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Product Integral for Lossless NMTLs

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Abstract

In solving the product integral for nonuniform multiconductor transmission lines (NMTLs), there are various special cases of interest leading to special solutions with special properties. For simple lossless NMTLs this paper develops a power (Taylor) series in complex frequency for the product integral, including for each of its four matrix blocks. The coefficients involve only the spatial form of the NMTL. This result is appropriate for low-frequency evaluation of the product integral.

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1. Introduction

This paper extends some of the results in [1 (Section 6)] concerning the blocks of the product integral (or matrizant) for lossless NMTLs (nonuniform multiconductor transmission lines). There it is shown that the diagonal blocks are even functions of the complex frequency, *s*, and the off-diagonal blocks are odd functions of *s*.

Summarizing, the telegrapher equations are

$$\begin{aligned}
&\frac{\partial}{\partial z}\left(\tilde{V}_n(z,s)\right) = -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right) + \left(\tilde{V}_n^{(s)'}(z,s)\right)\\
&\frac{\partial}{\partial z}\left(\tilde{I}_n(z,s)\right) = -\left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{V}_n(z,s)\right) + \left(\tilde{I}_n^{(s)'}(z,s)\right)\\
&s = \Omega + j\omega \equiv \text{two-sided Laplace-transform variable or complex frequency}\\
&\sim \; \equiv \text{two-sided Laplace transform}
\end{aligned} \tag{1.1}$$

For our present discussion the per-unit-length impedance and admittance matrices first take the simple forms

$$\begin{aligned}
\left(\tilde{Z}'_{n,m}(z,s)\right) &= s\left(L'_{n,m}(z)\right) = s\left(L'_{n,m}(z)\right)^T\\
\left(\tilde{Y}_{n,m}(z,s)\right) &= s\left(C'_{n,m}(z)\right) = s\left(C'_{n,m}(z)\right)^T
\end{aligned} \tag{1.2}$$

for a simple lossless (with reciprocity) NMTL.

The N-component voltage and current vectors and the N x N matrices are combined in supervector/ supermatrix form as

$$\begin{aligned}
&\frac{\partial}{\partial z}\begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ \left(Z_{n,m}\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix}\\
&= -\begin{pmatrix}\left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(z,s)\right) = \left(Y_{n,m}\right)\\ \left(Z_{n,m}\right) = \left(\tilde{Y}'_{n,m}(s)\right) & \left(0_{n,m}\right)\end{pmatrix} \odot \begin{pmatrix}\left(\tilde{V}_n(z,s)\right)\\ \left(Z_n\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix}\\
&\quad + \begin{pmatrix}\left(\tilde{V}_n^{(s)'}(z,s)\right)\\ \left(Z_{n,m}\right)\cdot\left(\tilde{I}_n^{(s)'}(z,s)\right)\end{pmatrix}
\end{aligned} \tag{1.3}$$

Here the normalizing impedance matrix is chosen as frequency independent with

$$\left(Z_{n,m}\right) = \left(Z_{n,m}\right)^T = \left(Y_{n,m}\right)^{-1} \tag{1.4}$$

This is solved via the supermatrizant differential equation

$$\begin{gathered}
\frac{\partial}{\partial z}\left(\left(\tilde{U}_{n,m}\left(z,z_0;s\right)\right)_{\upsilon,\upsilon'}\right) = \left(\left(\tilde{\Gamma}_{n,m}\left(z,s\right)\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{U}_{n,m}\left(z,z_0;s\right)\right)_{\upsilon,\upsilon'}\right) \\
\left(\left(\tilde{\Gamma}_{n,m}\left(z,s\right)\right)_{\upsilon,\upsilon'}\right) = -\begin{pmatrix} \left(1_{n,m}\right) & \left(Z'_{n,m}\left(z,s\right)\right) \cdot \left(Y_{n,m}\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{Y}'_{n,m}\left(z,s\right)\right) & \left(1_{n,m}\right) \end{pmatrix} \\
\left(\left(\tilde{U}_{n,m}\left(z,z_0;s\right)\right)_{\upsilon,\upsilon'}\right) = \left(\left(1_{n,m}\right)_{\upsilon,\upsilon'}\right) = \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(1_{n,m}\right) \end{pmatrix} \quad \text{(boundary condition)}
\end{gathered} \tag{1.5}$$

which is solved as the product integral

$$\left(\left(\tilde{U}_{n,m}\left(z,s\right)\right)_{\upsilon,\upsilon'}\right) = \prod_{z_0}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}\left(z',s\right)\right)_{\upsilon,\upsilon'}\right)dz'} \tag{1.6}$$

In terms of this the solution for voltage and current vectors is found as

$$\begin{gathered}
\begin{pmatrix} \left(\tilde{V}_n\left(z,s\right)\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(z,s\right)\right) \end{pmatrix} = \left(\left(\tilde{U}_{n,m}\left(z,z_0;s\right)\right)_{\upsilon,\upsilon'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n\left(z_0,s\right)\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n\left(z_0,s\right)\right) \end{pmatrix} \\
+\int_{z_0}^{z} \left(\left(\tilde{U}_{n,m}\left(z,z';s\right)\right)_{\upsilon,\upsilon'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(s)'}\left(z',s\right)\right) \\ \left(Z_{n,m}\right) \cdot \left(\tilde{I}_n^{(s)'}\left(z',s\right)\right) \end{pmatrix} dz'
\end{gathered} \tag{1.7}$$

It is then the properties of the product integral that we need to understand.

2. Power-Series Expansion of the Product Integral

A previous paper [2] considers an appropriate high-frequency solution of the NMTL equations, giving a generalized WKB solution. For low frequencies we can think of a power (Taylor) series around $s = 0$. There are various ways to approach this problem (perhaps all equivalent). For present purposes let us use the matrizant series which takes the form

$$\begin{aligned}
\left(\left(\tilde{U}_{n,m}(z, z_0; s) \right)_{\upsilon,\upsilon'} \right) &= \sum_{\ell=0}^{\infty} \left(\left(\tilde{U}_{n,m}(z, z_0; s) \right)_{\upsilon,\upsilon'} \right)_{\ell} \\
\left(\left(\tilde{U}_{n,m}(z, z_0; s) \right)_{\upsilon,\upsilon'} \right)_{0} &= \left(\left(1_{n,m} \right)_{\upsilon,\upsilon'} \right) \\
\left(\left(\tilde{U}_{n,m}(z, z_0; s) \right)_{\upsilon,\upsilon'} \right)_{\ell+1} &= \int_{z_0}^{z} \left(\left(\tilde{\Gamma}_{n,m}(z', s) \right)_{\upsilon,\upsilon'} \right) \odot \left(\left(\tilde{U}_{n,m}(z', s) \right)_{\upsilon,\upsilon'} \right)_{\ell} dz'
\end{aligned} \tag{2.1}$$

The ℓth term in this series is an ℓ-fold integral of the product integrand. Note that the zeros on the diagonal of the product integrand immediately imply

$$tr\left(\left(\left(\tilde{\Gamma}_{n,m}(z, s) \right)_{\upsilon,\upsilon'} \right) \right) = 0 \quad , \quad \det\left(\left(\left(\tilde{U}_{n,m}(z, z_0; s) \right)_{\upsilon,\upsilon'} \right) \right) = 1 \tag{2.2}$$

Consider the special case of a simple lossless NMTL characterized by

$$\begin{aligned}
\left(\tilde{Z}'_{n,m}(z, s) \right) &= s \left(L'_{n,m}(z) \right) \\
\left(\tilde{Y}'_{n,m}(z, s) \right) &= s \left(C'_{n,m}(z) \right)
\end{aligned} \tag{2.3}$$

where frequency and space are now separated. Then we can write

$$\begin{aligned}
\left(\left(\tilde{\Gamma}_{n,m}(z, s) \right)_{\upsilon,\upsilon'} \right) &= -s \left(\left(P_{n,m}(z) \right)_{\upsilon,\upsilon'} \right) \\
\left(\left(P_{n,m}(z) \right)_{\upsilon,\upsilon'} \right) &= \begin{pmatrix} \left(0_{n,m} \right) & \left(L'_{n,m}(z) \right) \cdot \left(Y_{n,m} \right) \\ \left(Z_{n,m} \right) \cdot \left(C'_{n,m}(z) \right) & \left(0_{n,m} \right) \end{pmatrix}
\end{aligned} \tag{2.4}$$

so that (2.1) becomes

$$\left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\upsilon,\upsilon'}\right) = \sum_{\ell=0}^{\infty} s^{\ell}(-1)^{\ell}\left(\left(Q_{n,m}(z,z_0)\right)_{\upsilon,\upsilon'}\right)$$
$$\left(\left(Q_{n,m}(z,z_0)\right)_{\upsilon,\upsilon'}\right)_0 = \left(\left(1_{n,m}\right)_{\upsilon,\upsilon'}\right) \tag{2.5}$$
$$\left(\left(Q_{n,m}(z,z_0)\right)_{\upsilon,\upsilon'}\right)_{\ell+1} = \int_{z_0}^{z}\left(\left(P_{n,m}(z')\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(Q_{n,m}(z',z_0)\right)_{\upsilon,\upsilon'}\right)_{\ell} dz'$$

Here we have a power series in s with supermatrix spatial coefficients.

Let us now separate out some of the terms in this series. Noting that $((P_{n,m})_{\upsilon,\upsilon'})$ can be regarded as an operator we can write

$$\left(\left(P_{n,m}(z(\bullet))\right)_{\upsilon,\upsilon'}\right) = \int_{z_0}^{z}\left(\left(P_{n,m}(z')\right)_{\upsilon,\upsilon'}\right) \odot (\bullet)dz' \tag{2.6}$$

where the operand is also a function of z', the operation being multiplication folowed by integration over z'. Then let us also define

$$\begin{aligned}&\left(R_{n,m}(z,(\bullet))_{\upsilon,\upsilon'}\right)\\ &\equiv \int_{z_0}^{z}\left(P_{n,m}(z_1)\right)_{\upsilon,\upsilon'} \odot \int_{z_0}^{z_1}\left(P_{n,m}(z_2)\right)_{\upsilon,\upsilon'} \odot (\bullet)dz_2dz_1\\ &= P_{n,m}(z,(\bullet)) \odot \left(P_{n,m}(z,(\bullet))\right)\end{aligned} \tag{2.7}$$

with this shorthand operator notation denoting successive operations beginning on the right and continuing to the left. For this purpose we note that

$$\begin{aligned}&\left(P_{n,m}(z_1)\right)_{\upsilon,\upsilon'} \odot \left(P_{n,m}(z_2)\right)_{\upsilon,\upsilon'}\\ &= \begin{pmatrix}\left(L'_{n,m}(z_1)\right) \bullet \left(C'_{n,m}(z_2)\right) & \left(0_{n,m}\right)\\ \left(0_{n,m}\right) & \left(Z_{n,m}\right)\bullet\left(C'_{n,m}(z_1)\right)\bullet\left(L'_{n,m}(z_2)\right)\bullet\left(Y_{n,m}\right)\end{pmatrix}\\ &= \left[\left(L'_{n,m}(z_2)\right)\bullet\left(C'_{n,m}(z_2)\right)\right] \oplus \left[\left(Z_{n,m}\right)\bullet\left(C'_{n,m}(z_1)\right)\bullet\left(L'_{n,m}(z_2)\right)\bullet\left(Y_{n,m}\right)\right]\end{aligned} \tag{2.8}$$

which is block diagonal. Successive application of the (2.7) operator then is also block diagonal with the upper left blocks separately combining without the lower right blocks, and conversely. So let us write

$$
\begin{aligned}
\left(\left(R_{n,m}(z,(\bullet))\right)_{\upsilon,\upsilon'}\right) &= \left(R_{n,m}(z,(\bullet))\right)_{1,1} \oplus \left(R_{n,m}(z,(\bullet))\right)_{2,2} \\
\left(\left(R_{n,m}(z,(\bullet))\right)_{\upsilon,\upsilon'}\right)^{\ell} &= \left(R_{n,m}(z,(\bullet))\right)_{1,1}^{\ell} \oplus \left(R_{n,m}(z,(\bullet))\right)_{2,2}^{\ell} \\
\left(\left(R_{n,m}(z,(\bullet))\right)_{\upsilon,\upsilon'}\right)^{0} &= \left(\left(1_{n,m}\right)_{\upsilon,\upsilon'}\right)
\end{aligned}
\tag{2.9}
$$

Consider the case of ℓ even. Then (2.1) gives zero for off-diagonal blocks leaving only diagonal blocks as

$$
\begin{aligned}
\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{1,1} &= \left(1_{n,m}\right) + \sum_{\ell=2}^{\infty,2} s^{\ell} \left(R_{n,m}(z,(\bullet))\right)_{1,1}^{\frac{\ell}{2}} \odot \left(1_{n,m}\right) \\
\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{2,2} &= \left(1_{n,m}\right) + \sum_{\ell=2}^{\infty,2} s^{\ell} \left(R_{n,m}(z,(\bullet))\right)_{2,2}^{\frac{\ell}{2}} \odot \left(1_{n,m}\right)
\end{aligned}
\tag{2.10}
$$

The second number above the summation sign indicates the successive values of ℓ by successively adding this number (giving only even terms in this case). This, of course, shows that the diagonal blocks are even functions of s, consistent with [1]. Note that all terms in (2.10) are N x N instead of 2N x 2N.

For ℓ odd, we note that we have terms of the form

$$
\begin{aligned}
&\left(\left(P_{n,m}(z,(\bullet))\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(R_{n,m}(z,(\bullet))\right)_{\upsilon,\upsilon'}\right)^{\frac{\ell-1}{2}} \\
&= \begin{pmatrix} \left(0_{n,m}\right) & \left(P_{n,m}(z,(\bullet))\right)_{1,2} \odot \left(R_{n,m}(z,(\bullet))\right)_{2,2}^{\frac{\ell-1}{2}} \\ \left(P_{n,m}(z,(\bullet))\right)_{2,1} \odot \left(R_{n,m}(z,(\bullet))\right)_{1,1}^{\frac{\ell-1}{2}} & \left(0_{n,m}\right) \end{pmatrix}
\end{aligned}
\tag{2.11}
$$

This leaves zero for the diagonal blocks. The off-diagonal blocks then are

$$
\begin{aligned}
\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{1,2} &= \sum_{\ell=1}^{\infty,2} s^{\ell} \left(P_{n,m}(z,(\bullet))\right)_{1,2} \odot \left(R_{n,m}(z,(\bullet))\right)_{1,1}^{\frac{\ell-1}{2}} \odot \left(1_{n,m}\right) \\
\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{2,1} &= \sum_{\ell=1}^{\infty,2} s^{\ell} \left(P_{n,m}(z,(\bullet))\right)_{2,1} \odot \left(R_{n,m}(z,(\bullet))\right)_{1,1}^{\frac{\ell-1}{2}} \odot \left(1_{n,m}\right)
\end{aligned}
\tag{2.12}
$$

The off-diagonal blocks are odd functions of s with coefficients involving only N x N terms (also consistent with [1]).

3. Special Case of Uniform MTL

As a check on the above formulae let $(L'_{n,m})$ and $(C'_{n,m})$ be independent of z. Then from (2.5) we have

$$\begin{aligned}\left(\left(Q_{n,m}(z,z_0)\right)_{\upsilon,\upsilon'}\right)_{\ell+1} &= \left(\left(P_{n,m}\right)_{\upsilon,\upsilon'}\right) \odot \int_{z_0}^{z} \left(\left(Q_{n,m}(z',z_0)\right)_{\upsilon,\upsilon'}\right)_{\ell} dz' \\ \left(\left(Q_{n,m}(z,z_0)\right)_{\upsilon,\upsilon'}\right) &= \left(\left(P_{n,m}\right)_{\upsilon,\upsilon'}\right)^{\ell} \int_{z_0}^{z}\int_{z_0}^{z_{\ell-1}}\cdots\int_{z_0}^{z_1} dz_1 \cdots dz_{\ell-1} = \frac{[z-z_0]^{\ell}}{\ell!}\left(\left(P_{n,m}\right)_{\upsilon,\upsilon'}\right)^{\ell}\end{aligned} \tag{3.1}$$

This gives a series for the exponential so that

$$\left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\upsilon,\upsilon'}\right) = \sum_{\ell=0}^{\infty}\frac{[-s[z-z_0]]^{\ell}}{\ell!}\left(\left(P_{n,m}\right)_{\upsilon,\upsilon'}\right)^{\ell} = e^{-s[z-z_0]\left(\left(P_{n,m}\right)_{\upsilon,\upsilon'}\right)} \tag{3.2}$$

which is a well-known solution for a uniform MTL.

The blocks also separate out. From (2.8) and (2.10) the diagonal blocks are

$$\begin{aligned}\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{1,1} &= \cosh\left(s[z-z_0]\left[\left(L'_{n,m}\right)\bullet\left(C'_{n,m}\right)\right]^{1/2}\right) \\ \left(\tilde{U}_{n,m}(z,z_0;s)\right)_{2,2} &= \cosh\left(s[z-z_0]\left[\left(Z_{n,m}\right)\bullet\left(C'_{n,m}\right)\bullet\left(L'_{n,m}\right)\bullet\left(Y_{n,m}\right)\right]^{1/2}\right) \\ &= \left(Z_{n,m}\right)\bullet\cosh\left(s[z-z_0]\left[\left(C'_{n,m}\right)\bullet\left(L'_{n,m}\right)\right]^{1/2}\right)\bullet\left(Y_{n,m}\right)\end{aligned} \tag{3.3}$$

Similarly the off-diagonal blocks are

$$\begin{aligned}\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{1,2} &= -\left(L'_{n,m}\right)\bullet\left[\left(C'_{n,m}\right)\bullet\left(L'_{n,m}\right)\right]^{-1/2}\bullet\sinh\left(s[z-z_0]\left[\left(C'_{n,m}\right)\bullet\left(L'_{n,m}\right)\right]^{1/2}\right)\bullet\left(Y_{n,m}\right) \\ \left(\tilde{U}_{n,m}(z,z_0;s)\right)_{2,1} &= -\left(Z_{n,m}\right)\bullet\left(C'_{n,m}\right)\bullet\left[\left(L'_{n,m}\right)\bullet\left(C'_{n,m}\right)\right]^{-1/2}\bullet\sinh\left(s[z-z_0]\left[\left(L'_{n,m}\right)\bullet\left(C'_{n,m}\right)\right]^{1/2}\right)\end{aligned} \tag{3.4}$$

Note that these terms can be grouped in various ways. One often collects terms as

$$\begin{aligned}\left(\tilde{\gamma}_{n,m}(s)\right) &= s\left[\left(L'_{n,m}\right)\bullet\left(C'_{n,m}\right)\right]^{1/2} \\ \left(\tilde{\gamma}_{n,m}(s)\right)^{T} &= s\left[\left(C'_{n,m}\right)\bullet\left(L'_{n,m}\right)\right]^{1/2}\end{aligned} \tag{3.5}$$

which is a matrix propagation constant.

4. Extension to Other Frequency Dependences

The form of the results in Section 2 admits of a generalization provided we can write, for the per-unit-length matrices, the factored forms

$$\begin{aligned}\left(\tilde{Z}'_{n,m}(z,s)\right) &= \tilde{f}_h(s)\left(L'_{n,m}(z)\right)\\ \left(\tilde{C}'_{n,m}(z,s)\right) &= \tilde{f}_e(s)\left(C'_{n,m}(z)\right)\end{aligned} \tag{4.1}$$

This can correspond to special cases of lossy/dispersive NMTLs, e.g., wires in a uniform lossy/dispersive medium.

In this case $\tilde{f}_h$ and $\tilde{f}_e$ take the place of s in the appropriate expressions. In particular, in (2.10) replace

$$s^{\ell} \rightarrow \tilde{f}_h^{\ell/2}(s)\,\tilde{f}_e^{\ell/2}(s) \text{ for both 1,1 and 2,2 blocks} \tag{4.2}$$

In (2.12) replace

$$\begin{aligned}s^{\ell} &\rightarrow \tilde{f}_h^{[\ell+1]/2}(s)\,\tilde{f}_e^{[\ell-1]/2}(s) \quad \text{for 1,2 block}\\ s^{\ell} &\rightarrow \tilde{f}_h^{[\ell-1]/2}(s)\,\tilde{f}_e^{[\ell+1]/2}(s) \quad \text{for 2,2 block}\end{aligned} \tag{4.3}$$

5. Concluding Remarks

Now we have the even/odd properties of the matrix blocks of the product integral for lossless NMTLs explicitely in terms of power series involving even/odd powers of s, as appropriate. This form is particularly suitable for low-frequency evaluation of the product integral. This complements the high-frequency form in [2].

These results are suggestive of solutions for other special cases of NMTLs, some of which may appear in the future.

References

1. C. E. Baum, “Reciprocity, Energy, and Norms for Propagation on Nonuniform Multiconductor Transmission Lines:, Interaction Note 583, April 2003.

2. C. E. Baum, “High-Frequency Propagation on Nonuniform Multiconductor Transmission Lines”, Interaction Note 589, October 2003.