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On Delta Functions, Part I: A Review of Various Representations and Properties of Dirac Delta Functions

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Abstract

The Dirac delta function[1] is widely used in pure and applied sciences, often as a mathematical representation for an idealized source function. A rigorous mathematical framework was provided by the theory of distributions[2, 3] for many operations that have long been performed on delta functions. This note, however, aims to put together the various representations and the properties of delta functions to serve the needs of a routine user who is not concerned with the rigor of the theory of distributions. Although much of the material in this note is drawn from the cited references, it is considered useful to compile the vast amount of information on this subject which is scattered in the literature. The subject of multidimensional delta functions is also briefly considered and will be dealt with in detail in an accompanying note.

Contents

I. Introduction

The Dirac delta function $\delta(x)$ introduced in Quantum mechanics by Dirac[1] in 1930 has existed in some form or another since much earlier times. The early indications of delta or impulse functions may be traced back to the works of Cauchy[4] and Poisson[5] in the early part of the last century (1815-1816). The delta functions arose when Cauchy and Poisson derived the Fourier integral theorem independently of not only each other but also of Fourier himself. Later Hermite[6] referred to the works of Cauchy and Poisson and made use of the impulse function. Kirchoff[7] was also acquainted with the impulse function in his formulation of Huygen's principle in the wave theory of light. The "heat source" of Lord Kelvin[8] is also relatable to the impulse function. It should be pointed out that the impulse function considered by all of the above consisted of a limiting form of certain types of sequence functions. Heaviside[9], towards the end of the last century, introduced an infinite series type of representation for the impulse function. In this context, the words impulse and delta will be used interchangeably. The name "delta" is for notational reasons and to justify "impulse," one might quote Heaviside[9]:

> "The function .... spots a single value of the arbitrary function in virtue of its impulsiveness."

This quotation has reference to an integral or sifting property of the delta function to be introduced in later sections. For an excellent account of the history of the delta function, the interested reader is referred to Van der Pol and Bremmer.[10]

Systematically, the delta function $\delta(x)$ was introduced by Dirac[1] in 1930 as a mathematical convenience. It evolved out of a need to mathematically represent the "state" of a dynamic variable in Quantum mechanics. When first introduced, it was called an "improper function" because it did not conform to the usual mathematical definition of a function which is required to have a definite value for each point in its domain. The theory of distributions or generalized functions aims to extend the definition of a function so that concepts like $\delta(x)$ can be put on a firm mathematical footing. So, strictly one should call $\delta(x)$ a delta distribution but the name delta function has now become a part of long tradition. Furthermore, for practical use of delta functions and their derivatives, what is important is their properties. Herein lies the justification for practical application of delta functions, as long as we ensure that no inconsistencies follow from their use.

The importance of delta functions in physical problems cannot be overemphasized. Delta functions often arise as idealized source functions, e.g., physics, fluid-flow, electromagnetic and acoustic problems. As an intermediate step in determining the system response for a given input condition, we often desire its response for idealized input sources. The mathematical representation of such idealized sources are typically spatial or temporal delta functions. It should be pointed out that such representations, albeit unphysical, are mathematically convenient. Theoretically, any input condition can be made up of an aggregate of delta functions and hence the importance of knowing the system response for a delta function input, commonly referred to as the Green's function. A knowledge of the Green's function is then used in determining the system response for an actual input condition. This will be illustrated in the following example

$$\left(\frac{d^2}{dx^2} - \gamma^2\right) f(x) = g(x) \tag{1.1}$$

This equation could be a mathematical model for any physical phenomenon; e.g., i) motion of a damped/an undamped harmonic oscillator, or ii) f(x) and g(x) could be proportional, respectively, to the magnetic vector potential and the incident electric field on the surface of a thin cylindrical conductor with γ being the propagation constant. In either case, the homogeneous solution is readily written down as

$$f_h(x) = A \sinh(\gamma x) + B \cosh(\gamma x) \tag{1.2}$$

In determining the particular integral, we may first find the delta function response by solving

$$\left(\frac{d^2}{dx^2} - \gamma^2\right) f_\delta(x) = \delta(x) \tag{1.3}$$

to be

$$f_\delta(x) = [-e^{-\gamma|x|}/(2\gamma)] \tag{1.4}$$

Using $f_\delta(x)$, the complete solution of equation 1.1 can now be written down as

$$f(x) = A \sinh(\gamma x) + B \cosh(\gamma x) - \frac{1}{2\gamma} \int_0^x g(x')\, e^{-\gamma|x-x'|}\, dx' \tag{1.5}$$

Thus, the delta function response or the Green's function for the differential operator in equation 1.1 is seen useful in constructing the total solution.

We shall conclude this section by noting the importance of the valuable impulse function, and in section II consider several representations and properties of the 1-dimensional delta function.

II. Definition and Examples of $\delta(x)$

It is nearly impossible to give a precise definition of the delta function that will satisfy both pure and applied mathematicians alike. For the purposes of this note outlined in the abstract, we will be content with the traditional definition available in applied mathematics and/or electromagnetic theory text books[11-15] wherein the delta function is often described as the limiting form of a sequence of functions which has certain assigned properties. Typically, the 1-dimensional delta function $\delta(x)$ is defined by

$$\delta(x) = \lim_{\epsilon \to 0} \delta_\epsilon(x) \tag{2.1}$$

and the three basic properties that the delta function is required to possess are

$$\left.\begin{array}{lll} \text{Property 1} & \delta(x) = 0 & \text{at } x \neq 0 \\ \text{Property 2} & \delta(x) = \infty & \text{at } x = 0 \\ \text{Property 3} & \int_{-\infty}^{\infty} \delta(x)\, dx = 1 & \end{array}\right\} \tag{2.2}$$

The difficulties in definition and usage are avoided by postponing the limit taking process until after whatever mathematical operation(s) is(are) to be performed on $\delta(x)$. For example, in checking for property 3, the limit taking process will be done after the integration is performed.

A. Sequence function representations

We shall consider four illustrative examples of sequence functions which in the limit tend to a 1-dimensional delta function, as given by

$$\delta(x) = \lim_{\epsilon \to 0} \delta_{\epsilon, i}(x) \ ; \ i = 1, 2, 3, 4 \tag{2.3}$$

where

$$\delta_{\epsilon,1}(x) = \begin{bmatrix} 1/(2\epsilon) \; ; & |x| < \epsilon \\ 0 \; ; & |x| > \epsilon \end{bmatrix} \quad , \quad \delta_{\epsilon,2}(x) = \left[\frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2} \right]$$

$$\delta_{\epsilon,3}(x) = \left[\frac{1}{\pi\epsilon} \frac{\sin(x/\epsilon)}{(x/\epsilon)} \right] \quad \text{and} \quad \delta_{\epsilon,4}(x) = \left[\frac{1}{\sqrt{\pi}\,\epsilon} e^{-(x^2/\epsilon^2)} \right] \qquad (2.4)$$

These sequence functions are plotted in figure 2.1 as a function of x with ϵ as the parameter. It is seen that in all four cases the functions become increasingly peaked in the immediate vicinity of the origin and tend to vanish everywhere else. In table 2.1 we also ensure that the sequence functions in the limit (or the delta functions) enclose a unit area. The sequence functions considered here are by no means exhaustive, since one can easily make up a large number of examples ensuring that the three basic properties are satisfied. Several shapes of sequence functions suggest themselves like triangles, inverted parabolas which become increasingly peaked and are suitably normalized to enclose unit areas. In fact Lebesgue[16] recognized that there exist numerous sequence functions $\delta_\epsilon(x)$. For instance,

$$\delta_\epsilon(x) = \frac{K(x/\epsilon)}{2\epsilon \int_0^\infty K(s)\, ds} \qquad \text{for} \quad x > 0 \qquad (2.5)$$

$$\delta_\epsilon(x) = \frac{K(x/\epsilon)}{2\epsilon \int_{-\infty}^0 K(s)\, ds} \qquad \text{for} \quad x < 0 \qquad (2.6)$$

Equation 2.6 is implied in equation 2.5 if $K(x)$ is an even function of x. The conditions that are required to be satisfied by $K(x)$ are that the integrals in the denominators of equations 2.5 and 2.6 be absolutely convergent and that $K(0) \neq 0$. For every choice of $K(x)$ that meets these requirements, one can get a sequence function or limiting form representation for a 1-dimensional delta function, like

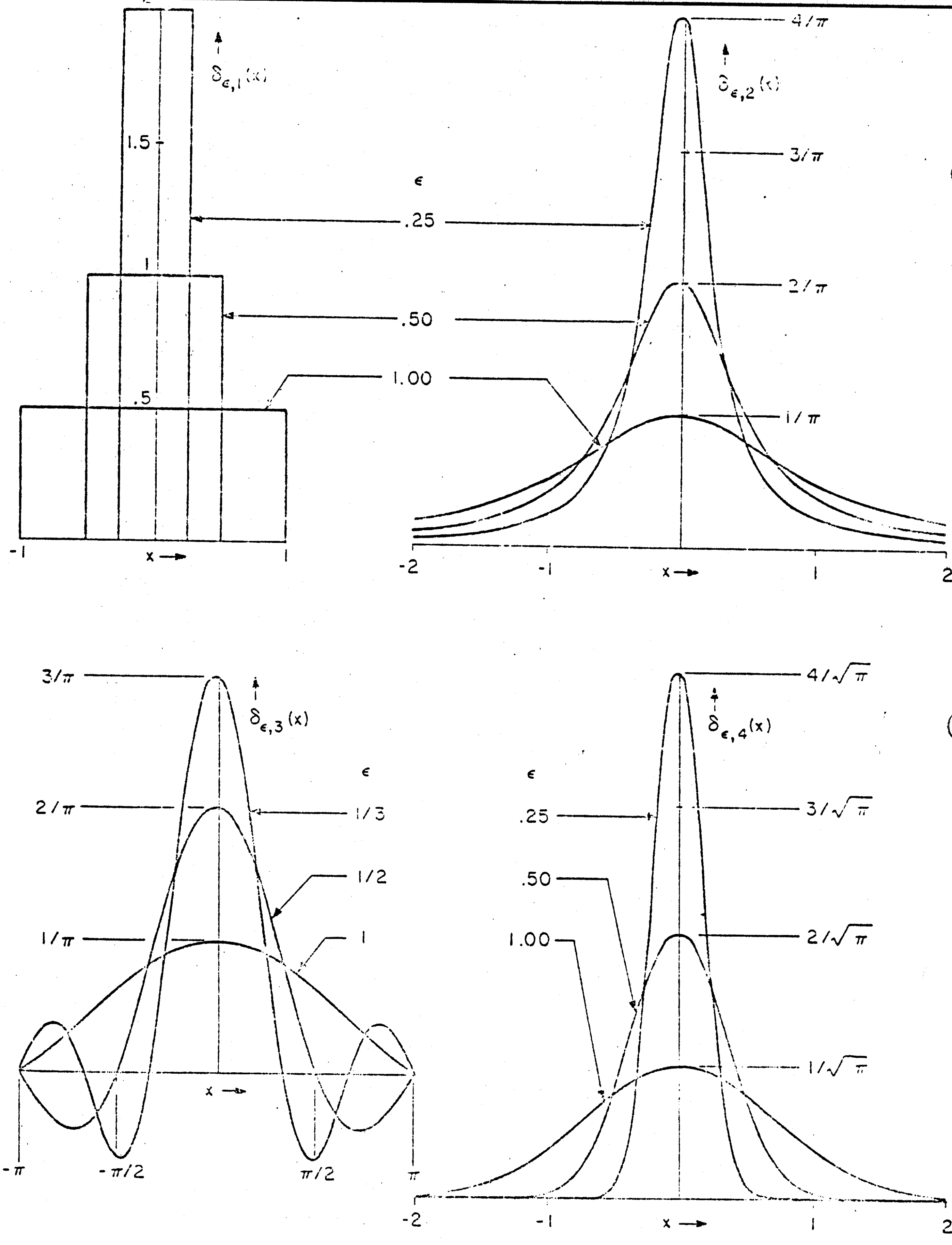

Figure 2.1. Four [illegible] Which T[illegible] in [illegible] Limit to a [illegible] Function

Table 2.1

Illustration of the Basic Properties of Four Possible Sequence Functions Which Tend in the Limit to a 1-Dimensional Delta Function

Example of the Sequence Function	The Three Basic Properties		
	No. 1 $\delta(x \neq 0) = 0$	No. 2 $\delta(x = 0) = \infty$	No. 3 $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$
$\delta_{\epsilon,1}(x) = \begin{cases} 1/(2\epsilon), & \lvert x \rvert < \epsilon \\ 0, & \lvert x \rvert > \epsilon \end{cases}$	$\lim_{\epsilon \to 0} [0] = 0$	$\lim_{\epsilon \to 0} \left[\frac{1}{2\epsilon}\right] = \infty$	$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon}\, dx = 1$
$\delta_{\epsilon,2}(x) = \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}$	$\lim_{\epsilon \to 0} \left[\frac{1}{\pi} \frac{\epsilon}{x^2}\right] = 0$	$\lim_{\epsilon \to 0} \left[\frac{1}{\pi\epsilon}\right] = \infty$	$\lim_{\epsilon \to 0} \left[\frac{2}{\pi} \tan^{-1} \frac{x}{\epsilon}\right]_0^{\infty} = 1$
$\delta_{\epsilon,3}(x) = \frac{1}{\pi\epsilon} \frac{\sin(x/\epsilon)}{(x/\epsilon)}$	$\lim_{\epsilon \to 0} \left[\frac{1}{\pi} \frac{\sin(x/\epsilon)}{x}\right] = 0$	$\lim_{\epsilon \to 0} \left[\frac{1}{\pi\epsilon}\right] = \infty$	$\lim_{\epsilon \to 0} \left[\frac{1}{\pi} \pi\right] = 1$
$\delta_{\epsilon,4}(x) = \frac{1}{\sqrt{\pi}\,\epsilon} e^{-x^2/\epsilon^2}$	$\lim_{\epsilon \to 0} \left[\frac{1}{\sqrt{\pi}} \frac{1}{(\epsilon + \frac{x^2}{\epsilon} + \cdots\cdots)}\right] = 0$	$\lim_{\epsilon \to 0} \left[\frac{1}{\sqrt{\pi}\,\epsilon}\right] = \infty$	$\lim_{\epsilon \to 0} \left[\frac{2}{\sqrt{\pi}\,\epsilon} \int_0^{\infty} e^{-x^2/\epsilon^2}\, dx\right] = \lim_{\epsilon \to 0} \left[\frac{2}{\sqrt{\pi}\,\epsilon} \frac{\Gamma(1/2)\epsilon}{2}\right] = 1$

$$\delta(x) = \lim_{\epsilon \to 0} \delta_{\epsilon, i}(x) = \lim_{\epsilon \to 0} \left[\frac{K_i(x/\epsilon)}{2\epsilon \int_0^\infty K_i(s)\, ds} \right] \tag{2.7}$$

The four examples of $\delta_\epsilon(x)$ that we considered earlier are all special cases of Lebesgue's definition. However, by no means are the sequence functions limited to those given by Lebesgue's definition. The last two examples ($i = 5, 6$) in table 2.2 illustrate this statement[10]. Several possible generating $[K_i(x)]$ and sequence $[\delta_{\epsilon, i}(x)]$ functions that may be used in equation 2.7 to represent a 1-dimensional delta function are listed in table 2.2. From table 2.2 it is also seen that the two cases $i = 3, 4$ can each lead to an infinite number of representations for $\delta(x)$.

B. Summation representations

As was pointed out in section I, Heaviside[9] in 1893 was probably the first one to provide an infinite series representation for the delta function of the type

$$\delta(u - v) = \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi u}{L}\right) \sin\left(\frac{n\pi v}{L}\right) \tag{2.8}$$

for real u and v such that $|u|, |v| < L$ but $\neq 0$.

It is, of course, no surprise that this series is divergent, otherwise $\delta(x)$ would be a bona fide function which it is not. We shall now outline a general scheme to generate numerous series or summation representations for $\delta(x)$.

It is seen that the sequence function representation of the earlier section may be usefully extended to yield summation representations. The procedure will consist of finding the Fourier series for the sequence functions $\delta_{\epsilon, i}(x)$ and then considering the limit of this series as $\epsilon \to 0$.

Table 2.2

Examples of Generating and Sequence Functions

i	Generating Function $K_i(x)$	Sequence Function $\delta_{\epsilon,i}(x)$
1	$u(x+1) - u(x-1)$	$\frac{1}{2\epsilon}\left[u\left(\frac{x+\epsilon}{\epsilon}\right) - u\left(\frac{x-\epsilon}{\epsilon}\right)\right]$
2	$1/(x^2+1)$	$\epsilon/[\pi(\epsilon^2+x^2)]$
3	$K_{3_n}(x) = \int_0^1 \cos(xs)(1-s)^n\, ds$	$\delta_{\epsilon,3_n}(x) = \frac{1}{\pi\epsilon} \mathrm{Re}\left[e^{-\omega} \frac{d^n}{d\omega^n}\left(\frac{e^{\omega}-1}{\omega}\right)\right]_{\omega=ix/\epsilon}$
	For example:	
	$n = 0 \rightarrow (\sin x)/x$	$\rightarrow \sin(x/\epsilon)/(\pi x)$
	$n = 1 \rightarrow (1-\cos x)/x^2$	$\rightarrow 2\epsilon \sin^2(x/2\epsilon)/(\pi x^2)$
	$n = 2 \rightarrow -2\left(1+\frac{\sin x}{x}\right)/x^2$	$\rightarrow 2[(x/\epsilon) - \sin(x/\epsilon)]/[\pi\epsilon(x/\epsilon)^3]$
4	$K_{4_m}(x) = \exp(-\lvert x\rvert^m)$, $m > 0$	$\delta_{\epsilon,4_m}(x) = \exp(-\lvert x/\epsilon\rvert^m)/[2\epsilon\Gamma(1/m)]$
	For example:	
	$m = 1 \rightarrow \exp(-\lvert x\rvert)$	$\rightarrow \exp(-\lvert x/\epsilon\rvert)/(2\epsilon)$
	$m = 2 \rightarrow \exp(-x^2)$	$\rightarrow \exp(-x^2/\epsilon^2)/(\sqrt{\pi}\,\epsilon)$
	etc.	
5	Not Applicable	$\frac{\sinh\epsilon}{2\pi(\cosh\epsilon - \cos x)}\{u(x+\pi) - u(x-\pi)\}$
6	Not Applicable	$\frac{1}{2\pi}\left[\frac{1-(1-\epsilon)^2}{1-2(1-\epsilon)\cos x + (1-\epsilon)^2}\right]\{u(x+\pi) - u(x-\pi)\}$

To illustrate the procedure, let us define the Fourier series of f(x), valid for $-\pi \le x \le \pi$ as

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{2.9}$$

where the Fourier coefficients are given by

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \end{aligned}\right\} \tag{2.10}$$

and consider an example $f(x) = \delta_{\epsilon,1}(x)$ given by equation 2.4. Since $\delta_{\epsilon,1}(x)$ is an even function of x, all b_n's are zero and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \delta_{\epsilon,1}(x)\cos(nx)\,dx \\ &= \frac{1}{\pi}\int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon}\cos(nx)\,dx \quad ; \quad \text{if } \epsilon < \pi \\ &= \frac{1}{n\pi\epsilon}\sin(n\epsilon) \quad ; \quad \text{with } a_o = \frac{1}{\pi} \end{aligned} \tag{2.11}$$

Therefore,

$$\delta_{\epsilon,1}(x) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{n\pi\epsilon}\sin(n\epsilon)\cos(nx)$$

and using equation 2.1, we get

$$\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos(nx) \qquad \text{for} \quad |x| \leq \pi \tag{2.12a}$$

$$\sum_{n=-\infty}^{\infty} \delta(x - 2n\pi) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos(nx) \qquad \text{for all } x \tag{2.12b}$$

By a suitable change of the variable x, it is possible to obtain Heaviside's formula of equation 2.8. Although the series representing $\delta(x)$ is necessarily divergent, it should not be concluded that the series is useless. Quite often, the series representation of the types given by equations 2.8 and 2.12 are useful when delta functions appear under the integral sign. We shall formally write down the general series representations for $\delta(x)$ using the results of table 2.2 and the fact $\delta(x) = \delta(-x)$ as,

$$\delta(x) = \text{Fourier Series (F.S.) of } \left[\lim_{\epsilon \to 0} \delta_{\epsilon, i}(x)\right]$$

$$= \lim_{\epsilon \to 0} [\text{F.S. of } \delta_{\epsilon, i}(x)]$$

or

$$\delta(x) = \lim_{\epsilon \to 0} \left[\frac{a_{o,i}}{2} + \sum_{n=1}^{\infty} a_{n,i}(\epsilon) \cos(nx)\right] \qquad \text{for} \quad |x| \leq \pi \tag{2.13a}$$

or

$$\sum_{m=-\infty}^{\infty} \delta(x - 2m\pi) = \lim_{\epsilon \to 0} \left[\frac{a_{o,i}}{2} + \sum_{n=1}^{\infty} a_{n,i}(\epsilon) \cos(nx)\right] \qquad \text{for all } x \tag{2.13b}$$

In equation 2.13, the Fourier coefficients $a_{n,i}(\epsilon)$ are given by

$$a_{n,i}(\epsilon) = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta_{\epsilon, i}(x) \cos(nx)\, dx \tag{2.14}$$

Since a large number of $\delta_{\epsilon, i}(x)$'s are possible from table 2.2, equations 2.13 and 2.14 can theoretically lead to numerous series representations for $\delta(x)$. We conclude this section by giving another example of series representation

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \quad ; \quad \text{for} \quad [-\pi \le (\phi - \phi') \le \pi] \tag{2.15a}$$

or

$$\sum_{m=-\infty}^{\infty} \delta(\phi - \phi' - 2m\pi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \quad ; \quad \text{all} \quad (\phi - \phi') \tag{2.15b}$$

which usually appears in problems with cylindrical geometries.

C. Differential and integral representations

It is observed that if we integrate $\delta(x)$ between the limits $-\infty$ and x, we get a unit step function

$$u(x) = \int_{-\infty}^{x} \delta(\xi)\, d\xi = \begin{cases} 0 & \text{if} \quad x < 0 \\ .5 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

Consequently, we have a differential representation,

$$\delta(x) = \frac{d}{dx} u(x) \tag{2.16}$$

We shall now proceed to express $\delta(x)$ as an integral. One way[17] of solving certain types of integral equations arising in physical problems involving finite regions is by the Eigenmode Expansion Method (EEM). For instance, if the integral equation is in scalar quantities and over a finite one-dimensional region, the resulting set of orthogonal functions is discreet. Their orthonormal (orthogonal and normalized) property can be expressed by

$$\int_a^b f_m(x)\, g_n(x)\, dx = \delta_{m,n} = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases} \tag{2.17}$$

where

$\delta_{m,n}$ is the Kroneker delta

m and n assume integer values, and

a and b are such that the integration is over a finite range.

Depending on problem symmetries and the nature of the operators from which they are derived, the orthonormal functions f(x) and g(x) may have a special relationship. But continuous systems of orthonormal functions do exist in problems involving infinite or semi-infinite regions (for example, either a or b $= \pm\infty$), in which case the analog of equation 2.17 will consist of the Dirac delta in place of the Kroneker delta.

$$\int_a^b f_\nu(x)\, g_{\nu'}(x)\, dx = \delta(\nu - \nu') \tag{2.18}$$

If both sides of equation 2.18 are integrated w.r.t. ν',

$$\int_a^b dx\, f_\nu(x) \int_{\nu_1}^{\nu_2} d\nu'\, g_{\nu'}(x) = \delta_{\nu_1, \nu, \nu_2} = \begin{cases} 0 & \nu < \nu_1 \\ 1 & \nu_1 < \nu < \nu_2 \\ 0 & \nu > \nu_2 \end{cases} \tag{2.19}$$

Equation 2.19 resembles equation 2.17 more closely than equation 2.18 and expresses the orthonormal condition for the case of continuous system of orthogonal functions. The delta in equation 2.19 is a logical extension of the Kroneker delta from integers to real number domain. Examples of the delta function expressed in terms of orthonormal functions are[15]

Example i:

$$\delta(z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(z-z')} \, dk \tag{2.20a}$$

In Laplace form

$$\delta(z - z') = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\gamma(z-z')} \, d\gamma$$

Since the integrand in above has no poles, σ can be chosen to be $= 0$, and hence

$$\delta(z - z') = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\gamma(z-z')} \, d\gamma \tag{2.20b}$$

Example ii:

$$\delta(\rho - \rho') = \int_{0}^{\infty} (k\rho) \, J_m(k\rho) \, J_m(k\rho') \, dk \tag{2.21}$$

where J_m is the Bessel function of first kind and order m.

Equations 2.20a and 2.21, which provide integral representations for the delta function, usually arise along with $\delta(\phi - \phi')$ of equation 2.15 in physical problems with cylindrical geometries. Equation 2.20a, sometimes referred to as the Fourier integral representation, suggests yet another way of deriving integral representations for $\delta(x)$ and its derivatives. This procedure consists of finding the Fourier Transform (F. T.) of the sequence functions $\delta_{\epsilon, i}(x)$ and then taking the limit as $\epsilon \to 0$. Let us define an F. T. pair as

$$\left. \begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega x} f(x) \, dx \\ f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} F(\omega) \, d\omega \end{aligned} \right\} \tag{2.22}$$

and consider an example in $f(x) = \delta_{\epsilon, 2}(x)$ of equation 2.4, therefore

$$\text{F. T. of } \delta_{\epsilon,2}(x) = \int_{-\infty}^{\infty} \left\{ \frac{\epsilon}{\pi(\epsilon^2 + x^2)} \right\} e^{-i\omega x} \, dx$$

$$= \exp(-|\omega|\epsilon)$$

This integral has been performed using equation 3.723.2.[18] So $\delta(x)$ can now be expressed as

$$\delta(x) = \lim_{\epsilon \to 0} \delta_{\epsilon,2}(x) = \lim_{\epsilon \to 0} \left[\frac{\epsilon}{\pi(\epsilon^2 + x^2)} \right]$$

$$= \lim_{\epsilon \to 0} \left[\text{F. T.}^{-1} \, e^{-|\omega|\epsilon} \right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \, e^{-|\omega|\epsilon} \, d\omega \right]$$

or

$$\delta(x) = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \, d\omega \right] = \left[\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\gamma x} \, d\gamma \right] = \left[\int_{-\infty}^{\infty} e^{i\omega 2\pi x} \, d\omega \right] \qquad (2.23)$$

Formally differentiating equation 2.23 n times w.r.t. x,

$$\left. \begin{aligned} 2\pi\delta'(x) &= \int_{-\infty}^{\infty} i\omega e^{i\omega x} \, d\omega \\ 2\pi\delta''(x) &= \int_{-\infty}^{\infty} (i\omega)^2 \, e^{i\omega x} \, d\omega \\ &\vdots \\ 2\pi\delta^{(n)}(x) &= \int_{-\infty}^{\infty} (i\omega)^n \, e^{i\omega x} \, d\omega \\ &= \frac{1}{i} \int_{-i\infty}^{i\infty} \gamma^n \, e^{\gamma x} \, d\gamma \end{aligned} \right\} \qquad (2.24)$$

where once again the derivatives of $\delta(x)$ can be represented by using sequence functions as

$$\delta^{(n)}(x) = \frac{d^n}{dx^n}\delta(x) = \frac{d^n}{dx^n}\left[\lim_{\epsilon\to 0}\delta_{\epsilon,i}(x)\right]$$

$$= \lim_{\epsilon\to 0}\left[\frac{d^n}{dx^n}\delta_{\epsilon,i}(x)\right] \quad ; \quad \begin{array}{l} n = 0, 1, 2 \cdots\cdots \\ i = 1, 2, 3 \cdots\cdots \end{array} \tag{2.25}$$

For example, using $\delta_{\epsilon,4}(x)$ of equation 2.4, a doublet is represented by

$$\delta'(x) = \lim_{\epsilon\to 0}\left[-\frac{2x}{\epsilon^3\sqrt{\pi}}e^{-x^2/\pi^2}\right] \tag{2.26}$$

Returning to the integral representation of $\delta(x)$ itself, if we find the F. T. of the sequence functions of table 2.2 to be $\delta_{\epsilon,i}(\omega)$, then

$$\delta(x) = \lim_{\epsilon\to 0}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\,\delta_{\epsilon,i}(\omega)\,d\omega\right] \tag{2.27}$$

Equation 2.27 provides a formula to derive integral representations for $\delta(x)$. In conclusion, it is observed that because of the evenness of $\delta(x)$, the F. T. in this section can be replaced by Fourier cosine transform, e. g.,

$$\delta(x) = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\,d\omega\right] = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\omega x)\,d\omega\right]. \tag{2.28}$$

D. Properties of delta functions

The practical usefulness of delta functions lies in their properties, some of which we will list below without giving proofs:

1. $\delta(ax) = \delta(-ax) = \frac{1}{|a|}\delta(x) \quad , \quad a \neq 0$

2 $\delta(ax - b) = \delta(b - ax) = \frac{1}{|a|}\delta\left(x - \frac{b}{a}\right) \quad , \quad a \neq 0$

3. If $(ad - bc) \neq 0$,[19]

$$\delta(ax_1 + bx_2)\,\delta(cx_1 + dx_2) = \frac{1}{|ad - bc|}\delta(x_1)\,\delta(x_2)$$

4. $f(x)\,\delta(x) = f(0)\,\delta(x)$

Example i: $x\delta(x) = 0$

5. Or, in general[20]

$$f(x)\,\delta^{(n)}(x) = \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!} f^{(k)}(0)\,\delta^{(n-k)}(x)$$

for $n = 1$,

$$f(x)\,\delta'(x) = f(0)\,\delta'(x) - f'(0)\,\delta(x)$$

Example i: $x\delta'(x) = -\delta(x)$

Example ii: $x^2\delta'(x) = 0$

6. $x\delta^{(m)}(x) = -m\delta^{(m-1)}(x) \quad ; \quad m \geq 0$

7. From[21]

$$x^n\delta^{(m)}(x) = \begin{cases} (-1)^n \frac{m!}{(m-n)!}\delta^{(m-n)}(x) & ; \quad m \geq n \\ 0 & ; \quad m < n \end{cases}$$

8. $\delta(x^2 - a^2) = \dfrac{\delta(x - a) + \delta(x + a)}{2a}$; $a > 0$

9. $\delta(\phi - \phi') = \dfrac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\phi - \phi')}$; $|(\phi - \phi')| \leq \pi$

Calling $(\phi - \phi') = x$

10. $\sum_{m=-\infty}^{\infty} \delta(x - 2m\pi) = \dfrac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx}$

which can be simplified to yield,

11. $\sum_{m=-\infty}^{\infty} \delta(x - 2m\pi) = \dfrac{1}{2\pi} + \dfrac{1}{\pi} \sum_{n=1}^{\infty} \cos(nx)$

Differentiating once w.r.t. x

12. $\sum_{m=-\infty}^{\infty} \delta'(x - 2m\pi) = -\dfrac{1}{\pi} \sum_{n=1}^{\infty} n \sin(nx)$

or, in general for $k = 0, 1, 2, 3 \cdots\cdots$ in (13) and (14),

13. $\sum_{m=-\infty}^{\infty} \delta^{(2k+1)}(x - 2m\pi) = \dfrac{(-1)^{k+1}}{\pi} \sum_{n=1}^{\infty} n^{2k+1} \sin(nx)$

14. $\sum_{m=-\infty}^{\infty} \delta^{(2k)}(x - 2m\pi) = \dfrac{\delta_{k,o}}{2\pi} + \dfrac{(-1)^{k}}{\pi} \sum_{n=1}^{\infty} n^{2k} \cos(nx)$

15. $\delta(x) = \dfrac{\nabla^2 |x|}{2} = \dfrac{1}{2} \dfrac{\partial^2}{\partial x^2} |x| = \dfrac{1}{2} \dfrac{\partial}{\partial x} \mathrm{sgn}(x)$

In 3-dimensional space,[21]

16. $\delta(\vec{r}) = -\frac{1}{4\pi} \nabla^2 \left(\frac{1}{r}\right) = -\frac{1}{4\pi} \nabla^2 \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right)$

In n-dimensional space,[21]

17. $\delta(r_H) = \frac{-1}{\Omega_n (n-2)} \nabla_H^2 \left(\frac{1}{|r_H|^{n-2}}\right)$

$$= \frac{-1}{\Omega_n (n-2)} \nabla^2 \left[\frac{1}{\left(\sum_{i=1}^{\infty} x_i^2\right)^{1/2}}\right]$$

where

Ω_n = Hyper surface area of a sphere of unit radius in n-D space

r_H = Position vector in n-D space

18. $\int_{-\infty}^{x} \delta(\xi - a)\, d\xi = u(x-a) = \begin{cases} 0 & ; \quad x < a \\ .5 & ; \quad x = a \\ 1 & ; \quad x > a \end{cases}$

or,

$$\delta(x - a) = \frac{d}{dx} u(x - a)$$

19. $\{\delta[y(x)]\}^{(k)} = \sum_i \frac{1}{|y'(x_i)|} \left(\frac{1}{y'(x)} \frac{\partial}{\partial x}\right)^k \delta(x - x_i)$

with $y(x_i) = 0$ and $y'(x_i) \neq 0$, and $k = 0, 1, 2 \cdots\cdots$

For example, $k = 0$

$$\delta[y(x)] = \sum_i \frac{\delta(x - x_i)}{|y'(x_i)|}$$

(Also see the following section.)

20. $\overset{\rightrightarrows}{\delta}(\vec{r} - \vec{r}\,') = \overset{\rightrightarrows}{1}\, \delta(\vec{r} - \vec{r}\,')$

where the l.h.s. is the dyadic delta function and $\overset{\rightrightarrows}{1}$ is an identity (or unit) dyad. $\overset{\rightrightarrows}{\delta}$ appears, e.g., in the characteristic equation {equation 3.5 of reference 17} for the eigenvalues, when one uses a dyadic Eigenmode Expansion Method (EEM) for solving certain types of integral equations.

We shall now define and use a functional notation,

$$\left\langle f(x)\,,\ g(x) \right\rangle = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx \qquad (2.29)$$

21. $\left\langle \delta(x)\,,\ \phi(x) \right\rangle = \left\langle \phi(x)\,,\ \delta(x) \right\rangle = \phi(0)$

22. $\left\langle \delta(x - x_o)\,,\ \phi(x) \right\rangle = \left\langle \delta(x)\,,\ \phi(x + x_o) \right\rangle = \phi(x_o)$

23. $\left\langle \delta'(x - x_o)\,,\ \phi(x) \right\rangle = -\,\phi'(x_o)$

24. $\left\langle \delta^{(k)}(x - x_o)\,,\ \phi(x) \right\rangle = (-1)^k\, \phi^{(k)}(x_o)$

25. $\left\langle \delta[y(x)]\,,\ f(x) \right\rangle = \sum_i \dfrac{f(x_i)}{|y'(x_i)|}$

where

$y(x_i) = 0$ and x_i's are in the range of integration.

26. $\left\langle \delta(x - a)\,,\ \delta(x - b) \right\rangle = \delta(a - b) = \delta(b - a)$

27. $\left\langle x^n\,,\ \delta(x - a) \right\rangle = a^n$.

We conclude this list by stating that although $\delta^2(x)$ is undefined, the convolution of two delta functions[20] is given by,

28. $\delta(x - x_1) * \delta(x - x_2) = \int_{-\infty}^{\infty} \delta(X - x_1)\, \delta(x - X - x_2)\, dX$

$$= \delta[x - (x_1 + x_2)] \quad .$$

In the above partial list of the properties of delta functions and their derivatives, we have included the delta function of a function of x. This forms the subject for the following section.

E. Delta function of a function of x

Using property 21 of the previous section,

$$\int_{-\infty}^{\infty} \delta(t)\, \phi(t)\, dt = \phi(0) \tag{2.30}$$

Treating the variable t as a function, $t = f(x)$ and hence $dt = f'(x)\, dx$, which gives

$$\int_{-\infty}^{\infty} \delta[f(x)]\, \phi[f(x)]\, f'(x)\, dx = \phi(0)$$

Calling $\quad \phi[f(x)]\, f'(x) = \psi(x)$

$$\int_{-\infty}^{\infty} \delta[f(x)]\, \psi(x)\, dx = \frac{\psi(x_o)}{|f'(x_o)|} \tag{2.31}$$

with $\quad f(x_o) = 0 \quad .$

If x_i's are simple roots of f(x) and, further, if $f'(x_i) \neq 0$,

$$\int_{-\infty}^{\infty} \delta[f(x)]\, \psi(x)\, dx = \sum_i \frac{\psi(x_i)}{|f'(x_i)|} \tag{2.32}$$

or,

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \tag{2.33}$$

The restrictions are $f(x_i) = 0$ and $f'(x_i) \neq 0$. As an example, let us consider $f(x) = \sin x$ for which $x_i = i\pi$, $|f'(x_i)| = |\cos x_i| = |\cos(i\pi)| = 1$, yielding

$$\delta[\sin x] = \sum_{i=0,1,2\cdots} \delta(x - i\pi) \tag{2.34}$$

Just like $\delta'(x)$, we can consider the derivative(s) of equation 2.33

$$\{\delta[f(x)]\}' = \delta'[f(x)]\, f'(x) \tag{2.35}$$

or, in general

$$\{\delta[f(x)]\}^{(k)} = \sum_i \frac{1}{|f'(x_i)|} \left(\frac{1}{f'(x)} \frac{\partial}{\partial x}\right)^k \delta(x - x_i) \tag{2.36}$$

with $f(x_i) = 0$ and $f'(x_i) \neq 0$.

III. Delta Function in a Multidimensional Space

The delta function can be defined in an n-dimensional space, and in this section we shall consider the special cases of n = 2 and 3.

A. 2-dimensional or the surface delta function

The surface delta function $\delta_s(\vec{r}_s - \vec{r}_{s_o})$ will be defined as an extension of the earlier description of $\delta(x - x_o)$.

$$\delta_s(\vec{r}_s - \vec{r}_{s_o}) = \delta_s[\vec{1}_x(x - x_o) + \vec{1}_y(y - y_o)]$$

$$\equiv \delta_s(x - x_o, y - y_o) = \delta(x - x_o)\,\delta(y - y_o) \tag{3.1}$$

The subscript s indicates the surface nature of the function and, if x and y have dimensions of length, then δ_s has the dimension of inverse area. This may be seen in the following integral property

$$\int_S \delta_s(\vec{r}_s - \vec{r}_{s_o})\, dS = \delta_{\vec{r}_{s_o}, S} = \begin{cases} 1 , & \vec{r}_{s_o} \in S \\ 0 , & \vec{r}_{s_o} \notin S \end{cases} \tag{3.2}$$

where $\delta_{\vec{r}_{s_o}, S}$ is a type of a Kroneker delta. Writing out equation 3.2 in more recognizable form

$$\int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy\ \delta_s(x - x_o, y - y_o) = \delta_{x_1, x_o, x_2}\ \delta_{y_1, y_o, y_2} \tag{3.3}$$

where the Kroneker delta on the r. h. s. is given by

$$\delta_{a, b, c} = \begin{cases} 0 , & b < a \\ 1 , & a < b < c \\ 0 , & b > c \end{cases} \tag{3.4}$$

Of course, in addition to the integral property of equation 3.3, the surface delta function vanishes everywhere except at one point in the 2-dimensional x, y space as given by

$$\delta_s(\vec{r}_s - \vec{r}_{s_o}) \equiv \delta_s(x - x_o, y - y_o) = \begin{cases} 0 \ , & \begin{bmatrix} \text{everywhere except at} \\ x = x_o \ \text{and} \ y = y_o \end{bmatrix} \\ \infty \ , & \text{at } x = x_o \ \text{and} \ y = y_o \ . \end{cases} \tag{3.5a}$$

The surface delta function is related to the linear or 1-dimensional delta functions according as,

$$\delta_s(x - x_o, y - y_o) = \delta(x - x_o)\,\delta(y - y_o) \tag{3.5b}$$

B. 3-dimensional or the volume delta function

In order to describe the 3-dimensional or the volume delta function, let us define a general curvilinear coordinate system (u, v, w) in which an elemental volume is given by[22]

$$dV = J(x, y, z; u, v, w)\ du\ dv\ dw \tag{3.6}$$

where the Jacobian J of the coordinate transformation from a Cartesian frame (x, y, z) to (u, v, w) is evaluated using

$$J(x, y, z; u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \tag{3.7}$$

The singular points of the coordinate system are given by the zeros of the Jacobian determinant. The Jacobian will be used in relating a 3-dimensional delta function to its 1-dimensional counterparts. For instance,

$$\delta_v(\vec{r} - \vec{r}_o) = \delta_v(u - u_o, v - v_o, w - w_o)$$

$$\delta_v(\vec{r} - \vec{r}_o)\, dx\, dy\, dz = \delta(u - u_o)\, \delta(v - v_o)\, \delta(w - w_o)\, du\, dv\, dw$$

Using equation 3.6

$$\delta_v(\vec{r} - \vec{r}_o) = \frac{1}{J(x, y, z; u, v, w)}\, \delta(u - u_o)\, \delta(v - v_o)\, \delta(w - w_o) \qquad (3.8)$$

For example,

i) Cartesian coordinates:

elemental volume $= dx\, dy\, dz$

$J(x, y, z; x, y, z) = 1$

J has no zeros and therefore

$$\delta_v(x - x_o, y - y_o, z - z_o) = \delta(x - x_o)\, \delta(y - y_o)\, \delta(z - z_o) \qquad (3.9)$$

ii) Cylindrical coordinates:

elemental volume $= \rho\, \partial\rho\, \partial\phi\, dz$

$J(x, y, z; \rho, \phi, z) = \rho$

Points on the z axis for which $\rho = 0$, and ϕ is ignorable are singular for this coordinate system. Therefore, using,[22]

$$\delta_v(\vec{r} - \vec{r}_o) = (1/\rho_o)\, \delta(\rho - \rho_o)\, \delta(\phi - \phi_o)\, \delta(z - z_o)$$

$$= \frac{\delta(\rho)\, \delta(z - z_o)}{\int_0^{2\pi} d\phi} = \frac{1}{2\pi\rho}\, \delta(\rho)\, \delta(z - z_o) \qquad (3.10)$$

iii) Spherical coordinates:

elemental volume $= r^2 \sin\theta\, \partial r\, \partial\phi\, \partial\theta$

$J(x, y, z; r, \phi, \theta) = r^2 \sin\theta$

Points on the axes for which $\theta = 0$ or π, and ϕ is ignorable are singular and therefore

$$\delta_v(\vec{r} - \vec{r}_o) = \frac{1}{r^2 \sin\theta} \delta(r - r_o)\, \delta(\phi - \phi_o)\, \delta(\theta - \theta_o)$$

$$= \delta(r - r_o)\, \delta(\theta - \theta_o) / \left[\int_0^{2\pi} r^2 \sin\theta \, d\phi \right]$$

$$= \frac{1}{2\pi r^2 \sin\theta} \delta(r - r_o)\, \delta(\theta - \theta_o) \qquad (3.11)$$

Next we shall write down an integral representation for the 3-dimensional delta function as

$$\delta_v(\vec{r} - \vec{r}_o) = \left[\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{ik\cdot(\vec{r}-\vec{r}_o)} \, d^3k \right]$$

$$= \left[\frac{1}{(2\pi i)^3} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\vec{\gamma}\cdot(\vec{r}-\vec{r}_o)} \, d^3\gamma \right] \quad ; \quad i\vec{k} = \vec{\gamma} \, , \qquad (3.12)$$

For example, in Cartesian coordinates,

$$\delta_v(\vec{r} - \vec{r}_o) = \delta(x - x_o)\, \delta(y - y_o)\, \delta(z - z_o)$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\vec{k}\cdot(\vec{r}-\vec{r}_o)} \, dk_x \, dk_y \, dk_z$$

$$= \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik_x(x-x_o)} \, dk_x \right] \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik_y(y-y_o)} \, dk_y \right] \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik_z(z-z_o)} \, dk_z \right] \qquad (3.13a)$$

$$= \left[\frac{1}{2\pi i} \int_{(\sigma - i\infty)}^{(\sigma + i\infty)} e^{\gamma_x(x-x_o)} \, d\gamma_x \right] \left[\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\gamma_y(y-y_o)} \, d\gamma_y \right] \left[\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\gamma_z(z-z_o)} \, d\gamma_z \right] \qquad (3.13b)$$

Since the integrands in equation 3.13b have no poles, the σ in the limits of integration can be set = 0 without any loss of generality, also

$$\vec{k} = k_x \vec{1}_x + k_y \vec{1}_y + k_z \vec{1}_z \tag{3.14a}$$

$$\vec{r} = x \vec{1}_x + y \vec{1}_y + z \vec{1}_z \tag{3.14b}$$

$$\vec{r}_o = x_o \vec{1}_x + y_o \vec{1}_y + z_o \vec{1}_z \tag{3.14c}$$

$$\vec{\gamma} = i \vec{k} = \gamma_x \vec{1}_x + \gamma_y \vec{1}_y + \gamma_z \vec{1}_z \tag{3.14d}$$

We conclude this section by stating that in an n-dimensional space, $\delta(r_H)$ may be represented using the ∇^2 Operator acting on $(1/r^{n-2})$. The general relation and an example for 3-dimensional space are found in properties 17 and 16 listed in section II. D.

IV. Integral Transforms of Delta Functions

We define an integral transform pair as,[23]

$$F(p) = \int_a^b f(x)\, K_1(x, p)\, dx \qquad (4.1a)$$

$$f(x) = \int_c^d F(p)\, K_2(x, p)\, dp \qquad (4.1b)$$

where $F(p)$ is said to be the integral transform of $f(x)$, and $f(x)$ the inverse integral transform of $F(p)$. $K_1(x, p)$ is the kernel of the transform and $K_2(x, p)$ of the inverse transform. Examples of integral transforms are Laplace, Fourier, Hankel and Mellin transforms which we shall consider individually.

A. Laplace transform

i) One sided Laplace transform:

$a = 0 \qquad b = \infty \qquad K_1(x, p) = e^{-px}$

$c = \sigma - i\infty \qquad d = \sigma + i\infty \qquad K_2(x, p) = e^{px}/(2\pi i)$

σ is so chosen that all the singularities of $F(p)$ lie to the left of the integration path in complex p plane.

ii) Two sided Laplace transform:

$a = -\infty \qquad b = -\infty \qquad K_1(x, p) = e^{-px}$

$c = \sigma - i\infty \qquad d = \sigma + i\infty \qquad K_2(x, p) = e^{px}/(2\pi i)$

The choice of σ is like in the one sided Laplace.

B. Fourier Sine and Cosine transform

$a = 0 \qquad b = \infty \qquad K_1(x, p) = \begin{matrix}\sin\\ \cos\end{matrix}(px)$

$c = 0 \qquad d = \infty \qquad K_2(x, p) = (2/\pi) \begin{matrix}\sin\\ \cos\end{matrix}(px)$

C. Fourier (exponential) transform

$a = -\infty \qquad b = \infty \qquad K_1(x, p) = \exp(\mp ipx)$

$c = -\infty \qquad d = \infty \qquad K_2(x, p) = \exp(\pm ipx)/(2\pi)$

D. Hankel transform

$a = 0 \qquad b = \infty \qquad K_1(x, p) = x J_n(px)$

$c = 0 \qquad d = \infty \qquad K_2(x, p) = p J_n(px)$

where $J_n(px)$ is the Bessel function of first kind and order n.

E. Mellin transform

$a = 0 \qquad b = \infty \qquad K_1(x, p) = x^{(p-1)}$

$c = \sigma - i\infty \qquad d = \sigma - i\infty \qquad K_2(x, p) = x^{-p}/(2\pi i)$

It is, of course, noted that the integral transform $F(p)$ of a function $f(x)$ does not always exist since appropriate restrictions on $f(x)$ have to be satisfied. We shall now set $f(x) = \delta(x - x_o)$ and obtain its integral transforms. For ease of notation, let us call

$D_{OSLT}(p)$ = One Sided Laplace Transform of $\delta(x - x_o)$

$D_{TSLT}(p)$ = Two Sided Laplace Transform of $\delta(x - x_o)$

$D_{FST}(p)$ = Fourier Sine Transform of $\delta(x - x_o)$

$D_{FCT}(p)$ = Fourier Cosine Transform of $\delta(x - x_o)$

$D_{FT}(p)$ = Fourier (Exponential) Transform of $\delta(x - x_o)$

$D_{HTN}(p)$ = nth order Hankel Transform of $\delta(x - x_o)$

$D_{MT}(p)$ = Mellin Transform of $\delta(x - x_o)$

The seven transforms listed above are given by

$$D_{OSLT}(p) = \int_0^\infty \delta(x - x_o)\, e^{-px}\, dx = \begin{cases} 0 & x_o < 0 \\ e^{-px_o} & x_o > 0 \end{cases} \tag{4.2}$$

$$D_{TSLT}(p) = \int_{-\infty}^\infty \delta(x - x_o)\, e^{-px}\, dx = e^{-px_o} \quad ; \quad \text{all } x_o \tag{4.3}$$

$$D_{FST}(p) = \int_0^\infty \delta(x - x_o) \sin(px)\, dx = \begin{cases} 0 & x_o < 0 \\ \sin(px_o) & x_o > 0 \end{cases} \tag{4.4}$$

$$D_{FCT}(p) = \int_0^\infty \delta(x - x_o) \cos(px)\, dx = \begin{cases} 0 & x_o < 0 \\ \cos(px_o) & x_o > 0 \end{cases} \tag{4.5}$$

$$D_{FT}(p) = \int_{-\infty}^\infty \delta(x - x_o)\, e^{\mp ipx}\, dx = e^{\mp ipx_o} \quad ; \quad \text{all } x_o \tag{4.6}$$

$$D_{HTN}(p) = \int_0^\infty \delta(x - x_o)\, x J_n(px)\, dx = \begin{cases} 0 & x_o < 0 \\ x_o J_n(px_o) & x_o > 0 \end{cases} \tag{4.7}$$

$$D_{MT}(p) = \int_0^\infty \delta(x - x_o)\, x^{p-1}\, dx = \begin{cases} 0 & x_o < 0 \\ x_o^{p-1} & x_o > 0 \end{cases} \tag{4.8}$$

The two-sided or bilateral Laplace transform has sometimes been defined as[10]

$$F(p) = p \int_{-\infty}^{\infty} f(x)\, e^{-px}\, dx \qquad (4.9a)$$

and

$$f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{F(p)}{p}\, e^{px}\, dp \qquad (4.9b)$$

If one uses the above definition rather than the more standard form of IV.A.ii), the bilateral transform of $\delta(x - x_o)$ is given by

$$D_{TSLT}(p) = p\, e^{-px_o} \qquad (4.10)$$

In all cases so far, a and b, which are the limits of integration in equation 4.1a, cover either an infinite or a semi-infinite range. This is not always essential. Finite integral transforms can also be defined[23]; for example $a = 0$, $b = \pi$ for finite Fourier sine or cosine transform and $a = 0$, $b = 1$ for finite Hankel transform. In such cases, the inversion formula given by equation 4.1b becomes an infinite sum rather than an integral. We shall not however determine the finite integral transforms of $\delta(x - x_o)$ since they do not differ significantly from the ones that have already been calculated.

V. Summary

The motivation for this work lies in the author's belief that a vast amount of information on this subject appears available but scattered in the literature. The delta function is given either a passing attention in several text books or a detailed and rigorous attention from a generalized functions framework in more recent (since say 1950) books. Both of these do not fill the needs of an engineer or a routine user of the delta function who may not be interested in the rigor of the generalized function theory. Keeping such users, of which the author is one, in mind, this note tries to put together, in a systematic way, a lot of known information in this area. After a brief introduction including a historical outline in section I, section II considers various representations and properties of the 1-dimensional delta function. In section III, larger dimensional spaces are considered with examples of Cartesian, cylindrical and spherical coordinate systems. Surface delta functions which could be useful in 2-dimensional problems, for example[24], are also described. In physical problems, the equations in which delta functions appear are usually transformed by using suitable integral transform(s) as necessary. With this in view, section IV deals with the integral transforms of $\delta(x - x_o)$.

It is believed that the topic of multidimensional delta functions, including that of delta function of a complex variable, deserves a more detailed study. This will form the subject for Part II of this report to be published at a later date.

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