Interaction Notes

Notes 604

January 14, 2008

Scattering From a Lossless Acoustic Sphere

Xi (Ronald) Chen[*], Carl E Baum and Thomas Hagstrom

Abstract

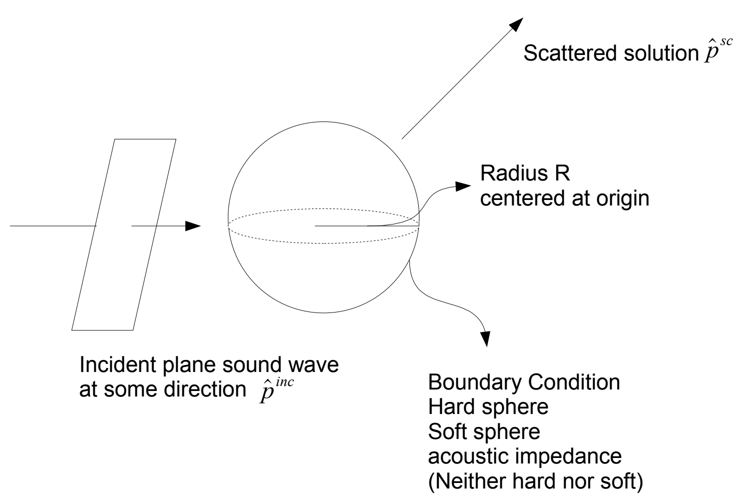
We solve a scattering problem from a lossless acoustic sphere with some different boundary conditions and an incident plane wave, focusing on the pole behaviour which we can derive from the explicit solution to the equation. Our interest in the pole behaviour comes from the singularity expansion method (SEM) used in solving electromagnetic scattering problems[2]. The acoustic sphere case can be viewed as a simplified analogue for us to understand the poles in the electromagnetic case. We show that second and third order poles can be formed by choosing some mathematically constructed impedance boundary condition as well as discussing the possibility of higher order poles. We use different approaches to construct the poles and we believe we can extend the same results to electromagnetic scattering problems.

[*]This work was supported by the Air Force Office of Scientific Research (AFOSR)

1 Introduction

An obstacle or inhomogeneity in the path of a sound wave causes *scattering* if secondary sound spreads out from it in a variety of directions. Such an inhomogeneity could be, for example, a fish in the ocean, a region of turbulence in the atmosphere, or a red corpuscle in a bloodstream. The smearing of propagation directions that results when a sound beam reflects from a rough surface is also recognized as scattering.

In this paper, we consider the following problem. An incident plane sound wave is propagating in some direction with the scatterer being a sphere. The scattered solution can be written explicitly using sphereical harmonics. However, the thing we are interested in is to use SEM to solve the problem. In another word, we want to study the pole behavior of the scattered solution. By putting different boundary conditions on the scatterer, we are able to construct not only simple poles but also 2nd or even higher order poles. Meanwhile, we always keep the impedance function satisfying the Foster theorem, so that it remains a lossless system. In subsequent work we plan to extend the results to the eletromagnetic case (not in this paper), which will still be lossless.



2 Introduction to the equations

Consider the linear acoustic equation [5]

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot v = 0 \tag{1}$$

$$\rho_0 \frac{\partial v}{\partial t} + \nabla p = 0 \tag{2}$$

$$p = c^2 \rho \quad , \quad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_0$$

where p is for acoustic pressure, ρ is for density, v is for fluid velocity, c is the speed of sound. We want to solve for the scattering solution to the above equation with an incident plane wave with some different but lossless boundary conditions on the sphere. We will use the expansion in terms of spherical harmonics to solve the equation in order to locate the poles. In fact, this is just the SEM in the acoustic case. Let's first derive the wave equation from the above system, so that later on we will treat the problem mathematically disregarding the physical meaning temporarily. Plug $p = c^2\rho$ into (1). We get

$$\frac{\partial p}{\partial t} + \rho_0 c^2 \nabla \cdot v = 0 \tag{3}$$

$$\rho_0 \frac{\partial v}{\partial t} + \nabla p = 0 \tag{4}$$

If we apply $\frac{\partial}{\partial t}$ to (3) and plug into (4), we get

$$\frac{\partial^2 p}{\partial t^2} + \rho_0 \nabla \cdot (-\frac{1}{\rho_0} \nabla p) = 0$$

$$\Rightarrow \quad \frac{\partial^2 p}{\partial t^2} - c^2 \nabla^2 p = 0 \tag{5}$$

similarly for v, we apply $\frac{\partial}{\partial t}$ to (4), and plug into (3) we get

$$\frac{\partial^2 v}{\partial t^2} + \frac{1}{\rho_0} \nabla(-\rho_0 c^2 \nabla \cdot v) = 0$$

$$\Rightarrow \quad \frac{\partial^2 v}{\partial t^2} - c^2 \nabla(\nabla \cdot v) = 0$$

assume v is irrotational i.e. $\nabla \times v = 0$, then $\nabla(\nabla \cdot v) = \nabla^2 v$. Then

$$\frac{\partial^2 v}{\partial t^2} - c^2 \nabla^2 v = 0 \tag{6}$$

Let u be either p or v,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0$$

taking the Laplace transformation we get

$$(\nabla^2 - \gamma^2)\hat{u} = 0, \quad where \quad \gamma = \frac{s}{c} \tag{7}$$

Assume the scatterer is a ball with radius 1 centered at the origin and that the impedance boundary condition has the following form

$$\frac{\partial \hat{u}}{\partial n} + \alpha(s)\hat{u} = 0 \quad , \quad at \quad r = 1 \tag{8}$$

where n is the outward normal.

Assume an incident plane wave is propagating in the z-direction i.e.

$$\hat{u}^{(inc)} = e^{-\gamma(0,0,1)\cdot(x,y,z)} = e^{-\gamma z} \tag{9}$$

We want to solve for the scattered solution $\hat{u}^{(sc)}$explicitly according to (7)(8)(9) and study the pole behavior of the scattered solution $\hat{u}^{(sc)}$.

3 Hard and Soft spheres

First of all, we want to relate the mathematical impedance function $\alpha(\gamma)$ to the actual acoustic impedance $Z_a(s)$. Units of acoustic impedance are $Pa \cdot s/m$ or $kg/(m^2 \cdot s)$ By definition

$$Z_a(s) = \frac{\hat{p}(r,s)}{\hat{v}(r,s) \cdot n_{in}} = \rho_a \hat{v}(s) = \left(\frac{\rho_a}{\hat{k}_a(s)}\right)^{\frac{1}{2}}$$

$\hat{p}, \hat{v}$ are the Laplace transformation of p and v, n_{in} is the inward normal. Take the Laplace transform of (2), we get $s\hat{v} + \frac{1}{\rho_0}\nabla\hat{p} = 0$. Then take the inner product with the outward normal direction n , we get

$$\begin{aligned} s\hat{v}\cdot n + \frac{1}{\rho_0}\frac{\partial \hat{p}}{\partial n} &= 0 \\ \Rightarrow \frac{\partial \hat{p}}{\partial n} &= -\rho_0 s\hat{v}\cdot n \end{aligned} \tag{10}$$

Then the acoustic impedance can be written as

$$Z(s) = \frac{\hat{p}}{\hat{v}\cdot n_{in}} = \frac{\hat{p}}{\frac{\partial \hat{p}}{\partial n}}\rho_0 s$$

In our formulation, if we plug (10) into (8), assuming we put the impedance on pressure (i.e. $u = p$ here), we derive

$$\frac{\hat{p}}{\hat{v}\cdot n_{in}} = -\frac{\rho_0 s}{\alpha(s)} = Z(s)$$

which relates the mathematical impedance condition to the real acoustic impedance boundary condition.

The infinite specific-acoustic-impedance limit $|Z| \to \infty$ corresponds to a hard (rigid) surface, the limit $|Z| \to 0$ corresponds to a soft (pressure-release) surface. Thus, for a hard sphere the mathematical impedance boundary condition becomes $\frac{\partial \hat{u}}{\partial n} = 0$ and $\hat{u} = 0$ for the soft sphere.

Follow the same computation as we did in [3]. We get

$$\hat{u}^{(inc)} = e^{-\gamma z} = e^{-\gamma r\cos\theta} = \sum_{n=0}^{\infty}(2n+1)(-1)^n i_n(\gamma r)P_n(\cos\theta)$$

$$\begin{aligned} \hat{u}^{(sc)} &= \sum_{n=0}^{\infty}\sum_{m=-n}^{m=n} a_{nm}(\gamma)k_n(\gamma r)Y_n^m(\theta,\varphi) \\ &= \sum_{n=0}^{\infty} a_n(\gamma)k_n(\gamma r)P_n(\cos\theta) \end{aligned}$$

Where $k_n(s)$, $i_n(s)$ are the modified Bessel functions, $a_n(\gamma)$ is the coefficient to be determined.

Applying the impedance boundary condition to hard and soft sphere respectively, at $r = 1$

$$\frac{\partial \hat{u}_{hard}^{(sc)}}{\partial n} = -\frac{\partial \hat{u}_{hard}^{(inc)}}{\partial n}$$

$$\hat{u}_{soft}^{(sc)} = -\hat{u}_{soft}^{(inc)}$$

n is the outward normal, i.e. r in our case(sphere).

We derive the scattered solutions as follows

$$\hat{u}_{hard}^{(sc)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\gamma i_n'(\gamma)]}{\gamma k_n'(\gamma)} k_n(\gamma r) P_n(\cos\theta)$$

$$\hat{u}_{soft}^{(sc)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\gamma i_n(\gamma)]}{k_n(\gamma)} k_n(\gamma r) P_n(\cos\theta)$$

Since the modified spherical Bessel functions k_n and its derivatives only have simple zeros [2], for both the hard and soft sphere scatterer, the solution only has simple poles.

4 Lossless loading of the sphere

4.1 Impedance boundary for lossless case

Consider the total energy in the usual mathematical way

$$E_D = \int\int_D\int \frac{1}{2}(u_t^2 + |\nabla u|^2)$$

$$\begin{aligned}
\frac{dE_D}{dt} &= \int\int_D\int u_t u_{tt} + \nabla u^T \cdot \nabla u_t \\
&= \int\int_D\int u_t(u_{tt} - \nabla^2 u) + \int_{\partial D} u_t u_\nu \\
&= \int\int_{\partial D} u_t u_\nu
\end{aligned}$$

and the net energy flux is

$$\iint\limits_{\partial D} s\hat{u}(s) \cdot \overline{\frac{\partial \hat{u}(y,s)}{\partial r}} \tag{11}$$

in our case D is a sphere. Plug in our impedance boundary condition. The net energy flux becomes

$$-\iint\limits_{\partial D} s\overline{\alpha(s)}|\hat{u}(y,s)|^2$$

For a lossless case, the total energy change should be zero, which implies $\int_{-i\infty}^{i\infty}$(11)=0⇒(11) should be odd ⇒ $\alpha(s)$ should be even, $Z(s)$ odd. Mathematically, this condition is sufficient to guarantee that energy is conserved. However, in order to extend our results later on, we need to put more constraints on $Z(s)$ or equivalent on $\alpha(s)$. Namely, we want $Z(s)$ to satisfy the Foster reactance theorem which we will discuss a little bit in a later section.

4.2 2nd order pole

The general expansion of the scattered solution can for our impedance condition can been written as

$$\hat{u}^{(sc)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\alpha(s)i_n(s) + si_n'(s)]}{\alpha(s)k_n(s) + sk_n'(s)} k_n(sr)P_n(\cos\theta)$$

Assume $c = 1$ here, so $\gamma = s$

The far field pattern solution

$$\hat{u}_{\infty}^{(sc)} = \frac{e^{-sr}}{r} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\alpha(s)i_n(s) + si_n'(s)]}{s(\alpha(s)k_n(s) + sk_n'(s))} P_n(\cos\theta)$$

In order to construct a second order pole we need both the denominator and the derivative of the denominator to the scattered solution to be zero at some specific $s = s_p$.

That is, the denominator=0

$$s_p\alpha(s_p)k_n(s_p) + s_p^2 \frac{d}{ds} k_n(s_p) = 0 \tag{12}$$

and the derivative of the denominator=0

$$\alpha\left(s_p\right)k_n\left(s_p\right)+s_p\left(\frac{d}{ds}\alpha\left(s_p\right)\right)k_n\left(s_p\right)+s_p\alpha\left(s_p\right)\frac{d}{ds}k_n\left(s_p\right)$$
$$+2\,s_p\frac{d}{ds}k_n\left(s_p\right)+{s_p}^2\frac{d^2}{ds^2}k_n\left(s_p\right)\quad=\quad 0 \qquad (13)$$

Note that we do not want to solve the above system of ODEs, because we only need the equations to hold at one specific s for some pre-chosen impedance boundary condition $\alpha(s)$

By using Bessel's equation itself we can replace

$$s^2\frac{d^2}{ds^2}k_n\left(s\right)=2\,s\frac{d}{ds}k_n\left(s\right)-\left(s^2+n\left(n+1\right)\right)k_n\left(s\right)$$

Solve (12) for $\frac{d}{ds}k_n\left(s\right)$ and plug into (13), we get

$$k_n\left(s_p\right)(s_p\left(\frac{d}{ds}\alpha\left(s_p\right)\right)+a(s_p)-\alpha\left(s_p\right)^2+{s_p}^2+n+n)=0$$

for $k_n(s_p)\neq 0$ we want

$$s_p\alpha'+s_p^2+\alpha+n(n+1)-\alpha^2=0$$

For $n=1$,$k_1(s)=\frac{\pi}{2}\frac{s+1}{s^2}e^{-s}$

Solve (12) for $\alpha(s_p)$, we get

$$\alpha(s_p)=\frac{{s_p}^2+2\,s_p+2}{s_p+1}$$

Solve (13) for $\alpha'(s_p)$, we get

$$\alpha'(s_p)=\frac{s_p\left(s_p+2\right)}{{s_p}^2+2\,s_p+1}$$

Note: We can not differentiate $\alpha(s_p)$ here, since it's just the value evaluated at s_p, not a function.

The solution we derived above simply means that, if we want to construct a 2nd order pole at $s=s_p$, the impedance function $\alpha(s)$ should be chosen

1. $\alpha(s)$ Even function

2. Value of $\alpha(s)$ at s_p equals $\frac{{s_p}^2+2\,s_p+2}{s_p+1}$

3. Value of $\alpha'(s)$ at s_p equals $\frac{s_p(s_p+2)}{{s_p}^2+2\,s_p+1}$.

Let's build a concrete example to verify the above results.

Choose $s_p = -2$, then $\alpha(s_p) = -2$, $\alpha'(s_p) = 0$

Assume $\alpha(s)$ has the following form

$$\alpha(s) = \frac{c_1 + c_2 s^2}{1 + c_3 s^2}$$

after some calculation we find that $\alpha(s) \equiv -2$

The denominator of the scattered solution $\hat{u}_\infty^{(sc)}$ for $n = 1$ is

$$-2sk_1(s) + s^2 k_1'(s) = \frac{\pi}{2}\frac{(s+2)^2}{s}e^{-s}$$

Thus, we got the 2nd order pole at $s = -2$.

4.3 3rd order pole

The process is similar to the 2nd order pole case. We want denominator, denominator',denominator"=0 at some $s = s_p$.

$$s_p\alpha\left(s_p\right)k_n\left(s_p\right) + {s_p}^2\frac{d}{ds}k_n\left(s_p\right) = 0 \tag{14}$$

$$\alpha\left(s_p\right)k_n\left(s_p\right)+s_p\left(\frac{d}{ds}\alpha\left(s_p\right)\right)k_n\left(s_p\right)+s_p\alpha\left(s_p\right)\frac{d}{ds}k_n\left(s_p\right)+2s_p\frac{d}{ds}k_n\left(s_p\right)+{s_p}^2\frac{d^2}{ds^2}k_n\left(s_p\right) = 0 \tag{15}$$

$$\begin{aligned}2\left(\frac{d}{ds}\alpha\left(s_p\right)\right)k_n\left(s_p\right) + 2\,\alpha\left(s_p\right)\frac{d}{ds}k_n\left(s_p\right) + 2\,\frac{d}{ds}k_n\left(s_p\right) + 2\,s_p\left(\frac{d}{ds}\alpha\left(s_p\right)\right)\frac{d}{ds}k_n\left(s_p\right)\\ + s_p\alpha\left(s_p\right)\frac{d^2}{ds^2}k_n\left(s_p\right) + s_p\left(\frac{d^2}{ds^2}\alpha\left(s_p\right)\right)k_n\left(s_p\right) + 4\,s_p\frac{d^2}{ds^2}k_n\left(s_p\right) + {s_p}^2\frac{d^3}{ds^3}k_n\left(s_p\right) = 0\end{aligned} \tag{16}$$

By computing the derivative of Bessel's equation we get

$$s^2\frac{d^3}{ds^3}k_n\left(s\right)=-4\,s\frac{d^2}{ds^2}k_n\left(s\right)-2\,\frac{d}{ds}k_n\left(s\right)+2\,sk_n\left(s\right)+\left(s^2+n\left(n+1\right)\right)\frac{d}{ds}k_n\left(s\right)$$

together with

$$s^2\frac{d^2}{ds^2}k_n\left(s\right)=2\,s\frac{d}{ds}k_n\left(s\right)-\left(s^2+n\left(n+1\right)\right)k_n\left(s\right)$$

and

$$\frac{d}{ds}k_n\left(s_p\right)=-\frac{\alpha\left(s_p\right)k_n\left(s_p\right)}{s_p}$$

equation (15), (16) reduces to

$$k_n\left(s_p\right)\left(\alpha\left(s_p\right)+s_p\frac{d}{ds}\alpha\left(s_p\right)-\left(\alpha\left(s_p\right)\right)^2+{s_p}^2+n^2+n\right)=0 \quad (17)$$

$$k_n\left(s_p\right)\left(2\,\frac{d}{ds}\alpha\left(s_p\right)+s_p\frac{d^2}{ds^2}\alpha\left(s_p\right)-2\,\left(\frac{d}{ds}\alpha\left(s_p\right)\right)\alpha\left(s_p\right)+2\,s_p\right)=0 \quad (18)$$

solve (14), (17), (18) for $\alpha, \alpha', \alpha''$ respectively. We find

$$\alpha(s_p)=-\frac{s_p\frac{d}{ds}k_n\left(s_p\right)}{k_n\left(s_p\right)}$$

$$\alpha'(s_p)=\frac{s_p\left(\frac{d}{ds}k_n\left(s_p\right)\right)k_n\left(s_p\right)+{s_p}^2\left(\frac{d}{ds}k_n\left(s_p\right)\right)^2-{s_p}^2\left(k_n\left(s_p\right)\right)^2-n^2\left(k_n\left(s_p\right)\right)^2-n\left(k_n\left(s_p\right)\right)^2}{s_p\left(k_n\left(s_p\right)\right)^2}$$

$$\begin{aligned}
\alpha''(s_p) &= 2\,\{-s_p\left(\frac{d}{ds}k_n\left(s_p\right)\right)\left(k_n\left(s_p\right)\right)^2-2\,k_n\left(s_p\right){s_p}^2\left(\frac{d}{ds}k_n\left(s_p\right)\right)^2+n^2\left(k_n\left(s_p\right)\right)^3\\
&+ n\left(k_n\left(s_p\right)\right)^3-{s_p}^3\left(\frac{d}{ds}k_n\left(s_p\right)\right)^3+{s_p}^3\left(\frac{d}{ds}k_n\left(s_p\right)\right)\left(k_n\left(s_p\right)\right)^2\\
&+ s_p\left(\frac{d}{ds}k_n\left(s_p\right)\right)n^2\left(k_n\left(s_p\right)\right)^2+s_p\left(\frac{d}{ds}k_n\left(s_p\right)\right)n\left(k_n\left(s_p\right)\right)^2\}/{s_p}^2\left(k_n\left(s_p\right)\right)^3
\end{aligned}$$

For $n=2$, $k_2(s)=\frac{1}{2}\,\frac{\pi\,e^{-s}\left(s^2+3\,s+3\right)}{s^3}$, we need

$$\alpha(s_p) = \frac{s_p^3 + 4s_p^2 + 9s_p + 9}{s_p^2 + 3s_p + 3}$$
$$\alpha'(s_p) = \frac{s_p(s_p^3 + 6s_p^2 + 12s_p + 6)}{(s_p^2 + 3s_p + 3)^2}$$
$$\alpha''(s_p) = \frac{6(3 + s_p^3 + 6s_p^2 + 9s_p)}{(s_p^2 + 3s_p + 3)^3}$$

Again, the solution we derived above simply means that, if we want to construct a third order pole at s, the impedance function $\alpha(s)$ should be chosen

1. $\alpha(s)$ Even function
2. Value of $\alpha(s)$ at s_p equals $\frac{s_p^3+4s_p^2+9s_p+9}{s_p^2+3s_p+3}$
3. Value of $\alpha'(s)$ at s_p equals $\frac{s_p(s_p^3+6s_p^2+12s_p+6)}{(s_p^2+3s_p+3)^2}$
4. Value of $\alpha''(s)$ at s_p equals $\frac{6(3+s_p^3+6s_p^2+9s_p)}{(s_p^2+3s_p+3)^3}$

Let us test our results again with $s_p = -4$. Thus, we require $\alpha(-4) = -\frac{27}{7}$, $\alpha'(-4) = \frac{40}{49}$, $\alpha''(-4) = -\frac{6}{343}$

Assume $\alpha(s)$ still has following form

$$\alpha(s) = \frac{c_1 + c_2 s^2}{1 + c_3 s^2}$$

we get

$$\alpha(s) = \frac{-\frac{47}{27} - \frac{11}{54}s^2}{1 + \frac{1}{54}s^2} = \frac{-11(s^2 + \frac{94}{11})}{s^2 + 54}$$

The denominator of the scattered solution $\hat{u}^{(sc)}$ for $n = 2$ is

$$\frac{\pi}{2} \frac{e^{-s}\left(s^2 + 3\,s + 12\right)\left(s + 4\right)^3}{s^2\left(54 + s^2\right)}$$

Thus, we got a 3rd order pole at $s = -4$.

4.4 High order poles

In principle, we can follow this procedure to get any order of pole at a specific location. However, the computation will become messier and messier. It will become more clear to construct high order poles if we try to view this process using taylor expansion around the pole s_p.

First, let's introduce some very nice properties of the modified Bessel function [4].

$$-s\frac{k_l'(s)}{k_l(s)} = s + 1 + \hat{S}_l(s)$$

where, for $l \neq 0$,

$$\begin{aligned}
\hat{S}_l(z) =& \frac{P_l(z)}{Q_l(z)} \\
P_l(z) =& \sum_{k=0}^{l-1} \frac{(2l-k)!}{k!(l-k-1)!}(2z)^k \\
Q_l(z) =& \sum_{k=0}^{l} \frac{(2l-k)!}{k!(l-k)!}(2z)^k
\end{aligned}$$

We also have the following beautiful continued fraction representation for

$$\hat{S}_l(z) = \frac{l(l+1)}{2}\frac{1}{z+1+\frac{l(l+1)-1\cdot 2}{4(z+2+\frac{l(l+1)-2\cdot 3}{4(z+3+\dots)})}}$$

Thus, it is possible for us to rewrite

$$k_l(s) = \frac{k_l^{(1)}(s)}{k_l^{(2)}(s)}\frac{e^{-s}}{s}$$

where, $k_l^{(1)}(s)$, $k_l^{(2)}(s)$ are just polynomials in s.

Suppose the impedance function $\alpha(s)$ has the following form

$$\alpha(s) = -c_0\frac{(s^2+a_0)(s^2+a_2)\cdots(s^2+a_k)}{(s^2+a_1)(s^2+a_3)\cdots(s^2+a_{2n-1})} = -c_0\frac{\alpha_1(s)}{\alpha_2(s)}$$

where, $c_0 > 0$, $0 \leq a_0 < a_1 < a_2 < \cdots < a_{2n-2} < a_{2n-1} < a_{2n} < \infty$, $k = 2n-2\,or\,2n$, and $\alpha_1(s)$, $\alpha_2(s)$ are of course polynomials.

For example, for a 2nd order pole we need 2 free parameters, thus $\alpha(s)$ can be chosen

$$\alpha(s) = -c_0(s^2 + a_0)$$

For a 3rd order pole we need 3 free parameters, thus $\alpha(s)$ can be chosen

$$\alpha(s) = -c_0 \frac{(s^2 + a_0)}{(s^2 + a_1)}$$

For a 4th order pole we need 4 free parameters, thus $\alpha(s)$ can be chosen

$$\alpha(s) = -c_0 \frac{(s^2 + a_0)(s^2 + a_2)}{(s^2 + a_1)}$$

The denominator to the scattered solution $\hat{u}^{(sc)}$ can now be written as

$$-\frac{k_n^{(1)}(s)}{k_n^{(2)}(s)} \frac{e^{-s}}{s} \Big(c_0 \frac{\alpha_1(s)}{\alpha_2(s)} + s + 1 + \hat{S}_n(s) \Big) \tag{19}$$

If we want to construct a $j-$th order pole at s_p, we just need to choose the correct power of $\alpha_1(s)$, $\alpha_2(s)$ and let $n = j - 1$, and rewrite the numerator using Taylor expansion around s_p.

$$k_n^{(1)}(s) \Big(c_0 \alpha_1(s) \hat{Q}_n(s) + \alpha_2(s) \hat{P}_n(s) \Big) = \sum_{i=0}^{j-1} \beta_i (s - s_p)^i + O((s - s_p)^j) \tag{20}$$

where $s + 1 + \hat{S}_n(s) = \frac{\hat{P}_n(s)}{\hat{Q}_n(s)}$

Choose c_0, a_0, a_1, $\ldots$ a_{j-2} so that $\beta_i = 0$ for $i = 0, \ldots j - 1$. Note that we always want c_0 and a_j to satisfy our assumptions.

Thus we construct a lossless impedance function $\alpha(s)$, which will produce a $j-$th order pole at s_p. One might ask: Can we always get a solution which satisfies all of our assumptions for arbitrary $s_p < 0$? At least, in our case, the answer is NO! There will be some restriction on s_p, i.e s_p can only be chosen from certain regions of the negative x-axis. (We have not yet considered complex poles.)

Let's go over the 3rd pole again using the Taylor method to have a better idea of the procedure and the restriction.

We are constructing a 3rd order pole at $s = s_p$, so $j = 3$, $n = 2$.

$$\alpha(s) = -c\frac{(s^2+a)}{(s^2+b)}$$

$$k_2(s) = \frac{e^{-s}}{s}\frac{(s^2+3s+3)}{s^2}$$

$$s+1+\hat{S}_n(s) = \frac{\hat{P}_n(s)}{\hat{Q}_n(s)} = \frac{s^3+4\,s^2+9\,s+9}{s^2+3\,s+3}$$

According to (20), the zeros are contained in

$$(s^2+3s+3)\Big(c(s^2+a)(s^2+3\,s+3)+(s^2+b)(s^3+4\,s^2+9\,s+9)\Big) \qquad (21)$$

expand (21) around $(s-s_p)$, we get the coefficient

$\beta_0 =$
${s_p}^7 + 18\,c{s_p}^3 + 7\,b{s_p}^4 + b{s_p}^5 + c{s_p}^6 + 6\,ca{s_p}^3 + 9\,c{s_p}^2 + 24\,{s_p}^5 + 7\,{s_p}^6 + 15\,ca{s_p}^2 + ca{s_p}^4 + 48\,{s_p}^4 + 6\,c{s_p}^5 + 54\,{s_p}^3 + 24\,b{s_p}^3 + 15\,c{s_p}^4 + 54\,bs_p + 18\,cas_p + 27\,{s_p}^2 + 27\,b + 48\,b{s_p}^2 + 9\,ca$

$\beta_1 =$
$54\,c{s_p}^2 + 7\,{s_p}^6 + 30\,cas_p + 4\,ca{s_p}^3 + 6\,c{s_p}^5 + 162\,{s_p}^2 + 18\,ca{s_p}^2 + 120\,{s_p}^4 + 30\,c{s_p}^4 + 18\,ca + 72\,b{s_p}^2 + 28\,b{s_p}^3 + 18\,cs_p + 42\,{s_p}^5 + 54\,s_p + 96\,bs_p + 192\,{s_p}^3 + 5\,b{s_p}^4 + 54\,b + 60\,c{s_p}^3$

$\beta_2 =$
$72\,bs_p + 6\,ca{s_p}^2 + 162\,s_p + 54\,cs_p + 15\,c{s_p}^4 + 18\,cas_p + 21\,{s_p}^5 + 105\,{s_p}^4 + 27 + 15\,ca + 60\,c{s_p}^3 + 48\,b + 240\,{s_p}^3 + 9\,c + 10\,b{s_p}^3 + 288\,{s_p}^2 + 90\,c{s_p}^2 + 42\,b{s_p}^2$

Solve $\beta_0 = \beta_1 = \beta_2 = 0$ for a, b, c in terms of s_p we get

$$a = \frac{{s_p}^6 + 12\,{s_p}^5 + 81\,{s_p}^4 + 315\,{s_p}^3 + 648\,{s_p}^2 + 648\,s_p + 216}{3\,{s_p}^4 + 28\,{s_p}^3 + 99\,{s_p}^2 + 153\,s_p + 96}$$

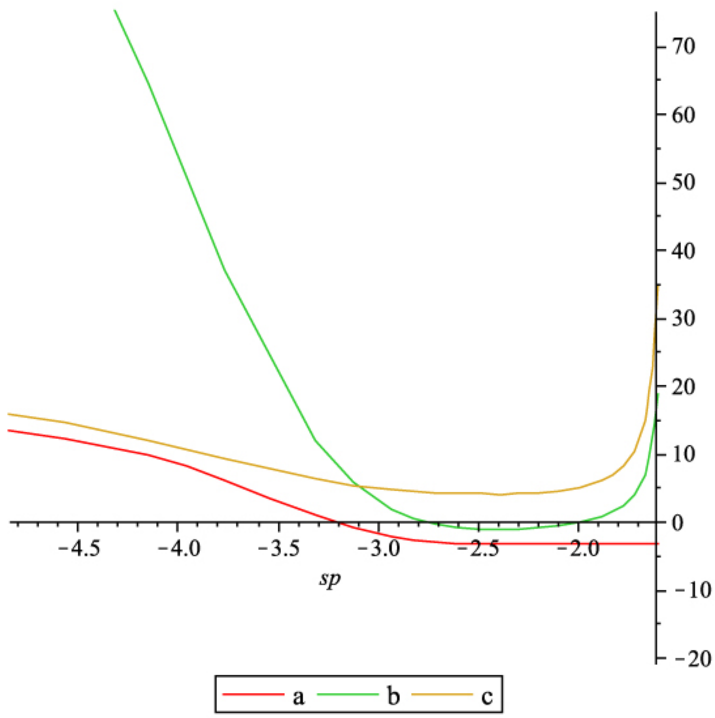
$$b = 3\,\frac{{s_p}^5 + 9\,{s_p}^4 + 35\,{s_p}^3 + 72\,{s_p}^2 + 72\,s_p + 24}{{s_p}^3 + 9\,{s_p}^2 + 27\,s_p + 24}$$

$$c = -\frac{3\,{s_p}^4 + 28\,{s_p}^3 + 99\,{s_p}^2 + 153\,s_p + 96}{{s_p}^3 + 9\,{s_p}^2 + 27\,s_p + 24}$$

If we choose $s_p = -4$, then $a = \frac{94}{11}$, $b = 54$, $c = 11$, clearly $c > 0$, and $0 \leq a < b < \infty$

$$\alpha(s) = \frac{-11(s^2 + \frac{94}{11})}{s^2 + 54}$$

which is the same as in section 4.3. In fact, if we choose $s_p = -3$, then a, b, c will not satisfy our assumptions. When we plot a, b, c in terms of s_p, we will see that in order to satisfy all the assumptions, s_p can only be chosen approximately $s_p < -3.2$. At $s_p = -3$, a will be negative. For more general cases, we can also use this graph to determine the range of s_p. For example, in this case, the range of s_p will be the interval where the graph of b is above a and all of the graphs of a, b, c are above the x-axis. For higher order poles, this procedure push on the available domain of s_p further to the left.



5 Interpretation of Results

Our prototype here for scattering is a constant-frequency plane wave proceeding in direction e_z. The overall acoustic pressure is written

$$\hat{p}(s,r,\theta) = \hat{p}^{(inc)}(s,r,\theta) + \hat{p}^{(sc)}(s,r,\theta)$$

where $\hat{p}^{(inc)}(s,r,\theta)$ is the incident plane wave and $\hat{p}^{(sc)}(s,r,\theta)$ is the scattered solution(wave's complex amplitude). The function $\hat{p}^{(sc)}(s,r,\theta)$ satisfies the Helmholtz equation and the Sommerfeld radiation condition.

For a hard surface of scatterer we require

$$\nabla \hat{p} = 0 \qquad \Rightarrow \quad \nabla \hat{p}^{(sc)} \cdot n = -\nabla \hat{p}^{(inc)} \cdot n$$

For a soft surface of scatterer we require

$$\hat{p} = 0 \qquad \Rightarrow \quad \hat{p}^{(sc)} \cdot n = -\hat{p}^{(inc)} \cdot n$$

where n is a unit normal vector pointing into the fluid.

If we assume the scatterer is a sphere centered at the origin with radius R, we have

$$\hat{p}^{(sc)}_{hard} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\gamma i_n'(\gamma R)]}{\gamma k_n'(\gamma R)} k_n(\gamma r) P_n(\cos\theta)$$

$$\hat{p}^{(sc)}_{soft} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\gamma i_n(\gamma)]}{k_n(\gamma R)} k_n(\gamma r) P_n(\cos\theta)$$

The poles to $\hat{p}^{(sc)}_{hard,soft}$ are all simple as discussed above. Since $\hat{v}^{(sc)}$ has exactly same zeros as $\hat{p}^{(sc)}$does, we will just talk about $\hat{p}^{(sc)}$ here.

If the sphere scatterer has a specific-acoustic-impedance (neither soft nor hard) condition (equivalent to $\alpha(s) = -2$)

$$Z(s) = \frac{\rho_0 s}{2}$$

$$\hat{p}^{(sc)} = \textstyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)[\alpha(\gamma R) i_n(\gamma R) + \gamma R i_n'(\gamma R)]}{\alpha(\gamma R) k_n(\gamma R) + \gamma R k_n'(\gamma R)} k_n(\gamma r) P_n(\cos\theta)$$

then $\hat{p}^{(sc)}$ will have a second order pole at $\gamma R = -2$ or $s = -\frac{2c}{R}$.

If we go back to the time domain, the solution associated with the 2nd order pole will have the slow decaying term look like (for simplicity $c = 1$, $R = 1$)

$$\frac{3\pi}{4r}\Big(-1 + 8\,(t-(r-2))e^{-2\,(t-(r-2))}\Big) H(t-(r-2))$$

where $H(t)$ is the heaviside function.

If the sphere scatterer has a specific-acoustic-impedance (neither soft nor hard) condition (equivalent to $\alpha(s) = \frac{-11(s^2+\frac{94}{11})}{s^2+54}$)

$$Z(s) = \frac{\rho_0 s(s^2+54)}{11(s^2+\frac{94}{11})}$$

then $\hat{p}^{(sc)}$ will have a third order pole at $\gamma R = -4$ or $s = -\frac{4c}{R}$.

In [6], Sancer showed that for acoustic scattering SEM poles are simple, but there is no contradiction. In that reference, the impedance condition is not considered in the proof and Sancer used a hard acoustic scatterer to derive the simple pole behavior for an arbitrary shape of scatterer, with which our results agree.

The reason why we choose our impedance function like so is because we want it to satisfy the Foster reactance theorem so that we can extend our result to a lossless network for the Electromagnetic case later on. The $Z(s)$ we derived above all satisfy the Foster reactance theorem. Let us briefly list the concepts and theorems of Foster's theorem [1] we use here, so that we can easily verify that the Foster theorem is satisfied.

A *positive real function* $F(s)$ is an analytic function of the complex variable $s = \sigma + j\omega$, which has the following properties:

1.$F(s)$ is regular for $\sigma > 0$

2.$F(\sigma)$is real

3.$\sigma \geq 0$ implies $Re[F(s)] \geq 0$

A *reactance function* is a positive real function that maps the imaginary axis into the imaginary axis.

Theorem: A *real rational function* of s is a *reactance function* if and only if all of its poles and zeros are simple, lie on the $j\omega$-axis, and alternate with each other.

In another word

$$\psi(s) = K\frac{s(s^2+\omega_1^2)(s^2+\omega_3^2)\cdots(s^2+\omega_{2n-1}^2)}{(s^2+\omega_0^2)(s^2+\omega_2^2)\cdots(s^2+\omega_k^2)}$$

is a reactance function, where $k = 2n-2 \, or \, 2n$, $K > 0$, $0 \leq \omega_0 < \omega_1 < \cdots < \omega_{2n-1} < \omega_{2n} < \infty$

Theorem: A *rational function* of s is *reactance function* if and only if it is the driving-point impedance or admittance of a lossless network.

6 Remarks on future work

The acoustic scattering is viewed as the scalar analogue of the pseudosymmetric argument developed for electromagnetic SEM. Using a sheet impedance condition on a sphere and other shape of scatterer is under consideration. It is reasonable to believe that high order poles are possible for such impedance boundary conditions.

References

[1] Norman Balabanian and Theodore A. Bickart. *Electrical Network Theory*. John Wiley and Sons, Inc, 1969.

[2] Carl E. Baum. *The Singularity Expansion Method*, volume 10 of *Topics in Applied Physics*. Springer-Verlag Berlin Heidelbery New York, 1976. Interaction Note 88.

[3] Xi (Ronald) Chen. Forward-scattering memos memo 2. 2007. Notes for Scalar Wave Equation in Scattering Problem.

[4] Thomas Hagstrom. *Radiation boundary conditions for the numerical simulation of waves*. Cambridge University Press, 1999.

[5] Allan D. Pierce. *An Introduction to Its Physical Principles and Applications*. McGraw-Hill, 1981.

[6] Maurice I. Sancer and A. D. Varvatsis. Toward an increased understanding of the singularity expansion method. 1980. Interaction Note 398.