Interaction Notes

Note 583

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Reciprocity, Energy, and Norms for Propagation on Nonuniform Multiconductor Transmission Lines

Carl E. Baum
Air Force Research Laboratory
Directed Energy Directorate

Abstract

The general solution of propagation on nonuniform multiconductor transmission lines is described by product integrals which may be difficult to evaluate except in numerical form. However, the conservation laws of electromagnetics concerning reciprocity and energy can be used to determine some of the analytic properties of these product integrals. Special results occur when the transmission lines are lossless.

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1. Introduction

An N-conductor (plus reference) nonuniform multiconductor transmission line (NMTL) is characterized by the telegrapher equations

$$
\begin{aligned}
\frac{\partial}{\partial z}\left(\tilde{V}_n(z,s)\right) &= -\left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{I}_n(z,s)\right) + \left(\tilde{V}_n^{(s)'}(z,s)\right) \\
\frac{\partial}{\partial z}\left(\tilde{I}_n(z,s)\right) &= -\left(\tilde{Y}'_{n,m}(z,s)\right) \cdot \left(\tilde{V}_n(z,s)\right) + \left(\tilde{I}_n^{(s)'}(z,s)\right)
\end{aligned}
\tag{1.1}
$$

The vectors have N components and the matrices are N x N. The various terms are

$$
\begin{aligned}
&\left(\tilde{V}_n(z,s)\right) \equiv \text{voltage vector} \\
&\left(\tilde{I}_n(z,s)\right) \equiv \text{current vector} \\
&\left(\tilde{V}_n^{(s)'}(z,s)\right) \equiv \text{per-unit-length voltage-source vector} \\
&\left(\tilde{I}_n^{(s)'}(z,s)\right) \equiv \text{per-unit-length current-source vector} \\
&\left(\tilde{Z}'_{n,m}(z,s)\right) = \left(\tilde{Z}'_{n,m}(z,s)\right)^{\mathsf{T}} \equiv \text{per-unit-length (longitudinal) impedance matrix} \\
&\left(\tilde{Y}'_{n,m}(z,s)\right) = \left(\tilde{Y}'_{n,m}(z,s)\right)^{\mathsf{T}} \equiv \text{per-unit-length (transverse) admittance matrix} \\
&\sim \; \equiv \text{two-sided Laplace transform over time } t \\
&s = \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency} \\
&z \equiv \text{spatial coordinate along transmission line}
\end{aligned}
\tag{1.2}
$$

The telegrapher equations are combined into a single equation with 2N-component vectors (supervectors) and 2N x 2N matrices (supermatrices) as

$$
\begin{aligned}
&\frac{\partial}{\partial z}\begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(s)\right) \end{pmatrix} \\
&= -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}_{n,m}(s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix} \odot \begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(s)\right) \end{pmatrix} \\
&\quad + \begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n^{(s)'}(z,s)\right) \end{pmatrix}
\end{aligned}
\tag{1.3}
$$

$$\left(\tilde{Z}_{n,m}(s)\right) = \left(\tilde{Z}_{n,m}(s)\right)^{\mathrm{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right)^{-1}$$

$\equiv$ normalizing impedance matrix chosen at our convenience (not a function of z)

The supermatrizant differential equation

$$
\begin{aligned}
&\frac{\partial}{\partial z}\left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\nu,\nu'}\right) \cdot \left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \\
&\left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\nu,\nu'}\right) = -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}_{n,m}(s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix} \\
&\left(\left(\tilde{U}_{n,m}(z_0,z_0;s)\right)_{\nu,\nu'}\right) = \left(\left(1_{n,m}\right)_{\nu,\nu'}\right) = \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(1_{n,m}\right) \end{pmatrix} \quad \text{(boundary condition)}
\end{aligned}
\tag{1.4}
$$

has a solution as the product integral [3]

$$\left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) = \prod_{z_0}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}(z',s)\right)_{\nu,\nu'}\right)dz'} \tag{1.5}$$

In terms of this the solution to (1.3) is

$$
\begin{aligned}
\begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(z,s)\right) \end{pmatrix} &= \left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(z_0,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(z_0,s)\right) \end{pmatrix} \\
&\quad + \int_{z_0}^{z} \left(\left(\tilde{U}_{n,m}(z,z';s)\right)_{\nu,\nu'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z',s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n^{(s)'}(z',s)\right) \end{pmatrix} dz'
\end{aligned}
\tag{1.6}
$$

In the present paper we do not include the presence of sources in our NMTL and concentrate on the source-free propagation, and in particular on the properties of the product integral.

We also consider the product integral

$$\left(\left(\tilde{T}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) = \prod_{z_0}^{z} e^{\left(\left(\tilde{G}_{n,m}(z',s)\right)_{\nu,\nu'}\right)dz'}$$
$$\left(\left(\tilde{G}_{n,m}(z,s)\right)_{\nu,\nu'}\right) = -\begin{pmatrix} (0_{n,m}) & \left(\tilde{Z}'_{n,m}(z,s)\right) \\ \left(\tilde{Y}'_{n,m}(z,s)\right) & (0_{n,m}) \end{pmatrix} \tag{1.7}$$

This corresponds to the differential equation (1.3) without the normalizing impedance $(\tilde{Z}_{n,m}(s))$. Defining

$$\left(\left(\tilde{\zeta}_{n,m}(s)\right)_{\nu,\nu'}\right) \equiv \begin{pmatrix} (1_{n,m}) & (0_{n,m}) \\ (0_{n,m}) & \left(\tilde{Z}_{n,m}(s)\right) \end{pmatrix} \tag{1.8}$$

the two forms are related by

$$\left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\nu,\nu'}\right) = \left(\left(\tilde{\zeta}_{n,m}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{G}_{n,m}(z,s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{\zeta}_{n,m}(s)\right)_{\nu,\nu'}\right)^{-1}$$
$$\left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) = \prod_{z_0}^{z} e^{\left(\left(\tilde{\zeta}_{n,m}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{G}_{n,m}(z',s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{\zeta}_{n,m}(s)\right)_{\nu,\nu'}\right)^{-1} dz'}$$
$$= \left(\left(\tilde{\zeta}_{n,m}(s)\right)_{\nu,\nu'}\right) \odot \left[\prod_{z_0}^{z} e^{\left(\left(\tilde{G}_{n,m}(z',s)\right)_{\nu,\nu'}\right)dz'}\right] \odot \left(\left(\tilde{\zeta}_{n,m}(s)\right)_{\nu,\nu'}\right)^{-1} \quad \text{(similarity rule)}$$
$$= \left(\left(\tilde{\zeta}_{n,m}(s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{T}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) \odot \left(\left(\tilde{\zeta}_{n,m}(s)\right)_{\nu,\nu'}\right)^{-1} \tag{1.9}$$

The form in (1.3) and (1.4) has consistent units for the elements of the vectors and matrices, while that in (1.7) is useful for what follows and can be related to the first form via (1.9). For later use we can write out the matrizant blocks in (1.9) as

$$
\begin{pmatrix} \left(\tilde{T}_{n,m}(z,z_0;s)\right)_{1,1} & \left(\tilde{T}_{n,m}(z,z_0;s)\right)_{1,2} \\ \left(\tilde{T}_{n,m}(z,z_0;s)\right)_{2,1} & \left(\tilde{T}_{n,m}(z,z_0;s)\right)_{2,2} \end{pmatrix}
= \begin{pmatrix} \left(\tilde{U}_{n,m}(z,z_0;s)\right)_{1,1} & \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{1,2} \\ \left(\tilde{Y}_{n,m}(s)\right)\cdot\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{2,1} & \left(\tilde{Y}_{n,m}(s)\right)\cdot\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{2,2}\cdot\left(\tilde{Z}_{n,m}(s)\right) \end{pmatrix} \tag{1.10}
$$

Note that since we have

$$
\begin{aligned}
tr\left(\left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\nu,\nu'}\right)\right) &= 0 \\
tr\left(\left(\left(\tilde{G}_{n,m}(z,s)\right)_{\nu,\nu'}\right)\right) &= 0
\end{aligned} \tag{1.11}
$$

we have the well-known general results

$$
\begin{aligned}
\det\left(\left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right)\right) &= 1 \\
\det\left(\left(\left(\tilde{T}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right)\right) &= 1
\end{aligned} \tag{1.12}
$$

This provides a general constraint on NMTL product integrals.

2. Reciprocity

Consider any two solutions of (1.1) denoted by superscripts 1 and 2. Then form (with zero distributed sources)

$$
\begin{aligned}
&\frac{d}{dz}\left[\left(\tilde{V}_n^{(1)}(z,s)\right)\cdot\left(\tilde{I}_n^{(2)}(z,s)\right)\right] \\
&\quad=\left[\frac{d}{dz}\left(\tilde{V}_n^{(1)}(z,s)\right)\right]\cdot\left(\tilde{I}_n^{(2)}(z,s)\right)+\left(\tilde{V}_n^{(1)}(z,s)\right)\left[\frac{d}{dz}\left(\tilde{I}_n^{(2)}(z,s)\right)\right] \\
&\quad=-\left(\tilde{I}_n^{(1)}(z,s)\right)\cdot\left(\tilde{Z}_n'(z,s)\right)\cdot\left(\tilde{I}_n^{(2)}(z,s)\right)-\left(\tilde{V}_n^{(1)}(z,s)\right)\cdot\left(\tilde{Y}_n'(z,s)\right)\cdot\left(\tilde{V}_n^{(2)}(z,s)\right) \\
&\quad=-\left(\tilde{I}_n^{(2)}(z,s)\right)\cdot\left(\tilde{Z}_n'(z,s)\right)\cdot\left(\tilde{I}_n^{(1)}(z,s)\right)-\left(\tilde{V}_n^{(2)}(z,s)\right)\cdot\left(\tilde{Y}_{n,m}'(z,s)\right)\cdot\left(\tilde{V}_n^{(1)}(z,s)\right) \\
&\quad=\frac{d}{dz}\left[\left(\tilde{V}_n^{(2)}(z,s)\right)\cdot\left(\tilde{I}_n^{(1)}(z,s)\right)\right]
\end{aligned}
\tag{2.1}
$$

This, of course, relies on the symmetry (reciprocity) of $(\tilde{Z}_{n,m}')$ and $(\tilde{Y}_{n,m}')$. Then we have

$$
\boxed{\frac{d}{dz}\left[\left(\tilde{V}_n^{(1)}(z,s)\right)\cdot\left(\tilde{I}_n^{(2)}(z,s)\right)-\left(\tilde{V}_n^{(2)}(z,s)\right)\cdot\left(\tilde{I}_n^{(1)}(z,s)\right)\right]=0}
\tag{2.2}
$$

which might be termed *differential reciprocity*. Integrating this we have

$$
\left(\tilde{V}_n^{(1)}(z,s)\right)\cdot\left(\tilde{I}_n^{(2)}(z,s)\right)-\left(\tilde{V}_n^{(2)}(z,s)\right)\cdot\left(\tilde{I}_n^{(1)}(z,s)\right)=\text{constant (independent of } z\text{)}
\tag{2.3}
$$

Stated another way we have

$$
\begin{aligned}
&\left(\tilde{V}_n^{(1)}(z_2,s)\right)\cdot\left(\tilde{I}_n^{(2)}(z_2,s)\right)-\left(\tilde{V}_n^{(2)}(z_2,s)\right)\cdot\left(\tilde{I}_n^{(1)}(z_2,s)\right) \\
&=\left(\tilde{V}_n^{(1)}(z_1,s)\right)\cdot\left(\tilde{I}_n^{(2)}(z_1,s)\right)-\left(\tilde{V}_n^{(2)}(z_1,s)\right)\cdot\left(\tilde{I}_n^{(1)}(z_1,s)\right)
\end{aligned}
\tag{2.4}
$$

for any two points z_1 and z_2 on the NMTL. This can also be rearranged as

$$
\begin{aligned}
&\left(\tilde{V}_n^{(1)}(z_2,s)\right)\cdot\left(\tilde{I}_n^{(2)}(z_2,s)\right)-\left(\tilde{V}_n^{(2)}(z_1,s)\right)\cdot\left(\tilde{I}_n^{(1)}(z_1,s)\right) \\
&=\left(\tilde{V}_n^{(1)}(z_2,s)\right)\cdot\left(\tilde{I}_n^{(2)}(z_2,s)\right)-\left(\tilde{V}_n^{(2)}(z_1,s)\right)\cdot\left(\tilde{I}_n^{(1)}(z_1,s)\right)
\end{aligned}
\tag{2.5}
$$

showing the symmetry on the interchange of the 1 and 2 labels.

Now consider the impedance properties of the NMTL. For this purpose let us attach a passive reciprocal impedance matrix (as in Fig. 2.1)

$$\left(\tilde{Z}_{n,m}^{(s)}(s)\right) = \left(\tilde{Z}_{n,m}^{(s)}(s)\right)^T = \left(\tilde{Y}_{n,m}^{(s)}(s)\right)^{-1} \tag{2.6}$$

at the beginning of the NMTL. Then look at the impedance matrix $(\tilde{Z}_{n,m}^{(\text{end})}(s))$ looking into the end ($z = \ell$) of the NMTL. With all reciprocal elements from (1.2) and (2.6) then this impedance matrix must also be symmetric, i.e.,

$$\left(\tilde{Z}_{n,m}^{(\text{end})}(s)\right) = \left(\tilde{Z}_{n,m}^{(\text{end})}(s)\right)^T = \left(\tilde{Y}_{n,m}^{(\text{end})}(s)\right)^{-1} \tag{2.7}$$

Noting the current convention, at the ends of the NMTL we have

$$\begin{aligned}
\left(\tilde{V}_n(0,s)\right) &= -\left(\tilde{Z}_{n,m}^{(s)}(s)\right) \cdot \left(\tilde{I}_n(0,s)\right) \\
\left(\tilde{V}_n(\ell,s)\right) &= -\left(\tilde{Z}_{n,m}^{(\text{end})}(s)\right) \cdot \left(\tilde{I}_n(\ell,s)\right)
\end{aligned} \tag{2.8}$$

Using the matrizant (product integral) we also have

$$\begin{aligned}
&\begin{pmatrix} \left(\tilde{V}_n(\ell,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(\ell,s)\right) \end{pmatrix} = \left(\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{\nu,\nu'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(0,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(0,s)\right) \end{pmatrix} \\
&= \begin{pmatrix} \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1} \cdot \left(\tilde{V}_n(0,s)\right) + \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(0,s)\right) \\ \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1} \cdot \left(\tilde{V}_n(0,s)\right) + \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(0,s)\right) \end{pmatrix} \\
&= \begin{pmatrix} \left[\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1} - \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}_{n,m}^{(s)}(s)\right)\right] \cdot \left(\tilde{V}_n(0,s)\right) \\ \left[\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1} - \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}_{n,m}^{(s)}(s)\right)\right] \cdot \left(\tilde{V}_n(0,s)\right) \end{pmatrix} \\
&= \begin{pmatrix} \left(\tilde{V}_n(\ell,s)\right) \\ -\left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}_{n,m}^{(\text{end})}(s)\right) \cdot \left(\tilde{V}_n(\ell,s)\right) \end{pmatrix}
\end{aligned} \tag{2.9}$$

Equating the two results for $(\tilde{V}_n(0,s))$ gives

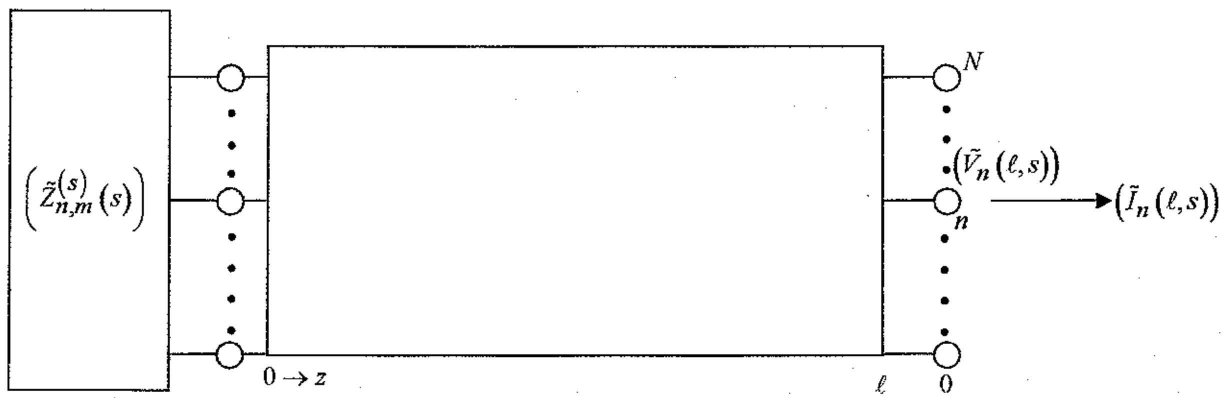

Fig. 2.1 NMTL with Reciprocal Termination at $z = 0$

$$\left[\left[\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1} - \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}^{(s)}_{n,m}(s)\right)\right]^{-1}\right.$$

$$\left. + \left[\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1} - \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}^{(s)}_{n,m}(s)\right)\right]^{-1} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}^{(\text{end})}_{n,m}(s)\right)\right] \cdot \left(\tilde{V}_n(\ell,s)\right)$$

$$= \left(0_n\right) \tag{2.10}$$

Varying $(\tilde{V}_n(\ell,s))$ over N independent choices gives

$$\begin{aligned} \left(\tilde{Y}^{(\text{end})}_{n,m}(s)\right) &= \left(\tilde{Z}^{(\text{end})}_{n,m}(s)\right)^{-1} \\ &= -\left(\tilde{Y}_{n,m}(s)\right) \cdot \left[\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1} - \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}^{(s)}_{n,m}(s)\right)\right] \\ &\quad \cdot \left[\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1} - \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}^{(s)}_{n,m}(s)\right)\right]^{-1} \end{aligned} \tag{2.11}$$

Since this must be symmetric we also have

$$\begin{aligned} &\left(\tilde{Y}^{(\text{end})}_{n,m}(s)\right) \cdot \left(\tilde{Z}^{(\text{end})}_{n,m}(s)\right)^{-1} \\ &= -\left[\left(\tilde{U}_{n,m}(\ell,0;s)\right)^{T}_{1,1} - \left(\tilde{Y}^{(s)}_{n,m}(s)\right) \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)^{T}_{1,2}\right]^{-1} \\ &\quad \cdot \left[\left(\tilde{U}_{n,m}(\ell,0;s)\right)^{T}_{2,1} - \left(\tilde{Y}^{(s)}_{n,m}(s)\right) \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)^{T}_{2,2}\right] \cdot \left(\tilde{Y}_{n,m}(s)\right) \end{aligned} \tag{2.12}$$

Equating the two results removes $(\tilde{Y}^{(\text{end})}_{n,m}(s))$ from the equation, giving constraints on the matrizant blocks (submatrices) due to reciprocity. This must hold for all symmetric $(\tilde{Y}^{(s)}_{n,m}(s))$. Note that the matrizant blocks are functions of $(\tilde{Z}_{n,m}(s))$.

We can simplify the above result by considering special cases of $(\tilde{Y}^{(s)}_{n,m}(s))$. Letting it be the zero matrix gives

$$\begin{aligned} &\left(\tilde{U}_{n,m}(\ell,0;s)\right)^{\mathrm{T}\,-1}_{1,1} \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)^{\mathrm{T}}_{2,1} \cdot \left(\tilde{Y}_{n,m}(s)\right) \\ &\quad = \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1} \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)^{-1}_{1,1} \end{aligned}$$

$$
\boxed{
\begin{aligned}
&\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1} \\
&\quad = \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1} \\
&\quad = \left[\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1}\right]^{\mathsf{T}}
\end{aligned}
}
\tag{2.13}
$$

Therefore, reciprocity implies that this product of three matrices is *symmetric*. Similarly by changing $(\tilde{Y}^{(s)}_{n,m}(s))$ to $(\tilde{Z}^{(s)}_{n,m}(s))^{-1}$ in (2.11) and (2.12) and converting to $(\tilde{Z}^{(s)}_{n,m}(s))$ by appropriate multiplication gives another form. Then setting this to zero gives

$$
\begin{aligned}
&\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2}^{\mathsf{T}\,-1} \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \\
&\quad = \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2} \cdot \left(\tilde{U}_{n,m}(\ell,o;s)\right)_{1,2}^{-1}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
&\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,2} \\
&\quad = \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,2}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2} \\
&\quad = \left[\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,2}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2}\right]^{\mathsf{T}}
\end{aligned}
}
\tag{2.14}
$$

This three-matrix product is then also *symmetric*. Note now that the end conditions on the NMTL have been removed and the z values of 0 and ℓ can be replaced by arbitrary $z_1 < z_2$.

One can go on to more combinations of the matrix blocks by equating (2.11) and (2.12), multiply to remove inverses, multiplying out the terms and removing (canceling) terms from the equalities in (2.13) and (2.14). These involve larger combinations than the simpler three-term products as

$$
\begin{aligned}
&\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}^{(s)}_{n,m}(s)\right) \\
&\quad + \left(\tilde{Y}^{(s)}_{n,m}(s)\right) \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,2}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1} \\
&= \left(\tilde{Y}^{(s)}_{n,m}(s)\right) \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1} \\
&\quad + \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}^{(s)}_{n,m}(s)\right) \\
&= \Big[\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{Y}^{(s)}_{n,m}(s)\right) \\
&\quad + \left(\tilde{Y}^{(s)}_{n,m}(s)\right) \cdot \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,2}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(\ell,0;s)\right)_{2,1}\Big]
\end{aligned}
\tag{2.15}
$$

This allows for various choices of $(\tilde{Y}^{(s)}_{n,m}(s))$, e.g.,

$$\left(\tilde{Y}^{(s)}_{n,m}(s)\right) = \left(\tilde{Y}_{n,m}(s)\right), \quad \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}^{(s)}_{n,m}(s)\right) = \left(\tilde{Y}^{(s)}_{n,m}(s)\right)\cdot\left(\tilde{Z}_{n,m}(s)\right) = \left(1_{n,m}\right) \tag{2.16}$$

which simplifies (2.15) considerably.

In the alternate form of the product integral in (1.7) we have similar results which we can find by replacing $(\tilde{Z}_{n,m}(s))$ by the identity giving

$$\boxed{\begin{aligned}\left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}_{1,1}\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1} &= \left[\left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}_{1,1}\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1}\right]^{\mathsf{T}}\\ \left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}_{1,2}\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2} &= \left[\left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}_{1,2}\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2}\right]^{\mathsf{T}}\end{aligned}} \tag{2.17}$$

Similarly (2.15) simplifies to

$$\begin{aligned}&\left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}_{1,1}\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2}\cdot\left(\tilde{Y}^{(s)}_{n,m}(s)\right)+\left(\tilde{Y}^{(s)}_{n,m}(s)\right)\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}_{1,2}\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1}\\ &=\left[\left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}_{1,1}\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2}\cdot\left(\tilde{Y}^{(s)}_{n,m}(s)\right)+\left(\tilde{Y}^{(s)}_{n,m}(s)\right)\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}_{1,2}\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1}\right]^{\mathsf{T}}\end{aligned} \tag{2.18}$$

This can be readily converted back to the other form of product integral via (1.9). Note that the simpler form in (2.18) must hold for all realizable symmetric $(\tilde{Y}^{(s)}_{n,m}(s))$.

Now manipulate (2.18) into the form

$$\begin{aligned}&\left(\tilde{a}_{n,m}(\ell,0;s)\right) \equiv \left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}_{1,1}\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2} - \left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}_{2,1}\cdot\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}\\ &\left(\tilde{a}_{n,m}(\ell,0;s)\right)\cdot\left(\tilde{Y}^{(s)}_{n,m}(s)\right) + \left(\tilde{Y}^{(s)}_{n,m}(s)\right)\cdot\left(\tilde{a}_{n,m}(\ell,0;s)\right)^{\mathsf{T}}\end{aligned} \tag{2.19}$$

First choosing

$$\left(\tilde{Y}_{n,m}(s)\right) \equiv \left(1_{n,m}\right) \Rightarrow \left(\tilde{a}_{n,m}(\ell,0;s)\right) = \left(\tilde{a}_{n,m}(\ell,0;s)\right)^{\mathsf{T}} \tag{2.20}$$

gives the general form

$$\left(\tilde{a}_{n,m}(\ell,0;s)\right) \cdot \left(\tilde{Y}^{(s)}_{n,m}(s)\right) = \left(\tilde{Y}^{(s)}_{n,m}(s)\right) \cdot \left(\tilde{a}_{n,m}(\ell,0;s)\right) \tag{2.21}$$

i.e., these two symmetric matrices commute. These two matrices then must have a common set of eigenvectors. Writing

$$\begin{gathered}\left(\tilde{Y}^{(s)}_{n,m}(s)\right) = \sum_{\beta=1}^{N} \tilde{y}^{(s)}_{\beta}(s)\left(\tilde{y}^{(s)}_{\beta}(s)\right)_{\beta}\left(\tilde{y}^{(s)}_{\beta}(s)\right)_{\beta} \\ \left(\tilde{y}^{(s)}_{\beta}(s)\right)_{\beta_1} \cdot \left(\tilde{y}^{(s)}_{\beta}(s)\right)_{\beta_2} = 1_{\beta_1\beta_2} \quad \text{(orthornormal)}\end{gathered} \tag{2.22}$$

We are free to choose the $(\tilde{y}^{(s)}_{n})$ many different ways, still giving a symmetric $(\tilde{Y}^{(s)}_{n,m})$. This implies that $(\tilde{a}_{n,m})$ has all possible sets of N orthogonal eigenvectors. In turn this implies that it is proportional to the identity.

$$\begin{gathered}\left(\tilde{a}_{n,m}(\ell,0;s)\right) = \tilde{a}(\ell,0;s)\left(1_{n,m}\right) \\ \left(1_{n,m}\right) = \sum_{\beta=1}^{N}\left(y^{(s)}_{n}(s)\right)_{\beta}\left(y^{(s)}_{n}(s)\right)_{\beta} \quad \text{(for all orthonormal choices)}\end{gathered} \tag{2.23}$$

Thus we have the restrictive result

$$\begin{gathered}\left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathrm{T}}_{1,1} \cdot \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2} - \left(\tilde{T}_{n,m}(\ell,0;s)\right)^{\mathrm{T}}_{2,1} \cdot \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2} \\ = \tilde{a}(\ell,0;s)\left(1_{n,m}\right)\end{gathered} \tag{2.24}$$

From (1.7) we also readily find

$$\tilde{a}(0,0;s) = 1 \tag{2.25}$$

Using (1.10) the result (2.24) can also be applied to $((\tilde{U}_{n,m}(\ell,o;s))_{\nu,\nu'})$, giving

$$
\begin{aligned}
&\left(\tilde{U}_{n,m}(\ell,0;s)\right)_{1,1}^{\mathsf{T}} \cdot \left(\tilde{Y}_{n,m}(s)\right) \cdot \left(\tilde{U}_{n,m}(z,z_0;s)\right)_{2,2} \cdot \left(\tilde{Z}_{n,m}(s)\right) \\
&\quad -\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{2,1}^{\mathsf{T}} \cdot \left(\tilde{U}_{n,m}(z,z_0;s)\right)_{1,2} \\
&\quad = \tilde{a}(\ell,0;s)\ \left(1_{n,m}\right)
\end{aligned}
\tag{2.26}
$$

Thus reciprocity implies that only three of the four matrix blocks are in some sense independent, the fourth being calculable from the other three plus a scalar. At this point we can note that the coordinates 0 and ℓ can be replaced by arbitrary z_1 and z_2.

3. Time-Domain Energy for Lossless NMTLs

Let us form

$$
\begin{aligned}
&\frac{\partial}{\partial z}\left[\left(V_n^{(1)}(z,t)\right) \bullet \left(I_n^{(2)}(z,t)\right)\right] \\
&= \left[\frac{\partial}{\partial z}\left(V_n^{(1)}(z,t)\right)\right] \bullet \left(I_n^{(2)}(z,t)\right) + \left(V_n^{(1)}(z,t)\right) \bullet \left[\frac{\partial}{\partial z}\left(I_n^{(2)}(z,t)\right)\right] \\
&= -\left[\frac{\partial}{\partial z}\left(I_n^{(1)}(z,t)\right)\right] \bullet \left(L'_{n,m}(z)\right) + \left(I_n^{(2)}(z,t)\right) + \left(V_n^{(1)}(z,t)\right) \bullet \left(C'_{n,m}(z)\right) \bullet \left[\frac{\partial}{\partial t}\left(V_n^{(2)}(z,t)\right)\right]
\end{aligned}
\tag{3.1}
$$

Setting the 1 and 2 solutions as the same gives (noting the symmetric matrices)

$$
\begin{aligned}
&\frac{\partial}{\partial z}\left[\left(V_n(z,t)\right) \bullet \left(I_n(z,t)\right)\right) \\
&\quad = -\frac{1}{2}\frac{\partial}{\partial t}\left[\left(I_n(z,t)\right) \bullet \left(L'_{n,m}(z,t)\right) \bullet \left(I_n(z,t)\right) + \left(V_n(z,t)\right) \bullet \left(C'_{n,m}(z,t)\right) \bullet \left(V_n(z,t)\right)\right] \\
&\left(V_n(z_2,t)\right) \bullet \left(I_n(z_2,t)\right) - \left(V_n(z_1,t)\right) \bullet \left(I_n(z_1,t)\right) \\
&\quad = -\frac{1}{2}\frac{\partial}{\partial t}\int_{z_1}^{z_2}\left[\left(I_n(z',t)\right) \bullet \left(L'_{n,m}(z')\right) \bullet \left(I_n(z',t)\right) + \left(V_n(z',t)\right) \bullet \left(C'_{n,m}(z')\right) \bullet \left(V_n(z',t)\right)\right]\right]dz'
\end{aligned}
\tag{3.2}
$$

This has a striking similarity to the usual electromagnetic Poynting vector theorem.

4. Frequency-Domain Energy

Beginning with

$$
\begin{aligned}
&\frac{d}{dz}\left[\left(\tilde{V}_n(z,s)\right) \bullet \left(\tilde{I}_n(z,-s)\right)\right] \\
&= \left[\frac{d}{dz}\left(\tilde{V}_n(z,s)\right)\right] \bullet \left(\tilde{I}_n(z,-s)\right) + \left(\tilde{V}_n(z,s)\right) \bullet \left[\frac{d}{dz}\left(\tilde{I}_n(z,-s)\right)\right] \\
&= -\left(\tilde{I}_n(z,s)\right) \bullet \left(\tilde{Z}'_{n,m}(z,s)\right) \bullet \left(\tilde{I}_n(z,-s)\right) - \left(\tilde{V}_n(z,s)\right) \bullet \left(\tilde{Y}'_{n,m}(z,-s)\right) \bullet \left(\tilde{V}_n(z,-s)\right)
\end{aligned}
\tag{4.1}
$$

we can integrate this to find

$$
\begin{aligned}
&\left(\tilde{V}_n(z_2,s)\right) \bullet \left(\tilde{I}_n(z_2,-s)\right) - \left(\tilde{V}_n(z_1,s)\right) \bullet \left(\tilde{I}_n(z_1,-s)\right) \\
&= -\int_{z_1}^{z_2} \left[\left(\tilde{I}_n(z,s)\right) \bullet \left(\tilde{Z}'_{n,m}(z,s)\right) \bullet \left(\tilde{I}_n(z,-s)\right) + \left(V_n(z,s)\right) \bullet \left(\tilde{Y}'_n(z,-s)\right) \bullet \left(\tilde{V}_n(z,-s)\right)\right] dz
\end{aligned}
\tag{4.2}
$$

Setting $s = j\omega$ we have the relation for real power.

$$
\begin{aligned}
&\mathrm{Re}\left(\left(\tilde{V}_n(z_2,j\omega)\right) \bullet \left(\tilde{I}_n(z_2,-j\omega)\right)\right) - \mathrm{Re}\left(\left(\tilde{V}_n(z_1,j\omega)\right) \bullet \left(\tilde{I}_n(z_1,-j\omega)\right)\right) \\
&= -\mathrm{Re}\left(\int_{z_1}^{z_2} \left[\left(\tilde{I}_n(z,j\omega)\right) \bullet \left(\tilde{Z}'_{n,m}(z,j\omega)\right) \bullet \left(\tilde{I}_n(z,-j\omega)\right)\right.\right. \\
&\qquad\qquad \left.\left. + \left(\tilde{V}_n(z,j\omega)\right) \bullet \left(\tilde{Y}'_{n,m}(z,-j\omega)\right) \bullet \left(\tilde{V}_n(z,-j\omega)\right)\right] dz\right)
\end{aligned}
\tag{4.3}
$$

For RMS values an additional factor of ½ appears in the above. Also note that, as real-valued time functions, the above functions of $-j\omega$ are the same as the conjugates of the functions of $-j\omega$.

For the lossless case we have

$$
\begin{aligned}
&\left(\tilde{V}_n(z_2,s)\right) \bullet \left(\tilde{I}_n(z_2,-s)\right) - \left(\tilde{V}_n(z_1,s)\right) \bullet \left(\tilde{I}_n(z_1,s)\right) \\
&= -s\int_{z_1}^{z_2} \left[\left(\tilde{I}_n(z,s)\right) \bullet \left(L'_{n,m}(z)\right) \bullet \left(\tilde{I}_n(z,-s)\right) - \left(\tilde{V}_n(z,s)\right) \bullet \left(C'_{n,m}(z)\right) \bullet \left(\tilde{V}_n(z,s)\right)\right] dz
\end{aligned}
\tag{4.4}
$$

Setting s = $j\omega$ we have (noting the real quadratic forms)

$$\mathrm{Re}\left(\left(\tilde{V}_n(z_2, j\omega)\right) \cdot \left(\tilde{I}_n(z_2, -j\omega)\right)\right) - \mathrm{Re}\left(\left(\tilde{V}_n(z_1, j\omega)\right) \cdot \left(\tilde{I}_n(z_1, -j\omega)\right)\right) = 0 \qquad (4.5)$$

which is, of course, the power-conservation law.

5. Implications of Lossless NMTL on Matrizant

As is well known a lossless impedance (or admittance) function is an odd function of s (with real coefficients) so that for $s = j\omega$ no real power enters or leaves the network. Also known as Foster's theorem, such reactance functions have the properties:

1. All poles and zeros are simple and lie on the $j\omega$ axis
2. Residues are real and positive.
3. The function has either a zero or a pole at $s = 0$ and at $s = \infty$ (for a finite number of circuit elements in the latter case).
4. The reciprocal of the function is also a reactance function.

Lossless, reciprocal impedance (or admittance) matrices also have special properties, as can be seen by forming

$$\left(\tilde{V}_n(s)\right) = \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(s)\right) \tag{5.2}$$

Setting $s = j\omega$ choose

$$\left(\tilde{I}_n(j\omega)\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{I}_n(j\omega) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.3}$$

by open-circuiting all but the nth port. Since no real power can enter (or leave) this port, then $\tilde{V}_n(j\omega)$ must be 90° out of phase with the current implying that

$$\tilde{Z}_{n,n}(j\omega) = \text{imaginary} \tag{5.4}$$

So all the diagonal elements are reactance functions and have the (5.1) properties.

For the off-diagonal elements take two nonzero currents, $\tilde{I}_n(j\omega)$ and $\tilde{I}_{n'}(j\omega)$, and form

$$\left(\tilde{V}_n(j\omega)\right) \bullet \left(\tilde{I}_n(-j\omega)\right) = \text{imaginary}$$

$$\tilde{V}_n(j\omega)\ \tilde{I}_n(j\omega)\ + \tilde{V}_{n'}(j\omega)\ \tilde{I}_{n'}(-j\omega) = \text{imaginary}$$

$$\begin{aligned}\tilde{V}_n(j\omega) &= \tilde{Z}_{n,n}(j\omega)\ \tilde{I}_n(j\omega) + \tilde{Z}_{n,n'}(j\omega)\ \tilde{I}_{n'}(j\omega)\\ \tilde{V}_{n'}(j\omega) &= \tilde{Z}_{n',n'}(j\omega)\ I_{n'}(-j\omega) + \tilde{Z}_{n',n}(j\omega)\ \tilde{I}_n(j\omega)\end{aligned} \tag{5.5}$$

Combining these gives

$$\begin{aligned}&\tilde{I}_n(j\omega)\ \tilde{Z}_{n,n}(j\omega)\ \tilde{I}_n(-j\omega) + \tilde{I}_{n'}(j\omega)\ \tilde{Z}'_{n,n'}(j\omega)\tilde{I}_n(-j\omega)\\ &+ \tilde{I}_{n'}(j\omega)\ \tilde{Z}_{n',n'}\ \tilde{I}_{n'}(-j\omega) + \tilde{I}_n(j\omega)\ \tilde{Z}_{n',n}(j\omega)\tilde{I}_{n'}(-j\omega)\\ &= \text{imaginary}\end{aligned} \tag{5.6}$$

Noting that the diagonal terms are imaginary, and a function times its conjugate is real, gives

$$\tilde{I}_{n'}(j\omega)\ \tilde{Z}_{n,n'}(j\omega)\ \tilde{I}_n(-j\omega) + \tilde{I}_n(j\omega)\ \tilde{Z}_{n',n}(j\omega)\tilde{I}_{n'}(j\omega) = \text{imaginary} \tag{5.7}$$

Applying matrix symmetry (reciprocity) gives

$$\left[\tilde{I}_{n'}(j\omega)\tilde{I}_n(-j\omega) + \tilde{I}_n(j\omega)\tilde{I}_{n'}(j\omega)\right]\tilde{Z}_{n,n'}(j\omega) = \text{imaginary} \tag{5.8}$$

Noting that the coefficient is the sum of conjugates and is therefore real, we have

$$\tilde{Z}_{n,n'}(j\omega) = \text{imaginary} \tag{5.9}$$

So all matrix elements are odd functions of s as

$$\left(\tilde{Z}_{n,m}(-s)\right) = -\left(\tilde{Z}_{n,m}(s)\right) \tag{5.10}$$

Recall that in Section 2 formulae were developed for the impedance matrices of terminated NMTLs. Constraining the NMTL to be lossless with a lossless termination leads to information concerning the matrizant blocks. From (2.11) setting the normalization to the identity gives

$$\begin{aligned}&\left[\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1} - \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2} \bullet \left(\tilde{Y}^{(s)}_{n,m}(s)\right)\right]\\ &\bullet \left[\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1} - \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2} \bullet \left(\tilde{Y}^{(s)}_{n,m}(s)\right)\right]^{-1} = \text{odd function of } s\end{aligned} \tag{5.11}$$

Taking $(\tilde{Y}_{n,m}^{(s)}(s))$ as open and short circuits (special degenerate cases of reactance functions) gives odd matrix functions

$$\boxed{\begin{array}{l} \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1}^{-1} = -\left(\tilde{T}_{n,m}(\ell,0;-s)\right)_{2,1} \bullet \left(\tilde{T}_{n,m}(\ell,0;-s)\right)_{1,1}^{-1} \\ \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}^{-1} = -\left(\tilde{T}_{n,m}(\ell,0;-s)\right)_{2,2} \bullet \left(\tilde{T}_{n,m}(\ell,0;-s)\right)_{1,2}^{-1} \end{array}} \tag{5.12}$$

These can also be rearranged as

$$\boxed{\begin{array}{l} \left(\tilde{T}_{n,m}(\ell,0;-s)\right)_{2,1}^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1} = -\left(\tilde{T}_{n,m}(\ell,0;-s)\right)_{1,1}^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1} \\ \left(\tilde{T}_{n,m}(\ell,0;-s)\right)_{2,2}^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2} = -\left(\tilde{T}_{n,m}(\ell,0;-s)\right)_{1,2}^{-1} - \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2} \end{array}} \tag{5.13}$$

Additional relations of this type can be found by rewriting (5.11) by replacing $(\tilde{Y}_{n,m}^{(s)})$ by $a(\tilde{Y}_{n,m}^{(s)})$ where a is real and small, as

$$\begin{array}{l} \left[\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1} - a\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2} \bullet \left(\tilde{Y}_{n,m}^{(s)}(s)\right)\right] \\ \quad \bullet \left[\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1} - a\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2} \bullet \left(\tilde{Y}_{n,m}^{(s)}(s)\right)\right] = \text{odd function of } s \end{array} \tag{5.14}$$

Next expand the inverse as a geometric series [4] for small a as

$$\begin{array}{l} \left[\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1} - a\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2} \bullet \left(\tilde{Y}_{n,m}^{(s)}(s)\right)\right]^{-1} \\ = \left[\left(1_{n,m}\right) + a\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1}^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2} \bullet \left(\tilde{Y}_{n,m}^{(s)}(s)\right)\right]^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1}^{-1} \\ = \left[\left(1_{n,m}\right) + a\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1}^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2} \bullet \left(\tilde{Y}_{n,m}^{(s)}(s)\right) + O\left(a^2\right)\right] \\ \quad \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1}^{1} \text{ as } a \to 0 \end{array} \tag{5.15}$$

Consider the various powers of a in the series expansion of (5.14). The a^0 term reproduces the first of (5.12). The coefficient of a^1 gives

$$\left[-\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2} \bullet \left(\tilde{Y}_{n,m}(s)\right) + \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1}^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2} \bullet \left(\tilde{Y}_{n,m}^{(s)}(s)\right)\right]$$

$$\bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1}^{-1} = \text{odd function of } s \tag{5.16}$$

An acceptable choice for $(\tilde{Y}_{n,m}^{(s)})$ as an odd function is of the form of a special capacitance matrix

$$\left(\tilde{Y}_{n,m}^{(s)}(s)\right) = sC\left(1_{n,m}\right) = -\left(\tilde{Y}_{n,m}^{(s)}(-s)\right) \tag{5.17}$$

giving

$$\left[\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2} - \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1}^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}\right] \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1}^{-1} \tag{5.18}$$

$$= \text{even function of } s$$

Higher powers of a give larger ensembles of the matrizant blocks as odd or even functions of s.

A similar result is found by rewriting (5.11) in the form

$$\left[b\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1} \bullet \left(\tilde{Z}_{n,m}^{(s)}(s)\right) - \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2}\right]$$

$$\bullet \left[b\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1} \bullet \left(\tilde{Z}_{n,m}^{(s)}(s)\right) - \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}\right]^{-1}$$

$$= \text{odd function of } s \tag{5.19}$$

Now take b as small and write

$$\left[b\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1} \bullet \left(\tilde{Z}_{n,m}^{(s)}(s)\right) - \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}\right]^{-1}$$

$$= -\left[-b\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1} \bullet \left(\tilde{Z}_{n,m}^{(s)}(s)\right) + \left(1_{n,m}\right)\right]^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}^{-1}$$

$$= -\left[\left(1_{n,m}\right) + b\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}^{-1} \bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1} \bullet \left(\tilde{Z}_{n,m}^{(s)}(s)\right) + O\left(b^2\right)\right] \tag{5.20}$$

$$\bullet \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}^{-1} \quad \text{as } b \to 0$$

Consider the various powers of b. The b^0 term reproduces the second of (5.12). The coefficient of b^1 with the choice

$$\left(\tilde{Z}_{n,m}^{(s)}(s)\right) = sL\left(1_{n,m}\right) = -\left(\tilde{Z}_{n,m}^{(s)}(-s)\right) \tag{5.21}$$

gives

$$\begin{aligned}&\left[\left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,1} - \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{2,2} \cdot \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}^{-1} \cdot \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,1}\right] \\ &\quad \cdot \left(\tilde{T}_{n,m}(\ell,0;s)\right)_{1,2}^{-1} \\ &= \text{even function of } s\end{aligned} \tag{5.22}$$

As we can see, lossless NMTL matrizants are significantly constrained by the lossless property. These results also apply to $(\tilde{U}_{n,m}(\ell,0;s)_{\nu,\nu'})$ by application of (1.9).

6. Direct Construction of Properties of Matrizant for Lossless NMTL

Now consider the properties of the matrizant in the simple lossless case from the definition of the product integral as a limit. For the simple lossless case we have

$$\begin{aligned}
\left(\tilde{Z}'_{n,m}(z,s)\right) &= s\left(L'_{n,m}(z)\right) \\
\left(\tilde{Y}'_{n,m}(z,s)\right) &= s\left(C'_{n,m}(z)\right) \\
\left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\nu,\nu'}\right) &= -s\begin{pmatrix} \left(0_{n,m}\right) & \left(L'_{n,m}(z)\right)\cdot\left(Y_{n,m}\right) \\ \left(Z_{n,m}\right)\cdot\left(C'_{n,m}(z)\right) & \left(0_{n,m}\right) \end{pmatrix}
\end{aligned} \tag{6.1}$$

where the normalizing impedance matrix is taken as constant (specifically frequency independent).

The product integral takes the forms [7]

$$\begin{aligned}
\left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\nu,\nu'}\right) &= \prod_{z_0}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}(z',s)\right)_{\nu,\nu'}\right)dz'} \\
&= \prod_{z_0}^{z}\left[\left(\left(1_{n,m}\right)_{\nu,\nu'}\right) + \left(\left(\tilde{\Gamma}_{n,m}(z',s)\right)_{\nu,\nu'}\right)dz\right] \\
&= \prod_{z_0}^{z}\begin{pmatrix} \left(1_{n,m}\right) & -s\left(L'_{n,m}(z')\right)\cdot\left(Y_{n,m}\right)dz' \\ -s\left(Z_{n,m}\right)\cdot\left(C'_{n,m}(z')\right)dz' & \left(1_{n,m}\right) \end{pmatrix}
\end{aligned} \tag{6.2}$$

In this latter form we take the product integral as a product of terms, one for each Δz from z_0 to z. Consider the even/odd in s properties of any of these terms as

$$\begin{pmatrix} \left(1_{n,m}\right) & -s\left(L'_{n,m}(z)\right)\cdot\left(Y_{n,m}\right)\Delta z \\ -s\left(Z_{n,m}\right)\cdot\left(C'_{n,m}(z')\right)\Delta z & \left(1_{n,m}\right) \end{pmatrix} = \begin{pmatrix} (\text{even}) & (\text{odd}) \\ (\text{odd}) & (\text{even}) \end{pmatrix} \tag{6.3}$$

this being the case for *all* such terms.

Take the dot product of any two adjacent terms in the continued products; this has the form

$$\begin{pmatrix}(\text{even}) & (\text{odd})\\ (\text{odd}) & (\text{even})\end{pmatrix} \cdot \begin{pmatrix}(\text{even}) & (\text{odd})\\ (\text{odd}) & (\text{even})\end{pmatrix}$$
$$= \begin{pmatrix}(\text{even})\cdot(\text{even}) + (\text{odd})\cdot(\text{odd}) & (\text{even})\cdot(\text{odd}) + (\text{odd})\cdot(\text{even})\\ (\text{odd})\cdot(\text{even}) + (\text{even})\cdot(\text{odd}) & (\text{odd})\cdot(\text{odd}) + (\text{even})\cdot(\text{even})\end{pmatrix} \tag{6.4}$$
$$= \begin{pmatrix}(\text{even}) & (\text{odd})\\ (\text{odd}) & (\text{even})\end{pmatrix}$$

By extension (induction) continue this process to all terms in the product giving the (6.4) result to the entire continued product. This holds for all $\Delta z > 0$. Take the limit as $\Delta z \to 0$ giving this as the exact form of the product integral.

We, therefore, have the general result for this special lossless case

$$\left.\begin{array}{l}\left(\tilde{U}_{n,m}(z, z_0; s)\right)_{1,1}\\ \left(\tilde{U}_{n,m}(z, z_0; s)\right)_{2,2}\end{array}\right\} = \text{even functions of } s$$
$$\left.\begin{array}{l}\left(\tilde{U}_{n,m}(z, z_0; s)\right)_{1,2}\\ \left(\tilde{U}_{n,m}(z, z_0; s)\right)_{2,1}\end{array}\right\} = \text{odd functions of } s \tag{6.5}$$

which is a yet more powerful result. This is based on the lossless properties of the incremental sections. Whereas, in Section 5, the results are based on the lossless properties of the entire NMTL as a whole, and, as such, are applicable to more general linear, reciprocal electromagnetic systems.

7. Bounds on Propagation on NMTLs

Consider the product integral (1.5) representing propagation on an NMTL. We have the norm (some associated matrix norm, later 2-norm)

$$
\begin{aligned}
\left\| \left(\left(\tilde{U}_{n,m}(z, z_0; s) \right)_{\nu,\nu'} \right) \right\| &= \left\| \prod_{z_0}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}(z', s) \right)_{\nu,\nu'} \right) dz'} \right\| \\
&= \left\| \prod_{z_0}^{z} \left[\left((1_{n,m})_{\nu,\nu'} \right) + \left(\left(\tilde{\Gamma}_{n,m}(z', s) \right)_{\nu,\nu'} \right) dz' \right] \right\| \\
&\leq \left\| \left((1_{n,m})_{\nu,\nu'} \right) + \left(\left(\tilde{\Gamma}_{n,m}(z_0, s) \right)_{\nu,\nu'} \right) \Delta z \right\| \left\| \left((1_{n,m})_{\nu,\nu'} \right) + \left(\left(\tilde{\Gamma}_{n,m}(z_0 + \Delta z; s) \right)_{\nu,\nu'} \right) \Delta z \right\| \\
&\quad \cdots \left\| \left((1_{n,m})_{\nu,\nu'} \right) + \left(\left(\tilde{\Gamma}_{n,m}(z_0 + [N-1]\Delta z, s) \right)_{\nu,\nu'} \right) \right\| \quad \text{for } \Delta z \to 0 \left(z - z_0 = N \Delta z \right) \\
&\qquad\qquad \text{(norm product rule)} \\
&\leq \left[1 + \left\| \left(\left(\tilde{\Gamma}_{n,m}(z_0, s) \right)_{\nu,\nu'} \right) \right\| \Delta z \right] \left[1 + \left\| \left(\left(\tilde{\Gamma}_{n,m}(z_0 + \Delta z; s) \right)_{\nu,\nu'} \right) \right\| \Delta z \right] \\
&\quad \cdots \left[1 + \left\| \left(\left(\tilde{\Gamma}_{n,m}(z_0 + [N-1]\Delta z, s) \right)_{\nu,\nu'} \right) \right\| \Delta z \right] \quad \text{(norm sum rule)} \\
&= \prod_{z_0}^{z} e^{\left\| \left(\left(\tilde{\Gamma}_{n,m}(z', s) \right)_{\nu,\nu'} \right) \right\| dz'} = e^{\int_{z_0}^{z} \left\| \left(\left(\tilde{\Gamma}_{n,m}(z', s) \right)_{\nu,\nu'} \right) \right\| dz'}
\end{aligned}
\tag{7.1}
$$

This reduces the problem to the norm of the propagation supermatrix (1.4). One can attempt to minimize this norm by appropriate choice of the normalizing impedance matrix. Note that all associated matrix norms have

$$
\left\| \left((1_{n,m})_{\nu,\nu'} \right) \right\| = 1 \tag{7.2}
$$

A special case of interest has $s = j\omega$ and the norm chosen as the 2-norm. Furthermore let the NMTL be lossless so that

$$
\begin{aligned}
\left(\tilde{Z}'_{n,m}(z, s) \right) &= s \left(L'_{n,m}(z) \right) \\
\left(\tilde{Y}'_{n,m}(z, s) \right) &= s \left(C'_{n,m}(z) \right)
\end{aligned}
\tag{7.3}
$$

A special case of interest is a lossless NMTL for which

$$\left(\tilde{Z}'_{n,m}(z,s)\right) = s\left(L'_{n,m}(z)\right) \quad , \left(\tilde{Y}'_{n,m}(z,s)\right) = s\left(C'_{n,m}(z)\right)$$

$$\left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\nu,\nu'}\right) = -s\begin{pmatrix} \left(0_{n,m}\right) & \left(L'_{n,m}(z)\right) \cdot \left(\tilde{Y}_{n,m}(s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(Y'_{n,m}(z)\right) & \left(0_{n,m}\right) \end{pmatrix} \tag{7.4}$$

Letting

$$s = j\omega \quad , \quad \left(Z_{n,m}\right) = \text{real, symmetric, and frequency independent} \tag{7.5}$$

We have the 2-norm [6]

$$\begin{aligned}
&\left\|\left(\left(1_{n,m}\right)_{\nu,\nu'}\right) + \left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{\nu,\nu'}\right)\Delta z\right\|_2 \\
&= \left[\text{max eigenvalue}\left(\left[\left(\left(1_{n,m}\right)_{\nu,\nu'}\right) + \left(\left(\Gamma_{n,m}(z,j\omega)\right)_{\nu,\nu'}\right)\Delta z\right]^{\dagger}\right.\right. \\
&\qquad \left.\left.\cdot \left[\left(\left(1_{n,m}\right)_{\nu,\nu'}\right) + \left(\left(\Gamma_{n,m}(z,j\omega)\right)_{\nu,\nu'}\right)\Delta z\right]\right)\right]^{1/2} \\
&= \left[\text{max eigenvalue}\left(\left(\left(1_{n,m}\right)_{\nu,\nu'}\right) + \left[\left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{\nu,\nu'}\right)^{\dagger} + \left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{\nu,\nu'}\right)\right]\Delta z\right)\right. \\
&\qquad \left. + O\left(\left(\Delta z\right)^2\right) \text{ as } \Delta z \to 0\right]^{1/2} \\
&= 1 + \frac{\Delta z}{2}\left[\text{max eigenvalue}\left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{\nu,\nu'}\right)^{\dagger} + \left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{\nu,\nu'}\right)\right] \\
&\qquad + 0\left(\left(\Delta z\right)^2\right) \text{ as } \Delta z \to 0
\end{aligned} \tag{7.6}$$

Thus we need to consider

$$\begin{aligned}
&\left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{\nu,\nu'}\right)^{\dagger} + \left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{\nu,\nu'}\right) \\
&= j\omega\begin{pmatrix} \left(0_{n,m}\right) & \left(L'_{n,m}(z)\right) \cdot \left(Y_{n,m}\right) - \left(C'_{n,m}(z)\right) \cdot \left(Z_{n,m}\right) \\ \left(Z_{n,m}\right) \cdot \left(C'_{n,m}(z)\right) - \left(Y_{n,m}\right) \cdot \left(l'_{n,m}(z)\right) & \left(0_{n,m}\right) \end{pmatrix}
\end{aligned} \tag{7.7}$$

This can be set to zero provided

$$
\begin{aligned}
&\left(L'_{n,m}(z)\right) \cdot \left(Y_{n,m}\right) - \left(C'_{n,m}(z)\right) \cdot \left(Z_{n,m}\right) = 0 \\
&\left(C'_{n,m}(z)\right)^{-1} \cdot \left(L'_{n,m}(z)\right) = \left(Z_{n,m}\right)^2 = \left(L'_{n,m}(z)\right) \cdot \left(C'_{n,m}(z)\right)^{-1} \\
&\left(L'_{n,m}(z)\right) \cdot \left(C'_{n,m}(z)\right) = \left(C'_{n,m}(z)\right) \cdot \left(L'_{n,m}(z)\right) \quad \text{(commute)}
\end{aligned}
\tag{7.8}
$$

This is a constant-characteristic impedance NMTL with the normalizing impedance matrix given as above. Furthermore the per-unit-length inductance and capacitance matrices commute implying common eigenvectors. Since these are also eigenvectors of the z-independent $\left(Z_{n,m}\right)$, then the eigenvectors must be z-independent except possibly in the case of degenerate eigenvalues of $\left(Z_{n,m}\right)$. Such commuting matrices have been treated in [1, 2].

With the constraint of (7.8) we have

$$
\begin{aligned}
&\left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{v,v'}\right)^{\mathsf{T}} + \left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{v,v'}\right) = \left(\left(0_{n,m}\right)\right) \\
&\left\|\left(\left(\tilde{U}_{n,m}(z,z_0;j\omega)\right)_{v,v'}\right)\right\|_2 \le \prod_{z_0}^{z} e^{0dz'} = 1
\end{aligned}
\tag{7.9}
$$

Applying this to the inverse matrizant and following the previous procedure also gives

$$
\left\|\left(\left(\tilde{U}_{n,m}(z_0,z;j\omega)\right)_{v,v'}\right)\right\|_2 \le 1
\tag{7.10}
$$

Taking the 2-norm of the product gives

$$
\begin{aligned}
&\left\|\left(\left(\tilde{U}_{n,m}(z,z_0;j\omega)\right)_{v,v'}\right) \odot \left(\left(\tilde{U}_{n,m}(z_0,z;j\omega)\right)_{v,v'}\right)\right\|_2 \\
&\quad = \left\|\left(\left(1_{n,m}\right)_{v,v'}\right)\right\|_2 = 1 \\
&\quad \le \left\|\left(\left(\tilde{U}_{n,m}(z,z_0;j\omega)\right)_{v,v'}\right)\right\|_2 \left\|\left(\left(\tilde{U}_{n,m}(z_0,z;j\omega)\right)_{v,v'}\right)\right\|_2
\end{aligned}
\tag{7.11}
$$

Combining these constraints gives

$$
\left\|\left(\left(\tilde{U}_{n,m}(z,z_0;j\omega)\right)_{v,v'}\right)\right\|_2 = \left\|\left(\left(\tilde{U}_{n,m}(z_0,z;j\omega)\right)_{v,v'}\right)\right\|_2 = 1
\tag{7.12}
$$

This can be compared to the general (unconstrained) case of the propagation supermatrix for which all-zero diagonal entries implies

$$\det\left(\left(\left(\tilde{U}_{n,m}(z,z_0;j\omega)\right)_{v,v'}\right)\right) = 1 \tag{7.13}$$

For the more general case of lossless NMTLs one could try to choose $(Z_{n,m})$ (symmetric) such that (7.7) is minimized in some sense over some range of z of interest. In 2-norm sense this means minimizing the maximum eigenvalue (magnitude) noting that (7.7) is Hermitian. In the sense of the continued product in (6.1) we then have

$$\left\|\left(\left(\tilde{U}_{n,m}(z,z_0;j\omega)\right)_{v,v'}\right)\right\|_2 \le e^{\int_{z_0}^{z} \frac{1}{2}\left[\text{max eigenvalue}\left(\left(\left(\tilde{\Gamma}_{n,m}(z',j\omega)\right)_{v,v'}\right)^{\dagger} + \left(\left(\tilde{\Gamma}_{n,m}(z',j\omega)\right)_{v,v'}\right)\right)\right]dz'} \tag{7.14}$$

as the real quantity to be minimized. This is accomplished by minimizing the integral in the exponent. One can show that the maximum eigenvalue is real and nonnegative by noting that (7.7) takes the form for eigenvalues

$$j\omega\begin{pmatrix} (0_{n,m}) & (a_{n,m}) \\ -(a_{n,m})^{\mathsf{T}} & (0_{n,m}) \end{pmatrix} \cdot \begin{pmatrix} (x_n) \\ (y_n) \end{pmatrix} = j\omega\psi\begin{pmatrix} (x_n) \\ (y_n) \end{pmatrix}$$

$$\begin{pmatrix} (a_{n,m}) \cdot (y_n) \\ -(a_{n,m})^{\mathsf{T}} \cdot (x_n) \end{pmatrix} = \psi\begin{pmatrix} (x_n) \\ (y_n) \end{pmatrix}$$

$$(a_{n,m})^{\mathsf{T}} \cdot (a_{n,m}) \cdot (y_n) = \psi (a_{n,m})^{\mathsf{T}} \cdot (x_n) = -\psi^2 (y_n)$$

$-\psi^2$ = eigenvalues of $(a_{n,m})^{\mathsf{T}} \cdot (a_{n,m})$ (real Hermitian, positive semidefinite)

≥ 0 (real, nonnegative)

$\psi = \pm$ [-real nonnegative]$^{1/2}$

$= \pm j$ [real nonnegative]$^{1/2}$ (7.15)

Thus take the most negative imaginary value of ψ, multiply by $j\omega$ to obtain the largest real positive eigenvalue. This implies

$$\frac{1}{2}\left[\text{max. eigenvalue}\left(\left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{v,v'}\right)^{\dagger} + \left(\left(\tilde{\Gamma}_{n,m}(z,j\omega)\right)_{v,v'}\right)\right)\right] \ge 0 \ \text{ for all } z \tag{7.16}$$

Appropriate choice of $(Z_{n,m})$ can minimize the integral of this. In turn

$$\left\|\left(\left(\tilde{U}_{n,m}(z,z_0;j\omega)\right)_{v,v'}\right)\right\|_2 \ge 1 \tag{7.17}$$

is minimized.

8. The Lossless Transmission Line: N = 1

Now specialize the foregoing results to the case of N = 1: a single-conductor (plus reference) transmission line. The matrizant blocks are then scalars. This should simplify the results, since the normalizing impedance is also a scalar and factors out of the equations. Note that the blocks now equal their own transposes. Multiplication now commutes. The forms of the matrizants in (1.10) now reduce to

$$\begin{pmatrix} \tilde{T}_{1,1}(z,z_0;s) & \tilde{T}_{1,2}(z,z_0;s) \\ \tilde{T}_{2,1}(z,z_0;s) & \tilde{T}_{2,2}(z,z_0;s) \end{pmatrix} = \begin{pmatrix} \tilde{U}_{1,1}(z,z_0;s) & \tilde{Z}(s)\tilde{U}_{1,2}(z,z_0;s) \\ \tilde{Y}(s)\tilde{U}_{2,1}(z,z_0;s) & \tilde{U}_{2,2}(z,z_0;s) \end{pmatrix} \tag{7.1}$$

The determinants in (1.12) also simplify to

$$\begin{aligned} \tilde{U}_{1,1}(z,z_0;s)\tilde{U}_{2,2}(z,z_0;s) - \tilde{U}_{1,2}(z,z_0;s)\tilde{U}_{2,1}(z,z_0;s) &= 1 \\ \tilde{T}_{1,1}(z,z_0;s)\tilde{T}_{2,2}(z,z_0;s) - \tilde{T}_{1,2}(z,z_0;s)\tilde{T}_{2,1}(z,z_0;s) &= 1 \end{aligned} \tag{7.2}$$

Comparing this to (2.24) we find for this special case

$$\tilde{a}(\ell,0;s) = 1 \tag{7.3}$$

Considering the reciprocity results in Section 2, we find that this case of N = 1 is the trivial case in which the appropriate equations become tautologies.

The lossless-case result in Sections 5 and 6 now simplifies for N = 1 with the additional constraint in (7.2).

9. Concluding Remarks

Considering a previous paper [5], there are various reciprocity and energy theorems applicable to electromagnetic fields and sources in free space (or a lossless uniform isotropic medium). The present paper considers the application of such concepts to NMTLs, with various results found. Perhaps the list of such results can be extended.

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