Interaction Notes

Note 605

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Dual-Polarization Scattering at Transmission-Line Junctions

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Abstract

This paper extends the forward-scattering theorem for a junction of two transmission lines to the case of four transmission lines. The transmission lines are arranged as two pairs, each pair simulating two orthogonal polarizations.

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1. Introduction

A recent paper [3] has considered the scattering of waves from a transmission-line junction. There it is shown that there is a forward-scattering theorem of a form similar to that in electromagnetic scattering [2]. In particular, a similar result applies for the case of a lossless scattering network, with the integral over incidence directions replaced by a sum of forward and backward scattered waves.

In this paper we extend the previous results to a more complicated scattering at a junction of transmission lines. In this case, we consider a transmission-line structure which mimics polarization, so as to make the problem even closer to the electromagnetic case. An incident wave is propagated as two orthogonal polarizations on a four-wire transmission line. This, in turn, scatters forward into a similar four-wire transmission line supporting two orthogonal polarizations. Of course there is also a backscattered wave on the wires supporting the incident wave.

2. Two Polarizations Incident on a One-Dimensional-Transmission-Line Junction

Extending from [3] let us consider scattering, from a four-[ort network, illustrated in Fig. 2.1. In this example we let the network have two "sides". Side 1 is where we have an incident wave with either of two "polarizations". These are taken as waves 1 and 2 incident. There are also waves 1 and 2 reflected, and waves 3 and 4 transmitted. Of course, we can equally have waves 3 and 4 incident, giving four scattered waves, in general. Thus we have a reversal matrix $(\tilde{R}_{n,m}(s))$ as 4 x 4, relating four incoming waves to four outgoing waves as

$$\left(\tilde{V}_n^{(out)}(s)\right) \equiv \begin{pmatrix} \tilde{V}_1^{(out)}(s) \\ \tilde{V}_2^{(out)}(s) \\ \tilde{V}_3^{(out)}(s) \\ \tilde{V}_4^{(out)}(s) \end{pmatrix} = \left(\tilde{R}_{n,m}(s)\right) \cdot \begin{pmatrix} \tilde{V}_3^{(in)}(s) \\ \tilde{V}_4^{(in)}(s) \\ \tilde{V}_1^{(in)}(s) \\ \tilde{V}_2^{(in)}(s) \end{pmatrix}$$

$$= \left(\tilde{R}_{n,m}(s)\right) \cdot \left(P_{n,m}\right) \cdot \left(\tilde{V}_n^{(in)}(s)\right) \tag{2.1}$$

$$\left(P_{n,m}\right) \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \quad \left(P_{n,m}\right)^2 = \left(1_{n,m}\right) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In this convention zero scattering is given by

$$\left(\tilde{R}_{n,m}(s)\right) = \left(1_{n,m}\right) \tag{2.2}$$

One can think of this case as a horizontally polarized incident wave passing through the junction with no loss, reflection, or depolarization, and similarly for the vertical polarization. This is analogous to a plane wave in free space propagating with no scattering.

As in previous papers [1] we define wave variables by

$$\left(\tilde{V}_n^{\binom{in}{out}}(s)\right) = \left(\tilde{V}_n(s)\right) \pm Z_c\left(\tilde{I}_n(s)\right)$$

(current positive into junction) (2.3)

where we have assumed that all four waves propagate on transmission lines of characteristic impedance

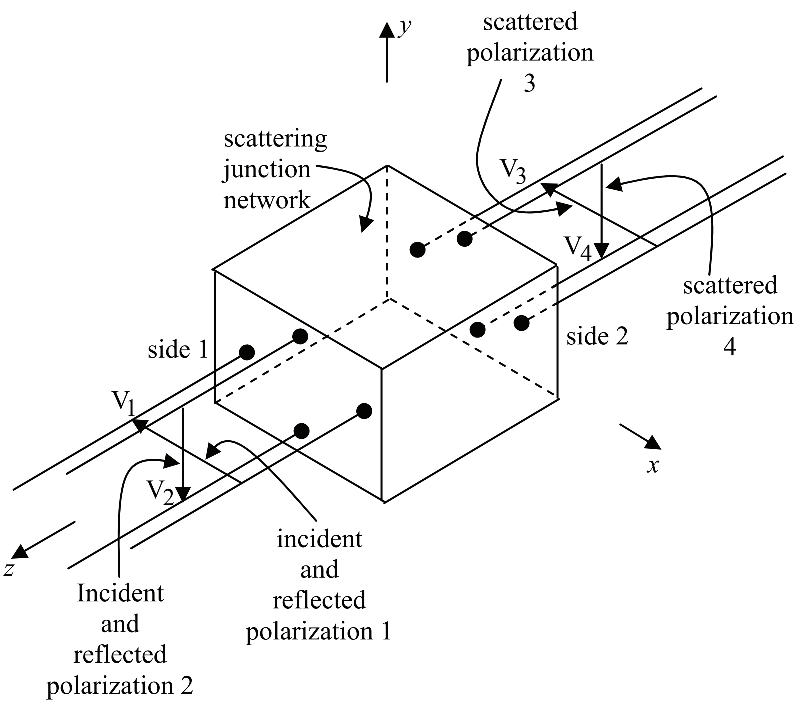

Fig. 2.1 Scattering From Junction in a Two-Conductor-Plus Reference, Ideal Lossless Transmission Line.

$$Z_c = Y_c^{-1} > 0 \tag{2.4}$$

which is the same for all four.

While we could let each transmission line be incident from four separate directions (such as on the four faces of a regular tetrahedron), the present configuration is more illustrative. Lines 1 and 2 are configured by symmetry to have two common symmetry planes so that waves on 1 do not couple to waves on 2, and conversely. The same applies to waves on 3 and 4 by the same symmetry. One can think of the waves on 1 and 2 as orthogonal to each other, and representing horizontal polarization for 1 (x component), and vertical polarization for 2 (y component) with propagation of the various waves in the $\pm z$ directions.

Now introduce some network at the junction. This can, in general, take any one of the four possible incident waves and scatter it into as many as all four outgoing waves as described by (2.1).

For a lossless scatterer $(\tilde{R}_{n,m}(j\omega))$ must be unitary (real pwoer in equals real power out) giving

$$
\begin{aligned}
&(\tilde{R}_{n,m}(j\omega)) \cdot (\tilde{R}_{n,m}(j\omega))^{\dagger} = (\tilde{R}_{n,m}(j\omega))^{\dagger} \cdot (\tilde{R}_{n,m}(j\omega)) = \left(1_{n,m}\right) \\
&\quad (\tilde{R}_{n,m}(j\omega))^{\dagger} = (\tilde{R}_{n,m}(-j\omega))^{T} \\
&\qquad \dagger = T^{*} \equiv \text{adjoint}
\end{aligned}
\tag{2.5}
$$

The analytic continuation of this out into the *s* plane is

$$
(\tilde{R}_{n,m}(s)) \cdot (\tilde{R}_{n,m}(-s))^{T} = (\tilde{R}_{n,m}(-s))^{T} \cdot (\tilde{R}_{n,m}(s)) = \left(1_{n,m}\right) \tag{2.6}
$$

which will be useful later. Defining

$$
(\tilde{R}'_{n,m}(s)) \equiv (\tilde{R}_{n,m}(s)) \cdot \left(P_{n,m}\right) \tag{2.7}
$$

we find that $(\tilde{R}'(j\omega))$ is also unitary with the same extension into the *s* plane, since $(P_{n,m})$ squares to the identity.

From the reversal matrix we can go to a scattering matrix which, in the present convention, represents the change introduced by the scatterer (a network other than straight through electrical connections). For this we have

$$
\begin{aligned}
&\left(\tilde{V}_n^{(sc)}(s)\right) = \left(\tilde{\Sigma}_{n,m}(s)\right) \cdot \begin{pmatrix} \tilde{V}_3^{(in)}(s) \\ \tilde{V}_4^{(in)}(s) \\ \tilde{V}_1^{(in)}(s) \\ \tilde{V}_2^{(in)}(s) \end{pmatrix} = \left(\tilde{\Sigma}'_{n,m}\right) \cdot \left(\tilde{V}_n^{(in)}(s)\right) \\
&\left(\tilde{\Sigma}'_{n,m}(s)\right) = \left(\tilde{\Sigma}_{n,m}(s)\right) \cdot \left(P_{n,m}\right)
\end{aligned}
\tag{2.8}
$$

The scattered waves, in this convention, are mixed with the other waves. So we write

$$
\begin{aligned}
\left(\tilde{V}_n^{(out)}(s)\right) &= \left(P_{n,m}\right) \cdot \left(\tilde{V}_n^{(in)}(s)\right) + \left(\tilde{V}_n^{(sc)}(s)\right) \\
&= \left(P_{n,m}\right) \cdot \left(\tilde{V}_n^{(in)}(s)\right) + \left(\tilde{\Sigma}'_{n,m}(s)\right) \cdot \left(\tilde{V}_n^{(in)}(s)\right) \\
&= \left[\left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(s)\right)\right] \cdot \left(P_{n,m}\right) \cdot \left(\tilde{V}_n^{(in)}(s)\right)
\end{aligned}
\tag{2.9}
$$

This gives

$$
\begin{aligned}
\left(\tilde{R}_{n,m}(s)\right) &= \left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(s)\right) \\
\left(\tilde{R}'_{n,m}(s)\right) &= \left(P_{n,m}\right) + \left(\tilde{\Sigma}'_{n,m}(s)\right)
\end{aligned}
\tag{2.10}
$$

3. Forward-Scattering Theorem

Additional properties for lossless scatterers include (from (2.6) and (2.10))

$$\begin{gathered}\left(\tilde{\Sigma}'_{n,m}(s)\right) + \left(\tilde{\Sigma}_{n,m}(-s)\right)^T + \left(\tilde{\Sigma}_{n,m}(s)\right) \bullet \left(\tilde{\Sigma}_{n,m}(-s)\right)^T = \left(0_{n,m}\right) \\ \left(\tilde{\Sigma}'_{n,m}(s)\right) \bullet \left(P_{n,m}\right) + \left(P_{n,m}\right) \bullet \left(\tilde{\Sigma}'_{n,m}(-s)\right)^T + \left(\tilde{\Sigma}'_{n,m}(s)\right) \bullet \left(\tilde{\Sigma}_{n,m}(-s)\right)^T = \left(0_{n,m}\right)\end{gathered} \tag{3.1}$$

This is the form that the forward-scattering theorem takes for the scattering network. The integral over directions of incidence and polarization [2] is replaced by the dot-product summation over the corresponding parameters here.

The usual network scattering matrix [2] can be calculated from the impedance matrix $(\tilde{Z}_{n,m}(s))$ of the junction as

$$\begin{aligned}\left(\tilde{S}_{n,m}(s)\right) &= \left[Y_c\left(\tilde{Z}_{n,m}(s)\right)+\left(1_{n,m}\right)\right]^{-1} \bullet \left[Y_c\left(\tilde{Z}_{n,m}(s)\right) - \left(1_{n,m}\right)\right] \\ &= \left[Y_c\left(\tilde{Z}_{n,m}(s)\right) - \left(1_{n,m}\right)\right] = \left[Y_c\left(\tilde{Z}_{n,m}(s)\right) + \left(1_{n,m}\right)\right]^{-1} \\ &= \left(\tilde{S}_{n,m}(s)\right)^T = \left(\tilde{R}'_{n,m}(s)\right) = \left(\tilde{R}'_{n,m}(s)\right)^T\end{aligned} \tag{3.2}$$

As before this matrix is unitary for lossless scatterers as

$$\begin{aligned}\left(\tilde{S}_{n,m}(j\omega)\right) \bullet \left(S_{n,m}(j\omega)\right)^{\dagger} &= \left(1_{n,m}\right) = \left(\tilde{S}_{n,m}(s)\right) \bullet \left(\tilde{S}_{n,m}(-s)\right)^T \\ &= \left(\tilde{S}_{n,m}(s)\right) \bullet \left(\tilde{S}_{n,m}(-s)\right) \\ \left(\tilde{S}_{n,m}(-s)\right) &= \left(S_{n,m}(s)\right)^{-1}\end{aligned} \tag{3.3}$$

Like in the case of plane waves in free space [4], the left-to-right scattering by the network equals the right-to-left scattering as

$$\begin{aligned}\left(\tilde{S}_{n,m}(s)\right)^T &= \left(\tilde{S}_{n,m}(s)\right) \\ \left(\tilde{R}_{n,m}(s)\right)^T &= \left(P_{n,m}\right) \bullet \left(\tilde{S}_{n,m}(s)\right) \\ &= \left(P_{n,m}\right) \bullet \left(\tilde{R}_{n,m}(s)\right) \bullet \left(P_{n,m}\right) \\ &= \begin{pmatrix} \tilde{R}_{3,3}(s) & \tilde{R}_{3,4}(s) & \tilde{R}_{3,1}(s) & \tilde{R}_{3,2}(s) \\ \tilde{R}_{4,3}(s) & \tilde{R}_{4,4}(s) & \tilde{R}_{4,1}(s) & \tilde{R}_{4,2}(s) \\ \tilde{R}_{1,3}(s) & \tilde{R}_{1,4}(s) & \tilde{R}_{1,1}(s) & \tilde{R}_{1,2}(s) \\ \tilde{R}_{2,3}(s) & \tilde{R}_{2,4}(s) & \tilde{R}_{2,1}(s) & \tilde{R}_{2,2}(s) \end{pmatrix} \\ \begin{pmatrix} \tilde{R}_{3,3}(s) & \tilde{R}_{3,4}(s) \\ \tilde{R}_{4,3}(s) & \tilde{R}_{4,4}(s) \end{pmatrix} &= \begin{pmatrix} \tilde{R}_{1,1}(s) & \tilde{R}_{2,1}(s) \\ \tilde{R}_{1,2}(s) & \tilde{R}_{2,2}(s) \end{pmatrix}\end{aligned} \tag{3.4}$$

This is seen in the scattering matrix symmetry as a wave into port *n* and out port *m* is the same as a wave into port *m* and out port *n*.

From (2.10) we now also have

$$\begin{aligned}
\left(\tilde{\Sigma}_{n,m}(s)\right)^T &= \left(\tilde{\Sigma}_{n,m}(s)\right) \\
\begin{pmatrix} \tilde{\Sigma}_{3,3}(s) & \tilde{\Sigma}_{3,4}(s) \\ \tilde{\Sigma}_{4,3}(s) & \tilde{\Sigma}_{4,4}(s) \end{pmatrix} &= \begin{pmatrix} \tilde{\Sigma}_{1,1}(s) & \tilde{\Sigma}_{2,1}(s) \\ \tilde{\Sigma}_{1,2}(s) & \tilde{\Sigma}_{2,2}(s) \end{pmatrix} \\
\left(\tilde{\Sigma}'_{n,m}(s)\right)^T &= \left(\tilde{\Sigma}'_{n,m}(s)\right)
\end{aligned} \tag{3.5}$$

which can be substituted into (3.1) for the forward scattering. Then (3.1) can be rewritten as

$$\begin{aligned}
\left(\tilde{\Sigma}_{n,m}(s)\right) + \left(\Sigma_{n,m}(-s)\right) + \left(\tilde{\Sigma}_{n,m}(s)\right) \bullet \left(\tilde{\Sigma}_{n,m}(-s)\right) &= \left(0_{n,m}\right) \\
\left(\tilde{\Sigma}'_{n,m}(s)\right) \cdot \left(P_{n,m}\right) + \left(P_{n,m}\right) \bullet \left(\tilde{\Sigma}_{n,m}(-s)\right) + \left(\tilde{\Sigma}'_{n,m}(s)\right) \bullet \left(\tilde{\Sigma}'_{n,m}(-s)\right) &= \left(0_{n,m}\right)
\end{aligned} \tag{3.6}$$

By our present construction, the forward scattering now includes two “polarizations”. So now the forward scattering is characterized by a 2 x 2 matrix. Referring to Fig. 2.1, we have two waves incident from the left (+*z*) representing two polarizations. These scatter into two waves propagating to the right (forward scattering, -*z*). This equals the right-to-left scattering by symmetry.

4. Application to Scattering Poles

Now consider a first-order scattering pole of the form (for s near s_α)

$$\left(\tilde{V}_n^{(sc)}(s)\right) = (a_n)\;[s-s_\alpha]^{-1} \tag{4.1}$$

In general, this represents four outgoing waves. As $s \to s_\alpha$ we have scattered waves with no incident wave, implying that the scattering matrix has poles at

$$\begin{aligned}
&\det\left(\left(\tilde{\Sigma}_{n,m}(s_\alpha)\right)^{-1}\right) = 0 \\
&\det\left(\left(\tilde{S}_{n,m}(s_\alpha)\right)^{-1}\right) = 0 = \det\left(\left(\tilde{S}_{n,m}(-s_\alpha)\right)\right) = 0 \\
&\det\left(Y_c\left(\tilde{Z}_{n,m}(s_\alpha)\right) + \left(1_{n,m}\right)\right) = \det\left(Y_c\left(\tilde{Z}_{n,m}(-s_\alpha)\right) - \left(1_{n,m}\right)\right) = 0
\end{aligned} \tag{4.2}$$

The locations in the s plane are then dependent on the details of $(\tilde{Z}_{n,m,}(s))$. Assuming a lossless reciprocal network, then we have an odd function of s as

$$\left(\tilde{Z}_{n,m}(s)\right) = -\left(\tilde{Z}_{n,m}(-s)\right) \tag{4.3}$$

So poles at s_α lead to zeros at $-s_\alpha$ (in the right half plane) as can be seen from the symmetrical form of the scattering matrix in (3.2). (So we can equally well do computations in both half planes.)

Having found some s_α of interest, we next need the natural mode (a_n). Substituting (4.1) in (2.1) and letting $s \to s_\alpha$ gives (for nondegenerate modes)

$$\left(\tilde{\Sigma}_{n,m}(s)\right) = (a_n)(a_n)[s-s_\alpha]^{-1} \text{ as } s \to s_\alpha \tag{4.4}$$

(For degenerate modes (same s_α) one has a sum of such terms.)

In (3.1) this becomes

$$\begin{aligned}
&(a_n)(a_n) + (a_n)(a_n) \bullet \left(\tilde{\Sigma}_{n,m}(-s_\alpha)\right)^T = \left(0_{n,m}\right) \\
&\left[\left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(-s_\alpha)\right)\right] \bullet (a_n)(a_n) = \left(0_{n,m}\right) \\
&\left[\left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(-s_\alpha)\right)\right] \bullet (a_n) = (0_n)
\end{aligned} \tag{4.5}$$

which gives another equation for s_α as

$$\det\left(\left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(-s_\alpha)\right)\right) = 0 = \det\left(\tilde{R}_{n,m}(-s_\alpha)\right) \tag{4.6}$$

This admits various solutions depending on the scattering lossless (LC) circuit. The "polarization" of such a wave is given by (a_n) for nondegenerate modes, including, in general, separate polarizations for left- and right-propagating waves (unless there are certain symmetries in the scattering network).

There are various lossless scattering networks that one might consider. A simple case has wave 1 only coupled to wave 3 with a series LC network in parallel connection across the transmission line. One can also similarly connect wave 2 to wave 4. In each case the calculations in [2] apply, including the case of second order poles.

One can also have more elaborate lossless scattering networks which scatter incident wave 1 into all four outgoing waves. In this case, the polarization can be "rotated" in various ways.

5. Concluding Remarks

So now we have a transmission-line scattering problem which more closely mimics the electromagnetic scattering problem, including polarization. In general, this involves four incoming waves and four outgoing waves, in the form of two pairs, each pair representing the two polarizations.

The previous paper [2] has shown that one can have second order scattering poles. In the present construction, this is also the case due to its relation to the previous case.

One can, of course, generalize the present case to a scattering network which connects 2N pairs of dually polarized transmission lines. Each pair can directly connect from some nth pair ($1 \leq n \leq N$) to some [n + N]th pair (forward scattering), thereby simulating 2N possible directions of incidence (n to n + N and n + N to n). Introducing a scattering 2N-port network at the junctions, the integral over 4π steradians in [2] is replaced by a sum over the 2N transmission lines. Such gives an even closer simulation of the electromagnetic-scattering case.

References

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