MATHEMATICS NOTES

Note 39

January 1975

Singularity Trajectories Under Parameter Variation

D. R. Wilton
R. J. Pogorzelski
University of Mississippi

ABSTRACT

A differential equation is derived for the trajectories in the Laplace domain of the singularities of a linear operator with respect to a varying parameter.

In the singularity expansion method recently proposed for representing solution to electromagnetic scattering problems [1-3], it is necessary to find the singularities of an operator $\mathcal{L}^{-1}$, which is the inverse of an operator $\mathcal{L}$, with respect to the Laplace transform variable s. The operator $\mathcal{L}$ appears in a linear equation of the form

$$\mathcal{L} f = g \tag{1a}$$

or

$$\mathcal{L} (x, x'; a, s) f(x'; a, s) = g(x; a, s), \tag{1b}$$

where in (1b) the spatial variables, as well as the transform variable and a parameter a, are exhibited explicitly. In the electromagnetic case, the operator $\mathcal{L}$ is usually an integro-differential operator and f is the induced current on the scatterer while g is the incident field. According to the singularity expansion method, the scatterer is completely characterized once one knows all the so-called natural resonant frequencies s_n and corresponding modal currents f_n which satisfy the homogeneous equation

$$\mathcal{L}_n f_n = 0 , \tag{2}$$

where $\mathcal{L}_n$ is the operator $\mathcal{L}$ evaluated at complex frequency $s = s_n$. Also needed are the homogeneous solutions h_n of the adjoint problem, defined below.

In the following, we derive a differential equation for the s-plane trajectory of a singularity having a as its independent variable. The equation is non-linear and must be solved numerically. We derive the required differential equation by first noting that by the alternative theorem, there must exist a non-trivial solution h_n to the homogeneous adjoint problem

$$\mathcal{M}_n h_n = 0 \tag{3}$$

where $\mathcal{M}_n$ is the operator $\mathcal{M}$ evaluated at $s=s_n$ and $\mathcal{M}$ is adjoint to $\mathcal{L}$, i.e.,

$$\left\langle \mathcal{L} f, g \right\rangle = \left\langle f, \mathcal{M} g \right\rangle \tag{4}$$

for a suitable inner product. Because of (2), we must have

$$\left\langle \mathcal{L}_n f_n, h_n \right\rangle = 0 . \tag{5}$$

Taking the total differential of both sides of (5) with respect to the variable a yields

$$\frac{ds_n}{da} = -\frac{\frac{\partial}{\partial a}\left\langle \mathcal{L}_n f_n, h_n \right\rangle}{\frac{\partial}{\partial s_n}\left\langle \mathcal{L}_n f_n, h_n \right\rangle}$$

$$= -\frac{\left\langle \left(\frac{\partial \mathcal{L}_n}{\partial a}\right) f_n + \mathcal{L}_n \left(\frac{\partial f_n}{\partial a}\right), h_n \right\rangle + \left\langle \mathcal{L}_n f_n, \frac{\partial h_n}{\partial a} \right\rangle}{\left\langle \left(\frac{\partial \mathcal{L}_n}{\partial s_n}\right) f_n + \mathcal{L}_n \left(\frac{\partial f_n}{\partial s_n}\right), h_n \right\rangle + \left\langle \mathcal{L}_n f_n, \frac{\partial h_n}{\partial s_n} \right\rangle} \tag{6}$$

where we have assumed that the singularity in question is a simple pole of $\mathcal{L}^{-1}$. Observing that

$$\left\langle \mathcal{L}_n \frac{\partial f_n}{\partial q}, h_n \right\rangle = \left\langle \frac{\partial f_n}{\partial q}, \mathcal{M}_n h_n \right\rangle = 0$$

and

$$\left\langle \mathcal{L}_n f_n, \frac{\partial h_n}{\partial q} \right\rangle = 0 , \quad q = a, s,$$

where we have used (2) - (4), one finally obtains the desired differential equation,

$$\frac{ds_n}{da} = - \frac{\left\langle \frac{\partial \mathcal{L}_n}{\partial a} f_n, h_n \right\rangle}{\left\langle \frac{\partial \mathcal{L}_n}{\partial s_n} f_n, h_n \right\rangle} \tag{7}$$

In a numerical procedure, (7) is used in conjunction with (2) and (3) so that f_n and h_n are up-dated as the parameter a is varied. An initial value of s_n for some value of a is also required.

We apply the method to a simple example of a loop antenna where the chosen parameter a is the wire radius. Wu [4] has obtained an approximate solution of an integral equation for the induced current for complex frequencies s=jkc, where k is the wavenumber. The integral equation for the loop current $I(\phi)$ takes the form [4]

$$\mathcal{L}[I(\phi')] = E_\phi^{inc}(\phi) \quad , \tag{8a}$$

where

$$\mathcal{L} = \frac{j\eta}{4\pi b} \int_{-\pi}^{\pi} d\phi' M(\phi-\phi) \tag{8b}$$

and η and b are the characteristic impedance of the medium and the large radius of the loop, respectively. The angle ϕ is the polar angle of points on the loop with the polar axis located at the loop center. Wu expands the current, the kernel, and the incident field in Fourier series

$$I(\phi', s) = \sum_{m=-\infty}^{\infty} I_m(s)\, e^{-jm\phi'} \quad , \tag{9a}$$

$$M(\phi-\phi') = \sum_{m=-\infty}^{\infty} a_m(s) e^{-jm(\phi-\phi')} \quad , \tag{9b}$$

$$E_\phi^{inc}(\phi, s) = \sum_{m=-\infty}^{\infty} E_{\phi m}(s) e^{-jm\phi} \quad , \tag{9c}$$

from which one readily obtains the Fourier coefficients of the current,

$$I_m(s) = -j\frac{2b}{\eta} \frac{E_{\phi m}(s)}{a_m(s)} \tag{10}$$

An approximate analytical expression for $a_m(s)$ is given by Wu. Each of the functions $a_m(s)$ has a distinct set of zeros, i.e., at $s=s_{mn}$,

$$a_m(s_{mn}) = 0, \quad n = 1, 2, \ldots, \tag{11}$$

from which we conclude that the corresponding modal currents are of the form $e^{-jm\phi'}$. Using the inner product definition [5]

$$\left\langle f, g \right\rangle = \int_0^{2\pi} f(\phi)g(\phi)d\phi \quad , \tag{12}$$

the operator $\mathcal{M}$ which is adjoint to $\mathcal{L}$ is easily found to be

$$\mathcal{M} = \frac{j\eta}{4\pi b} \int_{-\pi}^{\pi} d\phi' \; M(\phi'-\phi) \tag{13}$$

and, with this operator, at the natural resonances s_{mn}, Eq. (3) has homogeneous solutions of the form $e^{+jm\phi'}$. Hence, at a resonant frequency s_{mn}, using (8b) and the homogeneous solutions in (7) yields

$$\frac{ds_{mn}}{da} = - \frac{\frac{\partial a_m}{\partial a}}{\frac{\partial a_m}{\partial s_{mn}}} \tag{14}$$

In this example, (14) could have been directly obtained by finding the total derivative of (11), but in the more general case where the modal currents are also functions of the parameter a, (7) does not reduce to such a simple form and (2) and (3) must be used in the solution procedure. In the more general case, the method simultaneously provides the changing solutions of (2) and (3).

For m=2, the principal conjugate pair of natural frequencies for modes of the form $e^{\pm j2\phi'}$ is $sb/c \simeq \pm j2.0$, the approximation becoming exact for an infinitely thin loop. The table permits a comparison of singularities obtained from a numerical search for the roots of (11) using Muller's method [6] and the corresponding values obtained from the Runge-Kutta solution of the differential equation (14) using the value obtained from the root-finding procedure for $\Omega = 2\ell n\frac{2\pi b}{a} = 20.0$ as the starting value. These results were obtained with step sizes of unity in the parameter Ω.

Since all differential equation solving methods are subject to cumulative error, it may be necessary in some cases to determine more accurate starting values from time to time by the usual root-finding methods.

We also point out that the technique outlined here is applicable to any linear distributed parameter system where it is desired to study the system poles as a function of some system parameter.

Table 1.

Comparison of natural frequencies found by Muller's method for root-finding and by the Runge-Kutta numerical solution of the differential equation (14).

	$\frac{s_n b}{c}$, Muller's Method		$\frac{s_n b}{c}$, Runge-Kutta	
$\Omega = 2\ell n\frac{2\pi b}{a}$	Real	Imaginary	Real	Imaginary
20.0	-0.0688838	2.037518	-0.0688838	2.037518
19.0	-0.0743130	2.039852	-0.0743129	2.039853
18.0	-0.0806636	2.042475	-0.0806636	2.042475
17.0	-0.0881895	2.045432	-0.0881895	2.045432
16.0	-0.0972457	2.048778	-0.0972457	2.048778
15.0	-0.1083436	2.052563	-0.1083436	2.052564
14.0	-0.1222442	2.056820	-0.1222443	2.056821
13.0	-0.1401245	2.061508	-0.1401245	2.061508
12.0	-0.1638809	2.066364	-0.1638809	2.066364
11.0	-0.1966971	2.070466	-0.1966971	2.070466
10.0	-0.2440118	2.070842	-0.2440110	2.070842

REFERENCES

1. Baum, C. E., "On the singularity expansion method for the solution of electromagnetics problems," Interaction Note 88, December 1971.

2. Marin, L. and Latham, R. W., "Analytical properties of the field scattered by a perfectly conducting, finite body," Interaction Note 92, January 1972.

3. Tesche, F., "On the singularity expansion method as applied to electromagnetic scattering from thin-wires," Interaction Note 102, April 1972.

4. Wu, T. T., "Theory of the thin circular antenna," J. Math. Phys., Vol. 3, pp. 1301-1304, 1962. See also Collin, R. E. and F. J. Zucker (Eds.), Antenna Theory, Chapt. 11, McGraw-Hill, 1969.

5. Harrington, R. F., Field Computation by Moment Methods, Macmillan, 1968.

6. Conte, S. D., Elementary Numerical Analysis, McGraw-Hill, 1965.