian matrices and operators, orthogonality is most conveniently defined via a zero dot product (or symmetric product as distinguished from inner product) without conjugation of one of the vectors. This convention is adopted throughout.

For present purposes let us assume that one can construct N linearly independent $(r_n)_\beta$ and N linearly independent $(\ell_n)_\beta$ with the property

$$(r_n)_{\beta_1} \cdot (\ell_n)_{\beta_2} = 0 \quad \text{for } \beta_1 \neq \beta_2 \tag{5.3}$$

which can still allow for some cases of degeneracy ($\lambda_{\beta_1} = \lambda_{\beta_2}$ for $\beta_1 \neq \beta_2$). This assumption is equivalent to assuming that the right and left eigenvectors each span the N-dimensional complex space with the biorthogonalization property (5.3) even in cases of degeneracy. For a degeneracy (say p fold) with p equal eigenvalues we assume there is a set of p associated $(r_n)_\beta$ which are linearly independent, and likewise for a set of p of the $(\ell_n)_\beta$. Then a biorthogonal set of p pairs is constructed in a Gram-Schmidt type of manner.

With the assumption of spanning the space we can expand an arbitrary vector (N-dimensional complex) as

$$(a_n) = \sum_{\beta=1}^{N} \alpha_\beta (r_n)_\beta \tag{5.4}$$

Dot multiplication with $(\ell_n)_\beta$, using (5.3), gives

$$(\ell_n)_\beta \cdot (a_n) = \alpha_\beta (\ell_n)_\beta \cdot (r_n)_\beta \tag{5.5}$$

Choose the special case

$$(a_n) \equiv (\ell_n)_\beta^* \tag{5.6}$$

and note

$$(\ell_n)_\beta \cdot (\ell_n)_\beta^* = |(\ell_n)_\beta|^2 \neq 0 \tag{5.7}$$

(unless $(\ell_n)_\beta \equiv (0_n)$ which violates our assumption). Now since

$$\alpha_\beta = \frac{(\ell_n)_\beta \cdot (\ell_n)_\beta^*}{(\ell_n)_\beta \cdot (r_n)_\beta} \tag{5.8}$$

and since each α_β must exist (be finite) for any choice of (a_n) then we have

$$(\ell_n)_\beta \cdot (r_n)_\beta \neq 0 \tag{5.9}$$

This allows us to normalize our eigenvectors as a biorthonormal set

$$\left(\ell_n\right)_{\beta_1} \cdot \left(r_n\right)_{\beta_2} = 1_{\beta_1,\beta_2} \tag{5.10}$$

So a biorthogonal set spanning the space can be biorthonormalized.

For the special case of a symmetric matrix

$$\begin{aligned} (d_{n,m}) &= (d_{n,m})^T \\ T &\equiv \text{transpose} \\ d_{n,m} &= d_{m,n} \end{aligned} \tag{5.11}$$

and we can set

$$(\ell_n)_\beta \equiv (r_n)_\beta \equiv (e_n)_\beta \tag{5.12}$$

The assumption of spanning the space now gives an orthonormal set of eigenvectors with

$$\left(e_n\right)_{\beta_1} \cdot \left(e_n\right)_{\beta_2} = 1_{\beta_1,\beta_2} \tag{5.13}$$

Matrices for which the right and left eigenvectors are each a set of N linearly independent eigenvectors can then be written in the dyadic form

$$(d_{n,m}) = \sum_{\beta=1}^{N} \lambda_\beta (r_n)_\beta (\ell_n)_\beta \tag{5.14}$$

This form has all the properties of $(d_{n,m})$, not only by clearly satisfying the eigenvector/eigenvalue equations (5.1), but also by correctly giving $(d_{n,m}) \cdot (a_n)$ for all (a_n) due to the representation of (a_n) by (5.4) and (5.5) and the biorthonormalization

(5.10). If we have some function $F(\lambda)$ of a complex variable λ, with appropriate attention to a single-valued definition of F, we have the convenient result

$$F((d_{n,m})) = \sum_{\beta=1}^{N} F(\lambda_\beta)(r_n)_\beta (\ell_n)_\beta \tag{5.15}$$

The eigenvalues of $F((d_{n,m}))$ are then the N eigenvalues $F(\lambda_\beta)$ and the eigenvectors are unchanged. There are some convenient special cases

$$(1_{n,m}) = (d_{n,m})^o = \sum_{\beta=1}^{N} (r_n)_\beta (\ell_n)_\beta \quad \text{(identity)}$$

$$(d_{n,m})^{-1} = \sum_{\beta=1}^{N} \lambda_\beta^{-1} (r_n)_\beta (\ell_n)_\beta \quad \text{(inverse, if the matrix is non-singular)} \tag{5.16}$$

The eigenvectors can also be combined to form matrices as

$$(\ell_{n,m}) \equiv \begin{pmatrix} (\ell_m)_1 \\ (\ell_m)_2 \\ \vdots \\ (\ell_m)_N \end{pmatrix} \quad \text{(eigenvector rows)}$$

$$(r_{n,m}) \equiv ((r_n)_1, (r_n)_2, \ldots, (r_n)_N) \quad \text{(eigenvector columns)} \tag{5.17}$$

which by (5.10) are related as

$$(\ell_{n,m}) \cdot (r_{n,m}) = (1_{n,m}) \tag{5.18}$$

so that the left and right eigenvector matrices (normalized) are mutually inverse to each other. Hence we also have

$$(r_{n,m}) \cdot (\ell_{n,m}) = (1_{n,m}) \tag{5.19}$$

These eigenvector matrices can also be used to give a generalized form to the eigenvalue/eigenvector equations

$$\begin{aligned}
(d_{n,m}) \cdot (r_{n,m}) &= (r_{n,m}) \cdot (\Lambda_{n,m}) \\
(\ell_{n,m}) \cdot (d_{n,m}) &= (\Lambda_{n,m}) \cdot (\ell_{n,m}) \\
(\Lambda_{n,m}) &\equiv \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) \\
&= \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_N \end{pmatrix}
\end{aligned} \tag{5.20}$$

These also give

$$\begin{aligned}
(d_{n,m}) &= (r_{n,m}) \cdot (\Lambda_{n,m}) \cdot (\ell_{n,m}) \\
(\Lambda_{n,m}) &= (\ell_{n,m}) \cdot (d_{n,m}) \cdot (r_{n,m})
\end{aligned} \tag{5.21}$$

For symmetrical $(d_{n,m})$ we have

$$(\ell_{n,m}) \equiv (e_{n,m}) \equiv \begin{pmatrix} (e_m)_1 \\ (e_m)_2 \\ \vdots \\ (e_m)_N \end{pmatrix}$$

$$(r_{n,m}) = (e_{n,m})^T = (\ell_{n,m})^T$$

$$(e_{n,m}) \cdot (e_{n,m})^T = (1_{n,m}) = (e_{n,m})^T \cdot (e_{n,m}) \tag{5.22}$$

$$(e_{n,m})^T = (e_{n,m})^{-1}$$

VI. Spectral Radius

The spectral radius, often designated ρ, of a square matrix $(d_{n,m})$ is given by

$$\begin{aligned} \text{spectral radius} &\equiv \rho((d_{n,m})) \\ &\equiv \sup_{\beta} |\lambda_{\beta}| \\ &\equiv |\lambda((d_{n,m}))|_{max} \end{aligned} \qquad (6.1)$$

where $|\lambda|_{max}$ is defined as an eigenvalue of $(d_{n,m})$ with maximum magnitude.

VII. Hermitian Matrices

In the discussion of norms, hermitian matrices play a significant role. By a hermitian matrix is meant

$$(d_{n,m}) = (d_{n,m})^{\dagger}$$

$$\dagger \equiv T^* \equiv {}^*T \qquad (7.1)$$

$$\equiv \text{conjugate transpose or adjoint}$$

In another form

$$\text{Re}[(d_{n,m})] = (\text{Re}[(d_{n,m})])^T \quad \text{(symmetric)}$$

$$\text{Im}[(d_{n,m})] = -(\text{Im}[(d_{n,m})])^T \quad \text{(antisymmetric)} \qquad (7.2)$$

Note that only square matrices can be hermitian.

Applying (5.1) to a hermitian matrix we note that we can choose

$$(\ell_n)_\beta = (r_n)^*_\beta \qquad (7.3)$$

with

$$\lambda_\beta = \lambda^*_\beta \text{ , i.e., } \lambda_\beta \text{ real} \qquad (7.4)$$

Assuming N linearly independent eigenvectors, as in section 5, we have a property of conjugate orthogonality

$$\left(r_n\right)^*_{\beta_1} \cdot \left(r_n\right)_{\beta_2} = 0 \quad \text{for } \beta_1 \neq \beta_2 \qquad (7.5)$$

Then from (5.9) we can normalize the result as

$$\left(r_n\right)^*_{\beta_1} \cdot \left(r_n\right)_{\beta_2} = 1_{\beta_1,\beta_2} \qquad (7.6)$$

which makes the eigenvectors a conjugate orthonormal set of vectors.

As shown by Franklin [6] such a set of conjugate orthonormal eigenvectors, as in (7.6), can always be constructed for an arbitrary hermitian matrix. He also shows that for real hermitian matrices, i.e., for symmetric real matrices (and hence square), the eigenvectors can always be constructed as real orthonormal vectors. The eigenvalues are of course already real.

Related to hermitian matrices we have positive definite matrices

$$(a_n)^* \cdot (d_{n,m}) \cdot (a_n) \geq 0 \text{ for all } (a_n) \neq (0_n) \qquad (7.7)$$

and positive semidefinite matrices

$$(a_n)^* \cdot (d_{n,m}) \cdot (a_n) \geq 0 \text{ for all } (a_n) \qquad (7.8)$$

Merely requiring that the above quadratic form be real for all (a_n) gives

$$\begin{aligned} (a_n)^* \cdot (d_{n,m}) \cdot (a_n) &= [(a_n)^* \cdot (d_{n,m}) \cdot (a_n)]^* \\ &= (a_n) \cdot (d_{n,m})^* \cdot (a_n)^* \\ &= (a_n)^* \cdot (d_{n,m})^\dagger \cdot (a_n) \qquad (7.9) \end{aligned}$$

which implies

$$(d_{n,m}) = (d_{n,m})^\dagger \quad \text{(hermitian)} \qquad (7.10)$$

Expanding hermitian $(d_{n,m})$ in its eigenvectors as

$$(d_{n,m}) = \sum_{\beta=1}^{N} \lambda_\beta (r_n)_\beta (r_n)_\beta^* \qquad (7.11)$$

and substituting in (7.7) and (7.8) with (a_n) chosen as an arbitrary one of the eigenvectors leads to the standard result

$$\lambda_\beta > 0 \quad \text{for all } \beta \quad \text{(positive definite)}$$
$$\lambda_\beta \geq 0 \quad \text{for all } \beta \quad \text{(positive semidefinite)} \tag{7.12}$$

Positive definite matrices and positive semidefinite matrices are then subsets of the set of hermitian matrices.

Suppose now that $(d_{n,m})$ is an arbitrary complex $N \times M$ matrix. A hermitian matrix can be formed from this from either of the relations

$$[(d_{n,m})^\dagger \cdot (d_{n,m})]^\dagger = (d_{n,m})^\dagger \cdot (d_{n,m})$$
$$[(d_{n,m}) \cdot (d_{n,m})^\dagger]^\dagger = (d_{n,m}) \cdot (d_{n,m})^\dagger \tag{7.13}$$

i.e., both $(d_{n,m})^\dagger \cdot (d_{n,m})$ which is $M \times M$, and $(d_{n,m}) \cdot (d_{n,m})^\dagger$ which is $N \times N$, are hermitian. Note that both of these hermitian matrices in (7.13) are positive semidefinite since

$$(a_n)^* \cdot (d_{n,m})^\dagger \cdot (d_{n,m}) \cdot (a_n)$$
$$= [(d_{n,m}) \cdot (a_n)]^* \cdot (d_{n,m}) \cdot (a_n)$$
$$= |(d_{n,m}) \cdot (a_n)|^2 \geq 0 \tag{7.14}$$

and

$$(a_n)^* \cdot (d_{n,m}) \cdot (d_{n,m})^\dagger \cdot (a_n)$$
$$= [(d_{n,m})^\dagger \cdot (a_n)]^* \cdot (d_{n,m})^\dagger \cdot (a_n)$$
$$= |(d_{n,m})^\dagger \cdot (a_n)|^2 \geq 0 \tag{7.15}$$

An important property of hermitian matrices concerns the quadratic form $(a_n)^* \cdot (d_{n,m}) \cdot (a_n)$. For hermitian $(d_{n,m})$ we have the Rayleigh principle [6]

$$\lambda_{max}((d_{n,m})) = \sup_{(a_n)\neq(0_n)} \frac{(a_n)^* \cdot (d_{n,m}) \cdot (a_n)}{(a_n)^* \cdot (a_n)} \tag{7.16}$$

where λ_{max} is the maximum eigenvalue of $(d_{n,m})$. There is also a related result for the minimum eigenvalue of a hermitian matrix as

$$\lambda_{min}((d_{n,m})) = \inf_{(a_n)\neq(0_n)} \frac{(a_n)^* \cdot (d_{n,m}) \cdot (a_n)}{(a_n)^* \cdot (a_n)} \tag{7.17}$$

$$\inf \equiv \text{infimum} \equiv \text{greatest lower bound}$$

These results are related to the spectral radius (section 6) applied to hermitian matrices giving

$$\begin{aligned}
\text{spectral radius} &\equiv \rho((d_{n,m})) \\
&= |\lambda((d_{n,m}))|_{max} \\
&= \max[|\lambda_{max}| , |\lambda_{min}|] \\
&= \sup_{(a_n)\neq(0_n)} \left| \frac{(a_n)^* \cdot (d_{n,m}) \cdot (a_n)}{(a_n)^* \cdot (a_n)} \right| \\
&= \sup_{(a_n)\neq(0_n)} \frac{|(a_n)^* \cdot (d_{n,m}) \cdot (a_n)|}{(a_n)^* \cdot (a_n)}
\end{aligned} \tag{7.18}$$

The eigenvector matrices (5.17) take the form for hermitian matrices as

$$(\ell_{n,m}) = (r_{n,m})^{\dagger}$$
$$(r_{n,m})^{\dagger} \cdot (r_{n,m}) = (1_{n,m}) = (r_{n,m}) \cdot (r_{n,m})^{\dagger} \tag{7.19}$$

VIII. Some Common Associated Matrix Norms

Corresponding to various commonly defined vector norms (as for example in section 2) one can define the associated matrix norms.

Consider first the matrix norm associated with the vector euclidean norm, 2 norm, or magnitude. Let us define this accordingly as the matrix euclidean norm, 2 norm, or magnitude. This norm is also called the spectral norm. The magnitude symbol will be used to signify this particular matrix norm as

$$|(f_{n,m})| \equiv \|(f_{n,m})\|_2 \equiv \sup_{(a_n)\neq(0_n)} \frac{|(f_{n,m}) \cdot (a_n)|}{|(a_n)|} \tag{8.1}$$

Here $(f_{n,m})$ is a general complex rectangular $(N \times M)$ matrix; accordingly (a_n) is an M-dimensional complex vector. Note that in the definition the matrix is to the left of the vector. One then might think of this as a left norm; the right norm corresponds to the left norm of $(f_{n,m})^T$. The right and left associated matrix norms are not in general equal.

Squaring (8.1) we have

$$\begin{aligned} |(f_{n,m})|^2 &= \sup_{(a_n)\neq(0_n)} \frac{|(f_{n,m}) \cdot (a_n)|^2}{|(a_n)|^2} \\ &= \sup_{(a_n)\neq(0_n)} \frac{[(f_{n,m}) \cdot (a_n)]^* \cdot (f_{n,m}) \cdot (a_n)}{(a_n)^* \cdot (a_n)} \\ &= \sup_{(a_n)\neq(0_n)} \frac{(a_n)^* \cdot (f_{n,m})^\dagger \cdot (f_{n,m}) \cdot (a_n)}{(a_n)^* \cdot (a_n)} \end{aligned} \tag{8.2}$$

However this is just the Rayleigh principle (7.16) for the $M \times M$ hermitian matrix $(f_{n,m})^\dagger \cdot (f_{n,m})$ giving

$$|(f_{n,m})| = [\lambda_{max}((f_{n,m})^\dagger \cdot (f_{n,m}))]^{\frac{1}{2}} \tag{8.3}$$

Note the order $(f_{n,m})^\dagger \cdot (f_{n,m})$ since our definition of matrix norm is a left norm. Also note that all the eigenvalues λ_β of $(f_{n,m})^\dagger \cdot (f_{n,m})$ are non-negative ($\lambda_\beta \geq 0$ for $\beta = 1,2,\ldots,M$) by (7.12) since this is a positive semidefinite matrix by (7.14).

In the case that $(f_{n,m})$ is hermitian ($N \times N$) we have

$$\begin{aligned} |(f_{n,m})| &= [\lambda_{max}((f_{n,m})^\dagger \cdot (f_{n,m}))]^{\frac{1}{2}} \\ &= [\lambda_{max}((f_{n,m})^2)]^{\frac{1}{2}} \\ &= |\lambda((f_{n,m}))|_{max} \\ &= \rho((f_{n,m})) \equiv \text{spectral radius of } (f_{n,m}) \end{aligned} \tag{8.4}$$

where the result is used from (5.15) that the eigenvalues of $(f_{n,m})^2$ are the squares of the λ_β, the eigenvalues of $(f_{n,m})$. For hermitian matrices the magnitude, euclidean norm, or spectral norm is then just the spectral radius, a rather simple result.

There are other common matrix norms. Corresponding to the vector p norm, (2.7), there is an associated matrix p norm. For special cases we have the results [6]

$$\begin{aligned} \|(f_{n,m})\|_1 &= \max_{m=1,2,\ldots,M} \sum_{n=1}^{N} |f_{n,m}| \equiv \text{maximum column magnitude sum} \\ \|(f_{n,m})\|_\infty &= \max_{n=1,2,\ldots,N} \sum_{m=1}^{M} |f_{n,m}| \equiv \text{maximum row magnitude sum} \end{aligned} \tag{8.5}$$

These results apply to general complex $N \times M$ matrices.

There are some bounds that have been obtained concerning associated matrix norms. For general complex square ($N \times N$) matrices we have [4]

$$\|(f_{n,m})\| \geq \rho((f_{n,m})) = |\lambda((f_{n,m}))|_{max} \tag{8.6}$$

so that the spectral radius is a lower bound for all associated matrix norms (for square matrices). Note that equality in (8.6) is achieved if $(f_{n,m})$ is hermitian. Combining (8.6) with other matrix norms for square matrices (as in (8.5)) we have eigenvalue inequalities

$$|\lambda((f_{n,m}))|_{max} \leq \max_{m=1,2,...,N} \sum_{n=1}^{N} |f_{n,m}| \equiv \text{maximum column magnitude sum}$$

$$|\lambda((f_{n,m}))|_{max} \leq \max_{n=1,2,...,N} \sum_{m=1}^{M} |f_{n,m}| \equiv \text{maximum row magnitude sum} \tag{8.7}$$

Combining (8.3) with (8.6) we have

$$|\lambda((f_{n,m}))|_{max} \leq [\lambda_{max}((f_{n,m})^{\dagger} \cdot (f_{n,m}))]^{\frac{1}{2}} \tag{8.8}$$

IX. Eigenvector Norm for Vectors

After the "preliminaries" our odyssey has come to the eigenvector norm, a norm with potential application to the eigenmodes of electromagnetic scattering/antenna problems.

Consider a square $(N \times N)$ complex matrix $(d_{n,m})$ and assume that it is diagonalizable as in section 5 so that

$$\begin{aligned} (d_{n,m}) &= \sum_{\beta=1}^{N} \lambda_\beta (r_n)_\beta (\ell_n)_\beta \\ (\ell_n)_{\beta_1} \cdot (r_n)_{\beta_2} &= 1_{\beta_1,\beta_2} \end{aligned} \tag{9.1}$$

Expanding an arbitrary N-dimensional complex vector (a_n) as in (5.4) we have

$$\begin{aligned} (a_n) &= \sum_{\beta=1}^{N} \alpha_\beta (r_n)_\beta \\ \alpha_\beta &= (\ell_n)_\beta \cdot (a_n) \\ (\alpha_n) &= (\ell_{n,m}) \cdot (a_n) \end{aligned} \tag{9.2}$$

where the last expression combines the α_β coefficients as a vector.

Let us define something which will be referred to as the eigenvector norm (with respect to a set of $(r_n)_\beta$) by

$$\begin{aligned} \|(a_n)\|_e &\equiv \left[\sum_{\beta=1}^{N} |\alpha_\beta|^2\right]^{\frac{1}{2}} = \left[\sum_{\beta=1}^{N} |(\ell_n)_\beta \cdot (a_n)|^2\right]^{\frac{1}{2}} \\ &= |(\ell_{n,m}) \cdot (a_n)| \end{aligned} \tag{9.3}$$

The last equivalent form uses the matrix of left eigenvectors in (5.17) and thereby defines the eigenvector norm in terms of a magnitude or euclidean norm.

For (9.3) to be a vector norm conditions (2.1) must be satisfied. First we note that

$$\|(a_n)\|_e \equiv \left[\sum_{\beta=1}^{N} |\alpha_\beta|^2\right]^{\frac{1}{2}} = |(\alpha_n)| \geq 0 \tag{9.4}$$

Furthermore a value of 0 would imply that all $\alpha_\beta = 0$ which would imply $(a_n) = (0_n)$ in (9.2) since the eigenvectors span the N-dimensional complex space by hypothesis. Hence

$$\|(a_n)\|_e \geq 0 \text{ and } \|(a_n)\|_e = 0 \text{ iff } (a_n) = (0_n) \tag{9.5}$$

Second we have

$$\begin{aligned} \|\alpha(a_n)\|_e &= |(\ell_{n,m}) \cdot [\alpha(a_n)]| = |\alpha| \; |(\ell_{n,m}) \cdot (a_n)| \\ &= |\alpha| \; \|(a_n)\|_e \end{aligned} \tag{9.6}$$

For the third property we have

$$\begin{aligned} \|(a_n) + (b_n)\|_e &= |(\ell_{n,m}) \cdot [(a_n) + (b_n)]| \\ &= |(\ell_{n,m}) \cdot (a_n) + (\ell_{n,m}) \cdot (b_n)| \\ &\leq |(\ell_{n,m}) \cdot (a_n)| + |(\ell_{n,m}) \cdot (b_n)| \\ &= \|(a_n)\|_e + \|(b_n)\|_e \\ \|(a_n) + (b_n)\|_e &\leq \|(a_n)\|_e + \|(b_n)\|_e \end{aligned} \tag{9.7}$$

where both (a_n) and (b_n) are arbitrary N-component complex vectors. Here the inequality is obtained via the triangle inequality for the vector magnitude or euclidean norm, operating on transformed vectors. This points out that the eigenvector norm is the same as magnitude or euclidean norm in a new complex N-dimensional space defined by the transformation matrix $(\ell_{n,m})$ which maps vectors into this new space.

Fourth, since the α_β are continuous functions of (a_n) in (9.2), and since $\|(a_n)\|_e$ is a continuous function of the α_β in (9.3) we have

$$\|(a_n)\|_e \text{ depends continuously on } (a_n) \tag{9.8}$$

Thus $\|(a_n)\|_e$ satisfies all the requirements of a vector norm.

Note that the particular eigenvector norm being considered depends on the particular matrix being used to construct the eigenvectors. One can define various different eigenvector norms to suit one's purposes. To distinguish the different eigenvector norms one might subscript the norm with the matrix being used. For example, if one uses $(d_{n,m})^T$ we have

$$\|(a_n)\|_{e,(d_{n,m})^T} = \left[\sum_{\beta=1}^{N} |(r_n)_\beta \cdot (a_n)|^2\right]^{\frac{1}{2}} = |(a_n)\cdot(r_{n,m})| \tag{9.9}$$

showing that the right eigenvector norm just interchanges the roles of the left and right eigenvectors.

A special case of interest is for symmetric $(d_{n,m})$, for which both left and right eigenvectors are the same and are designated $(e_n)_\beta$; these eigenvectors form an orthonormal set as in (5.13). The norm for this important case is

$$\|(a_n)\|_e = \left[\sum_{\beta=1}^{N} |(e_n)_\beta \cdot (a_n)|^2\right]^{\frac{1}{2}}$$

$$= |(e_{n,m})\cdot(a_n)| = |(a_n)\cdot(e_{n,m})^{-1}| \tag{9.10}$$

Another special case of interest is for hermitian $(d_{n,m})$ which gives

$$\|(a_n)\|_e = |(r_{n,m})^\dagger \cdot (a_n)| = [(a_n)^* \cdot (r_{n,m}) \cdot (r_{n,m})^\dagger \cdot (a_n)]^{\frac{1}{2}}$$

$$= [(a_n)^* \cdot (1_{n,m}) \cdot (a_n)]^{\frac{1}{2}} = [(a_n)^* \cdot (a_n)]^{\frac{1}{2}}$$

$$= |(a_n)| \qquad (9.11)$$

Hence for hermitian $(d_{n,m})$ the eigenvector norm for vectors is the same as the magnitude or euclidean norm.

X. Eigenvector Norm for Matrices

The associated matrix norm corresponding to the eigenvector norm for vectors is defined for a general square complex matrix $(f_{n,m})$ by

$$\|(f_{n,m})\|_e \equiv \sup_{(a_n)\neq(0_n)} \frac{\|(f_{n,m}) \cdot (a_n)\|_e}{\|(a_n)\|_e} \tag{10.1}$$

Restricting the discussion to square matrices allows the eigenvector norms (in the numerator and denominator of the right side of (10.1)) to operate on vectors of the same dimensionality, and hence to use the same set of eigenvectors. For rectangular matrices one might use two different sets of vectors (eigenvectors if one wishes) of different dimensionalities, but these possibilities are not considered here. For our present purposes let us reserve $(d_{n,m})$ and its eigenvector decomposition (9.1) for our definition of the eigenvector norm.

For general $N \times N$ complex matrices $(f_{n,m})$ we have

$$\begin{aligned}
\|(f_{n,m})\|_e &= \sup_{(a_n)\neq(0_n)} \frac{\|(f_{n,m}) \cdot (a_n)\|_e}{\|(a_n)\|_e} \\
&= \sup_{(\ell_{n,m})\cdot(a_n)\neq(0_n)} \frac{|(\ell_{n,m}) \cdot (f_{n,m}) \cdot (a_n)|}{|(\ell_{n,m}) \cdot (a_n)|} \\
&= \sup_{(\ell_{n,m})\cdot(a_n)\neq(0_n)} \frac{|(\ell_{n,m}) \cdot (f_{n,m}) \cdot (r_{n,m}) \cdot (\ell_{n,m}) \cdot (a_n)|}{|(\ell_{n,m}) \cdot (a_n)|}
\end{aligned} \tag{10.2}$$

Now with the eigenvector norm of $(f_{n,m})$ represented in terms of magnitude, or euclidean or spectral norm, we have

$$\|(f_{n,m})\|_e = |(\ell_{n,m}) \cdot (f_{n,m}) \cdot (r_{n,m})|$$

$$= [\lambda_{max}((r_{n,m})^\dagger \cdot (f_{n,m})^\dagger \cdot (\ell_{n,m})^\dagger \cdot (\ell_{n,m}) \cdot (f_{n,m}) \cdot (r_{n,m}))]^{\frac{1}{2}} \tag{10.3}$$

Consider the special but important case of the eigenvector norm of $(d_{n,m})$ as

$$\begin{aligned} \|(d_{n,m})\|_e &= |(\ell_{n,m}) \cdot (d_{n,m}) \cdot (r_{n,m})| \\ &= |(\Lambda_{n,m})| \\ &= |\lambda(d_{n,m})|_{max} \equiv \max_{\beta=1,2,\ldots,N} |\lambda_\beta| \\ &= \rho((d_{n,m})) \end{aligned} \tag{10.4}$$

Notice with our diagonalized form of $(d_{n,m})$ and the functional form for functions of $(d_{n,m})$ in (5.15) we have

$$\begin{aligned} \|F((d_{n,m}))\|_e &= |F(\lambda_\beta)|_{max} \\ \|(d_{n,m})^{-1}\|_e &= [|\lambda_\beta|_{min}]^{-1} \end{aligned} \tag{10.5}$$

with the maximum and minimum being taken over all $\beta=1,2,\ldots,N$. So the eigenvector norm is a very natural one to use with expressions that involve the matrix $(d_{n,m})$ used to define the norm.

Let $(d_{n,m})$ be specialized to a hermitian matrix. Then for a general $N \times N$ matrix $(f_{n,m})$ we have

$$\begin{aligned} \|(f_{n,m})\|_e &= |(r_{n,m})^\dagger \cdot (f_{n,m}) \cdot (r_{n,m})| \\ &= [\lambda_{max}((r_{n,m})^\dagger \cdot (f_{n,m})^\dagger \cdot (r_{n,m}) \cdot (r_{n,m})^\dagger \cdot (f_{n,m}) \cdot (r_{n,m}))]^{\frac{1}{2}} \\ &= [\lambda_{max}((r_{n,m})^\dagger \cdot (f_{n,m})^\dagger \cdot (f_{n,m}) \cdot (r_{n,m}))]^{\frac{1}{2}} \end{aligned} \tag{10.6}$$

However

$$(r_{n,m})^\dagger = (r_{n,m})^{-1} \tag{10.7}$$

so that $(r_{n,m})^\dagger \cdot (f_{n,m})^\dagger \cdot (f_{n,m}) \cdot (r_{n,m})$ is a similarity transform of $(f_{n,m})^\dagger \cdot (f_{n,m})$ and both have the same eigenvalues, giving

$$\begin{aligned} \|(f_{n,m})\|_e &= [\lambda_{max}((f_{n,m})^\dagger \cdot (f_{n,m}))]^{\frac{1}{2}} \\ &= |(f_{n,m})| \end{aligned} \tag{10.8}$$

Hence for hermitian $(d_{n,m})$ the eigenvector norm of (square) matrices is the same as the magnitude, or euclidean or spectral norm.

XI. Relation of Eigenvector Norm to Euclidean Norm or Magnitude of Vectors for Case of Symmetric $(d_{n,m})$

Having defined the eigenvector norm of vectors in section 9, and having noted that, based on a hermitian matrix, the eigenvector norm is equivalent to the euclidean norm or magnitude (see (9.11)), let us relate the eigenvector norm to the vector magnitude for other types of eigenvector norms.

For the general eigenvector norm of vectors defined in section 9, we have the definition

$$\|(a_n)\|_e = |(\ell_{n,m}) \cdot (a_n)| \tag{11.1}$$

Using a property of the associated matrix norm (11.1) becomes

$$\|(a_n)\|_e \leq |(\ell_{n,m})| \; |(a_n)| \tag{11.2}$$

giving an upper bound for the eigenvector norm. Rewriting (a_n) via an identity as

$$(a_n) = (r_{n,m}) \cdot (\ell_{n,m}) \cdot (a_n) \tag{11.3}$$

and taking the vector magnitude (euclidean norm)

$$|(a_n)| = |(r_{n,m}) \cdot (\ell_{n,m}) \cdot (a_n)| \tag{11.4}$$

with an inequality for matrix norms gives a lower bound for the eigenvector norm as

$$\begin{aligned} |(a_n)| &\leq |(r_{n,m})| \; |(\ell_{n,m}) \cdot (a_n)| \\ &= |(r_{n,m})| \; \|(a_n)\|_e \end{aligned} \tag{11.5}$$

Combining these results we have lower and upper bounds as

$$|(r_{n,m})|^{-1}|(a_n)| \le \|(a_n)\|_e \le |(\ell_{n,m})|\ |(a_n)| \qquad (11.6)$$

These are general bounds; how tight they are depends on $(\ell_{n,m})$ and $(r_{n,m})$, and thereby on $(d_{n,m})$.

If $(d_{n,m})$ is symmetric with right and left eigenvectors $(e_n)_\beta$, then the eigenvector matrices are

$$(\ell_{n,m}) = (e_{n,m})$$
$$(r_{n,m}) = (e_{n,m})^T = (e_{n,m})^{-1} \qquad (11.7)$$

For the euclidean or spectral norm we have

$$|(e_{n,m})^T|^2 = \lambda_{max}((e_{n,m})^{T\dagger} \cdot (e_{n,m})^T)$$
$$= \lambda_{max}((e_{n,m})^{\dagger} \cdot (e_{n,m}))$$
$$= |(e_{n,m})|^2 \qquad (11.8)$$

where it is noted that a square matrix and its transpose have the same eigenvalues. The double inequality for the eigenvector norm then becomes

$$|(e_{n,m})|^{-1}|(a_n)| \le \|(a_n)\|_e \le |(e_{n,m})|\ |(a_n)| \qquad (11.9)$$

This brackets $\|(a_n)\|_e$ both above and below with respect to $|(a_n)|$ by a common factor $|(e_{n,m})|$. While (11.9) implies

$$|(e_{n,m})| \ge 1 \qquad (11.10)$$

one might like to have $|(e_{n,m})|$ close to one so that $\|(a_n)\|_e$ approximates $|(a_n)|$.

Consider the eigenvector norm (10.3) of the eigenvector matrix as

$$\|(e_{n,m})\|_e = |(e_{n,m}) \cdot (e_{n,m}) \cdot (e_{n,m})^T| = |(e_{n,m}) \cdot (1_{n,m})|$$

$$= |(e_{n,m})|$$

$$= [\lambda_{max}((e_{n,m})^\dagger \cdot (e_{n,m}))]^{\frac{1}{2}} \tag{11.11}$$

Conveniently then the eigenvector norm and euclidean or spectral norm of $(e_{n,m})$ are the same. Our problem then reduces to finding the spectral norm (spectral radius) of the hermitian matrix

$$(E_{n,m}) \equiv (e_{n,m})^\dagger \cdot (e_{n,m}) \tag{11.12}$$

Since $(E_{n,m})$ is positive semidefinite and all its eigenvalues are non-negative we can write

$$|(E_{n,m})| = \lambda_{max}((E_{n,m}))$$

$$= \sup_{(a_n)\neq(0_n)} \frac{(a_n)^* \cdot (E_{n,m}) \cdot (a_n)}{(a_n)^* \cdot (a_n)} \tag{11.13}$$

Rewriting $(E_{n,m})$ as

$$(E_{n,m}) = (e_{n,m})^\dagger \cdot (e_{n,m})$$

$$= \{(e_{n,m})^T - 2j\mathrm{Im}[(e_{n,m})^T]\} \cdot (e_{n,m})$$

$$= (e_{n,m})^\dagger \cdot \{(e_{n,m})^* + 2j\mathrm{Im}[(e_{n,m})]\}$$

$$= (1_{n,m}) - 2j\mathrm{Im}[(e_{n,m})^T] \cdot (e_{n,m})$$

$$= (1_{n,m}) + 2j(e_{n,m})^\dagger \cdot \mathrm{Im}[(e_{n,m})] \tag{11.14}$$

the norm becomes

$$|(E_{n,m})| = \lambda_{max} = 1 + \Delta$$

$$\Delta = \sup_{(a_n)\neq(0_n)} \frac{(a_n)^* \cdot (\Delta_{n,m}) \cdot (a_n)}{(a_n)^* \cdot (a_n)}$$

$$(\Delta_{n,m}) \equiv -2j\text{Im}[(e_{n,m})^T] \cdot (e_{n,m}) \tag{11.15}$$

$$= 2j(e_{n,m})^\dagger \cdot \text{Im}[(e_{n,m})]$$

where Δ is real and $(\Delta_{n,m})$ is hermitian.

Let us assume that $(e_{n,m})$ is approximately real, or equivalently that all the $(e_n)_\beta$ are approximately real. Then $(\Delta_{n,m})$ is proportional to the assumed small $\text{Im}[(e_{n,m})]$ to first order. Then one can say that Δ is proportional to the imaginary part of the $(e_n)_\beta$ to first or higher order. Note that

$$|(e_{n,m})| = \{|(E_{n,m})|\}^{\frac{1}{2}} = 1 + \frac{\Delta}{2} + O(\Delta^2) \quad \text{as } \Delta \to 0 \tag{11.16}$$

so that $\Delta/2$ gives the deviation of the eigenvector norm from the euclidean norm of vectors.

XII. Application to Integral Equations of Electromagnetic Scattering

One of the uses for the norm concepts concerns the response of electromagnetic scatterers/antennas which are described in terms of integral equations of the form

$$\left\langle \tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s) \; ; \; \tilde{\vec{J}}(\vec{r}\,',s) \right\rangle = \tilde{\vec{I}}(\vec{r},s)$$

$\tilde{\vec{J}}$ ≡ vector response current density

(12.1)

$\tilde{\vec{I}}$ ≡ vector incidence quantity (typically an electric or magnetic field distribution)

$\tilde{\overleftrightarrow{\Gamma}}$ ≡ dyadic kernel (related to a Green's function)

$\langle , \rangle$ indicates domain of integration over common spatial coordinates ($\vec{r}\,'$ above) with dot above comma indicating dot multiplication--can be surface, volume, or line integration depending on nature of body and approximations

$\vec{r},\vec{r}\,'$ ≡ space vectors

s ≡ complex frequency

In solving an integral equation as in (12.1) one can compute the eigenmodes and eigenvalues of the integral operator via

$$\left\langle \tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s) \; ; \; \tilde{\vec{j}}_\beta(\vec{r}\,',s) \right\rangle = \tilde{\lambda}_\beta(s) \, \tilde{\vec{j}}_\beta(\vec{r},s)$$

$$\left\langle \tilde{\vec{\mu}}_\beta(\vec{r},s) \; ; \; \tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s) \right\rangle = \tilde{\lambda}_\beta(s) \, \tilde{\vec{\mu}}_\beta(\vec{r}\,',s) \qquad (12.2)$$

$$\left\langle \tilde{\vec{\mu}}_{\beta_1}(\vec{r},s) \; ; \; \tilde{\vec{j}}_{\beta_2}(\vec{r},s) \right\rangle = 1_{\beta_1,\beta_2}$$

which takes the same form as for matrices as in (5.1). This method of solution is referred to as the eigenmode expansion method (EEM), and has many important features in describing electromagnetic scattering and antenna problems [1,2,8]. Here a discrete spectrum of eigenvalues is assumed.

In solving equations (12.1) and (12.2) approximations are often invoked for numerical computations. By expanding the response $\tilde{\vec{J}}$ in (12.1) in a set of basis functions, and the source $\tilde{\vec{I}}$ in a set of testing functions (usually orthogonal to the basis functions), and by truncating each of these sets of N functions we obtain the moment method (MoM) [7]. The coefficients of these functions form N-component complex vectors related by an $N \times N$ matrix in the form

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{J}_n(s)) = (I_n(s)) \tag{12.3}$$

This form is the same as for the vectors and matrices considered earlier in this note. Furthermore the matrix $(\tilde{\Gamma}_{n,m}(s))$ can be used to define eigenvectors and eigenvalues in N-component complex numerical form, directly corresponding (as a numerical approximation) to the distributed form (vector space functions) in (12.2).

In a more general sense we can look at the integral equation (12.1) to define our norms. Vector norms become functional norms with summation replaced by integration. The vector norms of section 2 may be generalized to functional norms with appropriate care. In particular, for norms that involve integration, such as the euclidean norm

$$\|\tilde{\vec{a}}(\vec{r},s)\|_2 = \left[\left\langle \tilde{\vec{a}}(\vec{r},s) \,;\, \tilde{\vec{a}}^*(\vec{r},s)\right\rangle\right]^{\frac{1}{2}} \tag{12.4}$$

we should generalize the definition in the first equation of (2.1) to allow a zero norm to imply a zero function "almost everywhere," i.e., the function be non-zero (such as at discrete $\vec{r}$) to the extent that the integral (as in (12.4)) is zero.

The eigenmode norm is the generalization of the eigenvector norm in terms of the eigenmodes in (12.2). Considering a vector space function $\tilde{\vec{a}}(\vec{r},s)$ defined over the same domain gives

$$\|\tilde{\vec{a}}(\vec{r},s)\|_e \equiv \left\{ \sum_{\beta} \left| \left\langle \tilde{\vec{\mu}}_\beta(\vec{r},s) \,;\, \tilde{\vec{a}}(\vec{r},s)\right\rangle \right|^2 \right\}^{\frac{1}{2}} \tag{12.5}$$

provided the sum converges. Now if $\tilde{\vec{a}}$ represents the difference of some quantity (such as the current density) on an object of interest from its desired or ideal functional form, then $\|\tilde{\vec{a}}(\vec{r},s)\|_e$ is a measure of the "error."

Analogous to (10.1) one can define the associated kernel (or operator) norm to the eigenmode norm. In the special case that the kernel is $\tilde{\overleftrightarrow{\Gamma}}$ we have, corresponding to (10.4), the result

$$\|\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s)\|_e = |\lambda(\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s))|_{sup} \tag{12.6}$$

provided the supremum exists.

There is a special case of interest for which $\tilde{\overleftrightarrow{\Gamma}}$ is symmetric, for the impedance or E-field integral equation, as [1,2,8]

$$\begin{aligned} \left\langle \tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s) \; ; \; \tilde{\vec{J}}(\vec{r}\,',s) \right\rangle &= \tilde{\vec{E}}_{inc}(\vec{r},s) . \\ \tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s) &= \tilde{\overleftrightarrow{Z}}^T(\vec{r},\vec{r}\,';s) \\ &= \tilde{\overleftrightarrow{Z}}(\vec{r}\,',\vec{r};s) \\ &= \tilde{\overleftrightarrow{Z}}^T(\vec{r}\,',\vec{r};s) \end{aligned} \tag{12.7}$$

In this case we can set

$$\tilde{\vec{\mu}}_\beta(\vec{r},s) \equiv \tilde{\vec{j}}_\beta(\vec{r},s) \tag{12.8}$$

As discussed in section 11, then the eigenmode norm approximates the euclidean norm for this symmetric case, provided $\tilde{\vec{j}}_\beta(\vec{r},s)$ can be considered almost real. Since the impedance integral equation has important properties for understanding electromagnetic response, the eigenmode norm in this symmetric form can be quite interesting for considering the errors in various approximations to the desired electromagnetic response.

XIII. Summary

This note has begun an inquiry into the application of norms to physical problems such as associated with electromagnetic scatterers/antennas.

Norms have important properties for bounding vectors, functions, etc. in some sense defined by the particular norm. Such concepts are important for bounding electromagnetic response over some range of parameters such as those associated with the excitation.

Norms also have application to estimating the "errors" in electromagnetic space functions, including both excitation and response functions. In the case of EMP simulation norms can be used to quantify simulation errors.

A particular norm, the eigenvector or eigenmode norm, has been introduced and its properties discussed. This norm has some special conveniences when used in conjunction with integral equations of electromagnetics if the eigenmodes have been generated from the same equation under consideration.

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