Interaction Notes

Note 614

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Propagation on Circulant Multiconductor Transmission Lines with Random Wire Interchanges

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Abstract

This paper extends the results for propagation on random lay cables. As with previous results we consider the case characterized by circulant matrices for which some deterministic results have been found.

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1. Introduction

In analyzing the propagation of waves on a nonuniform multiconductor cable it is instructive to look at certain aspects which are not statistical in nature [7] before tackling the more difficult statistical aspects. Here we have found, based on the canonical problem of a cable with a single set of *N* wires on a common radius circular cylinder (inside a reference cylindrical shell) with C_N symmetry) that the common mode is unaffected on interchange of wires along the cable. Furthermore, the differential modes for $N = 2$ and 3 are also unaffected (negligible reflections).

We need to understand, for higher *N*, what the reflections on wire interchange do to the differential-mode propagation. Such is the subject of this paper. Let us retain the above C_N symmetry for this purpose. This will lead to some statistical properties of the differential modes.

2. Product-Integral Representation

The general form of the *N*-conductor-plus-reference multiconductor-transmission-line (MTL) equations is (for voltage and current vectors)

$$\frac{d}{dz}\begin{pmatrix}\left(\tilde{V}_n(z,s)\right) \\ Z\left(I_n(z,s)\right)\end{pmatrix} = \left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{u,\mathrm{v}}\right) \odot \begin{pmatrix}\left(\tilde{V}_n(z_0,s)\right) \\ Z\left(I_n(z_0,s)\right)\end{pmatrix}$$

$$\left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{u,\mathrm{v}}\right) = -\gamma\left(\left(F_{n,m}(z)\right)_{u,\mathrm{v}}\right)$$

$$\left(\left(F_{n,m}(z)\right)_{u,\mathrm{v}}\right) = \begin{pmatrix}\left(0_{n,m}\right) & \left(f_{g_{n,m}}(z)\right) \\ \left(f_{g_{n,m}}(z)\right)^{-1} & \left(0_{n,m}\right)\end{pmatrix} \qquad \text{(real)}$$

$$\left(Z'_{n,m}(z,s)\right) = s\mu\left(f_{g_{n,m}}(z)\right) \equiv \text{per-unit-length impedance matrix } (N \times N)$$

$$\left(Y'_{n,m}(z,s)\right) = s\varepsilon\left(f_{g_{n,m}}(z)\right)^{-1} \equiv \text{per-unit-length capacitance matrix } (N \times N) \qquad (2.1)$$

$$\equiv se\left(g_{f_{n,m}}(z)\right)$$

$$\left(Z_{c_{n,m}}(z)\right) = Z\left(f_{g_{n,m}}(z)\right) \text{ characteristic impedance matrix}$$

For the *N* wires (plus reference) in a uniform isotropic medium. We also have

$$\left(f_{g_{n,m}}(z)\right) = \left(f_{g_{n,m}}(z)\right)^T \text{ (reciprocity) with nonnegative eigenvalues)}$$

$$\left(g_{f_{n,m}}(z)\right) \equiv \left(f_{g_{n,m}}(z)\right)^{-1}$$

$$Z = \left[\frac{\mu}{\varepsilon}\right]^{1/2} \equiv \text{wave impedance}$$

$$\mathrm{v} = [\mu\varepsilon]^{-1/2} \equiv \text{propagation speed}$$

$$\gamma = \frac{s}{\mathrm{v}} \equiv \text{propagation constant}$$

$\sim$ $\equiv$ two-sided Laplace transform over time (t)

$s = \Omega + j\omega =$ Laplace-transform variable or complex frequency

This is a special case of a lossless system. Appendix A gives some important properties of $((F_{n,m}(z))_{u,\mathrm{v}})$.

The solution of (2.1) takes the form

$$\begin{pmatrix} (\tilde{V}_n(z,s)) \\ Z(I_n(z,s)) \end{pmatrix} = \left((\tilde{U}_{n,m}(z,z_0;s))_{u,\mathrm{v}}\right) \odot \begin{pmatrix} (\tilde{V}_n(z_0,s)) \\ Z(I_n(z_0,s)) \end{pmatrix}$$
$$\left((\tilde{U}_n(z,z_0;s))_{u,\mathrm{v}}\right) = \prod_{z_0}^{z} e^{\left((\tilde{\Gamma}_{n,m}(z',s))_{u,\mathrm{v}}\right)dz'} \quad \text{(product integral)} \tag{2.3}$$

The product integral has many important properties [4-6, 8, 11] which will help us.

3. Interchanging Wires at Various Positions z_ℓ Along Multiconductor Transmission Line

At various positions z_ℓ along the multiconductor transmission line the positions of the various wires (excluding reference (or shield)) are interchanged. This is accomplished by a permutation matrix (Appendix B). Applying this to vectors as

$$\begin{gathered} (P_{n,m}) \cdot (\tilde{V}_n(z,s)) \\ (P_{n,m}) \cdot (\tilde{I}_n(z,s)) \end{gathered} \tag{3.1}$$

relocates each wire to a generally different position of some other wire, there being N such wire conditions. In the process the wire numbering is permuted according to $(P_{n,m})$ at the position z.

Going along from z_0 to z_1 where such a permutation occurs we have from (2.1)

$$
\begin{aligned}
&-\left(P_{n,m}\right)_1 \bullet \left[\gamma\left(f_{g_{n,m}}\right)\right] \bullet \left[Z\left(\tilde{I}_n\left(z_0,s\right)\right)\right] \\
&= -\gamma\left[\left(P_{n,m}\right)_1 \bullet \left(f_{g_{n,m}}\right) \bullet \left(P_{n,m}\right)_1^T\right] \bullet \left(P_{n,m}\right)_1 \bullet \left[Z\left(\tilde{I}_n\left(z_0,s\right)\right)\right]
\end{aligned}
\tag{3.2}
$$

This indicates that $(f_{g_{n,m}})$ is transformed as

$$
\left(f_{g_{n,m}}\right) \rightarrow \left(P_{n,m}\right)_1 \bullet \left(f_{g_{n,m}}\right) \bullet \left(P_{n,m}\right)_T \tag{3.3}
$$

and similarly as one goes to permutations at z_2, etc. From the voltage vector we find the same for

$$
\left(f_{g_{n,m}}\right)^{-1} \rightarrow \left(P_{n,m}\right)_1 \bullet \left(f_{g_{n,m}}\right)^{-1} \bullet \left(P_{n,m}\right)_1^T \tag{3.4}
$$

and similarly along the transmission line.

Recalling the propagation supermatrix from (2.1) and how it fits in the product integral (2.3) we can write

$$
\begin{aligned}
&\left(\left(\tilde{U}_{n,m}\left(z,z_0;s\right)\right)_{u,\mathrm{v}}\right) = \\
&\qquad \prod_{z_{M-1}}^{z_M} e^{\left(\left(\tilde{\Gamma}_{n,m}(z',s)\right)_{u,\mathrm{v}}\right)dz'} \odot\cdots\odot \prod_{z_0}^{z_1} e^{\left(\left(\tilde{\Gamma}_{n,m}(z',s)\right)_{u,\mathrm{v}}\right)dz'} \\
&\quad = e^{\int_{z_{M-1}}^{z_M}\left(\left(\tilde{\Gamma}_{n,m}(z',s)\right)_{u,\mathrm{v}}\right)dz'} \odot\cdots\odot e^{\int_{z_0}^{z_1}\left(\left(\tilde{\Gamma}_{n,m}(z',s)\right)_{u,\mathrm{v}}\right)dz'} \\
&\quad = e^{\left(\left(\tilde{\Gamma}_{n,m}(z_m,s)\right)_{u,\mathrm{v}}\right)\Delta z} \odot\cdots\odot e^{\left(\left(\tilde{\Gamma}_{n,m}(z_1,s)\right)_{u,\mathrm{v}}\right)\Delta z} \\
&z_\ell = \ell\Delta z + z_0 \ , \quad \ell = 1, 2, \ldots, M
\end{aligned}
\tag{3.5}
$$

where we have chosen $((\tilde{\Gamma}_{n,m}(z,s))_{u,\mathrm{v}}))\Delta z$ to be uniform in each section of common length Δz.

Assuming the Δz is small compared to radian wavelength we can write

$$e^{\left(\left(\tilde{\Gamma}_{n,m}(z_\ell,s)\right)_{u,\mathrm{v}}\right)\Delta z} = \left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) + \left(\left(\tilde{\Gamma}_{n,m}(z_\ell,s)\right)_{u,\mathrm{v}}\right)\Delta z + \mathrm{O}\left(\left[\gamma\Delta z\right]^2\right) \tag{3.6}$$

Inserting this in (3.2) we find

$$\begin{aligned}
&\left(\left(U_{n,m}(z_\ell,z_0;s)\right)_{u,\mathrm{v}}\right)\\
&= \left[\left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) + \left(\left(\tilde{\Gamma}_{n,m}(z_\ell,s)\right)_{u,\mathrm{v}}\right)\Delta z\right] \odot \cdots\\
&\quad \odot \left[\left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) + \left(\left(\tilde{\Gamma}_{n,m}(z_\ell,s)\right)_{u,\mathrm{v}}\right)\Delta z\right] + \mathrm{O}\left(\left[M\gamma\Delta z\right]^2\right)\\
&= \left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) + \sum_{\ell=1}^{M}\left(\left(\tilde{\Gamma}_{n,m}(z_\ell,s)\right)_{u,\mathrm{v}}\right)\Delta z + \mathrm{O}\left(\left[M\gamma\Delta z\right]^2\right)\\
&= \left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) + \gamma\left[z_\ell - z_0\right]\sum_{\ell=1}^{M}\frac{1}{M}\left(\left(F_{n,m}(z_\ell)\right)_{u,\mathrm{v}}\right) + \mathrm{O}\left(\left[M\gamma\Delta z\right]^2\right)\\
&= \left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) + \gamma\left[z_\ell - z_0\right] avg\left(\left(F_{n,m}(z_\ell)\right)_{u,\mathrm{v}}\right) + \mathrm{O}\left(\left[z_\ell = z_0\right]^2\right)
\end{aligned} \tag{3.7}$$

Here we have included a factor M^2 to account for the $\mathrm{O}(M^2)$ such times the quadratic factor appears.

Now $((F_{n,m}(z_\ell))_{u,\mathrm{v}})$ transforms like (3.2) through (3.4) giving

$$\begin{aligned}
\left(\left(F_{n,m}(z_{\ell+1})\right)_{u,\mathrm{v}}\right) &= \begin{pmatrix} \left(0_{n,m}\right) & \left(g_{f_{n,m}}(z_{0+1})\right) \\ \left(g_{f_{n,m}}(z_{0+1})\right) & \left(0_{n,m}\right) \end{pmatrix}\\
&= \begin{pmatrix} \left(0_{n,m}\right) & \left(P_{n,m}\right)_{\ell+1} \bullet \left(f_{n,m}(z_\ell)\right) \bullet \left(P_{n,m}\right)_{\ell+1}^T \\ \left(P_{n,m}\right)_{\ell+1} \bullet \left(g_{f_{n,m}}(z_\ell)\right) \bullet \left(P_{n,m}\right)_{\ell+1}^T & \left(0_{n,m}\right) \end{pmatrix}
\end{aligned} \tag{3.8}$$

So the summation in (3.7) looks like

$$
\sum_{\ell=1}^{M}\frac{1}{M}\begin{pmatrix}\left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right)\\ \left(g_{f_{n,m}}(z_\ell)\right) & \left(0_{n,m}\right)\end{pmatrix} \equiv \begin{pmatrix}\left(0_{n,m}\right) & \left(F_{g_{n,m}}\right)\\ \left(G_{f_{n,m}}\right) & \left(0_{n,m}\right)\end{pmatrix}
$$

$$
\begin{aligned}
\left(F_{g_{n,m}}\right) &= \sum_{\ell=1}^{M}\frac{1}{M}\left(f_{g_{n,m}}(z_\ell)\right)\\
\left(G_{f_{n,m}}\right) &= \sum_{\ell=1}^{M}\frac{1}{M}\left(g_{f_{n,m}}(z_\ell)\right)^{-1}\\
\left(g_{f_{n,m}}(z_\ell)\right) &\equiv \left(f_{g_{n,m}}(z_\ell)\right)^{-1}
\end{aligned} \tag{3.9}
$$

Now, appealing to Appendix B ((B.9) through (B. 11)), we have, assuming that all permutations $(P_{n,m})$ are equally likely,

$$
avg\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) = avg\left(\left(P_{n,m}\right)\bullet\left(f_{g_{n,m}}(z_0)\right)\bullet\left(P_{n,m}\right)^T\right)
$$

$$
= \begin{cases}\dfrac{1}{N}\displaystyle\sum_{N=1}^{N} f_{g_{n,n}} & \text{(diagonal elements)}\\[2ex] \dfrac{1}{N^2-N}\left[\displaystyle\sum_{n,m=1}^{N} f_{g_{n,m}} - \sum_{n=1}^{N} f_{g_{n,n}}\right] & \text{(off-diagonal elements)}\end{cases}
$$

$$
\mathrm{var}\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) = \mathrm{var}\left(\left(P_{n,m}\right)\bullet\left(f_{g_{n,m}}(z_0)\right)\bullet\left(P_{n,m}\right)^T\right)
$$

$$
= \begin{cases}\dfrac{1}{N}\left[\displaystyle\sum_{n,m=1}^{N} f_{g\,n,n}^2\right] - \left[\dfrac{1}{N}\displaystyle\sum_{n=1}^{N} f_{g\,n,n}\right]^2 & \text{(diagonal elements)}\\[2ex] \dfrac{1}{N^2-N}\left[\displaystyle\sum_{n,m,m'=1}^{N} f_{g_{n,m}} f_{g_{n,m'}}\right] - \left[\dfrac{1}{N^2-N}\displaystyle\sum_{\substack{n,m=1\\ n\neq m}}^{N} f_{g\,n,m}\right]^2 & \text{(off-diagonal elements)}\end{cases} \tag{3.10}
$$

In these formulae we have considered the geometric-factor matrix $(f_{g_{n,m}})$ which has negative off-diagonal terms (mutual inductance). The same formulas apply to its inverse $(g_{n,m}(z_\ell))$ which has positive off-diagonal terms (mutual capacitance).

4. Application to Circulant $(f_{g_{n,m}})$

As mentioned in Section 1, and illustrated in Fig. 4.1 let us consider the special case of N perfectly conducting wires of radius r_0 with centers on a common radius of Ψ_1 and surrounded by a perfectly conducting cylinder of radius Ψ_0. With N wires we have the wire centers positioned at

$$\begin{aligned} \phi_n &= n\phi_1 \\ \phi_1 &= \frac{2\pi}{N} \end{aligned} \tag{4.1}$$

From Appendix C we have the result for a single wire corresponding to n = 0 or N, but applying to any individual wire

$$f_{g_{1,1}} = f_{g_{n,n}} \simeq \frac{1}{2\pi}\left[\ell n\left(\frac{\Psi_0}{r_0}\right) + \ell n\left(\frac{\Psi_0}{\Psi_1}\right)\right] \tag{4.2}$$

From Appendix D we have the off-diagonal terms as

$$\begin{aligned} f_{g_{n,m}}\Big|_{n\neq m} &= f_{g_{m,n}}\Big|_{n\neq m} \\ &= -\frac{1}{4\pi}\ell n\left(\frac{\left[\frac{\Psi_0}{\Psi_1}\right]^4 + 1 - 2\left[\frac{\Psi_0}{\Psi_1}\right]^2 \cos\left(\frac{2\pi[n-m]}{N}\right)}{2\left[1-\cos\left(\frac{2\pi[n-m]}{N}\right)\right]}\right) \end{aligned} \tag{4.3}$$

For this we find

$$avg\left(\left(f_{g_{n,m}}(2_\ell)\right)\right) = \begin{cases} f_{g_{1,1}} & \text{(diagonal elements)} \\ \dfrac{1}{N-1}\displaystyle\sum_{m=2}^{N} f_{g_{1,m}} & \text{(off-diagonal elements)} \end{cases} \tag{4.4}$$

(independent of n). For the second term we have

$$f_{g_{n,m}}\Big|_{n\neq m} = -\frac{1}{4\pi}\ell n\left(\frac{\left[\frac{\Psi_0}{\Psi_1}\right]^4 + 1 - 2\left[\frac{\Psi_0}{\Psi_1}\right]^2\cos\left(\frac{2\pi[m-n]}{N}\right)}{2\left[1-\cos\left(\frac{2\pi[m-n]}{N}\right)\right]}\right) \tag{4.5}$$

An alternate representation is

$$\begin{aligned}&\frac{1}{N-1}\sum_{m=2}^{N} f_{1,m}\\ &= -\frac{1}{4\pi}\frac{1}{N-1}\ell n\left(\prod_{m=2}^{N}\ \frac{\left[\frac{\Psi_0}{\Psi_1}\right]^4 + 1 - 2\left[\frac{\Psi_0}{\Psi_1}\right]^2\cos\left(\frac{2\pi[m-1]}{N}\right)}{2\left[1-\cos\left(\frac{2\pi[m-1]}{N}\right)\right]}\right)\end{aligned} \tag{4.6}$$

In matrix form we can write

$$\begin{aligned}avg\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) &= f_{g_{1,1}}\left(1_{n,m}\right)\\ &\quad + \left[\frac{1}{N-1}\sum_{m=2}^{N} f_{g_{1,m}}\right]\left[\left(u_{n,m}\right) - \left(1_{n,m}\right)\right]\end{aligned} \tag{4.7}$$

For the variance we have

$$\begin{aligned}&\mathrm{var}\left(\left(f_{g_{n,m}}(z_\ell)\right)\right)\\ &= \begin{cases}\frac{1}{N}\left[\sum_{n,m=1}^{N} f_{g_{n,m}}^2\right] - \left[\frac{1}{N}\sum_{n=1}^{N} f_{g_{n,n}}\right] & \text{(diagonal elements)}\\ \frac{1}{N^2-N}\left[\sum_{n,m,m'=1}^{N} f_{g_{n,m}} f_{g_{n,m'}}\right] - \left[\frac{1}{N^2-N}\ \sum_{n,m=1}^{N} f_{g_{n,m}}\right] & \text{(off-diagonal elements)}\end{cases}\\ &= \begin{cases}\sum_{m=1}^{N} f_{g\,1,m}^2 - f_{g_{1,1}}^2\\ \frac{1}{N^2-N}\left[\sum_{n=1}^{N}\left[\sum_{m=1}^{N} f_{g_{n,m}}\left[\sum_{m'=1}^{N} f_{g_{n,m'}}\right]\right]\right] - \left[\frac{1}{N-1}\sum_{m=1}^{N} f_{g\,n,m}\right]^2\end{cases}\end{aligned}$$

$$= \begin{cases} \sum_{m=2}^{N} f_{g\,1,m}^{2} \\ \frac{N-2}{[N-1]^2}\left[\sum_{m=1}^{N} f_{g_{1,m}}\right]^2 \end{cases} \tag{4.8}$$

For interpreting this result we have used the properties of bicirculant matrices

$$\begin{aligned} sce \;=\; sre \;&\equiv\; \sum_{m=1}^{N} f_{g\,n,m} \;=\; \sum_{n=1}^{N} f_{g\,n,m} && \text{(independent of } n,\ m\text{)} \\ sce2 \;=\; sre2 \;&\equiv\; \sum_{m=1}^{N} f_{g\,n,m}^{2} \;=\; \sum_{n=1}^{N} f_{g\,n,m} && \text{(independent of } n.\ m\text{)} \\ de \;&\equiv\; f_{g_{n,n}} && \text{(independent of } n\text{)} \\ de2 \;&\equiv\; f_{g_{n,n}}^{2} && \text{(independent of } n\text{)} \end{aligned} \tag{4.9}$$

Then (4.7) can be written as

$$\begin{aligned} avg\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) \;&=\; de\left(1_{n,m}\right) \\ &\quad + \frac{1}{N-1}[sre \;-\; de]\left[\left(U_{n,m}\right) \;-\; \left(1_{n,m}\right)\right] \end{aligned} \tag{4.10}$$

and (4.8) can be written as

$$\begin{aligned} \mathrm{var}\left(\left(f_{n,m}(z_\ell)\right)\right) \;&=\; \begin{cases} sre2 \;-\; de2 & \text{(diagonal elements)} \\ \frac{N-2}{[N-1]^2}[sre]^2 & \text{(off-diagonal elements)} \end{cases} \\ &= [sre2 \;-\; de2]\ \left(1_{n,m}\right) \\ &\quad + \frac{N-2}{[N-1]^2}[sre]^2\left[\left(u_{n,m}\right) \;-\; \left(1_{n,m}\right)\right] \end{aligned} \tag{4.11}$$

Similar results apply for $(g_{f_{n,m}})$.

Consider a few cases. For N = 1 there is only one element giving

$$avg\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) = \left(f_{g_{1,1}}\right)$$
$$\text{var}\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) = (0) \tag{4.12}$$

noting that there are no permutations. For N = 2 we have

$$\begin{aligned}
avg\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) &= f_{g_{1,1}}\left(1_{n,m}\right) \\
&\quad + f_{g_{1,2}}\left[\left(u_{n,m}\right) - \left(1_{n,m}\right)\right] \\
&= \begin{pmatrix} f_{g_{1,1}} & f_{g_{1,2}} \\ f_{g_{1,2}} & f_{g_{1,1}} \end{pmatrix} = \left(f_{g_{n,m}}\right) \\
\text{var}\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) &= [sre2 - de2]\left(1_{n,m}\right) \\
&= f_{g_{1,2}}^2\left(1_{n,m}\right) \\
&= \begin{pmatrix} f_{g_{1,2}}^2 & 0 \\ 0 & f_{g_{1,2}}^2 \end{pmatrix}
\end{aligned} \tag{4.13}$$

For N = 3 we have

$$\begin{aligned}
avg\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) &= f_{g_{1,1}}\left(1_{n,m}\right) \\
&\quad + \frac{1}{2}\left[f_{g_{1,2}} + f_{g_{1,3}}\right]\left[\left(u_{n,m}\right) - \left(1_{n,m}\right)\right] \\
&= f_{g_{1,1}}\left(1_{n,m}\right) \\
&\quad + \frac{1}{4} f_{g_{1,2}}\left[\left(u_{n,m}\right) - \left(1_{n,m}\right)\right] \\
f_{g_{1,3}} &= f_{g_{1,2}} \qquad \text{(cyclic nature)} \\
\text{var}\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) &= \left[f_{g_{1,2}}^2 + f_{g_{1,3}}^2\right]\left(1_{n,m}\right) \\
&\quad + \frac{1}{4}[sre]^2\left[\left(u_{n,m}\right) - \left(1_{n,m}\right)\right] \\
&= 2f_{g_{1,2}}^2\left(1_{n,m}\right) \\
&\quad + \frac{1}{4}\left[f_{g_{1,1}} + 2f_{g_{1,2}}\right]^2\left[\left(u_{n,m}\right) - \left(1_{n,m}\right)\right]
\end{aligned} \tag{4.14}$$

These three cases correspond to the special cases in [7], in which the eigenmode propagation is not perturbed by wire interchanges. For N = 2, 3 we have merely a sign change some number of times on the voltages and currents in each of the differential eigenmodes.

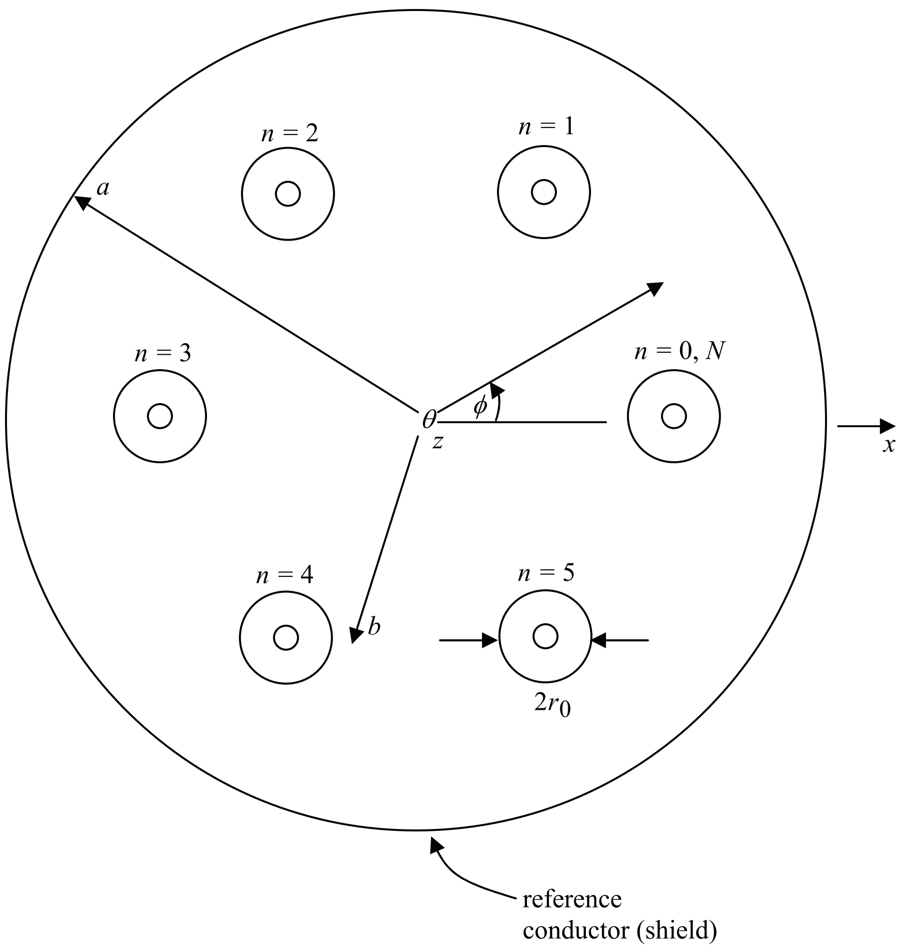

Fig. 4.1 MTL with C_N Symmetry: Example for N = 6.

5. Low-Frequency Form of the Product Integral

With M uniform sections of our MTL we can write our product integral in the form

$$\left(\left(\tilde{U}_{n,m}\left(z_M, z_0; s\right)\right)_{u,\text{v}}\right) = \prod_{z_0}^{z_M} e^{-\gamma\left(\left(F_{g_{n,m}}(z')\right)_{u,\text{v}}\right)dz'}$$

$$= \prod_{\ell=1}^{M} e^{-\gamma\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\text{v}}\right)\Delta z} \quad \text{(successive terms to left in dot-product sense)} \tag{5.1}$$

$$\Delta z = \frac{z_M - z_0}{M}$$

Since the propagation supermatrix is uniform in each section of length Δz we can evaluate each section analytically as an exponential matrix. This gives the product of M matrix terms (successive multiplications on the left).

The product in (5.1) can then be approximated from Appendix F as

$$\begin{aligned}\left(\left(\tilde{U}_{n,m}\left(z_M, z_0; s\right)\right)_{u,\text{v}}\right) &= \prod_{\ell=1}^{M}\left[\left(\left(1_{n,m}\right)_{u,\text{v}}\right) - \gamma\Delta z\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\text{v}}\right) + \mathrm{O}\left(\left[\gamma\Delta z\right]^2\right)\right] \\ &= \left(\left(1_{n,m}\right)_{u,\text{v}}\right) - \gamma\Delta z\sum_{\ell=1}^{M}\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\text{v}}\right) + \mathrm{O}\left(\left[M\gamma\Delta z\right]^2\right)\end{aligned} \tag{5.2}$$

There are $\mathrm{O}(M^2)$ quadratic terms. Thus we have for random permutations

$$\begin{aligned}&\left(\left(\tilde{U}_{n,m}\left(z_M, z_0; s\right)\right)_{u,\text{v}}\right) \\ &= \left(\left(1_{n,m}\right)\right)_{u,\text{v}} - \gamma\left[z_M - z_0\right]\sum_{\ell=1}^{M}\frac{1}{M}\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\text{v}}\right) + \mathrm{O}\left(\left[\gamma\left[z_M - z_0\right]\right]^2\right) \\ &= \left(\left(1_{n,m}\right)\right)_{u,\text{v}} - \gamma\left[z_M - z_0\right] avg\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\text{v}}\right) + \mathrm{O}\left(\left[\gamma\left[z_M - z_0\right]\right]^2\right)\end{aligned} \tag{5.3}$$

Now we have (from (3.10)) for random permutations

$$
\begin{aligned}
avg\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right) &= \begin{pmatrix} \left(0_{n,m}\right) & avg\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) \\ avg\left(\left(g_{f_{n,m}}(z_\ell)\right)\right) & \left(0_{n,m}\right) \end{pmatrix} \\
avg\left(\left(f_{g_{n,m}}(z_\ell)\right)\right) &= \begin{cases} f_{g_{1,1}} & \text{(diagonal elements)} \\ \dfrac{1}{N-1}\displaystyle\sum_{m=2}^{N} f_{g_{1,m}} & \text{(off-diagonal elements)} \end{cases} \\
avg\left(\left(g_{f_{n,m}}(z_\ell)\right)\right) &= \begin{cases} g_{f_{1,1}} & \text{(diagonal elements)} \\ \dfrac{1}{N-1}\displaystyle\sum_{m=2}^{N} g_{f_{1,m}} & \text{(off-diagonal elements)} \end{cases}
\end{aligned}
\tag{5.4}
$$

Using the exact results in Appendix F we can also use

$$
\begin{aligned}
\left(\left(\tilde{U}_{n,m}(z_M, z_0; s)\right)_{u,\mathrm{v}}\right) &= \prod_{\dagger=1}^{M} e^{-\gamma\left(\left(F_{n,m}(z_\ell)\right)_{u,\mathrm{v}}\right)\Delta z} \\
&= \prod_{\ell=1}^{M}\left[\left(1_{n,m}\right)\cosh(\gamma\Delta z) - \left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)\sinh(\gamma\Delta z)\right] \\
&= \cosh^M(\gamma z)\prod_{\ell=1}^{M}\left[\left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) - \left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)\tanh(\gamma\Delta z)\right]
\end{aligned}
\tag{5.5}
$$

which can be used for Monte-Carlo calculations. Expanding for small $M\gamma\Delta z = \gamma\left[z_M - z_0\right]$ we have

$$
\begin{aligned}
&\left(\left(\tilde{U}_{n,m}(z_M, z_0; s)\right)_{u,\mathrm{v}}\right) \\
&= \left[1 + \gamma^{-2}\left[z_M - z_a\right]^2\right]\left[\left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) - \left[\sum_{\ell=1}^{M}\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)\gamma\Delta z\left[1 + \mathrm{O}\left(\left[\gamma\Delta z\right]^2\right)\right]\right]\right] \\
&= \left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) - \left[\frac{1}{M}\sum_{\ell=1}^{M}\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)\right]\gamma\left[z_M - z_a\right] + \mathrm{O}\left(\gamma\left[z_M - z_a\right]^2\right)
\end{aligned}
\tag{5.6}
$$

which will find later use.

6. Low-Frequency Form of Voltage, Current, and Input Impedance

From Appendix E we have the result that the differential modes, and hence differential currents and voltages, have average value zero. So we write

$$
\begin{aligned}
\left(\tilde{V}_n^{(dif)}(z,s)\right) &= \left(\tilde{V}_n(z,s)\right) - V_{(z,s)}^{(avg)}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \\
\tilde{V}_{(z,s)}^{(avg)} &= \frac{1}{N}\sum_{n=1}^{N}\tilde{V}_n(z,s) \equiv \tilde{V}_{(z,s)}^{(c)} \\
\left(\tilde{I}_n^{(dif)}(z,s)\right) &= \left(\tilde{I}_n(z,s)\right) - I_{(z,s)}^{(avg)}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \\
\tilde{I}_{(z,s)}^{(avg)} &= \frac{1}{N}\sum_{n=1}^{N}\tilde{I}_n(z,s) \equiv \frac{1}{N}\tilde{I}_{(z,s)}^{(c)} \\
\tilde{V}_{(z,s)}^{(c)} &\equiv \text{common-mode voltage (average)} \\
\tilde{I}_{(z,s)}^{(c)} &\equiv \text{common-mode current (or total current)} \\
\frac{\tilde{V}_{(z,s)}^{(c)}}{\tilde{I}_{(z,s)}^{(c)}} &\equiv Z^{(c)} \equiv \text{common-mode impedance (independent of z and s)}
\end{aligned}
\tag{6.1}
$$

The common mode is unaffected by the wire interchanges and propagates deterministically along the transmission line. From Appendix C we have from (C.18)

$$
Z^{(c)} = Zf_g^{(N)} \tag{6.2}
$$

Setting up our gedanken experiment, let us set

$$
\left(\tilde{V}_n(z_\ell,s)\right) = \left(Z_{c_{n,m}(z_\ell)}\right)\cdot\left(\tilde{I}_n(z_\ell,s)\right) = Z\left(f_{g_{n,m}}(z_\ell)\right)\cdot\left(\tilde{I}_n(z_\ell,s)\right) \tag{6.3}
$$

For the common mode we have

$$
\begin{aligned}
\left(V_n^{(c)}(z,s)\right) &= V^{(c)}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \\
&= Z\left(F_{g_{n,m}}(z_\ell)\right) \bullet \frac{I^{(c)}(z,s)}{N}\begin{pmatrix}1\\\vdots\\\vdots\\1\end{pmatrix} \\
&= Z\, I^{(c)}(z,s)\frac{1}{N}\sum_{m=1}^{N} f_{g_{n,m}}(z_\ell) \qquad \text{(independent of } n\text{)} \\
&= Z\, f_g^{(N)} I^{(c)}(z,s)
\end{aligned}
\tag{6.4}
$$

We can see that the row (and column) sum (we have frequently encountered) is slosely related to the common-mode geometric factor $f_g^{(N)}$.

As we have seen (in [7]), there are no common-mode reflections on wire interchange. Hence, with no reflections back to the source we have

$$
\frac{\tilde{V}^{(c)}(0,s)}{\tilde{I}^{(c)}(0,s)} = Z\, f_g^{(N)} \tag{6.5}
$$

independent of frequency.

Now look at the differential-signal propagation. Note that, with the common mode not generating any differential modes on wire interchange, the reciprocity theorem assures us that the differential signals do not couple to the common mode. At any z we can expand the differential voltages and currents in terms of only the eigenmodes for $\beta = 1,2,\ldots,N-1$ (Appendix E). Each of these modes have zero average. For all z we can then say

$$
\begin{aligned}
\sum_{n=1}^{N} \tilde{V}_n^{(dif)}(z,s) &= 0 \\
\sum_{n=1}^{N} \tilde{I}_n^{(dif)}(z,s) &= 0
\end{aligned}
\tag{6.6}
$$

There can be various reflections and transmission losses at each z_ℓ, but the zero averages still hold.

Now let us write (using Appendix F)

$$
\begin{aligned}
&\left(\left(\tilde{U}_{n,m}(z_M, z_0; s)\right)_{u,\mathrm{v}}\right)^{-1} \odot \begin{pmatrix} \left(\tilde{V}_n^{(dif)}(z_\ell, s)\right) \\ Z\left(\tilde{I}_n^{(dif)}(z_\ell, s)\right) \end{pmatrix} = \begin{pmatrix} \left(\tilde{V}_n^{(dif)}(z_0, s)\right) \\ Z\left(\tilde{I}_n^{(dif)}(z_0, s)\right) \end{pmatrix} \\
&= \left(\left(\tilde{U}_{n,m}(z_0, z_M; s)\right)_{u,\mathrm{v}}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(dif)}(z_\ell, s)\right) \\ Z\left(\tilde{I}_n^{(dif)}(z_\ell, s)\right) \end{pmatrix} \\
&\left(\left(\tilde{U}_{n,m}(z_0, z_M; s)\right)_{u,\mathrm{v}}\right) = \prod_{\ell=M}^{1} e^{\gamma\left(\left(F_{n,m}(z_\ell)\right)_{u,\mathrm{v}}\right)\Delta z} \\
&= \prod_{\ell=M}^{1}\left[\left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right)\cosh(\gamma\Delta z) + \left(\left(F_{n,m}(z_\ell)\right)_{u,\mathrm{v}}\right)\sinh(\gamma\Delta z)\right]
\end{aligned}
\tag{6.7}
$$

noting the sign reversal on Δz in going from larger to smaller z_ℓ. Now constrain

$$
\left(\tilde{V}_n(z_M, s)\right) = \left(Z_{c_{n,m}}\right) \bullet \left(\tilde{I}_n(z_M, s)\right) = Z\left(f_{g_{n,m}}(z_M)\right) \bullet \left(\tilde{I}_n(z_M, s)\right) \tag{6.8}
$$

to properly terminate the transmission line.

Writing out (6.7) gives

$$
\begin{aligned}
&\left(\tilde{U}_{n,m}(z_0, z_M; s)\right)_{1,1} \bullet \left(\tilde{V}_n(z_\ell, s)\right) + \left(\tilde{U}_{n,m}(z_0, z_M; s)\right)_{1,2} \bullet Z\left(\tilde{I}_n^{(dif)}(z_M, s)\right) \\
&\qquad = \left(\tilde{V}_n^{(dif)}(z_0, s)\right) \\
&\left(\tilde{U}_{n,m}(z_0, z_M; s)\right)_{2,1} \bullet \left(\tilde{V}_n(z_\ell, s)\right) + \left(\tilde{U}_{n,m}(z_0, z_M; s)\right)_{2,2} \bullet Z\left(\tilde{I}_n^{(dif)}(z_M, s)\right) \\
&\qquad = Z\left(\tilde{I}_n^{(dif)}(z_0, s)\right)
\end{aligned}
\tag{6.9}
$$

Enforcing (6.8) gives

$$
\begin{aligned}
&\left[\left(\tilde{U}_{n,m}(z_0, z_M; s)\right)_{1,1} + \left(\tilde{U}_{n,m}(z_0, z_M; s)\right)_{1,2} \bullet \left(g_{f_{n,m}}(z_M)\right)\right] \bullet \left(\tilde{V}_n^{(dif)}(z_M, s)\right) = \left(\tilde{V}_n^{(dif)}(z_0, s)\right) \\
&\left[\left(\tilde{U}_{n,m}(z_0, z_M; s)\right)_{2,1} + \left(\tilde{U}_{n,m}(z_0, z_M; s)\right)_{2,2} \bullet \left(g_{f_{n,m}}(z_M)\right)\right] \bullet \left(\tilde{V}_n^{(dif)}(z_M, s)\right) = Z\left(\tilde{I}_n^{(dif)}(z_0, s)\right)
\end{aligned}
\tag{6.10}
$$

Eliminating ($\tilde{V}_n^{(dif)}(z_M, s)$) gives

$$
\begin{aligned}
&\left[\left(\tilde{U}_{n,m}(z_0,z_M;s)\right)_{1,1}+\left(\tilde{U}_{n,m}(z_0,z_M;s)\right)_{1,2}\bullet\left(g_{f_{n,m}}(z_M)\right)\right]^{-1}\bullet\left(\tilde{V}_n^{(dif)}(0,s)\right)\\
&=\left[\left(\tilde{U}_{n,m}(z_0,z_M;s)\right)_{2,1}+\left(\tilde{U}_{n,m}(z_0,z_M;s)\right)_{2,2}\bullet\left(g_{f_{n,m}}(z_M)\right)\right]^{-1}\bullet\left(g_{f_{n,m}}\right)\bullet\left(Z_{c_{n,m}}\right)\bullet\left(\tilde{I}_n^{(dif)}(0,s)\right)
\end{aligned}
\tag{6.11}
$$

The input impedance is then found from

$$
\begin{aligned}
\left(\tilde{V}_n^{(dif)}(0,s)\right) &= \left(\tilde{Z}_{n,m}^{(dif)}(0,s)\right)\bullet\left(\tilde{I}_n^{(dif)}(0,s)\right)\\
\left(\tilde{Z}_{n,m}^{(dif)}(0,s)\right) &= \left[\left(\tilde{U}_{n,m}(z_0,z_M;s)\right)_{1,1}+\left(\tilde{U}_{n,m}(z_0,z_M;s)\right)\bullet\left(g_{f_{n,m}}(z_M)\right)\right]\bullet\left(f_{g_{n,m}}(z_M)\right)\\
&\quad\bullet\left[\left(\tilde{U}_{n,m}(z_0,z_M;s)\right)_{2,1}\bullet\left(f_{g_{n,m}}(z_M)\right)+\left(\tilde{U}_{n,m}(z_0,z_M;s)\right)_{2,2}\right]^{-1}\bullet\left(f_{g_{n,m}}\right)\bullet\left(Z_{c_{n,m}}\right)\\
&=\left[\left(1_{n,m}\right)+\left[\prod_{\ell=M}^{1}\left(f_{g_{n,m}}(z_\ell)\right)\tanh(\gamma\Delta z)\right]\bullet\left(g_{f_{n,m}}(z_M)\right)\right]\bullet\left(f_{g_{n,m}}(z_M)\right)\\
&\quad\bullet\left[\left(1_{n,m}\right)+\left[\prod_{\ell=M}^{1}\left(g_{f_{n,m}}(z_\ell)\right)\tanh(\gamma\Delta z)\right]\bullet\left(f_{g_{n,m}}(z_M)\right)\right]^{-1}\bullet\left(g_{f_{n,m}}\right)\bullet\left(Z_{c_{n,m}}\right)
\end{aligned}
\tag{6.12}
$$

Now set

$$
\left(f_{g_{n,m}}(z_M)\right)\equiv\left(f_{g_{n,m}}\right)\ ,\quad \left(g_{f_{n,m}}(z_M)\right)\equiv\left(g_{f_{n,m}}\right)
\tag{6.13}
$$

This basically assures that wire n at the source connects to wire n at the load. Expanding for small $\gamma[z_M - z_0]$ gives

$$
\begin{aligned}
\left(\tilde{Z}_{n,m}^{(dif)}(s)\right) &= \left[\left(1_{n,m}\right)+\gamma[z_M-z_0]\left[\frac{1}{M}\sum_{\ell=1}^{M}\left(f_{g_{n,m}}(z_\ell)\right)\right]\bullet\left(g_{f_{n,m}}\right)+\mathrm{O}\left(M[\gamma\Delta z]^3\right)\right]\\
&\quad\bullet\left(f_{g_{n,m}}\right)\bullet\left[\left(1_{n,m}\right)+\gamma[z_M-z_0]\left[\frac{1}{M}\sum_{\ell=1}^{M}\left(g_{f_{n,m}}(z_\ell)\right)\right]\bullet\left(f_{g_{n,m}}\right)+\mathrm{O}\left(M[\gamma\Delta z]^3\right)\right]^{-1}\\
&\quad\bullet\left(g_{f_{n,m}}\right)\bullet\left(Z_{c_{n,m}}\right)\\
&=\left[\left(1_{n,m}\right)+\gamma[z_M-z_0]\left[\frac{1}{M}\sum_{\ell=1}^{M}\left(f_{g_{n,m}}(z_\ell)\right)\right]\left(g_{f_{n,m}}\right)+\mathrm{O}\left(M[\gamma\Delta z]^2\right)\right]\\
&\quad\bullet\left(f_{g_{n,m}}\right)\bullet\left[\left(1_{n,m}\right)-\gamma[z_M-z_0]\left[\frac{1}{M}\sum_{\ell=1}^{M}\left(g_{f_{n,m}}(z_\ell)\right)\right]\left(f_{g_{n,m}}\right)+\mathrm{O}\left(\gamma[z_M-z_0]^2\right)\right]\\
&\quad\bullet\left(g_{f_{n,m}}\right)\bullet\left(Z_{c_{n,m}}\right)
\end{aligned}
$$

$$= \left[\left(1_{n,m}\right) + \gamma\left[z_M - z_0\right]\left[\left[\frac{1}{M}\sum_{\ell=1}^{M}\left(f_{g_{n,m}}\left(z_\ell\right)\right)\right]\left(g_{f_{n,m}}\right) - \left(f_{g_{n,m}}\right) \bullet \sum_{\ell=1}^{M}\left(g_{f_{n,m}}\left(z_\ell\right)\right)\right]\right] + \mathrm{O}\left(\gamma\left[z_M - z_0\right]^2\right)\Bigg] \bullet \left(Z_{c_{n,m}}\right) \tag{6.14}$$

Here we see that at low frequencies, as we would expect,

$$\left(\tilde{Z}_{n,m}^{(in)}(s)\right) = \left(Z_{c_{n,m}}\right)\left[1 + \mathrm{O}\left(\gamma\left[z_M - z_0\right]\right)\right] \tag{6.15}$$

Since the cable is electrically short.

We can also look at the signal transfer to the load ($Z_{c_{n,m}}$). From (6.10)

$$\begin{aligned}
&\left[\cosh^M(\gamma\Delta z)\left(1_{n,m}\right) + \gamma\Delta z\left[\prod_{\ell=M}^{1}\left(f_{g_{n,m}}\left(z_\ell\right)\right)\right] \bullet \left(g_{f_{n,m}}\right)\right] \bullet \left(\tilde{V}_n^{(dif)}\left(z_M, s\right)\right) = \left(\tilde{V}_n(0,s)\right) \\
&\quad \left(\tilde{V}_n^{(dif)}\left(z_m, s\right)\right) \equiv \left(\tilde{T}_n^{(dif)}(s)\right) \bullet \left(\tilde{V}_n^{(dif)}(0,s)\right) \\
&\quad \left(\tilde{T}_n^{(dif)}(s)\right) \\
&\quad = \left[\cosh^M(\gamma\Delta z)\left(1_{n,m}\right) + \gamma\Delta z\left[\prod_{\ell=M}^{1}\left(f_{g_{n,m}}\left(z_\ell\right)\right)\right] \bullet \left(g_{f_{n,m}}\right)\right]^{-1} \\
&\quad = \left(1_{n,m}\right) - \gamma\left[z_M - z_0\right]\left[\frac{1}{M}\sum_{\ell=1}^{M}\left(f_{g_{n,m}}\left(z_\ell\right)\right)\right] \bullet \left(g_{f_{n,m}}\right) + \mathrm{O}\left(\gamma\left[z_M - z_0\right]^2\right)
\end{aligned} \tag{6.16}$$

Now from (3.10) for random permutations with the choice of (6.14) we have, noting the zero average values of the differential signals

$$\begin{aligned}
&avg\left(\left(\tilde{V}^{(dif)}\left(z_M, s\right)\right)\right) = avg\left(\left(\tilde{V}^{(dif)}\left(z_0, s\right)\right)\right) = \left(0_n\right) \\
&avg\left(\left(\tilde{T}^{(dif)}(s)\right)\right) \\
&\quad = \left(1_{n,m}\right) + \gamma\left[z_M - z_0\right]\Big[f_{g_{1,1}}\left(1_{n,m}\right)
\end{aligned}$$

$$+\frac{1}{N-1}\left[\sum_{m=2}^{N} f_{g_{1,m}}\right]\left(u_{n,m}\right)-\left(1_{n,m}\right)\Bigg]\bullet\left(g_{f_{n,m}}\right)$$

$$+\,\mathrm{O}\left(\left[\gamma\left[z_M - z_0\right]\right]^2\right)$$

$$avg\left(\left(\tilde{Z}^{(dif)}(s)\right)\right) = \Bigg[\left(1_{n,m}\right) = \gamma\left[z_M - z_0\right]\Bigg[f_{g_{1,1}}\left(1n,m\right)$$

$$+\frac{1}{N-1}\left[\sum_{m=2}^{N} f_{g_{1,m}}\right]\left[\left(u_{n,m}\right)-\left(1_{n,m}\right)\right]\Bigg]\bullet\left(g_{f_{n,m}}\right) \tag{6.17}$$

$$-\left(f_{g_{n,m}}\right)\bullet\Bigg[g_{f_{1,1}}\left(1_{n,m}\right)+\frac{1}{N-1}\left[\sum_{m=2}^{N} g_{f_{1,m}}\right]\left[\left(u_{n,m}\right)-\left(1_{n,m}\right)\right]\Bigg]$$

$$+\,\mathrm{O}\left(\gamma\left[z_M - z_0\right]\right)\Bigg]\bullet\left(Z_{c_{n,m}}\right)$$

For the variances we appeal to Appendices B and G. Since $\tilde{T}^{(dif)}$ starts with identity, we find from (G.5) that

$$\mathrm{var}\left(\left(\tilde{T}_{n,m}^{(dif)}(j\omega)\right)\right)$$

$$= \mathrm{O}\left(\left[k\left[z_M - z_0\right]\right]^2\right) \tag{6.18}$$

$$\gamma = jk = j\frac{\omega}{\mathrm{v}}$$

Similarly we find for the input impedance

$$\mathrm{var}\left(\left(\tilde{Z}_{n,m}^{(dif)}(j\omega)\right)\right)$$

$$= \mathrm{O}\left(\left[k\left[z_M - z_0\right]\right]^2\right) \tag{6.19}$$

So while the averages are first order in frequency, the variances are second order. To calculate these to second order the matrices in (6.14) and (6.17) need to be expanded through second order.

7. Concluding Remarks

So now we have a beginning on the theory of random-lay cables. While it strictly applies to random wire interchanges in a circulant system, one may expect that some features of this will lead toward properties of more general situations.

Appendix A. Properties of $((F_{n,m}(z))_{u,\text{v}})$

From Section 2 we have a 2N x 2N supermatrix

$$\left(\left(F_{n,m}(z)\right)_{u,\text{v}}\right) = \begin{pmatrix} \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z)\right) \\ \left(f_{g_{n,m}}(z)\right)^{-1} & \left(0_{n,m}\right) \end{pmatrix} \tag{A.1}$$

We quickly find

$$\begin{aligned} \left(\left(F_{n,m}(z)\right)_{u,\text{v}}\right)^2 &= \left(\left(F_{n,m}(z)\right)_{u,\text{v}}\right) \odot \left(\left(F_{n,m}(z)\right)_{u,\text{v}}\right) \\ &= \left(\left(1_{n,m}\right)_{u,\text{v}}\right) = \begin{bmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(1_{n,m}\right) \end{bmatrix} \\ 1_{n,m} &= \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases} \end{aligned} \tag{A.2}$$

So this supermatrix is a square root of the superidentity. From this we also find

$$\left(\left(F_{n,m}(z)\right)_{u,\text{v}}\right)^{-1} = \left(\left(F_{n,m}(z)\right)_{u,\text{v}}\right) \tag{A.3}$$

With zeros on the diagonal we have

$$tr\left(\left(F_{n,m}(z)\right)_{u,\text{v}}\right) = 0 = \text{sum of 2N eigenvalues} \tag{A.4}$$

For the determinant (A.2) gives

$$\det{}^2\left(\left(F_{n,m}(z)\right)_{u,\text{v}}\right) = 1 \tag{A.5}$$

We also have [9] with square $(A_{n,m})$ and $(D_{n,m})$

$$\det\begin{pmatrix} \left(0_{n,m}\right) & \left(B_{n,m}\right) \\ \left(C_{n,m}\right) & \left(D_{n,m}\right) \end{pmatrix} = \det\left(-\left(B_{n,m}\right) \bullet \left(C_{n,m}\right)\right) \tag{A.6}$$

(since the zero matrix commutes with all matrices of same size). Applied to $((F_{n,m}(z))_{u,\mathrm{v}})$ we have

$$\begin{aligned}
\det^2\left(\left(F_{n,m}(z)\right)_{u,\mathrm{v}}\right) &= \det\left(-\left(f_{g_{n,m}}(z)\right)\bullet\left(f_{g_{n,m}}(z)\right)^{-1}\right) \\
&= \det\left(-\left(1_{n,m}\right)\right) = (-1)^N \\
&= \begin{cases} 1 \text{ for N even} \\ -1 \text{ for N odd} \end{cases}
\end{aligned} \tag{A.7}$$

resolving the ambiguity in (A.4).

Since the eigenvalues of the superidentity are all 1 (2N of them), the eigenvalues of $((F_{n,m}(z))_{u,\mathrm{v}})$ are all ± 1. In order that the trace be zero, then we have

$$\begin{aligned}
\left(\left(F_{n,m}(z)\right)_{u,\mathrm{v}}\right) \odot \left(\left(x_n\right)_u\right)_\beta &= \lambda_\beta \left(\left(x_n\right)_u\right)_\beta \\
\lambda_\beta &= \pm 1 \quad \text{(N are +1 and N are -1)}
\end{aligned} \tag{A.8}$$

This is also consistent with (A.7), with the determinant as the product of the eigenvalues.

Appendix B. Permutation matrices

Consider N x N permutation matrices $(P_{n,m})$ of the form

$$\left(P_{n,m}\right) = \begin{pmatrix} 0\;0 & & \\ \vdots\;\vdots & & \\ 0\;\vdots & & \\ 1\;0 & \cdots\cdots\cdots & 0 \\ 0\;\vdots & & \\ \vdots\;0 & & \\ 0\;1\;0 & \cdots\cdots & 0 \\ \vdots\;0 & & \\ \vdots\;\vdots & & \end{pmatrix} \tag{B.1}$$

where each row has exactly one element 1 with N-1 elements 0, and similarly for columns. Applying this to our N-conductor transmission line, this corresponds to moving a wire at some position *m* to position *n* corresponding to the non zero elements of $(P_{n,m})$. Note that

$$\left(P_{n,m}\right) \bullet \left(P_{n,m}\right)^T = \left(1_{n,m}\right) \tag{B.2}$$

i.e., reordering followed by unreordering is an identity. From this we have

$$\left(P_{n,m}\right)^{-1} = \left(P_{n,m}\right)^T \tag{B.3}$$

so that the inverse is not singular.

Since (for general N x N matrices and $(P_{n,m})$ in particular)

$$\det\left(\left(P_{n,m}\right)^T\right) = \det\left(\left(P_{n,m}\right)\right) \tag{B.4}$$

then we have from (3.2)

$$\det\left(\left(P_{n,m}\right)\right) = \pm 1 \tag{B.5}$$

with both values appearing, depending on the particulars of the row/column reordering.

In the combination

$$\left(P_{n,m}\right) \bullet \left(Q_{n,m}\right) \bullet \left(P_{n,m}\right)^T \tag{B.6}$$

the diagonal elements are reordered as diagonal element. The off-diagonal elements are reordered as off-diagonal elements. This property allows us to state

$$\begin{aligned} tr\left(\left(P_{n,m}\right) \bullet \left(Q_{n,m}\right) \bullet \left(P_{n,m}\right)^T\right) &= tr\left(\left(Q_{n,m}\right)\right) \\ \det\left(\left(P_{n,m}\right) \bullet \left(Q_{n,m}\right) \bullet \left(P_{n,m}\right)^T\right) &= \det\left(\left(Q_{n,m}\right)\right) \end{aligned} \tag{B.7}$$

There is a list of the properties of such matrices in [2], such as

$$\left[\left(P_{n,m}\right)\bullet\left(Q_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right]^{-1}=\left(P_{n,m}\right)\bullet\left(Q_{n,m}\right)^{-1}\bullet\left(P_{n,m}\right)^T$$

$$\left[\left(P_{n,m}\right)\bullet\left(Q_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right]\bullet\left[\left(P_{n,m}\right)\bullet\left(R_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right] \tag{B.8}$$

$$=\left(P_{n,m}\right)\bullet\left(Q_{n,m}\right)\bullet\left(R_{n,m}\right)\bullet\left(P_{n,m}\right)^T$$

There are stochastic properties, assuming that all permutations are equally likely, including

$$avg\left(\left(P_{n,m}\right)\right) \equiv \text{expected value of } \left(P_{n,m}\right)$$

$$= \frac{1}{N}\left(u_{n,m}\right)$$

$$u_{n,m} = 1 \text{ for all } n,m$$

$$avg\left(\left(P_{n,m}\right)\bullet\left(Q_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right)$$

$$=\begin{cases}\frac{1}{N}\sum_{n} Q_{n,n} & \text{(diagonal elements)}\\ \frac{1}{N^2-N}\sum_{\substack{n,m\\ n\neq m}} Q_{n,m} & \text{(off-diagonal elements)}\end{cases}$$

$$\mathrm{var}\left(\left(P_{n,m}\right)\bullet\left(Q_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right) \tag{B.9}$$

$$=\begin{cases}\frac{1}{N}\left[\sum_{n} Q_{n,m}^2\right]-\left[\frac{1}{N}\sum_{n} Q_{n,n}\right]^2 & \text{(diagonal elements)}\\ \frac{1}{N^2-N}\left[\sum_{n,m,m'} Q_{n,m}Q_{n,m'}\right]-\left[\frac{1}{N^2-N}\sum_{\substack{n,m\\ n\neq m}} Q_{n,m}\right]^2 & \text{(off-diagonal elements)}\end{cases}$$

Note that variance is also defined as

$$\mathrm{var}\left(\left(R_{n,m}\right)\right) = avg\left(\left[\left(R_{n,m}\right)-avg\left(R_{n,m}\right)\right]^2\right) \tag{B.10}$$

The form that these take allows one to write

$$avg\left(\left(P_{n,m}\right)\bullet\left(Q_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right)$$

$$= \left[\frac{1}{N}\sum_n Q_{n,m}\right]\left(1_{n,m}\right) + \left[\frac{1}{N^2-N}\sum_{\substack{n,m\\n\neq m}} Q_{n,m}\right]\left[\left(u_{n,m}\right)-\left(1_{n,m}\right)\right]$$

$$\mathrm{var}\left(\left(P_{n,m}\right)\bullet\left(Q_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right) \tag{B.1}$$

$$= \left[\frac{1}{N}\left[\sum_{n,m} Q_{n,m}^2\right] - \left[\frac{1}{N}\sum_n Q_{n,m}\right]^2\right]\left(1_{n,m}\right)$$

$$+ \left[\frac{1}{N^2-N}\left[\sum_{n,m,m'} Q_{n,m}Q_{n,m'}\right] - \left[\frac{1}{N^2-N}\sum_{n,m} Q_{n,m}\right]^2\right]\left[\left(u_{n,m}\right)-\left(1_{n,m}\right)\right]$$

Appendix C. Geometric-Factor Matrix For N Equal Currents (Common Mode)

As in Fig. 4.1, let us consider N wires of radius r_0 at a common distance of Ψ_1 from the center of the surrounding circular cylindrical conductor (of radius Ψ_0). The wire centers are placed on angloes

$$\begin{aligned}\phi_n &= n\phi_0 \ , \ n = 1,\ldots,N \text{ or } n = 0,\ldots,N-1\\ \phi_0 &= \frac{2\pi}{N}\end{aligned} \tag{C.1}$$

noting that wire number zero and N correspond to the same wire.

From [1] we consider the conformal transformation defined by

$$\begin{aligned}\zeta &\equiv x + jy = \Psi e^{j\phi} \text{ (complex coordinate)}\\ w(\zeta) &= u(\zeta) + j\mathrm{v}(\zeta) \text{ (complex potential)}\\ \zeta &= \Psi_1\left[e^{-Nw'(\zeta)}+1\right]^{1/N}\\ w'(\zeta) &= -\frac{1}{N}\ell n\left(\left[\frac{\zeta}{\Psi_1}\right]^N - 1\right)\end{aligned} \tag{C.2}$$

This $w'(\zeta)$ corresponds to N wires in free space. To include the effect of the conducting boundary of radius Ψ_0 we find a potential due to image wires (of opposite charge) at a common radius

$$\Psi_1' = \frac{\Psi_0^2}{\Psi_1} \tag{C.3}$$

This corresponds to the symmetry of reciprocation [12]. So we map coordinates as

$$\zeta' = \frac{\Psi_0^2}{\zeta} \tag{C.4}$$

and find another potential as

$$w''(\zeta) = \frac{1}{N}\ell n\left(\left[\frac{\Psi_0^2}{\Psi_1\zeta}\right]^N - 1\right) \tag{C.5}$$

Adding these two potentials gives

$$\begin{aligned} w(\zeta) &= w'(\zeta) + w''(\zeta) \\ &= -\frac{1}{N}\ell n\left(\frac{\left[\frac{\zeta}{\Psi_1}\right]^N - 1}{\left[\frac{\Psi_0^2}{\Psi_1\zeta}\right]^N - 1}\right) \end{aligned} \tag{C.6}$$

Note that we require $2r_0$ to be small compared to the spacing between the wires, which is for large N

$$2r_0 \ll \frac{\Psi_1}{2\pi N} \;,\; r_0 \ll \frac{\Psi_1}{\pi N} \tag{C.7}$$

Similarly we need

$$r_0 \ll \Psi_0 - \Psi_1 \text{ (distance to the outer conductor)} \tag{C.8}$$

Consider the complex potential on

$$\zeta = \Psi_1 e^{j\phi} \;,\; |\zeta| = \Psi_1 \tag{C.9}$$

This gives

$$
\begin{aligned}
w(\zeta) &= -\frac{1}{N}\ell n\left(\frac{e^{jN\phi}-1}{e^{-jN\phi}-1}\right) \\
&= -\frac{1}{N}n\left(\left|\frac{e^{jN\phi}-1}{e^{-jN\phi}-1}\right|\right) - \frac{j}{N}\arg\left(\frac{e^{jN\phi}-1}{e^{-jN\phi}-1}\right) \\
&= -\frac{1}{N}\ell n(1) - \frac{j}{N}\arg\left(\frac{e^{jN\phi}-1}{e^{-jN\phi}-1}\right) \\
&= u\left(\Psi_1 e^{j\phi}\right) + j\mathrm{v}\left(\Psi_1 e^{j\phi}\right)
\end{aligned}
\tag{C.10}
$$

$$
u\left(\Psi_1 e^{j\phi}\right) = 0
$$

So the circle of radius Ψ_1 is an electric equipotential. The magnetic potential $\mathrm{v}(\zeta)$ on this circle phase wraps as one passes around on the circle.

Now with the outer conductor at zero potential we find the electric potential on a wire as (looking at the wire centered on $\phi = 0$)

$$
\begin{aligned}
\zeta &= \Psi_1 = +r_0 e^{j\phi'} \;\;, \;\; 0 \le \phi' \le 2\pi \\
u(\zeta) &= -\frac{1}{N}\ell n\left(\frac{\left|\left[1+\frac{r_0}{\Psi_1}e^{j\phi'}\right]^N - 1\right|}{\left|\left[\frac{\Psi_0}{\Psi_1}\right]^2\left[1+\frac{r_0}{\Psi_1}e^{-j\phi'}\right]^N - 1\right|}\right)
\end{aligned}
\tag{C.11}
$$

Expanding for small $\mathrm{N}r_0 / \Psi_1$ gives, keeping leading terms,

$$
\begin{aligned}
&\left[1+\frac{r_0}{\Psi_1}e^{j\phi'}\right]^N - 1 \\
&\quad = 1 + N\frac{r_0}{\Psi_1}e^{jN\phi'} + \cdots - 1 \\
&\quad = N\frac{r_0}{\Psi_1}e^{jN\phi'} + \mathrm{O}\left(N^2\left[\frac{r_0}{\Psi_1}e^{jN\phi'}\right]^2\right)
\end{aligned}
$$

$$\left|\left[1+\frac{r_0}{\Psi_1}e^{j\phi'}\right]^N - 1\right| = N\frac{r_0}{\Psi_1} + \mathrm{O}\left(N^2\left[\frac{r_0}{\Psi_1}\right]\right) \tag{C.12}$$

Note that this is asymptotic in small $\mathrm{N}r_0 / \Psi_1$, which requires the wire radius to be very small compared to the spacing.

Similarly we have

$$\begin{aligned} &\ell n\left(\left[\left[\frac{\Psi_0}{\Psi_1}\right]^2\left[1+\frac{r_0}{\Psi_1}e^{-j\phi'}\right]^{-1}\right]^N - 1\right) \\ &\quad = \ell n\left(\left[\frac{\Psi_0}{\Psi_1}\right]^2 - 1\right) + \mathrm{O}\left(N\frac{r_0}{\Psi_1}\right) \end{aligned} \tag{C.13}$$

for constant Ψ_0 / Ψ_1, again asymptotic for small $N r_0 / \Psi_1$.

Then from (C.9) the electric potential on the wire is approximately (for thin wires compared to spacing)

$$u(\zeta) \simeq \frac{1}{N}\left[\ell n\left(\frac{\Psi_1}{Nr_0}\right) + \ell n\left(\left[\frac{\Psi_0}{\Psi_1}\right]^{2N} - 1\right)\right] \tag{C.14}$$

Since the electric potential on the outer circle of radius Ψ_0 is zero we have

$$\Delta u \simeq \frac{1}{N}\left[\ell n\left(\frac{\Psi_1}{Nr_0}\right) + \ell n\left(\left[\frac{\Psi_0}{\Psi_1}\right]^{2N} - 1\right)\right] \tag{C.15}$$

The change in the magnetic potential around a wire is given by

$$\begin{aligned} \mathrm{v} &\simeq -\frac{1}{N}\left(\frac{Nr_0}{\Psi_1}e^{j\phi'}\right) = -\frac{1}{N}\phi' \\ \Delta\mathrm{v}(1) &= \frac{2\pi}{N} \end{aligned} \tag{C.16}$$

There are N such wire currents giving

$$\Delta \mathrm{v}(N) = N\, D\, \mathrm{v}(1) = 2\pi \tag{C.17}$$

The geometric factor for such N currents (total) is then

$$f_g^{(N)} \equiv \frac{\Delta u}{\Delta \mathrm{v}(N)} \simeq \frac{1}{2\pi N}\left[\ell n\left(\frac{\Psi_1}{N r_0}\right) + \ell n\left(\left[\frac{\Psi_0}{\Psi_1}\right]^{2N} - 1\right)\right] \tag{C.18}$$

As a special case of one wire we have

$$\begin{aligned} f_g^{(1)} &\simeq \frac{1}{2\pi}\left[\ell n\left(\frac{\Psi_1}{r_0}\right) + \ell n\left(\left[\frac{\Psi_0}{\Psi_1}\right]^{2} - 1\right)\right] \\ &\simeq f_{g_{1,1}} \quad \text{for} \quad \frac{N r_0}{\Psi_1} \to 0 \\ &\equiv f'_{g_{n,n}} \end{aligned} \tag{C.19}$$

Note that r_0 must be small compared to both Ψ_1 and $\Psi_0 - \Psi_1$, and hence small compared to Ψ_0. For a circular cylinder of radius Ψ_1 centered on the origin we have

$$f_g^{(cyl)} = \frac{1}{2\pi}\ell n\left(\frac{\Psi_0}{\Psi_1}\right) \tag{C.20}$$

which is a well-known result.

Appendix D. Off-Diagonal Terms of $(f_{g_{n,m}})$

Recall that the potential function for a single wire at ϕ = 0 or N is (from (C.6))

$$
\begin{aligned}
w(\zeta) &= -\frac{1}{N}\ell n\left(\frac{\dfrac{\zeta}{\Psi_1}-1}{\dfrac{\Psi_0^2}{\Psi_1\zeta}-1}\right)\\
u(\zeta) &= -\frac{1}{N}\ell n\left(\frac{\left|\dfrac{\zeta}{\Psi_1}-1\right|}{\left|\dfrac{\Psi_0^2}{\Psi_1\zeta}-1\right|}\right) \qquad \text{(D.1)}\\
\zeta_n &= \Psi_1 e^{j\frac{2\pi n}{N}} \quad , \quad n=1,\ldots,N-1
\end{aligned}
$$

This gives

$$
\begin{aligned}
u(\zeta_n) &= -\frac{1}{N}\ell n\left(\frac{\left|e^{j\frac{2\pi n}{N}}-1\right|}{\left|\left[\dfrac{\Psi_0}{\Psi_1}\right]^2 e^{-j\frac{2\pi n}{N}}-1\right|}\right)\\
&= \frac{1}{N}\ell n\left(\frac{2\left[1-\cos\left(\dfrac{2\pi n}{N}\right)\right]}{\left[\dfrac{\Psi_0}{\Psi_1}\right]^4+1-2\left[\dfrac{\Psi_0}{\Psi_1}\right]^2\cos\left(\dfrac{2\pi n}{N}\right)}\right) \qquad \text{(D.2)}\\
&= \Delta u_n
\end{aligned}
$$

From (C.17) we have the Δv going around a single wire as

$$
\Delta \mathrm{v}(1) = \frac{2\pi}{N} \qquad \text{(D.3)}
$$

so we have

$$
\begin{aligned}
f_{g_{0,m}}\Big|_{m\neq 0} &= \frac{\Delta u}{\Delta \mathrm{v}(1)} \\
&= \frac{1}{4\pi}\ell n\left(\frac{\left[\frac{\Psi_0}{\Psi_1}\right]^4 + 1 - 2\left[\frac{\Psi_0}{\Psi_1}\right]^2 \cos\left(\frac{2\pi m}{N}\right)}{2\left[1-\cos\left(\frac{2\pi m}{N}\right)\right]}\right) \\
&= f_{g_{m,0}}
\end{aligned}
\tag{D.4}
$$

Noting the bicyclic form of this matrix we have

$$
\begin{aligned}
f_{g_{n,m}}\Big|_{n\neq m} &= f_{g_{m,n}}\Big|_{n\neq m} \\
&= \frac{1}{4\pi}\ell n\left(\frac{\left[\frac{\Psi_0}{\Psi_1}\right]^4 + 1 - 2\left[\frac{\Psi_0}{\Psi_1}\right]^2 \cos\left(\frac{2\pi[n-m]}{N}\right)}{2\left[1-\cos\left(\frac{2\pi[n-m]}{N}\right)\right]}\right)
\end{aligned}
\tag{D.5}
$$

Besides being bicirculant, $(f_{g_{n,m}})$ has some additional properties.

$$
\begin{aligned}
\text{sre} &\equiv \sum_{m=1}^{N}\left(f_{g_{n,m}}\right) \equiv \text{ single row sum (independent of } n) \\
= \text{sce} &\equiv \sum_{n=1}^{N}\left(f_{g_{n,m}}\right) \equiv \text{ single column sum (independent of } m)
\end{aligned}
\tag{D.6}
$$

Interchanging rows and columns by a permutation matrix gives

$$
\begin{aligned}
\text{sre} &= \sum_{m=1}^{N}\left(P_{n,m}\right)\cdot\left(f_{g_{n,m}}\right)\cdot\left(P_{n,m}\right)^T \\
= \text{sce} &= \sum_{n=1}^{N}\left(P_{n,m}\right)\cdot\left(f_{g_{n,m}}\right)\cdot\left(P_{n,m}\right)^T
\end{aligned}
\tag{D.7}
$$

merely interchanging the order of the terms of the row and column sums. Similarly we have

$$
\text{sre}^2 \equiv \sum_{m=1}^{N} f_{g_{n,m}}^2 \quad \text{(independent of } n)
$$

$$= \text{sce}^2 \equiv \sum_{n=1}^{N} f_{g_{n,m}}^2 \quad \text{(independent of } m\text{)} \tag{D.8}$$

with the same permutation property as in (D.7).

Taking the inverse,

$$\left(g_{f_{n,m}}\right) = \left(f_{g_{n,m}}\right)^{-1} \tag{D.9}$$

this is also a bicirculant matrix, and the properties above apply to it as well.

Appendix E. Eigenmodes of Bicirculant Matrices

The bicirculant character of the geometric-factor matrix imparts much symmetry to this matrix. Its general properties are discussed in [3 (Appendices A and B). In particular the eigenmodes take the form

$$\left(w_n\right)_\beta = \left[\frac{2}{N}\right]^{1/2} \begin{pmatrix} \cos\left(\frac{2\pi\beta}{N}\right) \\ \vdots \\ \cos\left(\frac{2\pi[N-1]\beta}{N}\right) \\ 1 \end{pmatrix} \tag{E.1}$$

$$\beta = \begin{cases} 1, 2, \ldots, \frac{N}{2} - 1 & \text{for } N \text{ even} \\ 1, 2, \ldots, \frac{N-1}{2} & \text{for } N \text{ odd} \end{cases}$$

$$\left(w_n\right)_\beta = \left[\frac{2}{N}\right]^{1/2} \begin{pmatrix} \sin\left(\frac{2\pi\beta}{N}\right) \\ \vdots \\ \sin\left(\frac{2\pi[N-1]\beta}{N}\right) \\ 1 \end{pmatrix}$$

$$\left(w_n\right)_{\frac{N}{2}} = N^{-1/2}\begin{pmatrix}-1\\+1\\-1\\\vdots\\-1\\+1\end{pmatrix} \text{ for } N \text{ even} \tag{E.1}$$

$$\left(w_n\right)_0 = \left(w_n\right)_N = N^{-1/2}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \text{ (common mode)}$$

These have most of the eigenvalues λ_β applying to two modes, and we can have

$$\lambda_\beta = \lambda_{N-\beta} \text{ for } \beta = 1,2,\ldots,N \tag{E.2}$$

Note that $(w_n)_N$ is the common mode. In this normalization the common-mode current is

$$I_c = I_N = I_c \sum_{n=1}^{N} N^{1/2} w_{n;0} \tag{E.3}$$

This is treated separately in the analysis.

The differential modes correspond to

$$\beta = 1,\ldots,N-1 \tag{E.4}$$

Since the modes are biorthonormal we have

$$\left(w_n\right)_\beta \bullet \left(w_n\right)_{\beta'} = 1_{\beta,\beta'} = \begin{cases}1 \text{ for } \beta = \beta' \\ 1 \text{ for } \beta \neq \beta'\end{cases} \tag{E.5}$$

Selecting β' as N (the common mode) gives

$$\begin{aligned}\left(w_n\right)_\beta \bullet \left(w_n\right)_N &= N^{-1/2} \sum_{n=1}^{N} w_{n;\beta} = N^{-1/2}\, 1_{\beta;N} \\ &= 0 \text{ for } \beta \neq N\end{aligned} \tag{E.6}$$

and hence

$$\sum_{n=1}^{N} w_{n;\beta} = 0 \quad \text{for} \quad \beta \neq N \tag{E.7}$$

Thus all the differential modes have average value zero. Removing (subtracting) the common mode then leaves the sum of the remaining currents equal to zero since they are expanded in terms of N – 1 eigenmodes, all of which have average value zero.

The eigenvalues of bicirculant matrices are calculable from

$$\begin{aligned} \lambda_\beta &= (w_n)_\beta \cdot (w_n)_\beta \quad \text{for} \quad \beta = 1, \ldots, N \\ \left(f g_{n,m}\right) &= \sum_{\beta=1}^{N} \lambda_\beta (w_n)_\beta (w_n)_\beta \end{aligned} \tag{E.8}$$

The inverse matrix is given by

$$\begin{aligned} \left(g_{f_{n,m}}\right) \equiv \left(f g_{n,m}\right)^{-1} &= \sum_{\beta=1}^{N} \lambda_\beta^{-1} (w_n)_\beta (w_n)_\beta \\ \lambda_\beta^{-1} &= \text{eigenvalues} \quad \text{for } \beta = 1\ldots, N \end{aligned} \tag{E.9}$$

We can also note that for a bicirculant matrix, such as $(X_{n,m})$, we have

$$\begin{aligned} \left(X_{n,m}\right) &= \sum_{\beta=1}^{N} \lambda_\beta (w_n)_\beta (w_n)_\beta \\ \left(X_{n,m}\right)^* &= \sum_{\beta=1}^{N} \lambda_\beta^* (w_n)_\beta (w_n)_\beta \\ (w_n)_\beta &= \text{real vector} \end{aligned} \tag{E.10}$$

So we have

$$\begin{aligned} \left(X_{n,m}\right) \cdot \left(X_{n,m}\right)^* = \left(X_{n,m}\right)^* \cdot \left(X_{n,m}\right) &\quad \text{(commute)} \\ = \sum_{\beta=1}^{N} \lambda_\beta \lambda_\beta^* (w_n)_\beta (w_n)_\beta &= \text{real matrix} \end{aligned} \tag{E.11}$$

Appendix F. Terms in the Product Integral

From (5.1) we have an individual term in the product integral

$$e^{-\gamma\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)\Delta z} = \left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) + \sum_{p=1}^{\infty} \frac{1}{p!}[\gamma\Delta z]^p \left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right) \tag{F.1}$$

Note that

$$\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)^2 = \begin{pmatrix} \left(0_{n,m}\right) & \left(f_{g_{n,m}}(z_\ell)\right) \\ \left(f_{g_{n,m}}(z_\ell)\right)^{-1} & \left(0_{n,m}\right) \end{pmatrix}^2 = \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(1_{n,m}\right) \end{pmatrix} \tag{F.2}$$

for all z_ℓ, and likewise for any even p. Next we have

$$\begin{aligned} \left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)^p &= \left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right) \odot \left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)_{p-1} \\ &= \left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right) \quad \text{for odd } p \end{aligned} \tag{F.3}$$

This allows us to write

$$\begin{aligned} e^{-\gamma\left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)\Delta z} &= \left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right)\cosh(\gamma\Delta z) \\ &\quad - \left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)\sinh(\gamma\Delta z) \\ &= \left(\left(1_{n,m}\right)_{u,\mathrm{v}}\right) - \left(\left(F_{g_{n,m}}(z_\ell)\right)_{u,\mathrm{v}}\right)\gamma\Delta z \;=\; \mathrm{O}\left([\gamma\Delta z]^2\right) \\ &\qquad \text{as } \gamma\Delta z \to 0 \end{aligned} \tag{F.4}$$

Appendix G. Stochastic Properties of Complex Matrices

G.1. General

Consider a complex matrix of the form

$$\left(a_{n,m}\right) = \left(b_{n,m}\right) + j\left(c_{n,m}\right)$$
$$\left(b_{n,m}\right),\ \left(c_{n,m}\right) = \text{real N x N matrices} \qquad \text{(G.1)}$$

As before (Appendix B) the average is a linear operation

$$avg\left(\left(P_{n,m}\right)\bullet\left(a_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right)$$
$$= \left[\frac{1}{N}\sum_n a_{n,m}\right]\left(1_{n,m}\right) + \left[\frac{1}{N^2 - N}\sum_n a_{n,m}\right]\left[\left(u_{n,m}\right) - \left(1_{n,m}\right)\right] \qquad \text{(G.2)}$$

So we have

$$avg\left(\left(P_{n,m}\right)\bullet\left(b_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right) + j\ avg\left(\left(P_{n,m}\right)\bullet\left(c_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right) \qquad \text{(G.3)}$$

which is complex valued. Stated another (shorter) way we have

$$\underset{k}{avg}\left(\left(a_{n,m}\right)_k\right) = \left[\frac{1}{N}\sum_n a_{n,m}\right]\left(1_{n,m}\right) + \left[\frac{1}{N^2 - N}\sum_{\substack{n,m \\ n\neq m}} a_{n,m}\right]\left[\left(u_{n,m}\right) - \left(1_{n,m}\right)\right] \qquad \text{(G.4)}$$

For the variance we need some measure of the deviation of $(a_{n,m}) - avg(a_{n,m})$ which we would like to be real valued (like a 2-norm). For a complex vector (x_n) one definition is [10]

$$\text{var}\left(\left(x_n\right)\right) \equiv avg\left(\left[\left(x_n\right) - avg\left(\left(x_n\right)\right)\right]\bullet\left[\left(x_n\right) - avg\left(x_n\right)\right]^*\right) \qquad \text{(G.5)}$$

This gives a real-valued variance.

We need to extend the variance to complex-valued matrices. For this purpose we note that for our NMTL problem we have a random matrix which operates on (multiplies) a vector of sources which we can consider as a fixed (not random) complex valued vector (y_n) (N components). So let

$$(x_n)_k = (a_{n,m})_k \bullet (y_n) \tag{G.6}$$

represent the signals propagating down the transmission line. Our random variables enter through $(a_{n,m})_k$, which represents the kth choice of our transmission matrix.

The average value of $(x_n)_k$ is simply

$$\underset{k}{avg}\left((x_n)_k\right) = \left[\underset{k}{avg}(a_{n,m})_k\right] \bullet (y_n) \tag{G.7}$$

This is consistent with (G.2, (G.3), except to note that this works for complex-valued matrices as well. Note further that

$$\underset{k}{avg}\left((a_{n,m})_k\right) = \underset{k}{avg}\left((a_{n,m})_k^T\right) \tag{G.8}$$

i.e., it is symmetric. With

$$(a_{n,m})_k = (b_{n,m})_k + j(c_{n,m})_k \tag{G.9}$$

then we have

$$\underset{k}{avg}\left((a_{n,m})_k\right) = \left[\underset{k}{avg}(b_{n,m})_k\right] + j\left[\underset{k}{avg}(c_{n,m})_k\right] \tag{G.10}$$

both parts being symmetric.

The variance is more problematical. Using the definition in (G.4) we have the usual form for vectors [10]

$$\text{var}\left((x_n)_k\right) = \underset{k}{avg}\left(\left[(x_n)_k - \underset{k}{avg}\left((x_n)_{k'}\right)\right]^* \bullet \left[(x_n)_k - \underset{k}{avg}\left((x_n)_{k'}\right)\right]\right) \tag{G.11}$$

Substituting from (G.6) we have

$$\begin{aligned}
\text{var}\left((x_n)_k\right) &= \underset{k}{avg}\left((y_n)^* \bullet \left[(a_{n,m})_k - \underset{k'}{avg}\left((a_{n,m})_{k'}\right)\right]^T\right. \\
&\quad \left.\bullet \left[(a_{n,m})_k - \underset{k'}{avg}\left((a_{n,m})_{k'}\right)\right] \bullet (y_n)\right) \\
&= (y_n)^* \bullet (D_{n,m}) \bullet (y_n)
\end{aligned}$$

$$\begin{aligned}
(D_{n,m}) &= \underset{k}{avg}\left(\left[(a_{n,m})_k - \underset{k'}{avg}\left((a_{n,m})_{k'}\right)\right]^\dagger \bullet \left[(a_{n,m})_k - \underset{k'}{avg}\left((a_{n,m})_{k'}\right)\right]\right) \\
&= (D_{n,m})^\dagger \qquad \text{(Self adjoint)} \\
&= (E_{n,m}) + j(F_{n,m}) \\
(E_{n,m}) &= (E_{n,m})^T \qquad \text{(symmetric)} \\
(F_{n,m}) &= -(F_{n,m})^T \qquad \text{(skew symmetric)}
\end{aligned} \tag{G.12}$$

So one choice of variance for an N x N matrix is just $(D_{n,m})$. Note that if $(y_n) \bullet (D_{n,m}) \bullet (y_n)^*$ is used (equally valid) then $(D_{n,m})^*$ appears instead.

Breaking $(a_{n,m})_k$ into real and imaginary parts we have

$$
\begin{aligned}
&\left[\left(a_{n,m}\right)_k - \underset{k'}{avg}\left(\left(a_{n,m}\right)_{k'}\right)\right]^{\dagger} \bullet \left[\left(a_{n,m}\right)_k - \underset{k'}{avg}\left(\left(a_{n,m}\right)_{k'}\right)\right] \\
&= \left[\left(b_{n,m}\right)_k - \underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right]^{T} \bullet \left[\left(b_{n,m}\right)_k - \underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \\
&+ \left[\left(c_{n,m}\right)_k - \underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right]^{T} \bullet \left[\left(c_{n,m}\right)_k - \underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right] \\
&+ j\left[\left(b_{n,m}\right)_k - \underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right]^{T} \bullet \left[\left(c_{n,m}\right)_k - \underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right] \\
&- j\left[\left(c_{n,m}\right)_k - \underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right]^{T} \bullet \left[\left(b_{n,m}\right)_k - \underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \\
&= \left(E_{n,m}\right)_k + \left(F_{n,m}\right)_k
\end{aligned}
\tag{G.13}
$$

G.2 Bicirculant matrices

Noting that the averages are symmetric and the matrices are bicirculant we have

$$
\begin{aligned}
\left(E_{n,m}\right)_k &= \left[\left(b_{n,m}\right)_k - \underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right]^2 + \left[\left(c_{n,m}\right)_k - \underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right]^2 \\
&= \left(b_{n,m}\right)_k \bullet \left(b_{n,m}\right)_k + \left[\underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right]^2 - \left[avg\left(\left(b_{n,m}\right)_{k'}\right)\right] \bullet \left(b_{n,m}\right)_k - \left(b_{n,m}\right)_k \bullet \left[\underset{k}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \\
&+ \left(c_{n,m}\right)_k \bullet \left(c_{n,m}\right)_k + \left[\underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right]^2 - \left[avg\left(\left(c_{n,m}\right)_{k'}\right)\right] \bullet \left(c_{n,m}\right)_k - \left(c_{n,m}\right)_k \bullet \left[\underset{k}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right]
\end{aligned}
\tag{G.14}
$$

where the $(b_{n,m})_k$ are assumed bicirculant (hence symmetric). Noting the property of the average in (G.4) we have

$$
\begin{aligned}
&\left(b_{n,m}\right)_k \bullet \left[\underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \\
&= \left[\frac{1}{N}\sum_n b_{n,m;k}\right] \bullet b_{n,m;k} + \left[\frac{1}{N^2 - N}\sum_{\substack{n,m \\ n\neq m}} b_{n,m;k}\right]\left[\left[\sum_m b_{n,m;k}\right]\left(u_{n,m}\right) - \left(b_{n,m}\right)_k\right]
\end{aligned}
\tag{G.15}
$$

where for this result we have used the property of bicirculant matrices that all row and column sums are the same. Then the last term in (G.15) is just the transpose and

$$
\begin{aligned}
&\left(b_{n,m}\right)_k \bullet \left[\underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] + \left[\underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \bullet \left(b_{n,m}\right)_k \\
&= 2\left[\frac{1}{N}\sum_{n} b_{n,m;k}\right]\left(b_{n,m}\right)_k \\
&+ 2\left[\frac{1}{N^2-N}\sum_{\substack{n,m \\ n\neq m}} b_{n,m;k}\right]\left[-\left[\sum_{n} b_{n,m;k}\right]\left(u_{n,m}\right) - \left(b_{n,m}\right)_k\right]
\end{aligned}
\tag{G.16}
$$

Similarly we have

$$
\begin{aligned}
&\left[\underset{k'}{avg}\left(\left(b_{n,m}\right)_k\right)\right]^2 \\
&= 2\left[\frac{1}{N}\sum_{n} b_{n,m;k}\right]^2 \left(1_{n,m}\right) \\
&+ 2\left[\frac{1}{N}\sum_{n}\left(b_{n,m}\right)\right]\left[\frac{1}{N^2-N}\sum_{\substack{n,m \\ n\neq m}}\left(b_{n,m}\right)\right]\left[\left(u_{n,m}\right) - \left(1_{n,m}\right)\right] \\
&+ \left[\frac{1}{N^2-N}\sum_{n,m}\left(b_{n,m}\right)\right]^2 \left[[N-2]\left(u_{n,m}\right) + \left(1_{n,m}\right)\right]
\end{aligned}
\tag{G.17}
$$

So the bicirculant property significantly simplifies the result. So (G.15) is reduced to three symmetric matrices for $(b_{n,m})_k$ and three symmetric matrices for $(c_{n,m})_k$, exactly as in (G.18) with *b* replaced by *c*.

Next consider

$$\begin{aligned}
\left(F_{n,m}\right)_k &= \left[\left(b_{n,m}\right)_k - \underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \bullet \left[\left(c_{n,m}\right)_k - \underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right] \\
&\quad - \left[\left(c_{n,m}\right)_k - \underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right] \bullet \left[\left(b_{n,m}\right)_k - \underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \\
&= \left(b_{n,m}\right)_k \bullet \left(c_{n,m}\right)_k - \left(c_{n,m}\right)_k \bullet \left(b_{n,m}\right)_k \\
&\quad - \left(b_{n,m}\right)_k \bullet \left[\underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right] + \left(c_{n,m}\right)_k \bullet \left[\underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \\
&\quad - \left[\underset{k'}{avg}\left(b_{n,m}\right)_{k'}\right] \bullet \left(c_{n,m}\right)_{k'} + \left[\underset{k'}{avg}\left(c_{n,m}\right)_{k'}\right] \bullet \left(b_{n,m}\right)_{k'} \\
&\quad + \left[\underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \bullet \left[\underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right] - \left[\underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right] \bullet \left[\underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right]
\end{aligned} \tag{G.18}$$

Noting the average in (G.4) the last terms give

$$\begin{aligned}
&\left[\underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \bullet \left[\underset{k'}{avg}\left[\left(c_{n,m}\right)_{k'}\right]\right] - \left[\underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right] \bullet \left[\underset{k'}{avg}\left(\left(b_{n,m}\right)_{k'}\right)\right] \\
&\quad = 0
\end{aligned} \tag{G.19}$$

Similarly due the commuting property of $(1_{n,m})$ and $(u_{n,m})$ with bibirculant matrices we have

$$\begin{aligned}
&- \left(b_{n,m}\right)_k \bullet \left[\underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right] + \left[\underset{k'}{avg}\left(b_{n,m}\right)_{k'}\right] \bullet \left(b_{n,m}\right)_k \\
&\qquad = 0 \\
&\left(c_{n,m}\right)_{k'} \bullet \left[\underset{k'}{avg}\left(\left(c_{n,m}\right)_{k'}\right)\right] - \left[\underset{k'}{avg}\left(c_{n,m}\right)_{k'}\right] \bullet \left(b_{n,m}\right)_k \\
&\qquad = 0
\end{aligned} \tag{G.20}$$

This leaves

$$\begin{aligned}
\left(F_{n,m}\right)_k &= \left(b_{n,m}\right)_k \bullet \left(c_{n,m}\right)_k - \left(c_{n,m}\right)_k \bullet \left(b_{n,m}\right)_k \\
&= \left[\left(b_{n,m}\right)_k,\ \left(c_{n,m}\right)_k\right] \quad \text{(commutator)}
\end{aligned} \tag{G.21}$$

which is zero if the two matrices commute.

Note now that circulant matrices have the same eigenvectors (but generally different eigenvalues); the commutator is then zero (they commute) giving

$$\left(F_{n,m}\right)_k = \left(0_{n,m}\right) \tag{G.22}$$

Therefore,

$$\begin{aligned}&\left[\left(a_{n,m}\right)_k - \underset{k'}{avg}\left(\left(a_{n,m}\right)_{k'}\right)\right]^{\dagger} \bullet \left[\left(a_{n,m}\right) - \underset{k'}{avg}\left(\left(a_{n,m}\right)_{k'}\right)\right] \\ &\qquad = \left(E_{n,m}\right)_k \equiv \text{real matrix}\end{aligned} \tag{G.23}$$

From (G.15) then we need to compute the variances of $(\mathrm{b}_{n,m})$ and $(\mathrm{c}_{n,m})$ separately and add. This allows the alternate formulae in (B.11) to be used in each case (more compact).

As special cases we can have

$$\begin{aligned}\left(b_{n,m}\right) &= \left(1_{n,m}\right) \\ \left(c_{n,m}\right) &= k\left[z_M - z_0\right]\left(X_{n,m}\right) \quad \text{(real)} \\ \gamma &= jk\left(\text{from } s = j\omega\right)\end{aligned} \tag{G.24}$$

Then we have

$$\begin{aligned}\operatorname{var}\left(\left(P_{n,m}\right)\bullet\left(a_{n,m}\right)\bullet\left(P_{n,m}\right)^T\right) &= avg\left(\left[\left(c_{n,m}\right) - avg\left(c_{n,m}\right)\right]^2\right) \\ &= \operatorname{var}\left(\left(c_{n,m}\right)\right) \\ &= \left[k\left[z_M - z_0\right]\right]^2 \operatorname{var}\left(\left(X_{n,m}\right)\right)\end{aligned} \tag{G.25}$$

The reader can note that these formulae reduce to those in [2] for real $(a_{n,m})$.

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