Interaction Notes

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Band Ratio and Frequency-Domain Norms

Carl E. Baum
Air Force Research Laboratory
Directed Energy Directorate

and

Daniel H. Nitsch
German Armed Forces Institute for Protective Technologies
Munster, Germany

Abstract

In evaluating large-band-ratio frequency spectra of transient waveforms, one needs some appropriate measures, such as are obtained from norms. In this paper, after considering the response of resonant systems to such waveforms, appropriate weighted norms are introduced for such evaluation. These have the property of weighting the high-and low-frequency portions approximately equally. In turn one can define appropriate efficiencies.

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1. Introduction

In the development of impulse-radiating antennas (IRAs) very large band ratios (typically two decades) have been obtained [8]. This has led to some new definitions: hypoband (narrow band), mesoband (medium band), hyperband (at least one decade of band ratio) [2]. The old concept of ultra wideband based on 25 percent or greater bandwidth is in appropriate for hyperband systems, the new technology far surpassing the bandwidth concept, leading to band ratio. While some older antenna concepts (e.g., log periodic) can also have large band ratios, the IRAs in contradistinction are approximately dispersionless and are suitable for pulse radiation and reception.

Not all radiated pulse waveforms are as ideal as those from IRAs, having frequency spectra which are not even approximately flat over the frequency range of interest for coupling to complex electronic systems. How does one characterize the bandwidth or band ratio in such cases. The use of norms comes to mind, and some bounds have been obtained for bounding frequency-domain nortms by time-domain norms and conversely [6]. Window norms are also introduced to apply the appropriate p-norm integrals over only a portion of the time domain, or over only a portion of the Fourier transform (frequency spectrum) of the pulse [3]. This latter is related to bandwidth/ratio. This has been profitably employed in measurements [4, 7].

The present paper applies window norms to the frequency spectrum in a way that is consistent with the use of band ratio for hyperband pulses.

2. Efficiency of Hyperband Waveforms for Exciting Hypoband Responses

Let

$V^{(in)}(t) \equiv$ excitation voltage or field waveform

$V^{(out)}(t) \equiv$ response of target

$\tilde{T}(s) \equiv$ transfer function of target represented by hypoband (narrow band) filter

$$\tilde{V}^{(out)}(s) = \tilde{T}(s)\tilde{V}^{(in)}(s) \tag{2.1}$$

$\sim \equiv$ two-sided Laplace transform over time t

$s = \Omega + j\omega \equiv$ Laplace-transform variable or complex frequency

For the filter we assume the simple form (conjugate pair of first-order poles)

$$\begin{aligned}\tilde{T}(s) &= \frac{C}{[s-s_0]\left[s-s_0^*\right]} = \frac{C}{s^2 - 2\mathrm{Re}(s_0) + |s_0|^2} \\ &= \frac{C}{2j\omega_0}[s-s_0]^{-1} - \frac{C}{2j\omega_0}\left[s-s_0^*\right]^{-1} \\ C &= \text{real}\end{aligned} \tag{2.2}$$

Further assume that s_0 is near the $j\omega$ axis (hypoband)

$$\begin{aligned}s_0 &= \Omega_0 + j\omega_0\,,\ \Omega_0 < 0 \\ |\Omega_0| &<< \omega_0\end{aligned} \tag{2.3}$$

At $\omega = \omega_0$ we have

$$\tilde{T}(j\omega_0) = \frac{C}{[-\Omega_0][2j\omega_0 - \Omega_0]} \simeq -\frac{C}{2j\omega_0\Omega_0} \tag{2.4}$$

The usual $\Delta\omega$ bandwidth ($1/\sqrt{2}$ or 3 db) is then found via

$$\begin{aligned}\frac{\left|\tilde{T}\left(j\omega_0 \pm j\frac{\Delta\omega}{2}\right)\right|}{\left|\tilde{T}(j\omega_0)\right|} &= \frac{1}{\sqrt{2}} \\ &\simeq \left|\frac{2j\omega_0\Omega_0}{\left[j\frac{\Delta\omega}{2} - \Omega_0\right]2j\omega_0}\right| \simeq \frac{\Omega_0}{\left[\left[\frac{\Delta\omega}{2}\right]^2 + \Omega_0^2\right]^{1/2}} \\ \Delta\omega &\simeq 2|\Omega_0| << \omega_0\end{aligned} \tag{2.5}$$

Now assume that $\tilde{V}^{(in)}(s)$ is smooth (i.e., does not change appreciably, including phase) near s = $\pm j\omega_0$, giving

$$\tilde{V}^{(out)}(s) \simeq \frac{C}{2j\omega_0}\frac{\tilde{V}^{(in)}(j\omega_0)}{s-s_0} + \frac{C}{2j\omega_0\Omega_0}\frac{\tilde{V}^{(in)}(-j\omega_0)}{s-s_0^*}$$

$$\tilde{V}^{(out)}(t) \simeq \frac{C}{2j\omega_0}\left[\tilde{V}^{(in)}(j\omega_0)\,e^{s_0 t} - \tilde{V}^{(in)}(-j\omega_0)\,e^{s_0^* t}\right]u(t) \tag{2.6}$$

$$= \frac{C}{2j\omega_0}\left[\tilde{V}^{(in)}(j\omega_0)\,e^{j\omega_0 t} - \tilde{V}^{(in)}(-j\omega_0)\,e^{-j\omega_0 t}\right]e^{\Omega_0 t}u(t)$$

Noting that $|\Omega_0| << \omega_0$, we have an approximate peak of this as

$$V_p^{(out)} \simeq \left|\frac{C}{\omega_0}\right| \quad \left|\tilde{V}^{(in)}(j\omega_0)\right| \tag{2.7}$$

For convenience and normalization let us set

$$\left|\tilde{T}(j\omega_0)\right| \equiv 1 \quad , \quad \left|\frac{C}{2\omega_0\Omega_0}\right| = 1 \tag{2.8}$$

giving

$$V_p^{(out)} \simeq 2|\Omega_0| \left|\tilde{V}^{(in)}(j\omega_0)\right| = \Delta\omega \left|\tilde{V}^{(in)}(j\omega_0)\right| \tag{2.9}$$

As we can see small $\Delta\omega$ gives a small response to the incident waveform.

Of course, the transfer function is also a function of frequency, having different values for different resonances. So a more general form has

$$V_p^{(out)} = \Delta\omega \left|\tilde{T}(j\omega_0)\right| \left|\tilde{V}^{(in)}(j\omega_0)\right| \tag{2.10}$$

In this form we see that the peak response depends on the value of $|\tilde{T}V^{(in)}|$ at the various resonances of interest. For the transfer function one can choose a frequency range of interest such as around a GHz [1, 3]. How large $\tilde{T}(j\omega_0)$ can be at the resonances in this range is very complicated.

Consider next the variation of $\tilde{V}^{(in)}(j\omega)$. For an impulse-like waveform (such as from an impulse-radiating antenna (IRA)) this can be approximately constant, emphasizing the higher frequencies due to the general increase of $\Delta\omega$ with frequency. On the other hand, for a fast pulse (nanoseconds), perhaps with a fast rise time we can approximate the input by a step function as

$$\tilde{V}^{(in)}(j\omega) = \frac{V_0}{j\omega} \tag{2.11}$$

over some frequency band of interest. In this case we have

$$\begin{aligned} V_p^{(out)} &= f_{\omega_0} V_0 \left|\tilde{T}(j\omega_0)\right| \\ f_{\omega_0} &= \frac{\Delta\omega}{\omega_0} = \frac{2|\Omega_0|}{\omega_0} = Q^{-1} = \frac{1}{\pi N} \\ N &= \text{number of cycles to } e^{-1} \text{ for transient response} \end{aligned} \tag{2.12}$$

So for this type of excitation the response is inversely proportional to Q or $\Delta\omega / \omega_0$. If the various resonances have approximately the same Q, then frequencies of interest are weighted in the above form.

3. Application of Norms to Large-Band-Ratio Radiated Waveforms

The p-norm of the frequency spectrum of a waveform $f(t)$ is [6]

$$\left\| \tilde{f}(j\omega) \right\|_{p,\omega} = \left[\int_{-\infty}^{\infty} \left| \tilde{f}(j\omega) \right|^p d\omega \right]^{1/p} = \left[2\int_{0}^{\infty} \left| \tilde{f}(j\omega) \right|^p d\omega \right]^{1/p}$$

$$\tilde{f}\left(s^*\right) = \tilde{f}^*(s) \quad \text{(conjugate symmetry of real-valued time waveform)} \tag{3.1}$$

$$\tilde{f}(-j\omega) = \tilde{f}^*(j\omega)$$

Let us now also introduce a weighted norm [5] in the special form

$$\left\| \tilde{f}(j\omega) \right\|_{p,\ell n(\omega)} = \left[2\int_{0}^{\infty} \left| \tilde{f}(j\omega) \right|^p d(\ell n(\omega)) \right]^{1/p} = \left[2\int_{0}^{\infty} \left| \tilde{f}(j\omega) \right|^p \frac{d\omega}{\omega} \right]^{1/p} \tag{3.2}$$

For this logarithmic weighting we see the convenience of integrating over the positive ω axis.

In [3, 7] the concept of a window norm has been introduced. This applies here as

$$\left\| \tilde{f}(j\omega) \right\|_{p,\omega}^{(\omega_1,\omega_2)} \equiv \left[2\int_{\omega_1}^{\omega_2} \left| \tilde{f}(j\omega) \right|^p d\omega \right]^{1/p} \leq \left\| \tilde{f}(j\omega) \right\|_{p,\omega}$$

$$\left\| \tilde{f}(j\omega) \right\|_{p,\ell n(\omega)}^{(\omega,)} = \left[2\int_{0}^{\infty} \left| \tilde{f}(j\omega) \right|^p d(\ell n(\omega)) \right]^{1/p} = \left[2\int_{\omega_1}^{\omega_2} \left| \tilde{f}(j\omega) \right|^p \frac{d\omega}{\omega} \right]^{1/p} \leq \left\| \tilde{f}(j\omega) \right\|_{p,\ell n(\omega)} \tag{3.3}$$

$$0 < \omega_1 < \omega_2$$

Define a band ratio by

$$B \equiv \frac{\omega_2}{\omega_1} > 1 \tag{3.4}$$

where ω_1 and ω_2 are to be chosen later. Consider some incident waveform (single linear polarization)

$$E^{(inc)}(t) = E_0 f(t) \tag{3.5}$$

Then, similar to [7], let us define efficiencies

$$\begin{aligned} \eta_{p,\omega}^{(\omega_1,\omega_2)} &\equiv \frac{\left\|\tilde{f}(j\omega)\right\|_{p,\omega}^{(\omega_1,\omega_2)}}{\left\|\tilde{f}(j\omega)\right\|_{p,\omega}} \leq 1 \\ \eta_{p,\ell n(\omega)}^{(\omega_1,\omega_2)} &\equiv \frac{\left\|\tilde{f}(j\omega)\right\|_{p,\ell n(\omega)}^{(\omega_1,\omega_2)}}{\left\|\tilde{f}(j\omega)\right\|_{p,\ell n(\omega)}} \leq 1 \end{aligned} \tag{3.6}$$

Choosing some number (say 90%, or whatever) for these efficiencies we can say something about ω_1 and ω_2 and thereby infer band ratio.

Consider what we might call an IRA-like waveform, the spectrum of which we approximate as

$$\left|\tilde{f}(j\omega)\right| = \begin{cases} A\dfrac{\omega}{\omega_1} & \text{for } 0 \leq \omega \leq \omega_1 \\ A & \text{for } \omega_1 \leq \omega \leq \omega_2 \\ A\dfrac{\omega_2}{\omega} & \text{for } 0 \leq \omega_2 \leq \omega \end{cases} \tag{3.7}$$

Outside the approximately flat central portion there are 20 dB per decade rolloffs. One can choose other forms, but this will do for present illustration.

For the logarithmic form we have

$$\begin{aligned} \left[\left\|\tilde{f}(j\omega)\right\|_{p,\ell n(\omega)}^{(\omega_1,\omega_2)}\right]^p &= A^p \int_{\omega_1}^{\omega_2} d\ell n(\omega) = A^p \ell n\left(\frac{\omega_2}{\omega_1}\right) = A^p \ell n(B) \\ \left[\left\|\tilde{f}(j\omega)\right\|_{p,\ell n(\omega)}\right]^p &= A^p\left[\ell n(B) + \int_0^{\omega_1}\left[\frac{\omega}{\omega_1}\right]^p \frac{d\omega}{\omega} + \int_{\omega_2}^{\infty}\left[\frac{\omega}{\omega_2}\right]^{-p}\frac{d\omega}{\omega}\right] \\ &= A^p\left[\ell n(B) + p^{-1} + p^{-1}\right] = A^p\left[\frac{2}{p} + \ell n(B)\right] \\ \eta_{p,\ell n(\omega)}^{(\omega_1,\omega_2)} &= \left[\frac{\ell n(B)}{\dfrac{2}{p} + \ell n(B)}\right]^{1/p} = \left[1 + \frac{2}{p\ell n(B)}\right]^{-1/p} \end{aligned} \tag{3.8}$$

This is inverted as

$$B = \frac{\omega_2}{\omega_1} = \exp\left(\frac{2}{p}\left[\left[\eta_{p,\ell n(\omega)}^{(\omega_1,\omega_2)}\right]^{-p} - 1\right]^{-1}\right) \tag{3.9}$$

So the band ratio is found directly from the efficiency one may wish to specify. For example a two-decade B corresponds to roughly 0.9 for 2-norm efficiency.

For the other form we have

$$\left[\left\|\tilde{f}(j\omega)\right\|_{p,\ell n(\omega)}^{(\omega_1,\omega_2)}\right]^p = A^p \int_{\omega_1}^{\omega_2} d\omega = A^p[\omega_2 - \omega_1]$$

$$\begin{aligned}\left[\left\|\tilde{f}(j\omega)\right\|_{p,\omega}\right]^p &= A^p\left[[\omega_2 - \omega_1] + \int_0^{\omega_1}\left[\frac{\omega}{\omega_1}\right]^p d\omega + \int_0^{\omega_1}\left[\frac{\omega}{\omega_2}\right]^{-p} d\omega\right] \\ &= A^p\left[[\omega_2 - \omega_1] + \frac{\omega_1}{p+1} + \frac{\omega_2}{p-1}\right] \text{ for } p > 1 \\ &= A^p\left[\frac{p}{p-1}\omega_2 - \frac{p}{p+1}\omega_1\right]\end{aligned} \tag{3.10}$$

Note that $p > 1$ is necessary for the integral to ∞ to exist. Of course, one can assume a different form for the high-frequency rolloff. The efficiency is now

$$\eta_{p,\omega}^{(\omega_1,\omega_2)} = \left[\frac{\omega_2 - \omega_1}{\frac{p}{p-1}\omega_2 - \frac{p}{p+1}\omega_1}\right]^{1/p} = \left[\frac{p}{p-1}\frac{\omega_2}{\omega_2 - \omega_1} - \frac{p}{p+1}\frac{\omega_1}{\omega_2 - \omega_1}\right]^{-1/p} \tag{3.11}$$

Note that for $\omega_2 >> \omega_1$ we have

$$\eta_{p,\omega}^{(\omega_1,\omega_2)} \simeq \left[\frac{p-1}{p}\right]^{1/p} \tag{3.12}$$

independent of ω_1 and ω_2. This is problematical. Looking more closely we see that ω_2 is much more important than ω_1 in the formulae for the norms in (3.10). Comparison of ω_1 to ω_2 in this sense shows the lesser importance of ω_1 compared to ω_2. Instead of $\omega_2 - \omega_1$ one can use ω_2 / ω_1 which weights ω_2 and ω_1 equally. The logarithmic form in (3.8) and (3.9) seems more suited to large-band-ratio waveforms.

4. Concluding Remarks

In applying norms to the important parts of frequency spectra of large-band-ratio waveforms, there are various ways to approach this. In the present development we have introduced a weighted norm where the integration is with respect to $d\ell n(\omega) = \omega^{-1}d\omega$. This is shown to weight both the low- and high-frequency limits of the window norm equally. From this one can define an efficiency in norm sense for containing some fraction of the norm within the frequency limits.

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