Interaction Notes

Note 603

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Scattering at One-Dimensional-Transmission-Line Junctions

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Abstract

This paper considers the forward-scattering theorem (in electromagnetics) from the point of view of the scattering from a junction network (lossless) inserted in a transmission line. It is shown that second-order scattering poles can be produced.

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1. Introduction

A recent paper [6] has generalized the forward-scattering theorem with particular application to lossless bodies. In particular this gave new insight into the scattering natural frequencies and modes, also implying new ways to calculate them from the scattering operator in the right half of the complex *s*-plane (Laplace-transform plane). This gives new insight into the properties of the poles (natural frequencies, s_α) in the left-half plane and the associated natural scattering modes.

There are also various symmetries in the scattering operator [8] which are relevant here, in particular, the invariance of the forward scattering to the reversal of the direction of incidence. A question of interest concerns the order of the scattering poles, for lossless and/or perfectly conducting scatterers [4]. It is hoped that studying the scattering properties of scattering from junctions in transmission lines [3] will lead to a further understanding of these and other electromagnetic (EM) scattering questions. This, in the spirit of C. H. Papas, is a baby problem which may tell us a few things of significance.

So, as in Fig. 1.1, let us consider a canonical problem consisting of a junction in a transmission-line (single conductor plus reference) of a common characteristic impedance, Z_c, on both sides of the junction. Scattering from such a junction is only forward (subscript *f*) and backward (subscript *b*). The illustration is for a wave incident on the left side (side 1), but, of course, the symmetry of the problem (reciprocity)

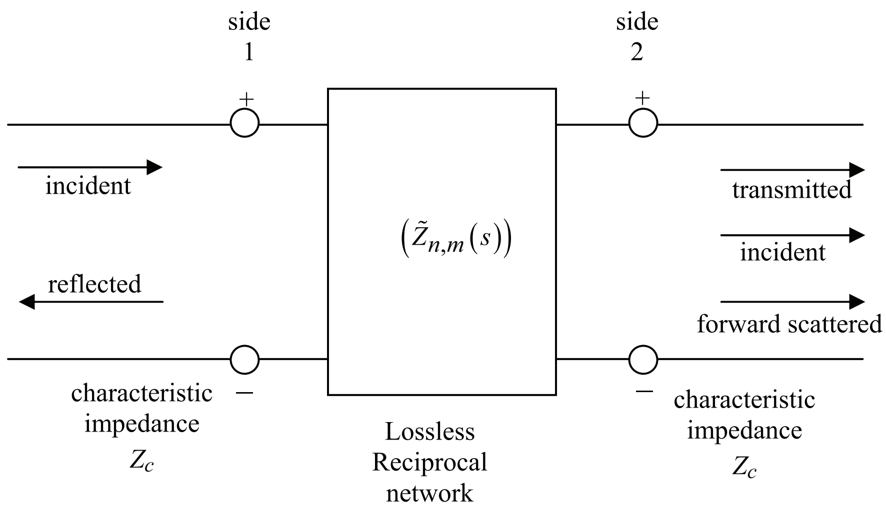
$$\left(\tilde{Z}_{n,m}(s)\right) = \left(\tilde{Z}_{n,m}(s)\right)^T \equiv \left(\tilde{Y}_{n,m}(s)\right)^{-1} \tag{1.1}$$

with identical left and right transmission lines (characteristic impedance, lossless, real, frequency independent) of impedance

$$Z_c = Y_c^{-L} \tag{1.2}$$

implies equal transmission in both directions.

In the present simplified problem there is only forward scattering and backscattering, other angles in 4π steradians (and second polarization) being absent.



Here incidence is assumed from the left,
but it can equally be from the right.

Fig. 1.1 Scattering From Junction in Single-Conductor Plus Reference, Ideal Lossless Transmission Line

2. Formulation of the Scattering Problem

Formulating this somewhat differently from a typical microwave-junction scattering problem, we let the removal of the junction network with straight-through connections correspond to no scattering. In this wasy we have a reversal matrix (incoming to outgoing) in the form

$$\begin{pmatrix} \tilde{V}_1^{(out)}(s) \\ V_2^{(out)}(s) \end{pmatrix} = \left(\tilde{R}_{n,m}(s)\right) \begin{pmatrix} \tilde{V}_2^{(in)}(s) \\ V_2^{(in)}(s) \end{pmatrix} = \left(\tilde{R}_{n,m}(s)\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^{(in)}(s) \\ \tilde{V}_2^{(in)}(s) \end{pmatrix} \tag{2.1}$$

For this zero-scattering case we have

$$\left(\tilde{R}_{n,m}(s)\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{(identify)} \tag{2.2}$$

Instead of indexing by ports, this is like indexing by wave direction as

$$\begin{aligned} &1 \Rightarrow \text{right propagating} \\ &2 \Rightarrow \text{left propagating} \end{aligned} \tag{2.3}$$

So now for a lossless scatterer $(\tilde{R}_{n,m}(j\omega))$ must be unitary (real power in equals real power out [7]) giving

$$\begin{aligned} &\left(\tilde{R}_{n,m}(j\omega)\right) \left(\tilde{R}_{n,m}(j\omega)\right)^{\dagger} = \left(\tilde{R}_{n,m}(j\omega)\right)^{\dagger} \left(\tilde{R}_{n,m}(j\omega)\right) = \left(1_{n,m}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\left(\tilde{R}_{n,m}(j\omega)\right)^{\dagger} = \left(\tilde{R}_{n,m}(-j\omega)\right)^{\dagger} \\ &\qquad \dagger = T* = \text{adjoint} \end{aligned} \tag{2.4}$$

This is a key point for a lossless scatterer. Defining

$$\left(\tilde{R}'_{n,m}(j\omega)\right) = \left(\tilde{R}_{n,m}(j\omega)\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.5}$$

we find that

$$\begin{aligned} \left(\tilde{R}'_{n,m}(j\omega)\right) \left(\tilde{R}'_{n,m}(j\omega)\right)^{\dagger} &= \left(\tilde{R}_{n,m}(j\omega)\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\tilde{R}_{n,m}(j\omega)\right)^{\dagger} \\ &= \left(\tilde{R}_{n,m}(j\omega)\right) \left(\tilde{R}_{n,m}(j\omega)\right)^{\dagger} = \left(1_{n,m}\right) \end{aligned} \tag{2.6}$$

so that $(\tilde{R}'_{n,m}(j\omega))$ is also unitary. By analytic continuation we can speak of a generalized kind of unitary matrix as

$$
\begin{aligned}
\left(\tilde{R}_{n,m}(s)\right) \quad \left(\tilde{R}_{n,m}(-s)\right)^T &= \left(1_{n,m}\right) \\
\left(\tilde{R}'_{n,m}(s)\right) \quad \left(\tilde{R}'_{n,m}(-s)\right)^T &= \left(1_{n,m}\right)
\end{aligned}
\tag{2.7}
$$

We go from this to a scattering matrix which represents the change due to the network at the junction as

$$
\begin{aligned}
\begin{pmatrix} \tilde{V}_1^{(sc)}(s) \\ \tilde{V}_2^{(sc)}(s) \end{pmatrix} &= \left(\tilde{\Sigma}_{n,m}(s)\right) \quad \begin{pmatrix} \tilde{V}_2^{(in)}(s) \\ \tilde{V}_1^{(in)}(s) \end{pmatrix} = \left(\tilde{\Sigma}'_{n,m}(s)\right) \quad \begin{pmatrix} \tilde{V}_1^{(in)}(s) \\ \tilde{V}_2^{(in)}(s) \end{pmatrix} \\
\left(\tilde{\Sigma}'_{n,m}(s)\right) &= \left(\tilde{\Sigma}_{n,m}(s)\right) \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{\Sigma}_{1,2}(s) & \tilde{\Sigma}_{1,1}(s) \\ \tilde{\Sigma}_{2,2}(s) & \tilde{\Sigma}_{2,1}(s) \end{pmatrix}
\end{aligned}
\tag{2.8}
$$

The scattered field, in this formulation, is mixed with the incident field. So we write

$$
\begin{aligned}
\begin{pmatrix} \tilde{V}_1^{(out)}(s) \\ \tilde{V}_2^{(out)}(s) \end{pmatrix} &= \begin{pmatrix} \tilde{V}_2^{(in)}(s) \\ \tilde{V}_1^{(in)}(s) \end{pmatrix} + \begin{pmatrix} \tilde{V}_1^{(sc)}(s) \\ \tilde{V}_2^{(sc)}(s) \end{pmatrix} \\
&= \begin{pmatrix} \tilde{V}_2^{(in)}(s) \\ \tilde{V}_1^{(in)}(s) \end{pmatrix} + \left(\tilde{\Sigma}_{n,m}(s)\right) \quad \begin{pmatrix} \tilde{V}_2^{(in)}(s) \\ \tilde{V}_1^{(in)}(s) \end{pmatrix} \\
&= \left[\left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(s)\right)\right] \quad \begin{pmatrix} \tilde{V}_2^{(in)}(s) \\ \tilde{V}_1^{(in)}(s) \end{pmatrix} \\
&= \left[\left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(s)\right)\right] \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} \tilde{V}_1^{(in)}(s) \\ \tilde{V}_2^{(in)}(s) \end{pmatrix}
\end{aligned}
\tag{2.9}
$$

We have then

$$
\begin{aligned}
\left(\tilde{R}_{n,m}(s)\right) &= \left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(s)\right) \\
\left(\tilde{R}'_{n,m}(s)\right) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \left(\tilde{\Sigma}'_{n,m}(s)\right)
\end{aligned}
\tag{2.10}
$$

which are unitary for lossless networks with $s = j\omega$. This, in turn, from (2.5) and (2.6) gives for lossless networks

$$
\begin{aligned}
&\left(\tilde{\Sigma}_{n,m}(s)\right) + \left(\tilde{\Sigma}_{n,m}(-s)\right)^T + \left(\tilde{\Sigma}_{n,m}(s)\right) \quad \left(\tilde{\Sigma}_{n,m}(-s)\right)^T = \left(0_{n,m}\right) \\
&\left(\tilde{\Sigma}'_{n,m}(s)\right) \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\tilde{\Sigma}'_{n,m}(-s)\right)^T + \left(\tilde{\Sigma}'_{n,m}(s)\right) \left(\tilde{\Sigma}'_{n,m}(-s)\right)^T = \left(0_{n,m}\right)
\end{aligned}
\tag{2.11}
$$

In this formulation forward scattering is described by

$$
\begin{aligned}
&\tilde{V}_1^{(in)}(s) \to \tilde{V}_2^{(sc)}(s) \Rightarrow \tilde{\Sigma}_{2,2}(s) = \tilde{\Sigma}'_{2,1}(s) \\
&\tilde{V}_2^{(in)}(s) \to \tilde{V}_1^{(sc)}(s) \Rightarrow \tilde{\Sigma}_{1,1}(s) = \tilde{\Sigma}'_{1,2}(s)
\end{aligned}
\tag{2.12}
$$

Note that incoming and outgoing waves can be separated as wave variables [3] in the form

$$
\begin{aligned}
&\left(\tilde{V}_1^{\binom{in}{out}}(s)\right) = \tilde{V}_1(s) \pm Z_c \tilde{I}_1(s) \\
&\left(\tilde{V}_2^{\binom{in}{out}}(s)\right) = \tilde{V}_1(s) \pm Z_c \tilde{I}_2(s)
\end{aligned}
\tag{2.13}
$$

where the positive direction for current is taken as *into* the junction (Fig. 2.1), so that

$$
\begin{pmatrix} \tilde{V}_1(s) \\ \tilde{V}_2(s) \end{pmatrix} = \left(\tilde{Z}_{n,m}(s)\right) \quad \begin{pmatrix} \tilde{I}_1(s) \\ \tilde{I}_2(s) \end{pmatrix}
\tag{2.14}
$$

From the wave variables the voltages and currents are reconstructed as

$$
\begin{aligned}
&\tilde{V}_1(s) = \frac{1}{2}\left[\tilde{V}_1^{(in)}(s) + \tilde{V}_1^{(out)}(s)\right] \quad , \quad \tilde{I}_1(s) = \frac{1}{2} Y_c \left[\tilde{V}_1^{(in)}(s) - \tilde{V}_1^{(out)}(s)\right] \\
&\tilde{V}_2(s) = \frac{1}{2}\left[\tilde{V}_2^{(in)}(s) + \tilde{V}_2^{(out)}(s)\right] \quad , \quad \tilde{I}_2(s) = \frac{1}{2} Y_c \left[\tilde{V}_2^{(in)}(s) - \tilde{V}_2^{(out)}(s)\right]
\end{aligned}
\tag{2.15}
$$

We can calculate the scattering matrices in terms of the impedance (or admittance) matrices [3] as

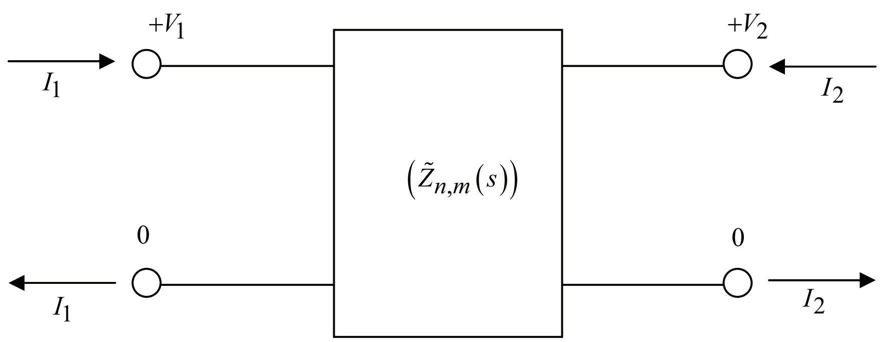


Fig. 2.1 Separation of Incoming and Outgoing Waves.

$$\begin{aligned}\left(\tilde{S}_{n,m}(s)\right) &= \left[Y_c\left(\tilde{Z}_{n,m}(s)\right) + \left(1_{n,m}\right)\right]^{-1} \quad \left[Y_c\left(\tilde{Z}_{n,m}(s)\right) - \left(1_{n,m}\right)\right] \\ &= \left[Y_c\left(\tilde{Z}_{n,m}(s)\right) - \left(1_{n,m}\right)\right] \quad \left[Y_c\left(\tilde{Z}_{n,m}(s)\right) + \left(1_{n,m}\right)\right]^{-1} \\ &= \left(\tilde{R}'_{n,m}(s)\right)\end{aligned} \tag{2.16}$$

noting that the matrices commute. Thus, we have

$$\begin{aligned}\begin{pmatrix}\tilde{V}_1^{(out)}(s) \\ \tilde{V}_2^{(out)}(s)\end{pmatrix} &= \left(\tilde{S}_{n,m}(s)\right) \quad \begin{pmatrix}\tilde{V}_1^{(in)}(s) \\ \tilde{V}_2^{(in)}(s)\end{pmatrix} \\ &= \left(\tilde{R}'_{n,m}(s)\right) \quad \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \quad \begin{pmatrix}\tilde{V}_2^{(in)}(s) \\ \tilde{V}_1^{(in)}(s)\end{pmatrix} \\ \left(\tilde{R}_{n,m}(s)\right) \equiv \left(\tilde{R}'_{n,m}(s)\right) \quad \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} &= \left(\tilde{S}_{n,m}(s)\right) \quad \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\end{aligned} \tag{2.17}$$

consistent with our previous definitions. As in the case of $(\tilde{R}_{n,m}(j\omega))$ for lossless scatterers, $(\tilde{S}_{n,m}(j\omega))$ is unitary, i.e.,

$$\left(\tilde{S}_{n,m}(j\omega)\right) \quad \left(\tilde{S}_{n,m}(j\omega)\right)^{\dagger} = \left(1_{n,m}\right) = \left(\tilde{S}_{n,m}(s)\right) \quad \left(\tilde{S}_{n,m}-(s)\right)^{T} \tag{2.18}$$

3. Scattering Symmetries

From (2.16) we find

$$\left(\tilde{S}_{n,m}(s)\right)^T = \left(\tilde{S}_{n,m}(s)\right) \tag{3.1}$$

due to the symmetry (reciprocity) of $(\tilde{Z}_{n,m}(s))$. Thus, the transmission of a wave from left to right is the same as from right to left. This symmetry does not extend to left-to-left versus right-to-right scattering. Of course, if the network characterized by $(\tilde{Z}_{n,m}(s))$ has left/right symmetry, then

$$\tilde{Z}_{1,1}(s) = \tilde{Z}_{2,2}(s) \tag{3.2}$$

and left-left and right-right scattering are equal, but this is not the general case.

For purposes of a forward-scattering result we find $(\tilde{R}_{n,m}(s))$ more convenient to use. In this case we have

$$\begin{aligned}
\left(\tilde{R}_{n,m}(s)\right)^T &= \begin{pmatrix} \tilde{R}_{1,1}(s) & \tilde{R}_{2,1}(s) \\ \tilde{R}_{1,2}(s) & \tilde{R}_{2,2}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\tilde{S}_{n,m}(s)\right)^T \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\tilde{S}_{n,m}(s)\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\tilde{R}_{n,m}(s)\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \tilde{R}_{2,2}(s) & \tilde{R}_{2,1}(s) \\ \tilde{R}_{1,2}(s) & \tilde{R}_{1,1}(s) \end{pmatrix} \\
\tilde{R}_{1,1}(s) &= \tilde{R}_{2,2}(s)
\end{aligned} \tag{3.3}$$

This describes the fact that left-to-right scattering equals right-to-left scattering. From (2.10) this gives

$$\begin{aligned}
\left(\tilde{\Sigma}_{1,1}(s)\right) &= \left(\tilde{\Sigma}_{2,2}(s)\right) \\
\left(\tilde{\Sigma}'_{1,2}(s)\right) &= \left(\tilde{\Sigma}'_{2,1}(s)\right)
\end{aligned} \tag{3.4}$$

4. Restriction to Lossless Networks

Lossless networks satisfy Foster's theorem. As discussed in [5] this implies

$$\left(\tilde{Z}_{n,m}(-s)\right) = -\left(\tilde{Z}_{n,m}(s)\right) \qquad \text{(odd function of } s\text{)} \tag{4.1}$$

For the single-port case we have scalar impedance and admittance functions with properties as previously summarized.

Returning to (2.16) we can note that

$$\left(\tilde{S}_{n,m}(-s)\right)^{-1} = \left(\tilde{S}_{n,m}(s)\right) \tag{4.2}$$

by substitution from (4.1). This is an alternate way to obtain (2.18) without appeal to analytic continuation.

5. Question of First-Order Poles

One of the continuing questions concerning the singularity expansion method (SEM) concerns the order of the scattering poles. As a practical matter, first-order poles are what one encounters. The case of a perfectly conducting sphere has only first-order scattering poles [1]. In [4] it is claimed that this is the case for perfectly conducting (finite size) scatterers. It has also been shown [2] that resistive-loaded bodies can have second-order poles (for a thin wire in this case). In [2] a synthesis procedure for impedance-loaded scatterers allows, in principle, for even higher-order scattering poles. Let us shed a little more light on this question.

For this purpose, consider the canonical problem in Fig. 5.1. In this case the LC network has

$$\left(\tilde{Z}_{n,m}(s)\right) = \left[sL + \frac{1}{sC}\right]\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \tilde{Z}(s)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{5.1}$$

This can be established by placing a unit current into either of the two ports and noting that V_1 and V_2 are the same. While this is a physically realizable network, it has the strange property

$$\det\left(\left(\tilde{Z}_{n,m}(s)\right)\right) = 0 \tag{5.2}$$

Independent of s.

The scattering matrix has poles at

$$\begin{aligned} \det\left(Y_c\left(Z_{n,m}(s_\alpha)\right) + \left(1_{n,m}\right)\right) &= 0 \\ &= \det\left(Y_c\tilde{Z}(s_\alpha)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \left[Y_c\tilde{Z}(s_\alpha) + 1\right]^2 - \left[Y_c\tilde{Z}(s_\alpha)\right]^2 \\ &= 2Y_c\tilde{Z}(s_\alpha) + 1 \end{aligned} \tag{5.3}$$

Rearranging, we have

$$\begin{aligned} 0 &= \frac{Z_c}{2} + \tilde{Z}(s_\alpha) \\ &= \frac{Z_c}{2} + s_\alpha L + \frac{1}{s_\alpha c} \\ 0 &= s_\alpha^2 LC + \frac{s_\alpha c Z_c}{2} + 1 \end{aligned} \tag{5.4}$$

Solving the quadratic gives

$$s_\alpha = \frac{-\frac{cZ_c}{2} \pm \left[\left[\frac{cZ_c}{2}\right]^2 - 4LC\right]^{1/2}}{2LC} \tag{5.5}$$

The two roots coalesce into a second-order zero at

$$\begin{aligned} \frac{cZ_c}{2} &= 2[LC]^{1/2} \\ Z_c &= 4\left[\frac{L}{C}\right]^{1/2} \\ s_\alpha &= -[LC]^{-1/2} \end{aligned} \tag{5.6}$$

which gives a second-order pole on the negative *Re(s)* axis (critical damping).

This example shows that it is possible to have a second-order scattering pole from a passive, lossless, reciprocal network. Due to the symmetry in Fig. 5.1, both the forward scattering and the backscattering have this second-order pole.

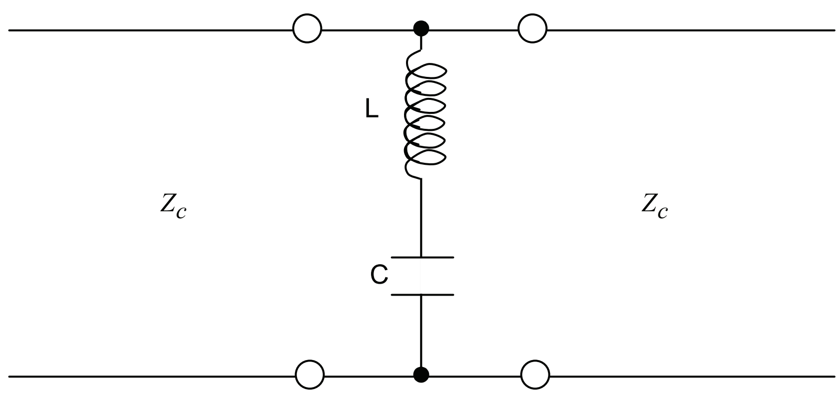


Fig. 5.1 Example of a Lossless Scattering Network for a Second-order Pole.

6. Forward-Scattering Theorem

Now, from Section 2, let us consider the forward scattering from the junction of the left and right transmission lines due to the network at this junction. Recall from Section 3 that forward scattering has

$$
\begin{aligned}
\tilde{S}_{1,2}(s) &= \tilde{S}_{2,1}(s) \\
\tilde{R}_{1,1}(s) &= \tilde{R}_{2,2}(s)
\end{aligned}
\tag{6.1}
$$

This is like the three-dimensional electromagnetic scattering in which the scattering dyadic is symmetric on reversal of the direction of incidence (with a transpose of the dyadic) [8].

Returning to (2.11) let there be a pole (first-order) at s_α. This takes the form

$$
\left(\tilde{\Sigma}_{n,m}(s)\right) = (a,b)(a,b)\left[s-s_\alpha\right]^{-1} \text{ as } s \to s_\alpha \tag{6.2}
$$

The (*a,b*) is the natural mode vector ($\vec{c}_\alpha(\vec{1}_r)$ in the electromagnetic case [6]). In (2.11) this gives

$$
\begin{aligned}
(a,b)(a,b) + (a,b)(a,b) \quad \left(\tilde{\Sigma}_{n,m}(-s_\alpha)\right)^T &= \left(0_{n,m}\right) \\
\left[\left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(-s_\alpha)\right)\right] \quad (a,b)(a,b) &= \left(0_{n,m}\right)
\end{aligned}
\tag{6.3}
$$

Dot multiplying on the right by (a^{-1},b^{-1}) gives

$$
\left[\left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(-s_\alpha)\right)\right] \quad (a,b) = (0,0) \tag{6.4}
$$

This has a solution iff

$$
\begin{aligned}
&\det\left(\left(1_{n,m}\right) + \left(\tilde{\Sigma}_{n,m}(-s_\alpha)\right)\right) = 0 \\
&= \left[1 + \tilde{\Sigma}_{1,1}(-s_\alpha)\right]\left[1 + \tilde{\Sigma}_{2,2}(-s_\alpha)\right] - \tilde{\Sigma}_{1,2}(-s_\alpha)\tilde{\Sigma}_{2,1}(-s_\alpha)
\end{aligned}
\tag{6.5}
$$

shich is an equation to be solved for s_α. Having found s_α, (6.4) can, in turn, be solved for the natural-scattering-mode vector (*a,b*) to within an arbitrary scalar multiplier.

This result is an analog of the result in [6], where the natural-mode vector is an integral (here a sum) over all 4π directions.

From (3.4) we now have

$$\begin{aligned}
&\left[1 + \tilde{\Sigma}_{1,1}(-s_\alpha)\right]a + \tilde{\Sigma}_{1,2}(-s_\alpha)b = 0 \\
&\tilde{\Sigma}_{2,1}(-s_\alpha)a + \left[1 + \tilde{\Sigma}_{2,2}(-s_\alpha)\right]b = 0 \\
&\tilde{\Sigma}_{2,1}(-s_\alpha)\frac{a}{b} = \tilde{\Sigma}_{1,2}(-s_\alpha)\frac{b}{a} \\
&\frac{b}{a} = \pm\left[\tilde{\Sigma}_{2,1}(-s_\alpha)/\tilde{\Sigma}_{1,2}(-s_\alpha)\right]^{1/2}
\end{aligned} \tag{6.6}$$

Which sign to choose depends on more information.

There are various lossless networks that one might place at the transmission-line junction. If we posit a left-right symmetry to the network, then (3.2) applies. In turn (2.1) implies

$$\begin{aligned}
&\tilde{S}_{1,1}(s) = \tilde{S}_{2,2}(s) \\
&\tilde{R}_{1,2}(s) = \tilde{R}_{2,1}(s) \\
&\tilde{\Sigma}_{1,2}(s) = \tilde{\Sigma}_{2,1}(s)
\end{aligned} \tag{6.7}$$

From (6.5) this gives

$$\frac{b}{a} = \pm 1 \tag{6.8}$$

For the network in Fig. 5.1, clearly by symmetry

$$\begin{aligned}
&\frac{b}{a} = 1 \\
&(a,b) = (1,1) \qquad \text{(acceptable choice)}
\end{aligned} \tag{6.9}$$

for the natural-scattering-mode vector.

7. Concluding Remarks

So now we have found that scattering from a lossless, reciprocal network at a transmission-line junction can have a second order s-plane pole. This can contribute to the discussion concerning 3-dimensional electromagnetic scattering from lossless, as well as perfectly conducting, targets.

There are various other cases to consider, such as lossless dielectric targets and perfectly conducting targets. Spherical geometry can help in this regard, giving special cases for consideration.

We would like to thank Peter Dorato, Tom Hagstrom, and Ronald Chen for discussions concerning this subject.

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