Interaction Notes

Note 611

30 September 2009

High-Frequency Multiconductor Transmission-Line Theory

Jürgen Nitsch and Sergey Tkachenko

Otto-von-Guericke University Magdeburg
Department of Electrical Engineering and Information Technology

Abstract

This work presents a thorough derivation of the full-wave transmission-line equations on the basis of Maxwell's theory. The multiconductor system is assumed to be composed of nonuniform thin wires. It is shown that the mixed potential integral equations are equivalent to generalized telegrapher equations. Novel, exact, and compact expressions for the multiconductor transmission-line parameters are derived, and it is shown how they are connected to radiation effects. Iteration and perturbation procedures are proposed for the solution of the generalized transmission-line equations.

This work was sponsored by the German Research Foundation (DFG), Contract Number: NI633/5-1.

1. Introduction

In electrical engineering conductors play an important role. They constitute a main part in every electrical and electronic system. Along them wanted signals are transmitted between devices and electronic components. In many cases they are also integrators for interfering (unwanted) signals. Therefore, from the EMC (Electromagnetic Compatibility) point of view it is essential to understand electromagnetic coupling processes to multiconductor transmission lines. This analysis is needed for the purpose of assessing and protecting an electrical system against inadvertent and intentional coupling.
At a first glance one may argue that the tools to handle this problem already exist: the classical transmission-line theory (cTLT) [1, and References given there]. This theory is a special case of Maxwell's theory and is known for more than hundred thirty years [2]. The speciality of this theory is its transverse electromagnetic mode (TEM mode) which exclusively propagates along transmission lines. Although, in reality conductors with an exclusive TEM mode do not exist, for many applications, at least when the wavelength is much larger than the cross-sectional size of the lines, the TEM mode is dominant. Other modes are also present, but they decay exponentially. Then cTLT gives excellent results that are in good agreement with experiments as well as with analytical calculations which take into account all modes. However, over the last few decades the number of electric and electronic components in industrial products, devices and installations has increased continuously. Since more and more of these components become concentrated in technical objects for information exchange, EMC becomes increasingly important as a product quality. Rising working frequencies, increasing communication – connected by wire or wireless – and the progressional miniaturization of electronic structures contribute to the promotion of this trend. Due to the increasing number of the build-in components the EMC analysis of the entire system becomes very complicated. On the one hand there is an increasing mutual interference of subsystems and on the other hand one needs increasing interconnection structures for the information and energy transfer. In a great number of cases these interconnection structures can mathematically be treated as nonuniform transmission lines. Due to this non-uniformity and to the very high working frequencies (up to several GHz) radiation phenomena (non-local interactions) occur and cTLT definitely becomes invalid and has to be replaced by another, more exact model, or a powerful software package must be used. Numerical simulations of complex systems are very time consuming and need large storage capacities. This disadvantage can partially be removed using a hybrid method that combines numerical field methods with an advanced transmission-line theory [3, 4]. During the last decade the authors have developed a full-wave transmission line theory which results from Maxwell's theory for thin conductors and thus is not restricted to cTLT conditions and most notably includes radiation effects.
There are several approaches to broaden the scope of cTLT ([5] and references therein), however, most of them only work in a particular low-frequency range. Many works [6-14] are dedicated to the correction of the telegrapher equations when transmission lines with non-uniformities or discontinuities are under investigation. The models encompass abrupt bends, round bends, vertical risers or the cross coupling between two skewed lines. For low frequencies (quasi-static case) the additional effects are modeled with the aid of lumped elements (L, C) or additional transmission lines that are connected to the actual line. Sometimes nonuniform lines are treated as if they were built of uniform segments, that is within each section the line is regarded to be uniform [15].

There have been attempts to solve nonuniform transmission-line problems with higher mode effects using the cTLT. This is done by modeling a physical effect, for example, radiation, which was previously excluded from the theory, by some additional term in the telegrapher equation [16].

Over the past few years considerable effort was put into the development of an extended transmission-line theory, which is not restricted to the TEM mode nor to any other mode. Instead of extending the existing theory, the preferred way to go is to derive generalized equations directly from Maxwell's theory. Some of the approaches deal with special geometric characteristics of the conductors, like sharp bends, vertical termination wires or field excitations that generate non –TEM modes [13-24]. Although these works show great improvements compared with cTLT, they do not provide a general procedure to handle arbitrary nonuniform transmission lines; they concentrate on one specific problem.

In [25] and [26] two different approaches are shown for the development of a generalized transmission-line theory, with interesting ideas. Unfortunately only the theoretical derivation is considered and no practical examples are presented.

A different approach is shown in [27]. In this case a proof is given that the current distribution of a nonuniform transmission line is governed by a second-order differential equation. Then from known linearly independent solutions for the currents for different terminations it is possible to construct the coefficients of the differential equations which then become the per-unit-length parameters. The required solutions can be obtained numerically, for example with the method of moments (MOM).

The most recent contribution to this topic is Ref. [28], where the mixed potential integral equation (MPIE) is chosen to derive a new transmission-line theory, the so-called transmission-line supertheory. There, one obtains, besides new telegrapher equations also equations for the determination of the per-unit-length parameters and the source terms. That theory is directly based on Maxwell's theory.

The present paper consolidates and extends the theoretical basis of previous publications [29, 27, 30]. Moreover, for lumped sources (located at the terminations of the conductors) the presented theory coincides with that of Ref. [28]. Exact equations for the parameter matrix and for the currents and potentials are given and solution procedures of these equations are discussed.

The outline of this paper is as follows: In Section 2 the mixed potential integral equations (MPIE) are derived for a multiconductor system, starting from Maxwell's equations. The nonuniform transmission lines are treated as thin wires and their geometrical arrangement is described by unit speed curves. It turns out that one can find with the aid of bijective mappings a single natural parameter "l" to which all conductor curves may be referred to. Then the MPIE can be brought into a structure which closely resembles that of the telegrapher equations.

Section 3 deals with the low-frequency limit of the MPIE and represents solutions of them in terms of the product integral or as matrizant [31]. It is explicitly shown that the inductance and capacitance matrices in their representation with respect to the general parameter "l" no longer have a physical meaning: they are not measurable quantities. Only their reference to their individual natural parameter (arc length) transfers them to measurable quantities. At the end of this section the relation between voltage and potential differences for non-uniform transmission lines is discussed.

In logical consequence Section 4 discusses the full-wave transmission-line equations (FWTL) and their solutions. These equations as presented here are novel and have not been discussed before. Novel iterative and perturbative approaches show how to find solutions for the

parameter matrices and for the currents propagating along the wires. It is emphasized that these parameter matrices, potentials, and currents include radiation effects of the considered transmission-line system.
How to calculate these radiation effects is shown in Section 5. The formalism which leads to the final result for the radiation power of the generalized transmission-line system uses the general MPIE themselves and very much resembles the methods used in quantum theory. A simple example gives a hint how to apply this general result.
Eventually, in the conclusion and outline the quintessence of the paper is summarized and prospects for the future research work are given.

2. Derivation of the Mixed-Potential Integral Equations (MPIE) for a Multiconductor System

Consider *N* nonuniform transmission lines above perfectly conducting ground which are excited by one or more lumped sources or by an exterior electromagnetic field (see Fig. 1)

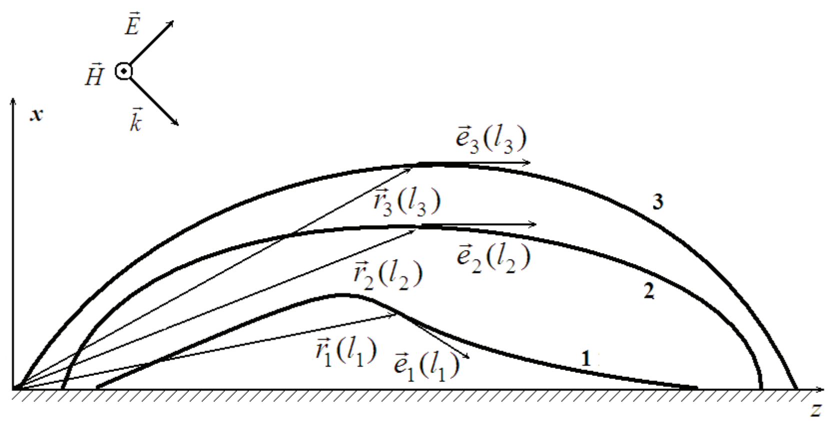

Fig. 1 Sketch of a nonuniform multi-conductor wiring system

. All *N* (*i=1,2,…N*) conductors are represented as unit speed curves (with natural parameters l_i) with the following geometrical functions:

$\vec{r}_i(l_i) := (x_i(l_i), y_i(l_i), z_i(l_i))$ - Cartesian coordinates along the i-th conductor ($0 \le l_i \le L_i$; L_i - length of conductor i)

$\vec{e}_i(l_i) := \dfrac{d\vec{r}_i}{dl_i}$: unit vector, tangential to the i-th conductor

$\tilde{\vec{r}}_i(l_i)$: mirrored vector of $\vec{r}_i(l_i)$ (1)

$\vec{e}_i(l_i) := \left((\vec{e}_i(l_i))_x, (\vec{e}_i(l_i))_y, -(\vec{e}_i(l_i))_z\right)$

$\vec{e}_{i,\perp}(l_i)$: unit vector orthogonal to $\vec{e}_i(l_i)$, e.g., $\vec{e}_{i,\perp}(l_i) = \vec{n}_i(l_i) = \dfrac{1}{\kappa_i(l_i)} \dfrac{\partial \vec{e}_i(l_i)}{\partial l_i}$

($\kappa_i(l_i)$:curvature of the of the i-th conductor)

Further it is assumed that the conductors (transmission lines) can be treated in the thin-wire approximation; i.e.:

- only axial currents are considered
- all currents are concentrated on the axes of the wires (2)
- all transmission lines (wires) are lossless (perfect conductors)

Then the scattered electrical potential can be written by means of the scalar Green's function of free space

$$g(r)=\frac{\exp(-jkr)}{r} \qquad (j \text{ is imaginary unit}) \tag{3}$$

as:

$$\varphi(\vec{r})=\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\int_0^{L_i} q_i'(l_i')\left(g(|\vec{r}-\vec{r}_i(l_i')|)-g(|\vec{r}-\tilde{\vec{r}}_i(l_i')|)\right)dl_i' \tag{4}$$

Observe that in (4) the mirrored charge per-unit-length $\tilde{q}_i'(l_i')=-q_i'(l_i')$ was taken into account. Using the continuity equation in differential form

$$q_i'(l_i)=-\frac{1}{j\omega}\frac{\partial I_i(l_i)}{\partial l_i} \qquad (I_i \text{ axial current along wire } i) \tag{5}$$

equation (4) can be rewritten in

$$\varphi(\vec{r})=-\frac{1}{j4\pi\varepsilon_0\omega}\sum_{i=1}^{N}\int_0^{L_i}\frac{\partial I_i(l_i')}{\partial l_i'}\left(g(|\vec{r}-\vec{r}_i(l_i')|)-g(|\vec{r}-\tilde{\vec{r}}_i(l_i')|)\right)dl_i' \tag{6}$$

Analogue to the scalar potential also the total magnetic vector potential $\vec{A}(\vec{r})$ is represented as

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\sum_{i=1}^{N}\int_0^{L_i} I_i(l_i')\left(\vec{e}_i(l_i')g(|\vec{r}-\vec{r}_i(l_i')|)-\tilde{\vec{e}}_i(l_i')g(|\vec{r}-\tilde{\vec{r}}_i(l_i')|)\right)dl_i' \tag{7}$$

Now with the aid of (6) and (7) one is prepared to calculate the total electrical field

$$\vec{E}^{total}(\vec{r})=\vec{E}^{scattered}(\vec{r})+\vec{E}^{exciting}(\vec{r})=-\nabla\varphi(\vec{r})-j\omega\vec{A}(\vec{r})+\vec{E}^{exc}(\vec{r}) \tag{8}$$

Of course, the gradient has to be adopted to curvilinear coordinates. Since perfectly conducting transmission lines are assumed the boundary condition for each conductor i reads:

$$\left(\vec{E}^{total}(\vec{r}=\vec{r}_i(l_i)+\vec{e}_{i,\perp}(l_i)a_i)\right)_{l_i}=0 \tag{9}$$

Here a_i denotes the (small) radius of transmission line i.

Assuming that the unit tangential vector along the surface of conductor i approximately equals that along the conductor axis then (9) can be simplified to:

$$\left(\vec{E}^{total}\right)_{l_i} = \vec{e}_i(l_i)\vec{E}^{total}(l_i) = 0 \tag{10}$$

The explicit expression for (10) can be obtained in curvilinear coordinates via (6)-(9) as

$$\frac{d\varphi_i(l_i)}{dl_i} + j\omega\frac{\mu_0}{4\pi}\sum_{j=1}^{N}\int_0^{L_j} I_j(l'_j)\left[\vec{e}_i(l_i)\cdot\vec{e}_j(l'_j)\,g_{ij}(l_i,l'_j;k)) - \vec{e}_i(l_i)\cdot\tilde{\vec{e}}_j(l'_j)\,\tilde{g}_{ij}(l_i,l'_j;k))\right]dl'_j = \\ = \vec{E}_i^{exc}(l_i)\cdot\vec{e}_i(l_i) \tag{11}$$

Here the abbreviations

$$g_{ij}(l_i,l'_j;k) := g\left(|\,\vec{r}_i(l_i) + \vec{e}_{i,\perp}(l_i)\,a_i - \vec{r}_j(l'_j)\,|\right)$$

and (12)

$$\tilde{g}_{ij}(l_i,l'_j;k) := g(|\,\vec{r}_i(l_i) + \vec{e}_{i,\perp}(l_i)\,a_i - \tilde{\vec{r}}_j(l'_j)\,|)$$

are used.
Similarly, on the boundary of conductor i equation (6) is rewritten in

$$\varphi_i(l_i) + \frac{1}{j4\pi\varepsilon_0\omega}\sum_{j=1}^{N}\int_0^{L_j}\frac{\partial I_j(l'_j)}{\partial l'_j}\left[g_{ij}(l_i,l'_j;k)) - \tilde{g}_{ij}(l_i,l'_j;k))\right]dl'_j = 0 \tag{13}$$

Equations (11) and (13) suggest to be written in a more compact matrix form. However, before this can be done, it is necessary to introduce bijective mappings

$$l_i = l_i(l) \quad \text{for } \forall\, l_i \quad (0 \le l \le L) \quad \text{with } l_i(0)=0 \text{ and } l_i(L)=L_i \tag{14}$$

A simple example for these mappings is: $l_i = \frac{L_i}{L}l$.

Then all quantities have to be represented with respect to this new parameter l like, e.g.,

$$\frac{d\varphi_i(l_i)}{dl_i} = \frac{d\varphi_i(l)}{dl}\left(\frac{dl_i(l)}{dl}\right)^{-1} \tag{15}$$

With this new single length-parameter one finally obtains

$$\frac{d}{dl}\underset{\downarrow}{\varphi} + j\omega\frac{\mu_0}{4\pi}\int_0^{L}\hat{g}_L(l,l';k)\cdot\underset{\downarrow}{I}(l')\,dl' = \underset{\downarrow}{v}_{exc}(l) \tag{16}$$

and

$$\int_0^{L}\hat{g}_C(l,l',k;)\cdot\frac{d}{dl'}\underset{\downarrow}{I}(l')\,dl' + j\omega 4\pi\varepsilon_0\,\underset{\downarrow}{\varphi}(l) = 0 \tag{17}$$

Here the column vectors $\underset{\downarrow}{\varphi}(l)$, $\underset{\downarrow}{I}(l)$, and $\underset{\downarrow}{v}_{exc}(l)$, and the matrices $\widehat{g}_L(l,l';k)$ and $\widehat{g}_C(l,l';k)$ are defined as follows:

$$\underset{\downarrow}{\varphi}(l) := \left[\varphi_1(l),\varphi_2(l),\dots,\varphi_N(l)\right]^T, \qquad \underset{\downarrow}{I}(l) := \left[I_1(l),I_2(l),\dots,I_N(l)\right]^T, \tag{18}$$

and

$$\underset{\downarrow}{v}_{exc}(l) := \left[\vec{E}^{exc.}(\vec{r}_1(l))\cdot\frac{d\vec{r}_1(l)}{dl}, \vec{E}^{exc.}(\vec{r}_2(l))\cdot\frac{d\vec{r}_2(l)}{dl},\dots,\vec{E}^{exc.}(\vec{r}_N\ (l))\cdot\frac{d\vec{r}_N(l)}{dl}\right]^T$$

$$\begin{aligned}\left[\widehat{g}_L(l,l',k)\right]_{i,j} := {} & \frac{d\vec{r}_i(l)}{dl}\cdot\frac{d\vec{r}_j(l')}{dl'}\, g(|\,\vec{r}_i(l)+\vec{e}_{i,\perp}(l)\,a_i-\vec{r}_j(l')\,|) - \\ & -\frac{d\vec{r}_i(l)}{dl}\cdot\frac{d\tilde{\vec{r}}_j(l')}{dl'}\, g(|\,\vec{r}_i(l)+\vec{e}_{i,\perp}(l)\,a_i-\tilde{\vec{r}}_j(l')\,|)\end{aligned} \tag{19}$$

and

$$\left[\widehat{g}_C(l,l';k)\right]_{i,j} := g(|\,\vec{r}_i(l)+\vec{e}_{i,\perp}(l)\,a_i-\vec{r}_j(l')\,|) - g(|\,\vec{r}_i(l)+\vec{e}_{i,\perp}(l)\,a_i-\tilde{\vec{r}}_j(l')\,|) \tag{20}$$

The coupled integro-differential equations (16) and (17) are the so called "mixed potential integro-differential equations (MPIE)" for a multiconductor transmission-line system.
If one goes through the derivation of the above equations, one observes that essentially Maxwell's equations in the Lorenz gauge applied to a multiconductor-line system are represented as MPIE system. Since, however, all quantities depend on the same geometrical parameter l, they do not have a direct physical meaning. Also the electrical potential $\underset{\downarrow}{\varphi}(l)$ only becomes a measurable quantity (voltage) if all closed contour integrals of the total electrical field vanish. In order to establish a relation to physical quantities the above equations are investigated in their low-frequency limit.

3. Low-Frequency Limit of the MPIE

In the low-frequency limit the wave number k approaches to zero ($k \to 0$) and the integrals in equations (16) and (17) can be simplified to

$$\left(\int_0^L \widehat{g}_L(l,l';0)\,dl'\right)\cdot \underset{\downarrow}{I}(l) =: \widehat{G}_L'^{0}(l)\cdot\underset{\downarrow}{I}(l) \tag{21}$$

$$\left(\int_0^L \widehat{g}_C(l,l';0)\,dl'\right)\cdot \frac{d}{dl}\underset{\downarrow}{I}(l) =: \widehat{G}_C'^{0}(l)\cdot\frac{d}{dl}\underset{\downarrow}{I}(l) \tag{22}$$

Here use was made of the strong weighting property of the Green's function (for a thin wire the phase-independent part of the Green's function has a sharp maximum when $|\,l-l'\,| \sim a = |\,dl_i(l)/dl\,|^{-1}\min\{a_1,a_2,\dots,a_N\}$). Therefore, the current vector $\underset{\downarrow}{I}(l)$ and its derivative can be pulled out of the integral, respectively. Then one easily gets the well-known result:

$$\frac{d}{dl}\underset{\downarrow}{\varphi}(l) + j\omega\widehat{L}'(l)\,\underset{\downarrow}{I}(l) = \underset{\downarrow}{v}_{exc.}(l) \tag{23}$$

and

$$\frac{d}{dl}\underset{\downarrow}{I}(l) + j\omega\widehat{C}'(l)\,\underset{\downarrow}{\varphi}(l) = 0 \tag{24}$$

with the per-unit-length inductance matrix $\widehat{L}'(l)$ and capacitance matrix $\widehat{C}'(l)$:

$$\widehat{L}'(l) = \frac{\mu_0}{4\pi}\widehat{G}_L'^0(l) \tag{25}$$

and

$$\widehat{C}'(l) = 4\pi\varepsilon_0\left[\widehat{G}_C'^0(l)\right]^{-1} \tag{26}$$

If one combines equations (23) and (24) to one equation one can write

$$\frac{d}{dl}\underset{\downarrow}{x}(l) + j\omega\widehat{P}_0(l)\,\underset{\downarrow}{x}(l) = \begin{pmatrix} \underset{\downarrow}{v}_{exc.} \\ \underset{\downarrow}{0} \end{pmatrix} \tag{27}$$

with the supervector

$$\underset{\downarrow}{x}(l) := \begin{pmatrix} \underset{\downarrow}{\varphi}(l) \\ \underset{\downarrow}{I}(l) \end{pmatrix} \tag{28}$$

and the supermatrix

$$\widehat{P}_0(l) := \begin{pmatrix} 0 & \widehat{L}'(l) \\ \widehat{C}'(l) & 0 \end{pmatrix} \tag{29}$$

Thus the MPIE equations have been casted into the form of the usual classical multiconductor transmission line equations (telegrapher equations). Note, however, that in general the elements of the inductance matrix and capacitance matrix are not measurable quantities. They depend on the local parameter l and therefore on the choice of the mappings (14). Different mappings lead to different matrix elements. Consider two bijective mappings of the unit speed parameters (natural parameters) which depend on the common parameter l and $\widetilde{l}$, respectively

$$l_i = l_i(l) \qquad \text{and} \qquad l_i = l_i(\widetilde{l}\,), \qquad i \in \{1,\ldots,N\} \tag{30}$$

then the corresponding relations of the inductance and capacitance matrices are

$$\widehat{L}'(\widetilde{l}) = \left[diag\left(\frac{dl_i}{d\widetilde{l}} \right) \right] \cdot \left[diag\left(\frac{dl_i}{dl} \right) \right]^{-1} \cdot \widehat{L}'(l) \tag{31}$$

and

$$\widehat{C}'(\widetilde{l}) = \left[diag\left(\frac{dl_i}{d\widetilde{l}} \right) \right] \cdot \left[diag\left(\frac{dl_i}{dl} \right) \right]^{-1} \cdot \widehat{C}'(l) \tag{32}$$

The dependency of all matrix elements on only one local parameter l or $\widetilde{l}$ (instead on many l_i) facilitates the solution procedure of the MPIE (27) and also resembles the structure of the classical single-conductor telegrapher equation. Once one has found a solution for $\underset{\downarrow}{x}(l)$ in (27) it has to be transformed with the aid of the inverse mappings of (14) into it dependencies of the N natural coordinates l_i, i.e.

$$\underset{\downarrow}{x}(l) := \begin{pmatrix} \underset{\downarrow}{\varphi}(l) \\ \underset{\downarrow}{I}(l) \end{pmatrix} = \begin{pmatrix} [\varphi_i(l(l_i))]^T \\ [I_i(l(l_i))]^T \end{pmatrix}, \qquad i \in \{1,\ldots,N\} \tag{33}$$

Thus, it remains to find a solution of (27). Formally this is rather simple. In [31] the solution is given as

$$\underset{\downarrow}{x}(l) = \widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}_0\} \cdot \underset{\downarrow}{x}_0(l) + \int_{l_0}^{l} \widehat{M}_{l'}^{l}\{-j\omega\widehat{P}_0\} \cdot \underset{\downarrow}{v}_{exc.}(l')dl' \tag{34}$$

The $2N \times 2N$ matrix $\widehat{M}_{l_0}^{l}$ is called matrizant, product integral or propagator [31]. In order to calculate the product integral

$$\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}_0\} = \lim_{\Delta l_\nu \to 0} \prod_{\nu} \exp\left(-j\omega\widehat{P}_0(\nu)\Delta l_\nu\right) \tag{35}$$

one has to know the parameter matrix $\widehat{P}_0(l)$. For some multiconductor configurations one may be able to calculate the integrals in (21) and (22) analytically. However, in most cases a numerical calculation is necessary. A third possibility is an approximate approach by an iteration procedure. Assume for a moment that there are no sources present. Then the differential equation for $\underset{\downarrow}{x}(l)$ reads:

$$\frac{d}{dl}\underset{\downarrow}{x}(l) + j\omega\widehat{P}_0(l)\underset{\downarrow}{x}(l) = 0 \tag{36}$$

If one inserts the solution

$$\underset{\downarrow}{x}(l) = \widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}_0\} \cdot \underset{\downarrow}{x}_0(l)$$

and its derivative $$\frac{d}{dl}\underset{\downarrow}{x}(l) = \frac{d}{dl}\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}_0\}\cdot\underset{\downarrow}{x}_0(l)$$

into (36) (for appropriate initial conditions $\underset{\downarrow}{x}_0(l)$) one obtains the parameter matrix as

$$\widehat{P}_0(l) = -\frac{1}{j\omega}\left(\frac{d}{dl}\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}_0\}\right)\cdot\left(\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}_0\}\right)^{-1} \tag{37}$$

If one knows $\widehat{P}_0(l)$ then equation (37) is an identity. Otherwise (37) is an implicit equation for $\widehat{P}_0(l)$. One could start a solution procedure for $\widehat{P}_0$ with some approximate values in the right-hand side of equation (37). As an example (this example includes classical transmission-line theory) one may choose [1]

$$\widehat{P}_0(l) = const$$

Then the matrizant becomes

$$\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}_0^{const}\} = \exp\{-j\omega(l-l_0)\widehat{P}_0^{const}\} \tag{38}$$

With this expression one gets through a first iteration using (37) a "first order" $\widehat{P}_0^{(1)}(l)$, which depends on the local parameter l. In this way one can continue the iteration procedure until convergence is reached. In praxis this is achieved after a few (one or two) iteration steps [5, 30, 32].

Since the matrizant for the homogeneous equation (36) also occurs in the solution (34) of the non-homogeneous equation (27) (see,.e.g. [31]), the above described way to derive a solution for the parameter matrix also remains valid in the presence of a source.

It should be noted that so far low frequencies were assumed ($k \to 0$). Therefore no radiation effects could occur. Also the term "potential" (instead of "voltage") was used in order to indicate that for arbitrary cable layings the voltage between two points on different wires does not correspond to the potential difference of these points. Voltage υ between two points is rather given by

$$\upsilon = \int_1^2 \vec{E}\cdot d\vec{s} \quad \left(\neq \varphi(2) - \varphi(1)\right)$$

However, there are exceptions. In classical transmission-line theory only TEM-modes for the current waves do propagate. In this case, or in cases when TEM-modes along certain parts of conductors dominate (quasi-TEM modes) voltage is given by a potential difference (for perpendicular line cross-sections).

In the following section the restriction to low frequencies is given up and solutions for the full-wave transmission-line equations are investigated.

4. Full-Wave Transmission Line Equations (FWTL) and Their Solutions

The objective of this section is to find the general solution of the MPIE (16) and (17), without any restrictions to frequencies. If one looks at the solution procedure of the foregoing section one recognizes that a basic facilitation to find the solution was the fact that it was allowed to put the current and its derivative out off the integrals. As a consequence one obtained equation (27) and could present the solution in matrizant form. Therefore, it is assumed that the dynamic equations (16) and (17) can also be presented as a first order differential equation with a general parameter matrix $\widehat{P}(l)$.

Then one obtains

$$\frac{d}{dl}\begin{pmatrix} \underset{\downarrow}{\varphi}(l) \\ \underset{\downarrow}{I}(l) \end{pmatrix} + j\omega \begin{pmatrix} \widehat{P}_{11}(l) & \widehat{P}_{12}(l) \\ \widehat{P}_{21}(l) & \widehat{P}_{22}(l) \end{pmatrix} \cdot \begin{pmatrix} \underset{\downarrow}{\varphi}(l) \\ \underset{\downarrow}{I}(l) \end{pmatrix} = \begin{pmatrix} \underset{\downarrow}{v}_{exc.}(l) \\ 0 \end{pmatrix} \tag{39}$$

This is, of course, an ansatz or a working hypothesis, and it has to be shown that this ansatz is consistent with the MPIE and leads to the general solution of the dynamic equations (16) and (17).

The above ansatz has the advantage that the solution is known as [31]

$$\begin{pmatrix} \underset{\downarrow}{\varphi}(l) \\ \underset{\downarrow}{I}(l) \end{pmatrix} = \begin{pmatrix} \left(\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}(l)\}\right)_{11} & \left(\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}(l)\}\right)_{12} \\ \left(\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}(l)\}\right)_{21} & \left(\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}(l)\}\right)_{22} \end{pmatrix} \cdot \begin{pmatrix} \underset{\downarrow}{\varphi}(l_0) \\ \underset{\downarrow}{I}(l_0) \end{pmatrix} +$$

$$\int_{l_0}^{l} \begin{pmatrix} \left(\widehat{M}_{l'}^{l}\{-j\omega\widehat{P}(l)\}\right)_{11} & \left(\widehat{M}_{l'}^{l}\{-j\omega\widehat{P}(l)\}\right)_{12} \\ \left(\widehat{M}_{l'}^{l}\{-j\omega\widehat{P}(l)\}\right)_{21} & \left(\widehat{M}_{l'}^{l}\{-j\omega\widehat{P}(l)\}\right)_{22} \end{pmatrix} \cdot \begin{pmatrix} \underset{\downarrow}{v}_{exc.}(l') \\ \underset{\downarrow}{0} \end{pmatrix} dl' \tag{40}$$

If one excludes exterior distributed sources and just permits lumped sources (loads) at the terminals of the multiconductor system, then the matrizant itself fulfills the equation

$$\frac{d}{dl}\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}\} + j\omega\widehat{P}(l)\cdot\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}\} = 0 \tag{41}$$

Remember that $\widehat{M}_{l_0}^{l}$ and $\widehat{P}$ are $2N \times 2N$ matrices and (41) can be resolved with respect to $\widehat{P}(l)$:

$$\widehat{P}(l) = -\frac{1}{j\omega}\frac{d}{dl}\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}\}\cdot\left(\widehat{M}_{l_0}^{l}\{-j\omega\widehat{P}\}\right)^{-1} \tag{42}$$

Thus one can conclude: if the mixed potential integral equations can be casted into the form of (42) then they are a solution of (39) without distributed sources ($\underset{\downarrow}{v}_{exc.}(l) = 0$). How this is achieved is shown in the next step.

At the end of this section arguments are given for the generality of solution (40) for the MPIE, and the inclusion of distributed sources ($\underset{\downarrow}{v}_{exc.}(l) \neq 0$) is discussed.

From the theory of linear homogeneous differential equations of first order one knows (e.g., [31]) that $\widehat{M}^{l}_{l_0}\{-j\omega\widehat{P}(l)\}$ in (41) is so-called normalized integral matrix of (39) (with $\underset{\downarrow}{v}_{exc.}(l) = 0$)), i.e., this integral matrix consists of $2N$ linearly independent column vectors, each of which is a solution of the homogeneous equation (39). Thus the normalized integral matrix (normalized matrizant) can be written as a $2N \times 2N$ matrix:

$$\widehat{M}^{l}_{l_0}\{-j\omega\widehat{P}(l)\} = \begin{pmatrix} \widehat{\varphi}_1(l) & \widehat{\varphi}_2(l) \\ \widehat{I}_1(l) & \widehat{I}_2(l) \end{pmatrix} \tag{43}$$

Here the quantities $\widehat{\varphi}_1(l)$, $\widehat{\varphi}_2(l)$, and $\widehat{I}_1(l)$, $\widehat{I}_2(l)$ are $N \times N$ matrices which fulfill the initial value condition:

$$\begin{aligned} \widehat{\varphi}_1(l_0) &= E, & \widehat{I}_1(l_0) &= 0 \\ \widehat{\varphi}_2(l_0) &= 0, & \widehat{I}_2(l_0) &= E \end{aligned} \tag{44}$$

and

$$\left(\widehat{M}^{l}_{l_0}\{-j\omega\widehat{P}\}\right)_{11} = \widehat{\varphi}_1(l), \qquad \left(\widehat{M}^{l}_{l_0}\{-j\omega\widehat{P}\}\right)_{12} = \widehat{\varphi}_2(l)$$

$$\left(\widehat{M}^{l}_{l_0}\{-j\omega\widehat{P}\}\right)_{21} = \widehat{I}_1(l), \qquad \left(\widehat{M}^{l}_{l_0}\{-j\omega\widehat{P}\}\right)_{12} = \widehat{I}_2(l)$$

All columns of $\widehat{\varphi}_1, \ldots \widehat{I}_2$ are linearly independent solutions of the MPIE equations (16) and (17). Therefore these equations are rewritten as ($l_0 = 0$):

$$\frac{d}{dl}\widehat{\varphi}_1(l) \quad + j\omega\frac{\mu_0}{4\pi}\int_0^L \widehat{g}_L(l,l',k)\cdot\widehat{I}_1(l')\,dl' = 0 \tag{46}$$

$$\int_0^L \widehat{g}_C(l,l',k)\cdot\frac{d}{dl'}\widehat{I}_1(l')\,dl' + j\omega 4\pi\varepsilon_0\,\widehat{\varphi}_1(l) = 0 \tag{47}$$

And, equivalently, for $\widehat{\varphi}_2$ and $\widehat{I}_2$

$$\frac{d}{dl}\widehat{\varphi}_2(l) \quad + j\omega\frac{\mu_0}{4\pi}\int_0^L \widehat{g}_L(l,l',k)\cdot\widehat{I}_2(l')\,dl' = 0 \tag{48}$$

$$\int_0^L \widehat{g}_C(l,l',k)\cdot\frac{d}{dl'}\widehat{I}_2(l')\,dl' + j\omega 4\pi\varepsilon_0\,\widehat{\varphi}_2(l) = 0 \tag{49}$$

Solving equations (47) and (49) for $\hat{\varphi}_1$ and $\hat{\varphi}_2$, respectively and inserting the results into equations (43) yields:

$$\hat{M}_0^l\{-j\omega\hat{P}\} = \begin{pmatrix} -\dfrac{1}{j\omega 4\pi\varepsilon_0}\displaystyle\int_0^L \hat{g}_C(l,l',k)\cdot\frac{d}{dl'}\hat{I}_1(l')\,dl' & -\dfrac{1}{j\omega 4\pi\varepsilon_0}\displaystyle\int_0^L \hat{g}_C(l,l',k)\cdot\frac{d}{dl'}\hat{I}_2(l')\,dl' \\ \hat{I}_1(l) & \hat{I}_2(l) \end{pmatrix} \tag{50}$$

Replacing the matrices $\hat{I}_1(l)$ and $\hat{I}_2(l)$ by their corresponding matrizant components one gets an implicit equation for the matrizant:

$$\hat{M}_0^l\{-j\omega\hat{P}\} = \begin{pmatrix} -\dfrac{1}{j\omega 4\pi\varepsilon_0}\displaystyle\int_0^L \hat{g}_C(l,l',k)\cdot\frac{d}{dl'}\left(\hat{M}_0^{l'}\{-j\omega\hat{P}\}\right)_{21} dl' & -\dfrac{1}{j\omega 4\pi\varepsilon_0}\displaystyle\int_0^L \hat{g}_C(l,l',k)\cdot\frac{d}{dl'}\left(\hat{M}_0^{l'}\{-j\omega\hat{P}\}\right)_{22} dl' \\ \left(\hat{M}_0^{l'}\{-j\omega\hat{P}\}\right)_{21} & \dfrac{d}{dl'}\left(\hat{M}_0^{l'}\{-j\omega\hat{P}\}\right)_{22} \end{pmatrix} \tag{51}$$

The inverse of this $2N\times 2N$ matrix is one factor of equation (42). Turning back to the essential equation (42) one still needs an equivalent expression for the derivative of the matrizant:

$$\frac{d}{dl}\hat{M}_0^l\{-j\omega\hat{P}\} = \begin{pmatrix} \dfrac{d\hat{\varphi}_1(l)}{dl} & \dfrac{d\hat{\varphi}_2(l)}{dl} \\ \dfrac{d\hat{I}_1(l)}{dl} & \dfrac{d\hat{I}_2(l)}{dl} \end{pmatrix} \tag{52}$$

Again, one uses two ((46) and (48)) of the MPIE in order to replace the derivatives $d\hat{\varphi}_1/dl$ and $d\hat{\varphi}_2/dl$. The result is:

$$\frac{d}{dl}\hat{M}_0^l\{-j\omega\hat{P}\} = \begin{pmatrix} -j\omega\dfrac{\mu_0}{4\pi}\displaystyle\int_0^L \hat{g}_L(l,l',k)\cdot\hat{I}_1(l')dl' & -j\omega\dfrac{\mu_0}{4\pi}\displaystyle\int_0^L \hat{g}_L(l,l',k)\cdot\hat{I}_2(l')dl' \\ \dfrac{d\hat{I}_1(l)}{dl} & \dfrac{d\hat{I}_2(l)}{dl} \end{pmatrix} \tag{53}$$

Inserting for the matrices $\hat{I}_1(l)$ and $\hat{I}_2(l)$ their corresponding matrizant components $(\hat{M}_0^l)_{21}$ and $(\hat{M}_0^l)_{22}$, respectively, yields

$$\frac{d}{dl}\hat{M}_0^l\{-j\omega\hat{P}\} =$$

$$\begin{pmatrix} -j\omega\frac{\mu_0}{4\pi}\int_0^L \widehat{g}_L(l,l',k)\cdot\left(\widehat{M}_0^{l'}\{-j\omega\widehat{P}\}\right)_{21}dl' & -j\omega\frac{\mu_0}{4\pi}\int_0^L \widehat{g}_L(l,l',k)\cdot\left(\widehat{M}_0^{l'}\{-j\omega\widehat{P}\}\right)_{22}dl' \\ \frac{d}{dl}\left(\widehat{M}_0^{l}\{-j\omega\widehat{P}\}\right)_{21} & \frac{d}{dl}\left(\widehat{M}_0^{l}\{-j\omega\widehat{P}\}\right)_{22} \end{pmatrix} \tag{54}$$

Finally, the matrix product of the matrizant and its derivative (see equation (42)) can be calculated to give an implicit equation for the parameter matrix $\widehat{P}(l)$:

$$\left[\widehat{P}(l)\right]_{2N,2N}=$$

$$-\frac{1}{j\omega}\begin{pmatrix} -j\omega\frac{\mu_0}{4\pi}\int_0^L \widehat{g}_L(l,l',k)\cdot\left(\widehat{M}_0^{l'}\{-j\omega\widehat{P}\}\right)_{21}dl' & -j\omega\frac{\mu_0}{4\pi}\int_0^L \widehat{g}_L(l,l',k)\cdot\left(\widehat{M}_0^{l'}\{-j\omega\widehat{P}\}\right)_{22}dl' \\ \frac{d}{dl}\left(\widehat{M}_0^{l}\{-j\omega\widehat{P}\}\right)_{21} & \frac{d}{dl}\left(\widehat{M}_0^{l}\{-j\omega\widehat{P}\}\right)_{22} \end{pmatrix}\cdot$$

$$\begin{pmatrix} -\frac{1}{j\omega 4\pi\varepsilon_0}\int_0^L \widehat{g}_C(l,l',k)\cdot\frac{d}{dl'}\left(\widehat{M}_0^{l'}\{-j\omega\widehat{P}\}\right)_{21}dl' & -\frac{1}{j\omega 4\pi\varepsilon_0}\int_0^L \widehat{g}_C(l,l',k)\cdot\frac{d}{dl'}\left(\widehat{M}_0^{l'}\{-j\omega\widehat{P}\}\right)_{22}dl' \\ \left(\widehat{M}_0^{l'}\{-j\omega\widehat{P}\}\right)_{21} & \frac{d}{dl'}\left(\widehat{M}_0^{l'}\{-j\omega\widehat{P}\}\right)_{22} \end{pmatrix}^{-1} \tag{55}$$

Now, one can make the following important conclusion: one can represent MPIE as a first order differential equation without sources if the necessary and sufficient condition (55) is fulfilled. Thus, it is mainly left to solve this equation which, of course, includes the physical dynamics. There are different ways to find a solution of (55). Firstly by iteration. Write the exact equation (55) as some functional:

$$\widehat{P}(l) = F\left\{\widehat{P}(l)\right\} \tag{56}$$

and iterate as

$$\widehat{P}^{(n+1)}(l) = F\left\{\widehat{P}^{(n)}(l)\right\} \quad \text{with some known } \widehat{P}^{(0)}(l)\text{, e.g., with}$$

$$\widehat{P}^{(0)}(l) = \widehat{P}_0(l) = \begin{pmatrix} 0 & \widehat{L}'(l) \\ \widehat{C}'(l) & 0 \end{pmatrix} \tag{57}$$

of section two. In many cases only $\widehat{P}^{(0)}(l)$ can be evaluated analytically. Following iteration steps must then be calculated numerically. This procedure was applied in Ref. [5]. Secondly, one may apply a simple perturbation theory. Then it is preferable to work in natural coordinates l_i instead in the system of only one coordinate l. The first perturbation solution is tried to find in the low-frequency limit $k \to 0$:

$$\frac{d\varphi_i(l_i)}{dl_i} + j\omega \sum_{j=1}^{N} \left[L'(l_i)\right]_{ij} I_j(l_j) = 0 \tag{58}$$

$$\sum_{j=1}^{N} \left[G_C^{'0}(l_i)\right]_{ij} \frac{dI_j(l_j)}{dl_j} + j\omega 4\pi\varepsilon_0 \varphi_i(l_i) = 0 \tag{59}$$

Remember, that the integrals for $\left[L'(l_i)\right]_{ij}$ and for $\left[G_C^{'0}(l_i)\right]_{ij}$ contain the differences between

$$g_{ij}(l_i,l_j;0) - \tilde{g}_{ij}(l_i,l_j;0) = \left(\sqrt{(\vec{r}(l_i) - \vec{r}(l_i))^2 + a_i^2}\right)^{-1} - \left(\sqrt{(\vec{r}(l_i) - \tilde{\vec{r}}(l_i))^2 + a_i^2}\right)^{-1} \tag{60}$$

and therefore diverge for $l_i = l_j$ if also $a_i = 0$. However, for small, but finite a_i one can approach these integrals as follows:

$$\left[G_C^{'0}(l_i)\right]_{ij} \cong \left[G_L^{'0}(l_i)\right]_{ij} \cong 2\ln\left(\frac{\tilde{R}_i}{a_i}\right) \cdot \delta_{ij} \tag{61}$$

In this equation the quantity $\tilde{R}_i$ has the dimension of a length and may be expressed-depending of the structure of the multiconductor system-by:

$$\tilde{R}_i \cong \begin{cases} R_i \text{ - radius of curvature of conductor } i \\ 2h_i - \text{height of conductor i from ground (reference plane)} \\ d_{ij}\,(>> a_i) - \text{distance between conductors } i \text{ and } j \end{cases} \tag{62}$$

Observe that in almost all multiconductor configurations all three quantities of the right hand side of (62) occur in the equations. Then $\tilde{R}_i$ is chosen as the minimum of them.

Since $\tilde{R}_i >> a_i$ one gets $2\ln(\tilde{R}_i / a_i) >> 1$. The non-diagonal elements in the approximation (61) are of the order of magnitude 1. The classical multiconductor transmission-line theory is an example for (61) (see Ref. [1]):

$$\left[G_{C,L}^{'0}(l_i)\right]_{ij} = 2\begin{cases} \ln(2h_i / a_i), & i = j \\ \ln\left(d_{ij} / \tilde{d}_{ij}\right) \sim 1, & i \neq j \end{cases} \tag{63}$$

Here $\tilde{d}_{ij}$ is the distance between the wire i and the mirrored wire j of the uniform multicomductor transmission line [1].

With the aid of the approximated matrix elements (61) the equations (58) and (59) simplify to:

$$\frac{d\varphi_i(l_i)}{dl_i} + j\omega\frac{\mu_0}{2\pi}\ln\left(\frac{\widetilde{R}_i}{a_i}\right)I_i(l_i) = 0 \tag{64}$$

$$\ln\left(\frac{\widetilde{R}_i}{a_i}\right)\frac{dI_i(l_i)}{dl_i} + j\omega 4\pi\varepsilon_0\varphi_i(l_i) = 0 \tag{65}$$

with the classical inductance and capacitance elements per-unit-length

$$L'_{0ii} = \frac{\mu_0}{2\pi}\ln\left(\frac{\widetilde{R}_i}{a_i}\right) \qquad \text{and} \qquad C'_{0ii} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{\widetilde{R}_i}{a_i}\right)} \tag{66}$$

and

$$L'_{0ii}C'_{0ii} = \mu_0\varepsilon_0 = \frac{1}{c^2} \qquad \text{(for lossless lines)} \qquad c\text{-speed of light} \tag{67}$$

Equations (64) and (65) are now combined to second-order differential equations for the currents $I_i(l_i)$.

$$\frac{d^2 I_i(l_i)}{dl_i^2} + k^2 I_i(l_i) = 0\,, \qquad (k = \omega/c) \tag{68}$$

This equation has two fundamental solutions (forward and backward running waves along each conductor i)

$$\left(I_i(l_i)\right)_1 = e^{-jkl_i} \qquad \text{and} \qquad \left(I_i(l_i)\right)_2 = e^{+jkl_i} \tag{69}$$

with their allocated matrices

$$\widehat{I}_1(l) = diag\; e^{-jkl_i(l)} = \left(\widehat{M}_0^l\{-j\omega\widehat{P}_{per}^{(0)}\}\right)_{21} \tag{70}$$

$$\widehat{I}_2(l) = diag\; e^{+jkl_i(l)} = \left(\widehat{M}_0^l\{-j\omega\widehat{P}_{per}^{(0)}\}\right)_{22} \tag{71}$$

The parameter matrix for zeroth-order in the perturbation expansion is given by

$$\widehat{P}_{per}^{(0)}(l) = \begin{pmatrix} 0 & diag(L'_{0ii}(l_i(l)) \\ diag(C'_{0ii}(l_i(l)) & 0 \end{pmatrix} \tag{72}$$

On the basis of the general equation (55) for the parameter matrix $\widehat{P}(l)$ one obtains an explicit equation for $\widehat{P}_{per}^{(1)}(l)$, the perturbation parameter matrix of first order:

$$\left[\widehat{P}^{(1)}_{per}(l)\right]=$$

$$-\frac{1}{j\omega}\begin{pmatrix} -j\omega\dfrac{\mu_0}{4\pi}\displaystyle\int_0^L \widehat{g}_L(l,l',k)\cdot diag(e^{-jkl_i(l')})\,dl' & -j\omega\dfrac{\mu_0}{4\pi}\displaystyle\int_0^L \widehat{g}_L(l,l',k)\cdot diag(e^{jkl_i(l')})\,dl' \\ -jk\,diag\left(\dfrac{dl_i(l)}{dl}e^{-jkl_i(l)}\right) & jk\,diag\left(\dfrac{dl_i(l)}{dl}e^{jkl_i(l)}\right) \end{pmatrix}\cdot$$

$$\begin{pmatrix} \dfrac{\eta_0}{4\pi}\displaystyle\int_0^L \widehat{g}_C(l,l',k)\cdot diag\left(\dfrac{dl_i(l')}{dl}e^{-jkl_i(l')}\right)dl' & -\dfrac{\eta_0}{4\pi}\displaystyle\int_0^L \widehat{g}_C(l,l',k)\cdot diag\left(\dfrac{dl_i(l')}{dl}e^{jkl_i(l')}\right)dl' \\ diag(e^{-jkl_i(l)}) & diag(e^{jkl_i(l)}) \end{pmatrix}^{-1}$$

(73)

This matrix now contains known expressions, is completely occupied, and already leads to radiation effects as is shown in the subsequent section. Only for the initial step of the perturbation calculation it was assumed that $k \to 0$. With increasing iteration steps also frequencies may increase. Frequently, already with the first order perturbation matrix $\widehat{P}^{(1)}_{per}(l)$ one finds sufficient results for the currents on the conductors (compared to measurements and to the results obtained by more sophisticated numerical programs).

In conclusion of this section it seems to be appropriate to add a remark concerning the inclusion of exterior (distributed) sources. Indeed, such sources can be also taken into consideration in the presented formalism. Then, however, the sources have to be renormalized in each iteration step. The interested reader is referred to Reference [5]. Another, more direct possibility for the inclusion of exterior sources is the extension to first order differential equations for three unknowns: Instead of equation (39) for the potential $\underset{\downarrow}{\varphi}$ and the current $\underset{\downarrow}{I}$ one could, e.g., use the three variables potential $\underset{\downarrow}{\varphi}$, charge per-unit-length $\underset{\downarrow}{q}$, and current $\underset{\downarrow}{I}$ [33]:

$$\frac{d}{dl}\begin{pmatrix}\underset{\downarrow}{\varphi}(l)\\ \underset{\downarrow}{q}(l)\\ \underset{\downarrow}{I}(l)\end{pmatrix} + \begin{pmatrix}\widehat{P}_{11}(l) & \widehat{P}_{12}(l) & \widehat{P}_{13}(l)\\ \widehat{P}_{21}(l) & \widehat{P}_{22}(l) & \widehat{P}_{23}(l)\\ \widehat{P}_{31}(l) & \widehat{P}_{32}(l) & \widehat{P}_{33}(l)\end{pmatrix}\cdot\begin{pmatrix}\underset{\downarrow}{\varphi}(l)\\ \underset{\downarrow}{q}(l)\\ \underset{\downarrow}{I}(l)\end{pmatrix} = \underset{\downarrow}{v}_{exc}(l) \qquad (39')$$

In this case there is no need to renormalize the sources. However, in this approach the parameter matrix depends on the exterior sources . In the theory of Ref. [5] the parameter matrix does not change when extending the MPIE from the homogeneous to the inhomogeneous form. The elements of the parameter matrix are considered as (unchangeable) material and geometrical properties of the conductors themselves and of their configuration.

Examples (see Ref. [33]) have shown successful application of $(39')$. However, further practicable investigations have to follow and need further research.

5. Radiation Losses

In the foregoing sections essential emphasis was laid on the estimation of the parameter matrix $\widehat{P}(l)$. Therefore, it seems to make sense to express also the radiation losses of the multiconductor-line system in terms of this matrix. Since the conductors are assumed to be lossless the only losses which can occur are due to radiation or to ohmic losses in the terminal loads. The emitted energy is part of the fed energy at the terminals of the conductors. The corresponding sources and/or loads are closely located (δ - function like) at the near and far ends of all lines.

From the text book literature (see, for example [34]) it is well known that the time averaged power W which is pumped into and absorbed by a system is given by (method of induced EMF)

$$W = \sum_{i=1}^{N} \frac{1}{2} \mathrm{Re}\left(\int_0^L E_{l_i}^{exc.}(l_i) I_i^*(l_i) dl_i \right) \tag{74}$$

In absence of radiation this equation would balance to zero. This is because all fed energy is absorbed in the loads. If, however, radiation takes place the expression (74) would give the radiation losses.

As mentioned, in the considered case (no exterior distributed sources) the excitation has some peculiarities (δ - functions) near the right and left terminals:

$$E_{l_i}^{exc.}(l_i) = U_i^{0,1} \delta(l_i - \Delta) + U_i^{0,2} \delta(l_i - L_i + \Delta) \tag{75}$$

with $0 \le l_i \le L_i$, and $\Delta \to 0$ is a small length.

The amplitudes $U_i^{0,k}$ ($i = 1,...N; k = 1,2$) can contain voltage sources as well as loads multiplied by current amplitudes [30, 32, 35]. With (75) the integration in (74) can be performed:

$$W = \frac{1}{2} \mathrm{Re} \sum_{i=1}^{N} \left(U_i^{0,1} I_i^*(\Delta) + U_i^{0,2} I_i^*(L_i - \Delta) \right) \tag{76}$$

Now, according to the equation (11), one can establish a connection between voltages and potentials:

$$\frac{d\varphi_i(l_i)}{dl_i} + j\omega \frac{\mu_0}{4\pi} \sum_{j=1}^{N} \int_0^{L_j} I_j(l'_j) \left[\vec{e}_i(l_i) \cdot \vec{e}_j(l'_j) g_{ij}(l_i, l'_j, k)) - \vec{e}_i(l_i) \cdot \tilde{\vec{e}}_j(l'_j) \tilde{g}_{ij}(l_i, l'_j, k)) \right] dl'_j =$$

$$= U_i^{0,1} \delta(l_i - \Delta) + U_i^{0,2} \delta(l_i - L_i + \Delta) \tag{77}$$

Integration along the interval $l_i \in [0, \Delta]$ yields:

$$\varphi_i(\Delta) - \underbrace{\varphi_i(0)}_{=0} + \underbrace{\int_0^{\Delta} j\omega \frac{\mu_0}{4\pi} dl_i \sum_{j=1}^{N} \int_0^{L_j} I_j(l'_j)[\ldots] dl'_j}_{\to 0} = U_i^{0,1} \tag{78}$$

The third term on the left-hand side limits to zero for very small Δ because the vector potential is a smooth function near the voltage sources.
Similarly, the integration along the interval $l_i \in [L_i - \Delta, L_i]$ results in:

$$\underbrace{\varphi_i(L_i)}_{=0} - \varphi_i(L_i - \Delta) + \underbrace{\int_{L_i - \Delta}^{L_i} j\omega \frac{\mu_0}{4\pi} dl_i \sum_{j=1}^{N} \int_0^{L_j} I_j(l'_j)[\ldots] dl'_j}_{\to 0} = U_i^{0,2} \tag{79}$$

Turning back to (76) the voltages are replaced by the adequate expressions for the potentials:

$$W = \frac{1}{2} \mathrm{Re} \sum_{i=1}^{N} \left(\varphi_i(\Delta) I_i^*(\Delta) - \varphi_i(L_i - \Delta) I_i^*(L_i - \Delta) \right) =$$

$$= -\frac{1}{2} \mathrm{Re} \sum_{i=1}^{N} \int_{\Delta}^{L_i - \Delta} \frac{d}{dl_i} \left(\varphi_i(l_i) I_i^*(l_i) \right) dl_i \tag{80}$$

From equation (80) one can clearly recognize that the radiated averaged power is the difference between the power which is fed into the system at the beginning of the conductors and the power which arrives at the ends of the conductors.
Note that the transfer to the general coordinate-system

$$l_i = l_i(l)$$

does not change the averaged power:

$$W = -\frac{1}{2} \mathrm{Re} \sum_{i=1}^{N} \int_{\Delta'}^{L - \Delta'} \frac{d}{dl} \left(\varphi_i(l) I_i^*(l) \right) dl = -\frac{1}{2} \mathrm{Re} \int_{\Delta'}^{L - \Delta'} \frac{d}{dl} \left(\underset{\downarrow}{I}^{+}(l) \cdot \underset{\downarrow}{\varphi}(l) \right) dl \tag{81}$$

where $\Delta' \to 0$, also small quantity like Δ.

$\underset{\downarrow}{I}^{+}(l) = [I_1(l), I_2(l), \ldots, I_N(l)]^*$ denotes the transposed complex conjugate value of $\underset{\downarrow}{I}(l)$.
Equation (81) can also be written in a symmetrical form

$$W = -\frac{1}{4} \int_{\Delta'}^{L - \Delta'} \frac{d}{dl} \left(\underset{\downarrow}{\varphi}^{+}(l) \cdot \underset{\downarrow}{I}(l) \cdot + \underset{\downarrow}{I}^{+}(l) \cdot \underset{\downarrow}{\varphi}(l) \right) dl \tag{82}$$

Knowing potential and current vectors, the radiated energy is calculated by (82). In order to see how the parameter matrix $\widehat{P}(l)$ contributes to this radiation, the derivatives in (82) are replaced with the aid of the linear equation

$$\frac{d}{dl}\begin{pmatrix} \underset{\downarrow}{\varphi}(l) \\ \underset{\downarrow}{I}(l) \end{pmatrix} + j\omega \begin{pmatrix} \widehat{P}_{11}(l) & \widehat{P}_{12}(l) \\ \widehat{P}_{21}(l) & \widehat{P}_{22}(l) \end{pmatrix} \cdot \begin{pmatrix} \underset{\downarrow}{\varphi}(l) \\ \underset{\downarrow}{I}(l) \end{pmatrix} = \underset{\downarrow}{0} \tag{83}$$

One gets

$$\frac{d}{dl}\underset{\downarrow}{\varphi}(l) = -j\omega\left(\widehat{P}_{11}(l)\cdot \underset{\downarrow}{\varphi}(l) + \widehat{P}_{12}(l)\cdot \underset{\downarrow}{I}(l) \right) \tag{84}$$

$$\frac{d}{dl}\underset{\downarrow}{I}(l) = -j\omega\left(\widehat{P}_{21}(l)\cdot \underset{\downarrow}{\varphi}(l) + \widehat{P}_{22}(l)\cdot \underset{\downarrow}{I}(l) \right) \tag{85}$$

and for the transposed complex conjugate quantities:

$$\frac{d}{dl}\underset{\downarrow}{\varphi}^{+}(l) = j\omega\left(\underset{\downarrow}{\varphi}^{+}(l)\cdot \widehat{P}_{11}^{+}(l) + \underset{\downarrow}{I}^{+}(l)\cdot \widehat{P}_{12}^{+}(l) \right) \tag{86}$$

$$\frac{d}{dl}\underset{\downarrow}{I}^{+}(l) = j\omega\left(\underset{\downarrow}{\varphi}^{+}(l)\cdot \widehat{P}_{21}^{+}(l) + \underset{\downarrow}{I}^{+}(l)\cdot \widehat{P}_{22}^{+}(l) \right) \tag{87}$$

Inserting equations (84) through (87) in the integral (82) and summing up appropriate terms, finally yields:

$$\begin{aligned} W \underset{\Delta'\to 0}{=} j\frac{\omega}{4}\int_{\Delta'}^{L-\Delta'} \Big[& \underset{\downarrow}{\varphi}^{+}(\widehat{P}_{22} - \widehat{P}_{11}^{+})\underset{\downarrow}{I} + \underset{\downarrow}{I}^{+}(\widehat{P}_{11} - \widehat{P}_{22}^{+})\underset{\downarrow}{\varphi} + \\ & + \underset{\downarrow}{I}^{+}(\widehat{P}_{12} - \widehat{P}_{12})\underset{\downarrow}{I} + \underset{\downarrow}{\varphi}^{+}(\widehat{P}_{21} - \widehat{P}_{21}^{+})\underset{\downarrow}{\varphi} \Big] dl \end{aligned} \tag{88}$$

This result clearly shows that the parameter matrix is involved in radiation effects. For the classical transmission-line theory the diagonal matrices $\widehat{P}_{11}$ and $\widehat{P}_{22}$ vanish, and the off-diagonal matrices are real-valued and symmetrical. Therefore, no radiation is present. If one only considers one single wire (plus reference) then (88) simplifies to (compare also Ref. [35]):

$$W \underset{\Delta'\to 0}{=} -\frac{\omega}{2}\int_{\Delta'}^{L-\Delta'} \left[\mathrm{Im}(P_{12})\,|I|^{2} + \mathrm{Im}(P_{21})\,|\varphi|^{2} + \mathrm{Im}(\varphi(P_{11} - P_{22}^{*})I^{*}) \right] dl \tag{89}$$

It is interesting to observe that only imaginary parts appear in the integral: For the off-diagonal elements the imaginary parts of these parameters itself and for the diagonal elements the imaginary parts of a product of these parameters with current and potential.

6. Conclusion and future work

The focus of this paper is the full-wave description of the electromagnetic interaction of multiconductor transmission lines. These lines are fed by lumped sources and terminated by lumped loads. Also the excitation by distributed sources (electromagnetic field excitation) is discussed.

The derivation of this novel theory starts with the mixed potential integral equations which are based on Maxwell's equations. In order to cast these equations into a system of first-order differential equations for the unknown currents and potentials one has to solve integral equations for the per-unit-length parameter matrix which occurs as coefficient matrix in the first-order differential equations. The resulting equations are the extended telegrapher equations. Because there are no known general analytical solutions for these equations iterative or perturbative procedures are suggested which give very good results even after one iteration.

The position-dependent per-unit-length parameters become complex and frequency-dependent. The extended telegrapher equtions with these parameters correctly model the propagation of electromagnetic waves along nonuniform multiconductor transmission lines, the radiation of electromagnetic energy, and also the coupling of external fields for all frequencies. There is no restriction to the quasi TEM mode like in the classical transmission-line theory.

Future enhancements of the theory include the generalization of the theory for thick nonuniform wires. This would allow not only to take into account radiation and nonuniformity effects, but also to calculate the skin and proximity effects in the conductors. Further, it is of interest to investigate the propagation of current waves through wiring systems having stochastical straggling of their geometrical characteristics. In this case one could use the reduction of the FWTL equations to a Schrödinger-type equation of second order [32]. Then the statistical characteristics of corresponding "potentials" in the Schrödinger equation are defined by the statistical characteristics of the stochastical geometry of the wiring.

The coupling of an electromagnetic field to wiring systems located in resonators changes radically compared with the typically considered coupling to these systems in free space [36-38]. Cavity resonances affect the strength of the coupling, in particular if non-linear elements are present. Then demodulation effects occur and field strengths may increase up to several tenfold [39]. This is an important aspect if EMC has to be guaranteed for electrical and electronic components in, e.g., computer housings, cars, aircraft and satellites.

7. References

1. Tesche, F.M.; Ianoz, M.V.; Karlsson, T: *EMC Analysis Methods and Computational Models*, John Willey and Sons, 1997.

2. Heaviside, O.: *Electromagnetic Theory*, Volume II, Ernest Benn Ltd., London, UK, 1925. First publication in 1899.

3. Sabath, F: „Ein hybrides Verfahren zur Berechnung des abgestrahlten Feldes von Leiterplatten“ , Universitaet Hannover, Dissertation, Oktober 1998.
4. Görisch, A.: „Netzwerkorientierte Modellierung und Simulation elektrischer Verbindungsstrukturen mit der Methode der partiellen Elemente“, Otto-von-Guericke-Universitaet Magdeburg, Dissertation, May 2002.
5. Haase, H.; Nitsch, J.; Steinmetz,T.: “Transmission-Line Supertheory: A new Approach to an Effective Calculation of Electromagnetic Interactions”, *URSI Radio Science Bulletin (Review of Radio Science)*, 307, pp.33-60, 2003.
6. King, R.P.W.: *Transmission Line Theory*, Mc Graw-Hill Book Company, New York, 155.
7. Lam, J.: “Equivalent Lumped Parameters for a Bend in a Two-Wire Transmission Line: Part I: Inductance”, *Interaction Note 303*, December 1976.
8. Lam, J.: “Equivalent Lumped Parameters for a Bend in a Two-Wire Transmission Line: Part II: Capacitance”, *Interaction Note 304*, January 1977.
9. Giri, D.V.; Chang, S.K. and Tesche, F.M.: “A Coupling Model for a Pair of Skewed Transmission Lines”, *Interaction Note 349*, September 1978.
10. Green, H.E and Cashman J.D.: “End Effect in Open-Circuited Two-Wire Transmission Lines”, *IEEE Transactions on MTT*, **34**, 1986, 180.
11. Getsinger, W.J.: “End-effects in quasi-TEM transmission lines”, *IEEE Transactions on MTT*, **41**, 1993, 666.
12. Degauque, P. and Zeddam, A.: “Remarks on the transmission-line approach to determining the current induced on above-ground cables”, *IEEE Transactions on EMC*, **30**, 1988, 77.
13. Tesche, F.M. and Brändli, B.R.: “Observations of the adequacy of transmission-line coupling model for long overhead cables”, in Proceedings of the International Symposium on EMC, Rome, Italy, September 1994, p. 374.
14. Liu, W. and Kami, Y. “Vertical riser effects of a finite-length transmission line”, in Proceedings of the International Symposium on EMC, Tokyo, Japan, 1999.
15. Omid, M.; Kami, Y. and Haykawa, M.: “Field coupling to nonuniform and uniform transmission lines”, *IEEE Transactions on EMC*, **39**, 1997, 201-211.
16. Wendt, D.O. and ter Haseborg, J.L. “Description of electromagnetic effects in the transmission line theory via concentrated and distributed linear elements”, in Proceeding of the IEEE AP-S International Symposium and URSI North American Radio Science Meeting, volume 4, Montreal, Quebec, Canada, July 1977, pp. 2330-2333.
17. Nakamura, T.; Hayashi, N.; Fukuda, D. and Yokokawa, S. ”Radiation from the transmission line with an acute bend”, *IEEE Transactions on EMC*, **37**, 1995, 317-325.
18. Tkachenko, S.; Rachidi, F. and Ianoz, M. “On the theory of high-frequency wave propagation along non-uniform transmission lines”, in Proceedings of the International Symposium on EMC, Magdeburg, Germany, October 1999.
19. Tkachenko, S.; Rachidi, F. and Ianoz, M. “High-frequency electromagnetic field coupling to long terminated lines”, , *IEEE Transactions on EMC*, 4**3**, 2001, 117-129.
20. Tkachenko, S. and Nitsch, J. „Investigation of high-frequency coupling with uniform and non-uniform lines: comparison of exact analytical results with those of different approximations”, in Proceedings of the 24th General Assembly of the International Union of Radio Science Maastricht, Netherlands August 2002.
21. Tkachenko, S.; Rachidi, F.; Nitsch, J., and Steinmetz, T. „Electromagnetic field coupling to nonuniform transmission lines: treatment of discontinuities”, in Proceeding of 15th International Zurich Symposium and Technical Exhibition on EMC, Zurich, Switzerland, February 2003, pp. 603-607.

22. Nitsch, J. and Tkachenko, S. „Source dependent transmission line parameters – plane wave vs TEM excitation”, *Interaction Note 577*, October 2002.
23. Nitsch, J. and Tkachenko, S. “Complex-valued transmission-line parameters and their relation to the radiation resistance”, *IEEE Transactions on EMC,* **46**, 2004, 477-487.
24. Nitsch, J. and Tkachenko, S. “Telegrapher equations for arbitrary frequencies and modes - radiation of an infinite, lossless transmission line”, *Radio Science*, **39**, 2004, RS2026, doi:10.1029/2002RS002817.
25. Sturm, R.J. “On the theory of nonuniform multiconductor transmission lines: an approach to irregular wire configurations”, *Interaction Note 541*, June 1998.
26. Sturm, R.J. “The treatment of multiconductor nonuniform transmission lines: a comparison of high and low characteristic impedance lines”, in Proceedings of the International Symposium on EMC, Magdeburg, Germany, October 1999, pp. 231-234.
27. Mei, K.K. “Theory of Maxwellian circuits”, Radio Science Bulletin, **305**, 2003, 6-13.
28. Nitsch, J.; Gronwald, F.; Wollenberg, G., *Radiating nonuniform transmission-line systems and the partial element equivalent circuit method,* John Willey and Sons, 2009.
29. Rachidi, F.; Tkachenko, S. (eds.): *Electromagnetic field interaction with transmission lines: From Classical Theory to HF Radiation Effects*, WIT Press, 2008.
30. Nitsch, J. and Tkachenko, S “Newest development in transmission line theory and applications”, *Interaction Note 592*, September 2004.
31. Gantmacher, F.R.: *The Theory of Matrices*, Volume 2, Chelsea Publishing Company, New York, USA, 1984.
32. Nitsch, J. and Tkachenko, S. “Propagation of Current Waves along Quasi-Periodical Thin-Wire Structures: Accounting of Radiation Losses”, *Interaction Notes, Note 601*, 23 May 2006.
33. Nitsch, J.; Tkachenko, S. and Rachidi, F. „Generalization of the full-wave transmission line theory for loaded lines with distributed excitation”, in Proceedings of the PIERS Symposium, Beijing, March 2009.
34. Miller E.K., “PCs for AP and Other EM Reflections”, *IEEE Antennas and Propagation Magazine*, **41**, 82-86, 1999.
35. Nitsch, J. and Tkachenko, S. “Global and Modal Parameters in the Generalized Transmission Line Theory and Their Physical Meaning”, *Radio Science Bulletin*, **312**, March 2005, pp.21-31.
36. Tkachenko, S.V.; Vodopianov, G.V. and Martinov, L.M. "Electromagnetic field coupling to an electrically small antenna in a rectangular cavity", in Proceeding of the 13th International Zurich Symposium and Technical Exhibition on Electromagnetic Compatibility, Zurich, Switzerland, February 1999, pp.379-384.
37. Tkachenko, S., Gronwald F., Nitsch J. and Steinmetz T., „Electromagnetic field coupling to transmission lines inside of resonators” in Proceeding of the 5th International Symposium on Electromagnetic Compatibility and Electromagnetic Ecology (EMC-2003), St.-Petersburg, Russia, September, 2003, pp. 68-74 (Rus.).
38. Tkachenko, S.; Krauthäuser, H.G.; Gronwald, F.; and Nitsch, J. „High frequency electromagnetic field coupling to small antennas in rectangular resonator”, in Proceedings of the International Conference on Electromagnetics in Advanced Applications (ICEAA09) September, 2009 Torino, Italy.
39. Krauthäuser H.G.; Tkachenko, S. and Nitsch, J. „Strong linear and non-linear coupling to system-cavity modes from repetitive high frequency illumination“, in Proceedings of the International Conference on Electromagnetics in Advanced Applications (ICEAA01) September, 2001 Torino, Italy.