Interaction Notes

Note 592

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Newest Developments in Transmission-Line Theory and Applications

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Abstract

For a thin wire model we establish a connection between the global (physical) representation of the parameters for a generalised (full-wave) transmission line theory, investigated earlier, and the modal representation of the parameters, which are contained in the coupling equations for each mode. These parameters are complex-valued, frequency- and gauge-dependent, and they depend on the local coordinate or on the modal number, respectively. With the concept of generalised transmission-line (TL) parameters it is shown that a thick wire can be treated as a multiconductor transmission line.

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I. Introduction

Linear structures constitute an essential part in electrical and electronic circuits, switch-boards, devices, systems and buildings. Therefore, it is unavoidable to treat the electromagnetic interaction of electromagnetic fields with these structures, and in recent days this interaction has to take into account higher and higher frequencies, currently reaching up to several GHz. Thus the usual, classical transmission line theory does no longer cover modern requirements for complex systems and the need for a great accuracy has led to the need for better models. Such models have to be capable to model very complex geometries, like finite lines, nonuniform line conditions with curvature and torsion, including periodic structures. They have to include radiation losses and non-TEM coupling at higher frequencies. Even for thick cables or cable-bundles at high frequencies azimuthal current distributions will became of remarkable amplitude and have therefore also to be treated in an extended new transmission-line theory.

In the present paper we describe a new model which is based on the exact system of the electric field integral equations for the "current and potential" pair, which can be cast in the form of telegrapher equations. Therefore, most of the existing techniques to solve such equations can be applied. In general, the new theory is computationally more efficient than other full-wave methods in certain applied problems. Other advantages are the possibilities to derive a physical interpretation of the new line parameters and to establish a relation between thick wires and multiconductor lines. The new line parameters become complex-valued, gauge-dependent, and they depend on the local coordinate and on frequency. In the modal representation of these parameters their imaginary parts are related to the modal radiation resistances. They can be transformed into their global (physical) representation with the aid of lengthy (mathematical) expressions.

New results are also presented for a thick transmission line above perfectly conducting ground. In this case the usual telegrapher equations are completed by an additional (third) equation for the angle-component of the current. The inductance per unit length and the capacitance per unit length become matrices of (in general) infinite dimensions. Besides the "longitudinal" inductance matrix we also obtain an "azimuthal" one. The three coupled telegrapher equations for the longitudinal component of the current, the azimuthal component of the current, and the scalar potential can be reduced to the usual two equations. In these equations, however, the line matrices become modified again. Another interesting result deals with the proximity effect. Evaluating this effect we can show, that a thick wire formally can be described by multiconductor TL equations. A short comparison with the result of Sommerfeld [19] will be performed.

II. Global Parameters of the Generalised TL Theory in the Thin-Wire Model

We consider a thin wire of arbitrary geometric form, $\vec{r}(l)$, near the perfectly conducting ground (see, for example, a semi-circular loop in Fig. 1 a), which can be loaded and excited by an external field $\vec{E}^i(\vec{r})$ as well as by point sources U_1^0. It is assumed that the line is connected with the ground plane at both ends. Using the reflection principle in electrodynamics it is possible to show, that the problem of "half-loop" excitation is equivalent to the problem of excitation of the complete "closed loop" in the entire space, but with the sources and loads symmetrized with respect to the ground plane (see Fig.1 b).

This corresponds to a change $\vec{E}^i(\vec{r}) \to \vec{E}^e(\vec{r}) = \vec{E}^i(\vec{r}) + \vec{E}^r(\vec{r})$, where $\vec{E}^i(\vec{r})$ and $\vec{E}^r(\vec{r})$ are the incident and reflected electric fields, respectively. $\vec{E}^e(\vec{r})$ is the exciting electric field. All fields are continued into the entire space. To facilitate our calculation we will consider the symmetrized problem.

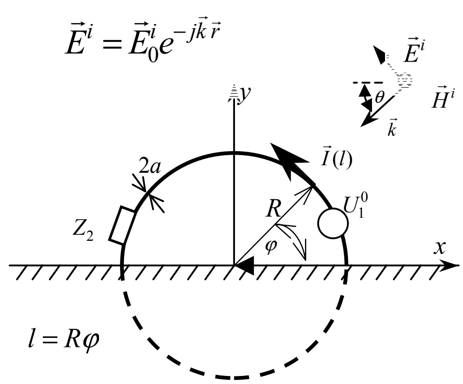

Fig.1a: Excitation of a semi-circular loop.

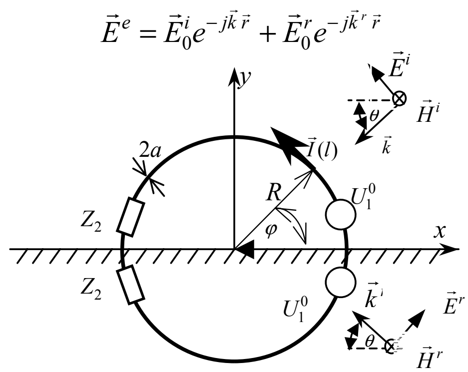

Fig.1b: Equivalent symmetric excitation of a complete circular loop.

The induced current $I(l)$ and the charge density $q(l)$ are sources of the scattered electric field, $\vec{E}_l^{sc}(l)$. The tangential component of the total electric field $\vec{E}_{tot} = \vec{E}^e + \vec{E}^{sc}$ on the surface of the wire has to be zero. As usual in the thin-wire approximation, we assume that the current and charge are distributed along the wire axis and consider the current tangential component only. This yields the boundary condition:

$$E_l^{sc}(l) + E_l^e(l) = 0 \tag{1}$$

Here $E_l^{sc}(l) = \vec{e}_l(l)\cdot\vec{E}^{sc}(l)$ and $E_l^e(l) = \vec{e}_l(l)\cdot\vec{E}^e(l)$, where $\vec{e}_l(l) = \partial\vec{r}(l)/\partial l$ is the unit tangential vector of the curve which describes the wire axis.

The scattered tangential electric field is calculated with the aid of the scalar potential $\Phi(l)$ and the tangential component of the vector potential $A_l(l)$:

$$E_l^{sc}(l) = -j\omega A_l(l) - \partial\Phi(l)/\partial l \tag{2}$$

with (in the Lorenz gauge):

$$\Phi(j\omega, l) = \frac{1}{4\pi\varepsilon_0}\int_0^L g(\vec{r}(l), \vec{r}(l'))\, q(l')dl' ; \quad \vec{A}(j\omega, l) = \frac{\mu_0}{4\pi}\int_0^L \vec{e}_l(l')\, I(l') g(\vec{r}(l), \vec{r}(l'))dl' \tag{3 a,b}$$

$$g(l, l') = \frac{e^{-jkR}}{R} = \exp\left(-jk\sqrt{(\vec{r}(l) - \vec{r}(l'))^2 + a^2}\right) \Big/ \sqrt{(\vec{r}(l) - \vec{r}(l'))^2 + a^2} \tag{4}$$

The function $g(l, l')$ is the scalar Green's function along the line, a the radius of the wire, and $L = \oint dl'$ is the length of the complete closed loop.

Using the continuity equation for the linear charge density and the total tangential current

$$j\omega q(l) + \partial I(l)/\partial l = 0 \tag{5}$$

we can rewrite equations (3 a,b) into a system of integro-differential equations for the pair of functions "current and potential":

$$\begin{cases} \dfrac{\partial \Phi(l)}{\partial l} + j\omega \dfrac{\mu_0}{4\pi} \displaystyle\int_0^L \vec{e}(l)\cdot\vec{e}(l')\, g(l,l')\, I(l')\, dl' = E_l^e(l) \\ \displaystyle\int_0^L g(l,l') \dfrac{\partial I(l')}{\partial l'} dl' + j\omega 4\pi\varepsilon_0\, \Phi(l) = 0 \end{cases} \tag{6 a,b}$$

Now, in order to define the global generalised transmission line parameters, we consider an excitation of the transmission line by a point source U_0^1, located at the beginning of the line. The line is assumed to be loaded by a lumped impedance, Z_2, at the far end. Then the source $E_l^i(l)$ can be written as [1]

$$E^i{}_l(l) = U_1^0 \delta(l-\Delta) - Z_2 I(l)\delta(l - L/2 + \Delta) \qquad \text{where } \Delta \to 0 \tag{7}$$

The corresponding symmetrized exciting tangential electric field looks like

$$\begin{aligned} E^e{}_l(l) = U_1^0 \delta(l-\Delta) + U_1^0 \delta(l+\Delta) - Z_2 I(l)\delta(l - L/2 + \Delta) - Z_2 I(l)\,\delta(l - L/2 - \Delta) \approx \\ 2U_1^0 \delta(l) - 2Z_2 I(L/2)\delta(l - L/2) \end{aligned} \tag{8}$$

Let now the functions $Y(l,l')$, $K(l,l')$ ($[Y] = \Omega^{-1}$, $[K] = 1$) be solutions of the system (6 a,b) for the current and the potential with the δ *(l-l')*-source of unit amplitude located in the point l'.
Then it is possible to show that

$$Y(l',l) = Y(l,l'), \qquad K(l',l) = -K(l,l') \tag{9}$$

Due to the linearity of the considered problem we can write a solution for the total induced current as

$$I(l) = 2U_1^0\, Y(l,0) - 2Z_2\, I(L/2)\, Y(l, L/2) \tag{10}$$

Having this, it is easy to find the unknown current in the point $L/2$ from eq. (10):

$$I(L/2) = 2U_1^0 Y(L/2,0) \big/ \big(1 + 2Z_2 Y(L/2, L/2)\big) \tag{11}$$

With that we write for the total current:

$$I(l) = \widetilde{U}_1^0\, Y(l,0) + \widetilde{U}_2^0\, Y(l, L/2) \tag{12}$$

with $\quad \widetilde{U}_1^0 = 2U_1^0, \qquad \widetilde{U}_2^0 = -2Z_2 I(L/2) \qquad (13)$

It is obvious that any other solution with voltage-like lumped non-uniformities in the termination points (with excitation in point 2 and lumped load in point 1, or with voltage sources located in the points 1 and 2) can be presented in the form (12). For the potential $\Phi(l)$ along the wire we find a similar equation:

$$\Phi(l) = \widetilde{U}_1^0\, K(l,0) + \widetilde{U}_2^0\, K(l, L/2) \tag{14}$$

Now we are ready to look for a system of differential equations for the potential and current in TL-like form:

$$\begin{aligned}\frac{d\Phi(l)}{dl} + j\omega P_{12}(l)\, I(l) + j\omega P_{11}(l)\,\Phi(l) = 0 \\ \frac{dI(l)}{dl} + j\omega P_{21}(l)\,\Phi(l) + j\omega P_{22}(l)\, I(l) = 0\end{aligned} \quad \text{or} \quad \frac{d}{dl}\begin{bmatrix}\Phi(l) \\ I(l)\end{bmatrix} = -j\omega \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix}\begin{bmatrix}\Phi(l) \\ I(l)\end{bmatrix} \tag{15}$$

Substitution of (12) and (14) into (15) and setting the factors in front of the independent constants $\widetilde{U}_1^0$ and $\widetilde{U}_2^0$ to zero yields the elements of the parameter matrix $[P(l)]$:

$$P_{11}(l) = (j\omega)^{-1}\big(K'(l,0)Y(l,L/2) - Y(l,0)\,K'(l,L/2)\big)\big(Y(l,0)K(l,L/2) - Y(l,L/2)\,K(l,0)\big)^{-1} \tag{16}$$
$$P_{12}(l) = -(j\omega)^{-1}\big(K'(l,0)K(l,L/2) - K(l,0)\,K'(l,L/2)\big)\big(Y(l,0)K(l,L/2) - Y(l,L/2)\,K(l,0)\big)^{-1} \tag{17}$$
$$P_{21}(l) = (j\omega)^{-1}\big(Y'(l,0)Y(l,L/2) - Y(l,0)\,Y'(l,L/2)\big)\big(Y(l,0)K(l,L/2) - Y(l,L/2)\,K(l,0)\big)^{-1} \tag{18}$$
$$P_{22}(l) = -(j\omega)^{-1}\big(Y'(l,0)K(l,L/2) - K(l,0)\,Y'(l,L/2)\big)\big(Y(l,0)K(l,L/2) - Y(l,L/2)\,K(l,0)\big)^{-1} \tag{19}$$

The prime at the capital letters *Y* and *K* indicates differentiation with respect to *l*. Thus we have shown that the system of integro-differential equations (6 a,b) with lumped excitation can be reduced to the differential equations (15) with parameters (16)-(19). These parameters (global parameters in a generalised TL theory or the parameters of Maxwellian circuits) are complex-valued and also describe the radiation of the system. They dependent on the geometry of the system and therefore on the local parameter *l* along the line. This fact was established earlier in [2, 3] with the method of product integrals and (up to a notation) in [4], by processing the numerical solutions for current and potential with the Method of Moments. We note that in the early seventieths it was suggested to describe the insulated dipole antenna in a relatively dense medium as a section of a transmission line with distributed radiation loss appearing as a part of the series impedance per unit length [5].
Also the parameter matrix $[P(l)]$ in (15) depends on the gauge of the potentials. For the Coulomb gauge, e.g., it has a different form (because in this case the function $K(l,l')$ is different). One of the ways to obtain gauge-independent parameters is to consider a second order equation for the current $I(z)$ (in [4] this form was given and the parameters were defined on the basis of the analysis of numerical solutions).

$$I''(l) + \mathrm{U}(l)\, I'(l) + \mathrm{T}(l)\, I(l) = 0 \tag{20}$$

The l-dependent coefficients $\mathrm{U}(l)$ and $\mathrm{T}(l)$ can be found by substitution of eq. (12) into eq. (20) and by setting the factors in front of the independent constants $\widetilde{U}_1^0$ and $\widetilde{U}_1^0$ equal to zero. A simpler derivation for them follows from eq. (15). In both cases we obtain the result:

$$\mathrm{U}(l) = -d\ln\big(P_{21}(l)\big)/d\,l + j\omega\big(P_{11}(l) + P_{22}(l)\big) = -d\ln\big(W(l)\big)/dl \tag{21}$$

$$\mathrm{T}(l) = j\omega P_{21}(l)\frac{d}{dz}\left(\frac{P_{22}(l)}{P_{21}(l)}\right) - \omega^2 \det[P(l)] = \frac{Y''(l,0)Y'(l,L/2) - Y'(l,0)Y''(l,L/2)}{W(l)} \qquad (22)$$

where the function $W(l)$ is a Wronskian-like determinant

$$W(l) = \begin{vmatrix} Y'(l,0) & Y(l,0) \\ Y'(l,L/2) & Y(l,L/2) \end{vmatrix} = Y'(l,0)\,Y(l,L/2) - Y(l,0)Y'(l,L/2) \qquad (23)$$

Observe that the parameters in eq.(20) solely can be expressed by the function Y and its first and second derivatives.

III. Modal Parameters in the Thin-Wire Model

By virtue of the fact that all functions in equations (6 a,b) are periodic (with period L) with respect to the distances l and l', we can solve this system by a Fourier series expansion using the complete function system $\left\{e^{-jk_m l}\right\}$,

where $k_m = m \cdot 2\pi/L$, $\qquad m = \ldots -2, -1, 0, 1, 2, \ldots \qquad$ (24 a)

We use the matrix notation

$\left[e^{-jk_m l}\right]_{m_1} = \exp(-jk_{m_1} l)$, $\quad [1]_{m_1} = 1 \qquad$ (24 b)

and introduce modal amplitudes and modal functions for the exciting field and the induced potential and current:

$$E_l^e(l) = \sum_{m=-\infty}^{\infty} E_{l,m}^e \, e^{-jk_m l} = \left[e^{-jk_m l}\right]^T \cdot \left[E_{l,m}^e\right] = [1]^T \cdot \left[E_{l,m}^e(l)\right] \qquad (25)$$

$$\Phi(l) = \sum_{m=-\infty}^{\infty} \Phi_m \, e^{-jk_m l} = \left[e^{-jk_m l}\right]^T \cdot [\Phi_m] = [1]^T \cdot [\Phi_m(l)] \qquad (26)$$

$$I(l) = \sum_{m=-\infty}^{\infty} I_m \, e^{-jk_m l} = \left[e^{-jk_m l}\right]^T \cdot [I_m] = [1]^T \cdot [I_m(l)] \qquad (27)$$

$$[\Phi_m]_{m_1} = \Phi_{m_1}, \quad [\Phi_m(l)]_{m_1} = \Phi_{m_1} \exp(-jk_{m_1} l) \qquad (28)$$

$$[I_m]_{m_1} = I_{m_1}, \quad [I_m(l)]_{m_1} = I_{m_1} \exp(-jk_{m_1} l) \qquad (29)$$

Also for the integrals containing the scalar Green's functions we write the expansions:

$$\left[G^L\right]_{m,m_1} = \frac{1}{L}\int_0^L dl \int_0^L dl' \vec{e}_l(l) \cdot \vec{e}_l(l') g(l,l') e^{-jk_{m_1} l' + jk_m l} = \frac{1}{L}\int_0^L dl e^{-j(k_{m_1} - k_m) l} \int_0^L d\xi \vec{e}_l(l) \cdot \vec{e}_l(l+\xi) g(l,l+\xi) e^{-jk_{m_1}\xi} \qquad (30)$$

$$\left[G^C\right]_{m,m_1} = \frac{1}{L}\int_0^L dl \int_0^L dl' \, g(l,l') e^{-jk_{m_1} l' + jk_m l} = \frac{1}{L}\int_0^L dl e^{-j(k_{m_1} - k_m) l} \int_0^L d\xi \, g(l,l+\xi) e^{-jk_{m_1}\xi} \qquad (31)$$

Applying the orthogonality property of the functions $e^{-jk_m l}$ we derive a system of first-order differential equations for the modal column-matrices of the potential and the current:

$$\begin{cases} \partial[\Phi_m(l)]/\partial l + j\omega[L'(l)]\cdot[I_m(l)] = [E^e_{l,m}(l)] \\ \partial[I_m(l)]/\partial l + j\omega[C'(l)]\cdot[\Phi_m(l)] = 0 \end{cases} \qquad (32\text{ a,b})$$

Formally these equations look like a classical system of Telegrapher equations for a non-uniform multiconductor transmission line, and each mode corresponds to one "wire".
In eqs. (32 a,b) we have introduced per-unit length matrices (of infinite dimension) for the inductance and capacitance and their modal functions $[L'(l)]$, $[C'(l)]$, and modal amplitudes $[L']$, $[C']$:

$$[L'] = \frac{\mu_0}{4\pi}[G^L], \qquad [L'(l)]_{m,m_1} = \frac{\mu_0}{4\pi}[G^L]_{m,m_1}\exp(-j(k_m - k_{m_1})l) \qquad (33\text{ a,b})$$

$$[C'] = 4\pi\varepsilon_0[G^C]^{-1}, \qquad [C'(l)]_{m,m_1} = 4\pi\varepsilon_0\left([G^C]^{-1}\right)_{m,m_1}\exp(-j(k_m - k_{m_1})l) \qquad (34\text{ a,b})$$

Different from the classical line parameters, these parameters depend on the mode indices, they are complex-valued and frequency and gauge dependent (here the Lorenz gauge is used). Their connection with radiation phenomena is analysed in a subsequent section.
Equations (32 a,b) can be combined into one second order differential equation for the column matrix of the modal current $[I_m(l)]$ (Pocklington equation). Its solution is obtained as:

$$I(l) = [e^{-jk_m l}]^T\cdot[I_m], \qquad [I_m] = [Z']^{-1}\cdot[E^e_{l,m}] \qquad (35\text{ a,b})$$

Here

$$[Z'] = j\omega[L'] + \frac{[k_m]\cdot[C']^{-1}\cdot[k_m]}{j\omega}, \; [Z']_{m,m_1} = \frac{1}{4\pi\varepsilon_0 j\omega}\left\{k_m k_{m_1}\cdot[G^C]_{m,m_1} - k^2\cdot[G^L]_{m,m_1}\right\}, \; [k_m]_{m,m_1} = k_m\cdot\delta_{m,m_1}$$

(36 a,b,c)

$[Z']$ is the per-unit length impedance matrix (for modal amplitudes) which relates the current column matrix with the column matrix of the scattered field

$$[E^{sc}_{l,m}] = -[Z'(j\omega)]\cdot[I_m], \qquad E^{sc}(l) = [e^{-jk_m l}]^T\cdot[E^{sc}_{l,m}] \qquad (37\text{ a,b})$$

With this field and the current $I(l)$ we can present the differential-(complex) power density along the line by virtue of the induced-EMF (IEMF) method (in analogy to [6], where the time averaged real part of this value was considered):

$$P_{IEMF}(l) = -E^{sc}_l(l)\,I^*(l) = \left\{[e^{-jk_m l}]^T\cdot[E^{sc}_l]\right\}\cdot\left\{[e^{-jk_m l}]^T\cdot[I_m]\right\}^* \qquad (38)$$

The total radiated energy $\overline{W}$ (time averaged) can be obtained by integration of (38) along the wire from 0 to $L/2$. Using the symmetry property of the current $I(l) = I(-l)$ (which is caused by the symmetrical reflection of the semi-loop and the symmetrization of the sources and loads) and the orthogonality property of the functions $e^{-jk_m l}$ we find:

$$\overline{W} = \frac{L}{4}\mathrm{Re}\left\{[I_m]^{T*}\cdot[Z'(j\omega)]\cdot[I_m]\right\} \tag{39}$$

On the other hand, the power which is radiated by the current distribution (27) in the loop, can be evaluated in the standard way: Calculate the vector potential for the far field and the corresponding electric and magnetic fields, their Poynting vector, and, finally, integrate over angles. After a lengthy and cumbersome calculation one can show that the total radiation from current (27) is expressed by the same equation (39) (in the thin-wire approximation).

Now we establish the connection between the modal and global parameters, investigated in the previous Section. To do that we consider a lumped δ - source with unit amplitude V_0 located in the point with coordinate l_1. For such a source the exciting tangential field $E_l^{e,\delta}(l)$ and its Fourier transform can be calculated as

$$E_l^{e,\delta}(l) = V_0\delta(l-l_1)\,, \qquad \left[E_{l,m}^{e,\delta}\right] = \frac{V_0}{2\pi}\left[e^{jk_m l_1}\right] \qquad V_0 = 1V\,, \tag{40 a,b,c}$$

Using eqs. (32 a,b) and (35 a,b) the corresponding functions for the current and the potential along the wire then become:

$$Y(l,l_1) = \frac{V_0}{2\pi}\left[e^{-jk_m l}\right]^T\cdot[Z']^{-1}\cdot\left[e^{jk_m l_1}\right], \qquad K(l,l_1) = \frac{V_0}{2\pi\,\omega}\left[e^{-jk_m l}\right]^T\cdot[C']^{-1}\cdot[k_m]\cdot[Z']^{-1}\cdot\left[e^{jk_m l_1}\right] \tag{41 a,b}$$

The substitution of these functions into (16)-(19) yields the desired matrix for the global parameters $[P(l)]$ and establishes the connection between the modal and global parameters.

For the case of a finite closed wire with high symmetry (e.g. circular loop in space, horizontal loop parallel to the ground [7], vertical semi-loop near a perfectly conducting ground [1]; for all these curves the curvature $K = const$, and torsion $b = 0$) the scalar Green´s function $g(l,l')$ and the scalar product of the tangential vectors $\vec{e}_l(l)\cdot\vec{e}_l(l')$ only depend on the difference $l - l'$ of their arguments:

$$g(l,l') = g(l-l')\,, \qquad \vec{e}_l(l)\cdot\vec{e}_l(l') = \cos(\varphi(l-l')) \tag{42 a,b}$$

This leads to a diagonal matrix representation of modal parameters.

$$\left[G^L\right]_{m,m_1} = \delta_{m,m_1}\cdot\int_0^L d\xi\cos(\varphi(\xi))g(l,l+\xi)e^{-jk_{m_1}\xi} \qquad \left[G^C\right]_{m,m_1} = \delta_{m,m_1}\cdot\int_0^L d\xi\, g(l,l+\xi)e^{-jk_{m_1}\xi} \tag{43 a,b}$$

For example, in the case of a vertical semi-loop near a perfectly conducting ground eqs. (43 a,b) together with (33),(34) yield:

$$L'_m = \mu_0 R\left(g_{m+1}(k,R,a) + g_{m-1}(k,R,a)\right)/8\pi \qquad C'_m = 4\pi\varepsilon_0/\left(R\,g_m(k,R,a)\right) \tag{44 a,b}$$

Using (44) together with (35) and (36) we obtain the well-known modal solution for the current distribution in a vertical semi-loop [8, 9]. The function g_m in (44) corresponds to the known function in the analytical solution of the diffraction problem for the circular loop in free space ([10], [11], [12]).

$$g_m(k,R,a) = \int_0^{2\pi} \frac{e^{jm\varphi - jk\sqrt{4R^2\sin^2(\varphi/2)+a^2}}}{\sqrt{4R^2\sin^2(\varphi/2)+a^2}} d\varphi = 2\pi \int_0^{\infty} \frac{e^{-a\sqrt{k_\rho^2-k^2}}}{\sqrt{k_\rho^2-k^2}} k_\rho J_m^2(k_\rho R) dk_\rho \qquad (45)$$

$$g_m(k,R,a) \cong 2/R\left(\ln(2R/a) - \gamma - \psi(1/2-|m|)\right) + \pi/R \int_0^{2kR} \left(E_{2|m|}(x) - jJ_{2|m|}(x)\right)dx \qquad (46)$$

$E_{2|m|}(x)$ is the Lommer-Weber-function [11,13].

In Figs. 3a and 3b the spatial dependence of two global parameters is shown for this configuration: $P_{12}(l)$ - the global “inductance” and $P_{21}(l)$ - the global “capacitance”.

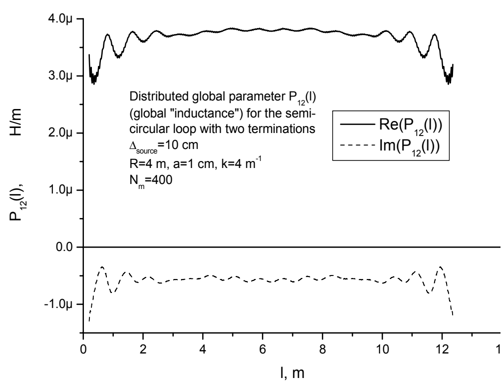

Fig.3 a

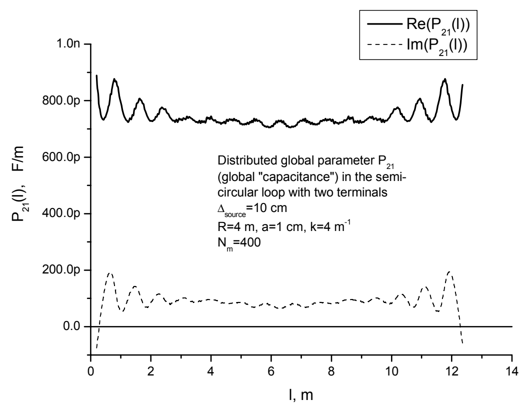

Fig. 3 b

Fig. 3: Distributed global parameters for the semi-circular vertical loop terminated at both ends.

The method of modal parameters can also be applied to the case of an infinite wire of arbitrary form. However, in this case we have to integrate over all the modal states instead to summarize. One example of a system of high symmetry (translation symmetry) is an infinite straight wire parallel to the ground (for this case the curvature $K = 0$, torsion $b = 0$). For this configuration the corresponding TL-like equations decouple. We studied the modal inductances and capacitances in different gauges in detail [14, 15]. There, the method of modal parameters (which are connected with radiation) allowed us to establish a physical meaning of the global parameters. In the cases of high symmetry the method gives us the possibility to obtain simple analytical equations for the induced currents and potentials. In addition, this method is efficient for lower frequencies, when $\lambda <\sim L$, and we have to use only a few number of modes to get very satisfying results.(Further simplification is possible, for example, for smooth wires [16].) Also, the modal approach can be generalized for an important practical system: A horizontal thick wire near the ground which is considered in the next Section.

IV. Modal Parameters for the Horizontal Thick Wire

In this section we consider a thick, lossless, cylindrical, and horizontal wire near the perfectly conducting ground which is excited by a plane electromagnetic wave (see Fig. 4). The exciting electromagnetic field induces a surface current density $\vec{i}\,(\varphi, z)$ on the surface of the wire with two components: the axial current density, $i_z(\varphi, z)$, and the azimuthal current density, $i_\varphi(\varphi, z)$,

$$\vec{i}\,(\varphi, z) = \vec{e}_\varphi(\varphi)\, i_\varphi(\varphi, z) + \vec{e}_z\, i_z(\varphi, z), \quad \text{where } \vec{e}_\varphi(\varphi) = (-\sin\varphi, \cos\varphi, 0)\,, \vec{e}_z = (0,0,1) \qquad (47)$$

It depends on two variables of the surface of the wire: the cylindrical coordinates φ and z. This fact requires a modal expansion depending on two sets of parameters, due to the symmetry of the system: the pure translation symmetry along the z-axis and the approximately axial symmetry around the z-axis. Corresponding to these symmetries it is possible to use for the Fourier expansion the following complete system of modal functions $\left\{\exp(-jk'z)\cdot\exp(-jm\varphi)\big/\sqrt{2\pi}\right\}$, where the first exponent is a representation of the translation group, the second one a representation of the rotation group. The real number k' and the integer m are parameters of these representations.

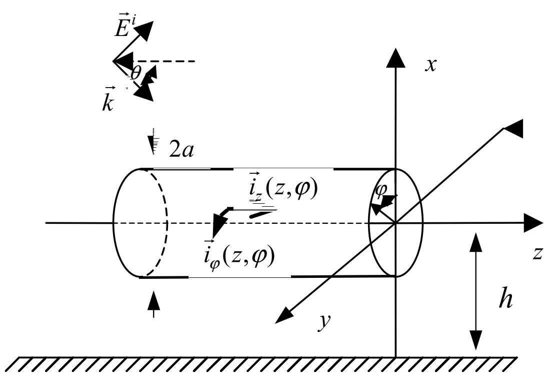

Fig. 4: Plane wave coupling to an infinite thick wire.

For the case of plane wave excitation two components of the exciting tangential electric field on the surface of the wire have to be taken into account (including reflection from the ground plane):

$$E^e_{z,k_1}(\varphi, z) = \sum_{m=-\infty}^{\infty} E^e_{z,k_1,m} \frac{e^{-jm\varphi}}{\sqrt{2\pi}} e^{-jk_1 z} = \left[e^{-jk_m l}\big/\sqrt{2\pi}\right]^T \cdot \left[E^e_{z,k_1,m}\right] = [1]^T \cdot \left[E^e_{z,k_1,m}(l)\right] \qquad (48)$$

$$E^e_{\varphi,k_1}(\varphi, z) = \sum_{m=-\infty}^{\infty} E^e_{\varphi,k_1,m} \frac{e^{-jm\varphi}}{\sqrt{2\pi}} e^{-jk_1 z} = e^{-jk_1 z}\left[e^{-jk_m l}\big/\sqrt{2\pi}\right]^T \cdot \left[E^e_{\varphi,k_1,m}\right] = [1]^T \cdot \left[E^e_{\varphi,k_1,m}(l,z)\right] \qquad (49)$$

where $k_1 := k\cdot\cos(\theta)$,and θ is the angle of incidence.

The Fourier components for the case of a vertically-polarized plane wave are given by:

$$E^e_{z,k_1,m} = E^i\sqrt{2\pi}\; 2\,j\sin(\theta)\sin\left(kh\sin(\theta) + m\,\pi/2\right)J_m\left(ka\sin(\theta)\right) \qquad (50)$$

$$E^{e}_{\varphi,k_1,m} = -E^{i}\sqrt{2\pi}\,2j\frac{\cos(\theta)}{\sin(\theta)}\cos\left(kh\sin(\theta)+m\,\pi/2\right)\frac{k_m}{k}J_m\left(ka\sin(\theta)\right) \qquad (51)$$

$$k_m = m\cdot 2\pi/L = m/a\,,\ l = a\varphi\ ,\ L = 2\pi a \qquad (52)$$

We note that for the plane wave excitation we only have one axial mode, $e^{-jk_1 z}$, in the Fourier expansion.
Then we can write a system of electric field integro-differential equations for the current components and for the scalar potential (in the Lorenz gauge) on the surface of the wire. For that we have to use a zero-boundary condition for the two tangential components of the total (exciting and scattered) electric field, and to integrate the surface density of the induced charge over the surface of the wire to obtain the expression for the scalar potential, and, finally, to apply the continuity equation for the induced current densities and charge $q(\varphi,z)$. For convenience we introduce normalized values for the current densities (which have the dimension of the current).

$$I_{\varphi}(\varphi,z) = 2\pi a\, i_{\varphi}(\varphi,z)\,; \qquad I_{z}(\varphi,z) = 2\pi a\, i_{z}(\varphi,z) \qquad (53)$$

These currents and the scalar potential are then expanded into Fourier series on the surface of the wire using the chosen modal system:

$$I_{\varphi}(l,z) = e^{-jk_1 z}\sum_{m=-\infty}^{\infty} I_{\varphi,k_1,m}\,e^{-jk_m l}\big/\sqrt{2\pi} = e^{-jk_1 z}\left[e^{-jk_m l}\big/\sqrt{2\pi}\right]^{T}\cdot\left[I_{\varphi,k_1,m}\right] = [1]^{T}\cdot\left[I_{\varphi,k_1,m}(l)\right] \qquad (54)$$

$$\left\lfloor I_{\varphi,k_1,m}\right\rfloor_{m_1} = I_{\varphi,k_1,m_1}\,; \qquad \left[I_{\varphi,k_1,m}(l,z)\right]_{m_1} = I_{\varphi,k_1,m_1}e^{-jk_1 z}\exp(-jk_{m_1}l)\big/\sqrt{2\pi} \qquad (55)$$

$$I_{z}(l,z) = e^{-jk_1 z}\sum_{m=-\infty}^{\infty} I_{z,k_1,m}\,e^{-jk_m l}\big/\sqrt{2\pi} = e^{-jk_1 z}\left[e^{-jk_m l}\big/\sqrt{2\pi}\right]^{T}\cdot\left[I_{z,k_1,m}\right] = [1]^{T}\cdot\left[I_{z,k_1,m}(l)\right] \qquad (56)$$

$$\left\lfloor I_{z,k_1,m}\right\rfloor_{m_1} = I_{z,k_1,m_1}\,; \qquad \left[I_{z,k_1,m}(l,z)\right]_{m_1} = I_{z,k_1,m_1}e^{-jk_1 z}\exp(-jk_{m_1}l)\big/\sqrt{2\pi} \qquad (57)$$

$$\Phi(l,z) = e^{-jk_1 z}\sum_{m=-\infty}^{\infty}\Phi_{k_1,m}\,e^{-jk_m l}\big/\sqrt{2\pi} = e^{-jk_1 z}\left[e^{-jk_m l}\big/\sqrt{2\pi}\right]^{T}\cdot\left[\Phi_{k_1,m}\right] = [1]^{T}\cdot\left[\Phi_{k_1,m}(l)\right] \qquad (58)$$

$$\left\lfloor \Phi_{k_1,m}\right\rfloor_{m_1} = \Phi_{k_1,m_1}\,; \qquad \left[\Phi_{k_1,m}(l,z)\right]_{m_1} = \Phi_{k_1,m_1}e^{-jk_1 z}\exp(-jk_{m_1}l)\big/\sqrt{2\pi} \qquad (59)$$

In eqs. (54) through (59) we have introduced both, modal amplitudes and modal functions.
In our last step we substitute (54)-(59) into the system of the electric-field integral equations for the currents and the potential and obtain, after a longer calculation, the following system for the modal amplitudes (60 a,b,c) and the modal functions (61 a,b,c), respectively:

$$\begin{cases} -jk_1\left[\Phi_{k_1,m}\right] + j\omega\left[L'_z\right]\cdot\left[I_{z,k_1,m}\right] = \left[E^{e}_{z,k_1,m}\right] \\ -j\left[k_m\right]\cdot\left[\Phi_{k_1,m}\right] + j\omega\left[L'_{\varphi}\right]\cdot\left[I_{\varphi,k_1,m}\right] = \left[E^{e}_{\varphi,k_1,m}\right] \\ -j\left[k_m\right]\cdot\left[I_{\varphi,k_1,m}\right] - jk_1\left[I_{z,k_1,m}\right] + j\omega\left[C'\right]\cdot\left[\Phi_{k_1,m}\right] = 0 \end{cases} \qquad (60\ a,b,c)$$

$$\begin{cases} \frac{\partial\left[\Phi_{k_1,m}(l,z)\right]}{\partial z} + j\omega\left[L_z(l)\right]\cdot\left[I_{z,k_1,m}(l,z)\right] = \left[E^e_{z,k_1,m}(l,z)\right] \\ \frac{\partial\left[\Phi_{k_1,m}(l,z)\right]}{\partial l} + j\omega\left[L_\varphi(l)\right]\cdot\left[I_{\varphi,k_1,m}(l,z)\right] = \left[E^e_{\varphi,k_1,m}(l,z)\right] \\ \frac{\partial\left[I_{\varphi,k_1,m}(l,z)\right]}{\partial l} + \frac{\partial\left[I_{z,k_1,m}(l,z)\right]}{\partial z} + j\omega\left[C(l)\right]\cdot\left[\Phi_{k_1,m}(l,z)\right] = 0 \end{cases} \tag{61 a,b,c}$$

In eqs. (60 b,c) $\left[k_m\right]_{m_1,m_2} = k_{m_1}\delta_{m_1,m_2}$ is a diagonal matrix.

In eqs. (60) and (61) we have introduced the per-unit length capacitance matrix and per-unit length inductance matrices (all of infinite dimension) for the axial current and for the azimuthal current in their modal amplitude-($[C'],[L'_z],[L'_\varphi]$) and their modal functions ($[C'(l)],[L'_z(l)],[L'_\varphi(l)]$)-representation:

$$\left[L'_z\right] = \frac{\mu_0}{4\pi}\left[G_z\right]; \qquad \left[L'_z(l)\right]_{m,m_1} = \frac{\mu_0}{4\pi}\left[G_z\right]_{m,m_1}\exp(-j(k_m - k_{m_1})l) \tag{63}$$

$$\left[L'_\varphi\right] = \frac{\mu_0}{4\pi}\left[G_\varphi\right]; \qquad \left[L'_\varphi(l)\right]_{m,m_1} = \frac{\mu_0}{4\pi}\left[G_\varphi\right]_{m,m_1}\exp(-j(k_m - k_{m_1})l) \tag{64}$$

$$\left[C'\right] = 4\pi\varepsilon_0\left[G_z\right]^{-1}; \qquad \left[C'(l)\right]_{m,m_1} = 4\pi\varepsilon_0\left(\left[G_z\right]^{-1}\right)_{m,m_1}\exp(-j(k_m - k_{m_1})l) \tag{65}$$

$$\left[L'_z\right]\cdot\left[C'\right] = \left[L'_z(l)\right]\cdot\left[C'(l)\right] = \frac{1}{c^2}\left[U\right]; \qquad \left[U\right] \text{ is the unit matrix} \tag{66}$$

The matrices $[G_z]$ and $[G_\varphi]$ appear (as well as in (30)-(31))) after the calculation of the Fourier representation of the kernels of the electric field integral equations using Graf's addition theorem for cylindrical functions [13]:

$$\left[G_z\right]_{m,m_1} = -j\pi\left\{H^{(2)}_m(\tilde{k}a)\,J_m(\tilde{k}a)\delta_{m,m_1} - (-1)^{m+m_1}J_m(\tilde{k}a)\,J_{m_1}(\tilde{k}a)H^{(2)}_{m+m_1}(2\tilde{k}h)\right\} \tag{67}$$

$$\left[G_\varphi\right]_{m,m_1} = -j\pi\left\{\frac{H^{(2)}_{m+1}(\tilde{k}a)\,J_{m+1}(\tilde{k}a) + H^{(2)}_{m-1}(\tilde{k}a)\,J_{m-1}(\tilde{k}a)}{2}\cdot\delta_{m,m_1} - \frac{\left[J_{m+1}(\tilde{k}a)\,J_{m_1-1}(\tilde{k}a) + J_{m-1}(\tilde{k}a)\,J_{m_1+1}(\tilde{k}a)\right]}{2}(-1)^{m+m_1}H^{(2)}_{m+m_1}(2\tilde{k}h)\right\} \tag{68}$$

The system (61) formally looks like the system (32) for the modal current and the potential functions. However, it includes three matrix equations instead of two in eq. (32). This is caused by the necessity to satisfy a zero-boundary condition for the two tangential components of the total electric field on the surface of the wire. The formal solution of eqs. (60)-(61) under consideration of eqs. (53)-(54) yields the current density distribution along the wire.

We note, that the integro-differential equations for the axial and azimuthal components of the induced current density in the Pocklington-like form, i.e., with excluded scalar potential, was obtained in the early seventieths [8] for the toroidal wire ("thick circular wire "). The importance (in certain cases) of the consideration of the azimuthal current components for the coupling of a pair of skewed transmission lines was specified in [17].

From a calculational point of view, it is not necessary to consider a large number of terms in the parameter matrices, because they decrease as $1/m!$ for finite ka. This is in particular true for frequencies which are important in modern high-frequency ($ka \sim 1$) applications. Therefore, the current distribution along a thick wire, including the proximity effect, can be described by matrices of finite dimension for an arbitrary excitation.

In many cases the axial component of the induced current, $I_z(\varphi, z)$, is of interest only. For example, the total axial current is equal to the corresponding Fourier amplitude with zero index:

$$I_z^{tot}(z) = \int_0^{2\pi} I_z(\varphi, z)\, a\, d\varphi = I_0(z) \tag{69}$$

Then the TL-like modal system for the axial current and the potential with renormalized capacitance and current source can be obtained from eq. (61) excluding the azimuthal component $\left[I_{\varphi,k_1,m}(l,z)\right]$:

$$\begin{cases} \dfrac{\partial\left[\Phi_{k_1,m}(l,z)\right]}{\partial z} + j\omega\left[L'_z(l)\right]\cdot\left[I_{z,k_1,m}(l,z)\right] = \left[E^e_{z,k_1,m}(l,z)\right] \\ \dfrac{\partial\left[I_{z,k_1,m}(l,z)\right]}{\partial z} + j\omega\left\{\left[C'(l)\right] - \dfrac{\left[k_m\right]\cdot\left[L'_\varphi(l)\right]^{-1}\cdot\left[k_m\right]}{\omega^2}\right\}\cdot\left[\Phi_{k_1,m}(l,z)\right] = \dfrac{\left[k_m\right]\cdot\left[L'_\varphi(l)\right]^{-1}}{\omega}\cdot\left[E^e_{\varphi,k_1,m}(l,z)\right] \end{cases} \tag{70}$$

One can say that in eq. (70) the thick wire is considered formally as a multiconductor system, if we assume that each mode corresponds to one wire.

After a straightforward matrix calculation we obtain an explicit analytical solution for the axial current. The current amplitude, for example, is:

$$\begin{gathered} \left[I_{z,k_1,m}\right] = j\omega\left\{\left(k_1^2 - k^2\right)\left[U\right] + \left[k_m\right]\cdot\left[L'_\varphi\right]^{-1}\cdot\left[k_m\right]\cdot\left[L'_z\right]\right\}^{-1}\cdot \\ \left\{\left\{\left[C'\right] - \frac{\left[k_m\right]\cdot\left[L'_\varphi\right]^{-1}\cdot\left[k_m\right]}{\omega^2}\right\}\cdot\left[E^e_{z,k_1,m}\right] + k_1\frac{\left[k_m\right]\cdot\left[L'_\varphi\right]^{-1}}{\omega^2}\cdot\left[E^e_{\varphi,k_1,m}\right]\right\} \end{gathered} \tag{71}$$

All equations mentioned above for the current and the potential, (60) through (71), have been obtained for an excitation of general form and for arbitrary modal amplitudes of the exciting electric field $E^e_{z,k_1,m}$, $E^e_{\varphi,k_1,m}$ in (48)-(49). However, in our case of excitation (vertically-polarized plane wave) with modal amplitudes (50)-(51) a further essential simplification is possible. One can show that for these amplitudes the following relation holds:

$$\left[E^e_{\varphi,k_1,m}\right] = -\frac{k_1}{k^2 - k_1^2}\left[k_m\right]\cdot\left[E^e_{z,k_1,m}\right] \tag{72}$$

After substitution of eq.(72) into (71) and some matrix manipulations, taking into account (66), we can achieve a simpler expression for the modal current in (71):

$$\left[I_{z,k_1,m}\right]=\frac{k}{jc(k^2-k_1^2)}\left[L'_z\right]^{-1}\cdot\left[E^e_{z,k_1,m}\right] \tag{73}$$

Using eq. (72) it is also possible to show that for the vertically-polarized plane wave the azimuthal current amplitude is absent:

$$\left[I_{\varphi,k_1,m}\right]=0 \tag{74}$$

This fact now has a fundamental physical impact: We deal with a TM wave, i.e. the z-component H_z^e of the exciting magnetic field is zero. It is proved (see, for example [18]) that in every system of ideal conductors with cylindrical surfaces of arbitrary form (not necessarily circular cylinders) the solution of Maxwell's equations decouple into two systems of waves: TM waves and TE waves. Since the azimuthal current is connected with the z component of the magnetic field, it is zero for the TM polarization.
Some numerical examples for the azimuthal angle dependence of the axial current induced by the vertically-polarized plane wave are presented in Fig. 5.

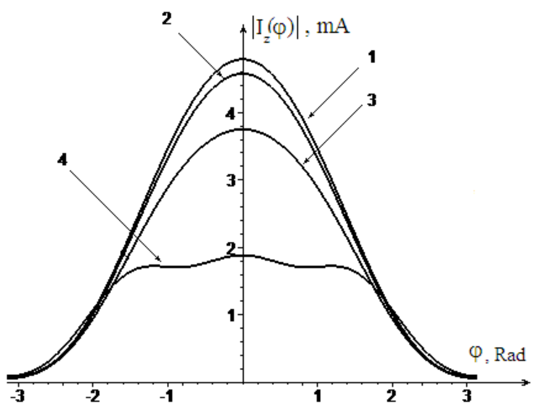

Fig.5: Azimuthal angle dependence of the axial current $I_z(\varphi,z)=2\pi\, a\, i(\varphi,z)$. $E^i=1$ V/m, $\theta=\pi/4$, $a=0.05$ m, $h=0.1$ m, $z=0$. Curve 1- $f=0.1$ GHz; 2- $f=1$ GHz; 3- $f=3$ GHz; 4 - $f=10$ GHz.

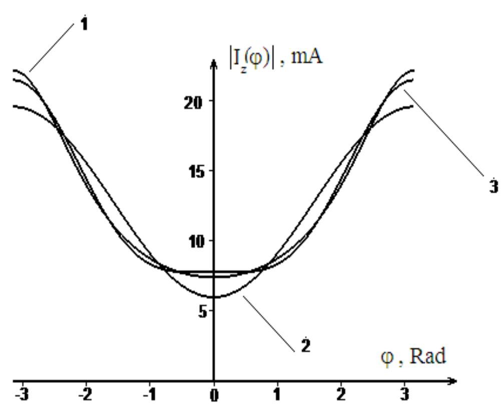

Fig. 6: Azimuthal angle dependence of the axial current $I_z(\varphi,z)=2\pi\, a\, i(\varphi,z)$ for the TEM wave. $\Phi_0=1$ V/m, $a=0.05$ m, $h=0.1$ m, $z=0$. Curve 1 – (75); 2 – modal solution, 3 terms; 3- modal solution, 5 terms. (The difference between the exact solution and modal solution with 7 terms is not recognized).

To check our approach we compare our results for the azimuthal distribution of the axial current, $I_z(\varphi,z)$, with the known analytical result for the current distribution of a TEM wave, propagating along a thick wire near the ground [19]. In this case the electric and magnetic fields have only transverse components, the electric field distribution in the transverse plane has the electrostatic form, and the scalar potential for a fixed z does not depend on the azimuthal angle φ and has a constant value Φ_0. In our notation this solution has the following form:

$$I_z(\varphi,z)=\frac{\Phi_0 e^{-jkz}}{\eta_0}\frac{2\pi\sqrt{h^2-a^2}}{\ln\left(\dfrac{h+\sqrt{h^2-a^2}}{a}\right)}\cdot\frac{1}{h+a\cos(\varphi)} \tag{75}$$

To compare our results with (75), in a first step we rewrite the system (60 a,b,c) for the TM polarization taking into account (74):

$$\begin{cases} -jk_1\left[\Phi_{k_1,m}\right]+j\omega\left[L'_z\right]\cdot\left[I_{z,k_1,m}\right]=\left[E^e_{z,k_1,m}\right] \\ -jk_1\left[I_{z,k_1,m}\right]+j\omega\left[C'\right]\cdot\left[\Phi_{k_1,m}\right]=0 \end{cases} \tag{76 a,b}$$

Noting that the z- component of the electric field for the TEM wave is zero, we obtain instead of (75) a uniform system of linear equations. The condition of the existence of a non-trivial solution of this system then gives:

$$-k_1^2+\omega^2\left[C'\right]\cdot\left[L'_z\right]=0\text{, or, using (66): } k_1^2=\omega^2/c^2=k^2 \tag{77}$$

From eq. (76b) we then derive:

$$\left[I_{z,k,m}\right]=\frac{\omega}{k}\left[C'\right]\cdot\left[\Phi_{k,m}\right]=\frac{4\pi}{\eta_0}\left[G_z\right]^{-1}\cdot\left[\Phi_{k,m}\right] \tag{78}$$

where the matrix $\left[G_z\right]$ is taken for $k_1 \to k$.

Remember that for the TEM wave the scalar potential $\Phi(\varphi,z)$ on the wire does not depend on the angle. Therefore we obtain:

$$\left[\Phi_{k,m}\right]_m=\Phi_0\sqrt{2\pi}\,\delta_{m,0} \tag{79}$$

Comparing the first three terms of a Fourier series for eq.(75) (with respect to the angle φ) with the first three terms calculated with the aid of (78),(79), and (57) for small $a/2h$ one can show, that eq. (75) as well as eqs. (78),(79), and (57) lead to the same result. One can see from the numerical example, presented in Fig. 6, that for the case $h-a\sim a$ (when the distance between the surface of the wire and the ground plane is about the radius of the wire) a good agreement is achieved, even for only a few modal terms. This example demonstrates the possibility to describe the proximity effect by the modal method.

In conclusion of this section we offer a criterium for the notation of a "thick wire". One of the principal physical features specific for a thick wire is the azimuthal current density, which is induced by the azimuthal component of the electric field for the TE polarization. For the criterium of the thickness of the wire we can consider the ratio between the maxima of the absolute values of the current density components for the TE polarization and for the TM polarization: $\max\left|I_\varphi^{TE}(\varphi)\right|\big/\max\left|I_z^{TM}(\varphi)\right|$. The calculation of the azimuthal component $I_\varphi^{TE}(\varphi)$, which we do not describe here, has shown that this ratio is small for $ka<<1$ and is about 1 for $ka\sim 1$, for different heights of the wire. Thus, for values of the parameter $ka\geq 1$ the wire should be considered as a thick wire.

V. Conclusion

The connection between the global TL parameters (parameters of Maxwellian circuits, which describe the exact solution for the induced current of the Maxwell equations for a thin wire of arbitrary geometric form) and the current and scalar potential constituting functions of the wire (for lumped sources, which are familiar in antenna theory) is established. These response functions, in turn, are connected to the so-called modal transmission line parameters. They are introduced as matrices in the Fourier representation of the electric field integral equation for the current and the scalar potential. They are complex-valued, depend on the used gauge and are connected with radiation. The solution for an arbitrary wire can be formally derived from these modal parameters. For high-symmetry cases which are characterized by constant "Differential Geometry Parameters" of the thin wire like, e.g., curvature and/or torsion, the corresponding matrices of the modal parameters become diagonal.

The method of modal parameters is generalized for the practically important case of the excitation of a thick wire near ground. In this case the modal representation of the electric field integral equation generally includes an additional equation for the azimuthal component of the induced current. However, a TL-like system for the axial current and the potential which is similar to the usual TL system for multiconductor wires can be derived.

We have shown that the obtained system (60)-(61) allows an additional simplification for TM or TE waves: For each such polarization the system decouples into a system of two matrix equations. Moreover, the solution of the corresponding homogeneous system coincides with the solution for the axial current distribution for a TEM wave propagating along a thick cylindrical wire, which was obtained at the beginning of the twentieth century [19]. Corresponding calculations for the TE polarization as well as an investigation of the global parameters for thick wires and of modal parameters for the curved thick wire will be a theme of further papers.

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