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An interpolation Technique for Analyzing Sections of Nonuniform Multiconductor Transmission Lines, Part 2

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Abstract

This paper extends previous results by reducing the diagonalization of a $2N \times 2N$ matrix to that of diagonalizing an $N \times N$ propagation matrix. A simple direct product extends this to the $2N \times 2N$ case. In addition, a judicious choice of average per-unit-length impedance and admittance matrices can improve the accuracy of the perturbation result accounting for the nonuniform character of the transmission line.

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1. Introduction

A recent paper [6] introduces a simple interpolation technique for evaluating the product integral representing the propagation of waves on a nonuniform multiconductor transmission line (NMTL). For a given (say ℓth) section of the line (for $z_\ell \le z \le z_{\ell+1}$) the technique involves defining an average value of the impedance-per-unit-length and admittance-per-unit-length matrices. This allows one to decompose the propagation supermatrix into a relatively large constant supermatrix plus a relatively small varying supermatrix (function of z). The product integral of the first is analytically exactly solvable. Using the sum rule the product integral including the second is approximately solvable by a perturbation involving the usual sum integral. An important aspect is the fact that only one supermatrix, the constant one, needs to be diagonalized, this being done only once for the entire interval.

Summarizing from [6] we have the supermatrizant differential equation

$$
\begin{aligned}
&\frac{\partial}{\partial z}\left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\upsilon,\upsilon'}\right) = \left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{U}_{n,m}(z,z_0;s)\right)_{\upsilon,\upsilon'}\right)\\
&\left(\left(\tilde{\Gamma}_{n,m}(z,s)\right)_{\upsilon,\upsilon'}\right) = -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}_{n,m}(s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\left(\tilde{Y}'_{n,m}(s)\right) & \left(0_{n,m}\right) \end{pmatrix}\\
&\qquad \equiv \text{propagation supermatrix}\\
&\left(\left(\tilde{U}_{n,m}(z_0,z_0;s)\right)_{\upsilon,\upsilon'}\right) = \left(\left(1_{n,m}\right)_{\upsilon,\upsilon'}\right) = \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(1_{n,m}\right) \end{pmatrix}\\
&\qquad \text{(boundary condition)}\\
&\left(\tilde{Z}_{n,m}(s)\right) \equiv \left(\tilde{Y}_{n,m}(s)\right)^{-1} \equiv \text{normalizing impedance matrix (symmetric, positive real)}\\
&\qquad\qquad \text{to be chosen at our convenience}\\
&\left(\tilde{Z}'_{n,m}(z,s)\right) = \left(\tilde{Z}'_{n,m}(z,s)\right)^T \equiv \text{longitudinal impedance-per-unit-length matrix}\\
&\left(\tilde{Y}'_{n,m}(z,s)\right) = \left(\tilde{Y}'_{n,m}(z,s)\right)^T \equiv \text{transverse admittance-per-unit-length matrix}\\
&\sim\ \equiv \text{two-sided Laplace transform over time } t\\
&s \equiv \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency}
\end{aligned}
\tag{1.1}
$$

While this is quite general, we first observe that this will be applied to each of the sections (labeled by ℓ) of the line. The normalizing impedance matrix will be separately chosen for each section. With solutions for all sections the matrizants for each section are multiplied by a supermatrix involving the normalizing impedance matrices for adjacent sections [6 ((4.8))] which need not concern us here.

2. Solution for a Section of Transmission Line

For the ℓth section of the NMTL set

$$\left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}(z,s)\right)_{\upsilon,\upsilon'}\right) = \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right) + \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(z,s)\right)_{\upsilon,\upsilon'}\right) \tag{2.1}$$

The constant (with respect to z) term has

$$\left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) = \prod_{z_\ell}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right)dz'} = e^{\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right)[z-z_\ell]} \tag{2.2}$$

The sum rule of the product integral then gives

$$\begin{aligned}
\prod_{z_\ell}^{z} e^{\left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}(z',s)\right)_{\upsilon,\upsilon'}\right)dz'} &= \left(\left(\tilde{G}^{(\ell,0)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{G}_{n,m}^{(\ell,1)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) \\
\left(\left(\tilde{G}_{n,m}^{(\ell,1)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) &= \prod_{z_\ell}^{z} e^{\left(\left(\tilde{H}_{n,m}^{(\ell)}(z',z_\ell;s)\right)_{\upsilon,\upsilon'}\right)dz'} \\
\left(\left(\tilde{H}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) &\equiv \left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right)^{-1} \odot \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(z,s)\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right)
\end{aligned} \tag{2.3}$$

At this point one can proceed as in [6] to diagonalize the constant supermatrix. However, let us delve more deeply into this to simplify the results.

3. Diagonalization of $\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right)$

First we need to appropriately define this constant matrix. Previously this was done via

$$
\begin{aligned}
\left(\tilde{Z}'^{(\ell,avg)}_{n,m}(s)\right) &\equiv \frac{1}{2}\left[\left(\tilde{Z}'_{n,m}(z_{\ell+1},s)\right) + \left(\tilde{Z}'_{n,m}(z_\ell,s)\right)\right] \\
\left(\tilde{Y}'^{(\ell,avg)}_{n,m}(s)\right) &\equiv \frac{1}{2}\left[\left(\tilde{Y}'_{n,m}(z_{\ell+1},s)\right) + \left(\tilde{Y}'_{n,m}(z_\ell,s)\right)\right]
\end{aligned}
\tag{3.1}
$$

For present purposes let us introduce another definition

$$
\begin{aligned}
\left(\tilde{Z}'^{(\ell,avg)}_{n,m}(s)\right) &= \frac{1}{z_{\ell+1}-z_\ell}\int_{z_\ell}^{z_{\ell+1}} \left(\tilde{Z}'_{n,m}(z,s)\right)dz \\
\left(\tilde{Y}'^{(\ell,avg)}_{n,m}(s)\right) &= \frac{1}{z_{\ell+1}-z_\ell}\int_{z_\ell}^{z_{\ell+1}} \left(\tilde{Y}'_{n,m}(z,s)\right)dz
\end{aligned}
\tag{3.2}
$$

If the original per-unit-length matrices are of the form a constant matrix plus z times a constant matrix, these two definitions reduce to the same thing. Note also that the both forms of average give symmetric (reciprocity) positive real (p.r.) matrices. With this we have for the constant part of the propagation matrix

$$
\begin{aligned}
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right) &= -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}'^{(\ell,avg)}_{n,m}(s)\right)\cdot\left(\tilde{Y}^{(\ell)}_{n,m}(s)\right) \\ \left(\tilde{Z}^{(\ell)}_{n,m}(s)\right)\cdot\left(\tilde{Y}'^{(\ell,avg)}_{n,m}(s)\right) & \left(0_{n,m}\right) \end{pmatrix} \\
\left(\tilde{Z}^{(\ell)}_{n,m}(s)\right) &= \left(\tilde{Z}^{(\ell)}_{n,m}(s)\right)^T = \left(\tilde{Y}_{n,m}(s)\right)^{-1} \\
&\equiv \text{normalizing impedance matrix for } \ell\text{th section}
\end{aligned}
\tag{3.3}
$$

Note that all the impedance and admittance matrices are N x N. The supermatrices are then 2N x 2N.

The form of this propagation supermatrix is exactly the form for a uniform MTL. Using the results of [1] we then use the form of the normalizing impedance matrix as the characteristic impedance matrix which separates the propagation into left- and right-propagating waves. So we form

$$\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) = \left[\left(\tilde{Z}'^{(\ell,avg)}_{n,m}(s)\right) \cdot \left(\tilde{Y}'^{(\ell,avg)}_{n,m}(s)\right)\right]^{1/2}$$

(N x N, p.r. square root)

$$\begin{aligned}
\left(\tilde{Z}'^{(\ell)}_{n,m}(s)\right) &= \left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) \cdot \left(\tilde{Y}'^{(\ell,avg)}_{n,m}(s)\right)^{-1} \\
&= \left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)^{-1} \cdot \left(\tilde{Z}'^{(\ell,avg)}_{n,m}(s)\right) \\
\left(\tilde{Y}'^{(\ell)}_{n,m}(s)\right) &= \left(\tilde{Y}'^{(\ell,avg)}_{n,m}(s)\right) \cdot \left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)^{-1} \\
&= \left(\tilde{Z}'^{(\ell,avg)}_{n,m}(s)\right)^{-1} \cdot \left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) \\
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right) &= -\begin{pmatrix} (0_{n,m}) & \left(\gamma_{n,m}^{(\ell)}(s)\right) \\ \left(\gamma_{n,m}^{(\ell)}(s)\right) & (0_{n,m}) \end{pmatrix} \\
&= -\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned} \tag{3.4}$$

(See [5 (Appendix A)] for conventions used concerning the direct (or Kronecker) product and other supermatrix operations.) From this we have

$$\begin{aligned}
\left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) &= e^{\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right)[z-z_\ell]} \\
&= e^{-\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}[z-z_\ell]} \\
&= \begin{pmatrix} \cosh\left(\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)[z-z_\ell]\right) & -\sinh\left(\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)[z-z_\ell]\right) \\ -\sinh\left(\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)[z-z_\ell]\right) & \cosh\left(\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)[z-z_\ell]\right) \end{pmatrix} \\
&= \cosh\left(\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)[z-z_\ell]\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad - \sinh\left(\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)[z-z_\ell]\right) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned} \tag{3.5}$$

Instead of directly diagonalizing the 2N x 2N matrices, consider the N x N matrices. From [1] we have the eigenvalue/eigenvector equations (N x N)

$$\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)^2 \cdot \left(\tilde{\mathrm{v}}_{\mathrm{n}}^{(\ell)}(s)\right)_\beta = \tilde{\gamma}_\beta^{(\ell)^2}(s)\left(\tilde{\mathrm{v}}_{\mathrm{n}}^{(\ell)}(s)\right)_\beta$$
$$\left(\tilde{i}_n^{(\ell)}(s)\right)_\beta \cdot \left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)^2 = \tilde{\gamma}_\beta^{(\ell)^2}(s)\left(\tilde{i}_{\mathrm{n}}^{(\ell)}(s)\right)_\beta \tag{3.6}$$

Defining $\tilde{\gamma}_\beta^{(\ell)}(s)$ by the p.r. square root we then summarize

$$\left(\tilde{\gamma}_{n,m}^{(\ell)^2}(s)\right) = \sum_{\beta=1}^{N} \tilde{\gamma}_\beta^{(\ell)}(s)\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta \left(\tilde{i}_n^{(\ell)}(s)\right)_\beta$$
$$\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_{\beta_1} \cdot \left(\tilde{i}_n^{(\ell)}(s)\right)_{\beta_2} = 1_{\beta_1,\beta_2} \quad \text{(biorthonormal)} \tag{3.7}$$
$$\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta = \left(\tilde{Z}_{n,m}^{(\ell)}(s)\right) \cdot \left(\tilde{i}_n^{(\ell)}(s)\right)_{\beta_2} = \left(\tilde{i}_n^{(\ell)}(s)\right)_\beta \cdot \left(\tilde{Z}_{n,m}^{(\ell)}(s)\right)$$
$$\left(\tilde{Z}_{n,m}^{(\ell)}(s)\right) = \sum_{\beta=1}^{N}\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta \left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta \quad , \qquad \left(\tilde{Y}_{n,m}^{(\ell)}(s)\right) = \sum_{\beta=1}^{N}\left(\tilde{i}_n^{(\ell)}(s)\right)_\beta \left(\tilde{i}_n^{(\ell)}(s)\right)_\beta$$
$$\left(\tilde{Z}_{n,m}^{(\ell,avg)}(s)\right) = \sum_{\beta=1}^{N}\tilde{\gamma}_\beta^{(\ell)}(s)\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta \left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta , \quad \left(\tilde{Y}_{n,m}^{(\ell,avg)}(s)\right) = \sum_{\beta=1}^{N}\tilde{\gamma}_\beta^{(\ell)}(s)\left(\tilde{i}_n^{(\ell)}(s)\right)_\beta \left(\tilde{i}_n^{(\ell)}(s)\right)_\beta$$

Introducing

$$\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \left(1_{n,m}\right) & \left(1_{n,m}\right) \\ \left(1_{n,m}\right) & -\left(1_{n,m}\right) \end{pmatrix} = \frac{1}{\sqrt{2}}\left(1_{n,m}\right) \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$= \left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)^{-1} \tag{3.8}$$
$$\det\left(\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)\right) = (-1)^N$$
$$tr\left(\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)\right) = 0$$

we have the similarity transformation

$$\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)^{-1}$$
$$= -\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) \otimes \left[\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right]$$
$$= -\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) \end{pmatrix} \tag{3.9}$$
$$= \left[-\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)\right] \oplus \left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)$$

This is precisely the form of the propagation supermatrix for separation of the right-going (+z) waves (upper left block) from the left-going (-z) waves (lower right block) [1].

It is now rather straightforward to diagonalize this last form using the eigenvectors of $(\tilde{\gamma}_{n,m}^{(\ell)}(s))$ [5 (Appendix A)] as

$$\left[-\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \odot \left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta \otimes q\left(a_v\right)_q\right]$$
$$= -\tilde{\gamma}_\beta^{(\ell)}(s)\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta \otimes q\begin{pmatrix} a_1 \\ -a_2 \end{pmatrix}_q \tag{3.10}$$
$$q = \pm 1$$
$$\left(a_v\right)_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \left(a_v\right)_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The 2N eigenvalues are just the $-q\tilde{\gamma}_\beta^{(\ell)}(s)$. Similarly we have

$$\left[\left(\tilde{i}_n^{(\ell)}(s)\right)_\beta \otimes q\left(a_\upsilon\right)_q\right] \odot \left[-\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)_\beta \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = -\tilde{\gamma}_\beta^{(\ell)}(s) \otimes q\begin{pmatrix} a_1 \\ -a_2 \end{pmatrix}_q \tag{3.11}$$

So we can write

$$-\left(\tilde{\gamma}_{n,m}^{(\ell)}\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sum_{\beta=1}^{N} \sum_{q=\pm 1} -\tilde{\gamma}_\beta^{(\ell)}(s)\, q\left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta \otimes \left(a_\upsilon\right)_q\right]\left[\left(\tilde{i}_n^{(\ell)}(s)\right)_\beta \otimes \left(a_\upsilon\right)_q\right]$$

$$\left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta \otimes \left(a_\upsilon\right)_q\right]\left[\left(\tilde{i}_n^{(\ell)}(s)\right) \otimes \left(a_\upsilon\right)_q\right] \quad \text{(dyadic product)}$$
$$= \left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta \left(\tilde{i}_n^{(\ell)}(s)\right)_\beta\right] \otimes \left[\left(a_\upsilon\right)_q \left(a_\upsilon\right)_q\right]$$

$$\sum_{q=\pm 1} \left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta \otimes \left(a_\upsilon\right)_q\right]\left[\left(\tilde{i}_n^{(\ell)}(s)\right)_\beta \otimes \left(a_\upsilon\right)_q\right]$$

$$= \left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s) \right)_\beta \left(\tilde{i}_n^{(\ell)}(s) \right)_\beta \right] \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s) \right)_{\beta_1} \otimes (a_\upsilon)_{q_1} \right] \odot \left[\left(\tilde{i}_n^{(\ell)}(s) \right)_{\beta_2} \otimes (a_\upsilon)_{q_2} \right] = 1_{\beta_1,\beta_2} 1_{q_1 q_2} \tag{3.12}$$

(biorthonormal)

In turn we have from (3.9)

$$\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s) \right)_{\upsilon,\upsilon'} \right) = -\left(\tilde{\gamma}_{n,m}^{(\ell)}(s) \right) \otimes \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \left(\left(Q_{n,m} \right)_{\upsilon,\upsilon'} \right)^{-1} \odot \left[-\left(\tilde{\gamma}_{n,m}^{(\ell)}(s) \right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \odot \left(\left(Q_{n,m} \right)_{\upsilon,\upsilon'} \right)^{-1}$$

$$= \sum_{\beta=1}^{N} \sum_{q=\pm 1} -\tilde{\gamma}_\beta^{(\ell)}(s)_q \left[\tilde{\mathrm{v}}_n^{(\ell)}(s) \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot (a_\upsilon)_q \right] \left[\left(\tilde{i}_n^{(\ell)}(s) \right)_\beta \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot (a_\upsilon)_q \right]$$

$$= \sum_{\beta=1}^{N} \sum_{q=\pm 1} -\tilde{\gamma}_\beta^{(\ell)}(s)_q \left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s) \right)_\beta \otimes (b_\upsilon)_q \right] \left[\left(\tilde{i}_n^{(\ell)}(s) \right)_\beta \otimes (b_\upsilon)_q \right] \tag{3.13}$$

$$(b_\upsilon)_q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot (a_\upsilon)_q$$

$$(b_\upsilon)_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (b_\upsilon)_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot (a_\upsilon)$$

$$\left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s) \right)_{\beta_1} \otimes (b_\upsilon)_{q_1} \right] \odot \left[\left(\tilde{i}_n^{(\ell)}(s) \right)_{\beta_2} \otimes (b_\upsilon)_{q_2} \right] = 1_{\beta_1,\beta_2} 1_{q_1,q_2}$$

(biorthonormal)

This exhibits the eigenvalues and eigensupervectors.

The above results can be applied to the product integral as well. As a special case of the similarity rule we have various forms as

$$\left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z, z_\ell; s) \right)_{\upsilon,\upsilon'} \right)$$

$$= \left(\left(Q_{n,m} \right)_{\upsilon,\upsilon'} \right)^{-1} \odot \left[\prod_{z_\ell}^{z} e^{-\left((Q_{n,m})_{\upsilon,\upsilon'} \right) \odot \left[\left(\tilde{\gamma}_{n,m}^{(0)}(s) \right) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \odot \left((Q_{n,m})_{\upsilon,\upsilon'} \right)^{-1} dz'} \right] \odot \left(\left(Q_{n,m} \right)_{\upsilon,\upsilon'} \right)^{-1}$$

$$= \left(\left(Q_{n,m} \right)_{\upsilon,\upsilon'} \right)^{-1} \odot \left[\prod_{z_\ell}^{z} e^{\left[\left[\left(-\tilde{\gamma}_{n,m}^{(\ell)}(s) \right) \right] \oplus \left(\tilde{\gamma}_{n,m}^{(\ell)}(s) \right) \right] dz'} \right] \odot \left(\left(Q_{n,m} \right)_{\upsilon,\upsilon'} \right)^{-1}$$

$$=\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)\odot\left[\left[\prod_{z_\ell}^{z} e^{-\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)dz'}\right]\oplus\left[\prod_{z_\ell}^{z} e^{\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right)dz'}\right]\right]\odot\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)^{-1}$$

$$=\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)\odot\left[e^{-\left(\gamma_{n,m}^{(\ell)}(s)\right)[z-z_\ell]dz'}\oplus e^{\left(\gamma_{n,m}^{(\ell)}(s)\right)[z-z_\ell]}\right]\odot\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)^{-1} \tag{3.14}$$

$$=\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)\odot e^{-\left(\gamma_{n,m}^{(\ell)}(s)\right)\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix}[z-z_\ell]}\odot\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)$$

$$=\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)\odot\left[\sum_{\beta=1}^{N}\sum_{q=\pm} e^{-\gamma_{n,m}^{(\ell)}(s)q[z-z_\ell]}\left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta\otimes\left(a_\upsilon\right)_q\right]\left[\left(\tilde{i}_n^{(\ell)}(s)\right)_\beta\otimes\left(a_\upsilon\right)_q\right]\right]\odot\left(\left(Q_{n,m}\right)_{\upsilon,\upsilon'}\right)^{-1}$$

This has the eigenvectors as in (3.12). Carrying out the $\odot$ multiplication gives

$$\left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) =\sum_{\beta=1}^{N}\sum_{q=\pm1} e^{-\tilde{\gamma}_\beta^{(\ell)}(s)q[z-z_\ell]}\left[\left(\tilde{\mathrm{v}}_n^{(\ell)}(s)\right)_\beta\otimes\left(b_\upsilon\right)_q\right]\left[\left(\tilde{i}_n^{(\ell)}(s)\right)_\beta\otimes\left(b_\upsilon\right)_q\right] \tag{3.15}$$

This has the eigenvectors as in (3.13).

Summarizing this section we have the first product integral completely diagonalized in terms of the N pairs of eigenvectors of an N x N matrix. Extension to those of the 2N x 2N supermatrices is done in closed analytic form.

4. Solution for $\left(\left(\tilde{G}^{(\ell,1)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right)$

Returning to (2.1) and (3.3) we now have

$$\begin{gathered}
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(z,s)\right)_{\upsilon,\upsilon'}\right) = \left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}(z,s)\right)_{\upsilon,\upsilon'}\right) - \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right) \\
= -\begin{pmatrix} \left(0_{n,m}\right) & \left(\Delta\tilde{Z}'^{(\ell)}_{n,m}(z,s)\right)\cdot\left(\Delta\tilde{Y}^{(\ell)}_{n,m}(s)\right) \\ \left(\tilde{Z}^{(\ell)}_{n,m}(s)\right)\cdot\left(\Delta\tilde{Y}'^{(\ell)}_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix} \\
\left(\Delta\tilde{Z}'^{(\ell)}_{n,m}(z,s)\right) \equiv \left(\tilde{Z}'_{n,m}(z,s)\right) - \left(\tilde{Z}'^{(\ell,avg)}_{n,m}(s)\right) \\
\left(\Delta\tilde{Y}'^{(\ell)}_{n,m}(z,s)\right) \equiv \left(\tilde{Y}'_{n,m}(z,s)\right) - \left(\tilde{Y}'^{(\ell,avg)}_{n,m}(s)\right)
\end{gathered} \tag{4.1}$$

For later use note that the definition of average in (3.2) implies

$$\int_{z_\ell}^{z_{\ell+1}} \left(\Delta\tilde{Z}'^{(\ell)}_{n,m}(z,s)\right)dz = \left(0_{n,m}\right) \quad , \quad \int_{z_\ell}^{z_{\ell+1}} \left(\Delta\tilde{Y}'^{(\ell)}_{n,m}(z,s)\right)dz = \left(0_{n,m}\right) \tag{4.2}$$

Another form (4.1) takes is

$$\begin{gathered}
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(z,s)\right)_{\upsilon,\upsilon}\right) = -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{\zeta}^{(\ell)}_{n,m}(s)_{scc}\left(\tilde{\gamma}^{(\ell)}_{n,m}(s)\right)\right) \\ \left(\tilde{\gamma}^{(\ell)}_{n,m}(s)\right)\cdot\left(\tilde{\xi}^{(\ell)}_{n,m}(s)\right) & \left(0_{n,m}\right) \end{pmatrix} \\
\left(\tilde{\zeta}^{(\ell)}_{n,m}(z,s)\right) \equiv \left(\Delta\tilde{Z}'^{(\ell)}_{n,m}(z,s)\right)\cdot\left(\tilde{Z}'^{(\ell,avg)}_{n,m}(s)\right)^{-1} \\
\left(\tilde{\xi}^{(\ell)}_{n,m}(z,s)\right) \equiv \left(\tilde{Y}'^{(\ell,avg)}_{n,m}(s)\right)^{-1}\cdot\left(\Delta\tilde{Y}'^{(\ell,avg)}_{n,m}(z,s)\right)
\end{gathered} \tag{4.3}$$

which has the correction matrices normalized by the average matrices, these normalized matrices assumed small (compared to the identity).

Now we can compute

$$
\begin{aligned}
&\left(\left(\tilde{H}_{n,m}^{(\ell)}\left(z,z_{\ell};s\right)\right)_{\upsilon,\upsilon'}\right) \\
&= \left(\left(\tilde{G}_{n,m}^{(\ell,0)}\left(z,z_{\ell};s\right)\right)_{\upsilon,\upsilon'}\right)^{-1} \odot \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}\left(z,s\right)\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{G}^{(\ell,0)}\left(z,z_{\ell};s\right)\right)_{\upsilon,\upsilon'}\right)
\end{aligned}
\tag{4.4}
$$

using the constant (z independent) eigenvectors from Section 3, we still have a variable 2N x 2N matrix in (4.4) with the various elements having in general different variation with z.

In the spirit of [6], one approach is to approximate the correction matrices as constant matrices times scalar functions of z. So we approximate as in [6 (Section 3)]

$$
\begin{aligned}
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}\left(z,s\right)\right)_{\upsilon,\upsilon'}\right) &\simeq f^{(\ell)}(z)\left(\left(\tilde{C}_{n,m}^{(\ell)}\left(s\right)\right)_{\upsilon,\upsilon'}\right) \\
f^{(\ell)}\left(z_{\ell}\right)\left(\left(\tilde{C}_{n,m}^{(\ell)}\left(s\right)\right)_{\upsilon,\upsilon'}\right) &= \left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}\left(z_{\ell},s\right)\right)_{\upsilon,\upsilon'}\right) - \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}\left(s\right)\right)_{\upsilon,\upsilon'}\right) \\
f^{(\ell)}\left(z_{\ell+1}\right)\left(\left(\tilde{C}_{n,m}^{(\ell)}\left(s\right)\right)_{\upsilon,\upsilon'}\right) &= \left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}\left(z_{\ell+1},s\right)\right)_{\upsilon,\upsilon'}\right) - \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}\left(s\right)\right)_{\upsilon,\upsilon'}\right)
\end{aligned}
\tag{4.5}
$$

$$
\left(\left(\tilde{C}_{n,m}^{(\ell)}\left(s\right)\right)_{\upsilon,\upsilon'}\right) = \left[f^{(\ell)}\left(z_{\ell+1}\right) - f^{(\ell)}\left(z_{\ell}\right)\right]^{-1}\left[\left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}\left(z_{\ell+1},s\right)\right)_{\upsilon,\upsilon'}\right) - \left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}\left(z_{\ell},s\right)\right)_{\upsilon,\upsilon'}\right)\right]
$$

$$
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}\left(s\right)\right)_{\upsilon,\upsilon'}\right) = \left[f^{(\ell)}\left(z_{\ell+1}\right) - f^{(\ell)}\left(z_{\ell}\right)\right]^{-1}\left[f^{(\ell)}\left(z_{\ell+1}\right)\left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}\left(z_{\ell},s\right)\right)_{\upsilon,\upsilon'}\right) - f^{(\ell)}\left(z_{\ell}\right)\left(\left(\tilde{\Gamma}_{n,m}^{(\ell)}\left(z_{\ell+1},s\right)\right)_{\upsilon,\upsilon'}\right)\right]
$$

The choice of

$$
f^{(\ell)}\left(z_{\ell+1}\right) \equiv \frac{1}{2} \quad , \quad f^{(\ell)}\left(z_{\ell}\right) \equiv -\frac{1}{2}
\tag{4.6}
$$

is a special case corresponding to the definition of average in (3.1). This still leaves the choice of $f^{(\ell)}(z)$ subject to the above boundary conditions. In [6] this is

$$
f^{(\ell)}(z) = \frac{2z - z_{\ell+1} - z_{\ell}}{z_{\ell+1} - z_{\ell}}
\tag{4.7}
$$

However, various other choices are possible, including the more accurate approximation discussed in [7]. Note that with smooth $f^{(\ell)}(z)$ there are no step discountinuities at the ends of the interval where the approximation of the propagation supermatrix becomes exact.

At this point we can evaluate $\left(\left(\tilde{H}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right)$ in (4.4) using (3.13) (with the constant eigenvectors) and the approximation in (4.7). Integrating over $f^{(\ell)}(z)$ and the exponential eigenvalues (as in [6 (Section 4)] gives a perturbation result for $\left(\left(\tilde{G}_{n,m}^{(\ell,1)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right)$, valid for small $\left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\upsilon,\upsilon'}\right)$, using the first two terms in the matrizant series [4] as

$$\begin{aligned}
&\left(\left(\tilde{G}_{n,m}^{(\ell,1)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) \\
&\simeq \left(\left(1_{n,m}\right)_{\upsilon,\upsilon'}\right) + \int_{z_\ell}^{z} f^{(\ell)}(z')\left(\left(\tilde{G}_{n,m}^{(\ell)}(z',z_\ell;s)\right)_{\upsilon,\upsilon'}\right)^{-1} \odot \left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{G}_{n,m}^{(\ell)}(z',z_\ell;s)\right)_{\upsilon,\upsilon'}\right) dz' \qquad (4.8) \\
&\quad + O\left(\text{largest eigenvalue of } \left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\upsilon,\upsilon'}\right)^{\dagger} \odot \left(\left(\tilde{C}_{n,m}^{(\ell)}(s)\right)_{\upsilon,\upsilon'}\right)\right)
\end{aligned}$$

Note now that

$$\begin{aligned}
&\left(\left(\tilde{G}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right)^{-1} \odot \left(\left(\tilde{C}_{n,m}(s)\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{G}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) \\
&= \sum_{\beta=1}^{N} \sum_{q=\pm1} \sum_{\beta'=1}^{N} \sum_{q'=\pm1} e^{\left[\tilde{\gamma}_{\beta}^{(\ell)}(s)-\tilde{\gamma}_{\beta'}^{(\ell)}(s)\right][z-z_\ell]} \tilde{D}_{\beta,q;\beta',q'}(s)\left[\left(\tilde{v}_{n}^{(\ell)}(s)\right)_{\beta} \otimes (b_\upsilon)_q\right]\left[\left(\tilde{i}_{n}^{(\ell)}(s)\right)_{\beta'} \otimes (b_\upsilon)_{q'}\right] \qquad (4.9) \\
&\tilde{D}_{\beta,q';\beta',q'}(s) = \left[\left(\tilde{i}_{n}^{(\ell)}(s)\right)_{\beta} \otimes (b_\upsilon)_q\right] \odot \left(\left(\tilde{C}_{n,m}(s)\right)_{\upsilon,\upsilon'}\right) \odot \left[\left(\tilde{v}_{n}^{(\ell)}(s)\right)_{\beta'} \otimes (b_\upsilon)_q\right]
\end{aligned}$$

(4-index scalars)

In turn this gives

$$\begin{aligned}
&\left(\left(\tilde{G}_{n,m}^{(\ell,1)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) \\
&\approx \left(\left(1_{n,m}\right)_{\upsilon,\upsilon'}\right) + \sum_{\beta'=1}^{N}\sum_{q=\pm1}\sum_{\beta'=1}^{N}\sum_{q'=\pm1}\left[\int_{z_\ell}^{z} f(z') e^{\left[\tilde{\gamma}_{\beta}^{(\ell)}(s)-\tilde{\gamma}_{\beta'}^{(\ell)}(s)\right][z'-z_\ell]} dz'\right] \qquad (4.10) \\
&\quad \tilde{D}_{\beta,q;\beta',q'}(s)\left[\left(\tilde{v}_{n}^{(\ell)}(s)\right)_{\beta} \otimes (b_\upsilon)_q\right]\left[\left(\tilde{i}_{n}^{(\ell)}(s)\right)_{\beta'} \otimes (b_\upsilon)_{q'}\right]
\end{aligned}$$

Depending on one's choice for $f(z')$, the integration over z' is readily performed. The choice in (4.7) is considered in [6].

Looking at the combination $\tilde{\gamma}_{\beta}^{(\ell)}(s)-\tilde{\gamma}_{\beta'}^{(\ell)}(s)$ we can see by expanding the exponential a first order correction for low frequencies.

5. Case of Lossless NMTLs

Let the per-unit-length matrices have the typical inductive and capacitive character of lossless NMTLs as

$$\begin{aligned}\left(\tilde{Z}'_{n,m}(z,s)\right) &\equiv s\left(L'_{n,m}(z)\right) \quad , \quad \left(\zeta_{n,m}(z)\right) = \left(\Delta L'^{(\ell)}_{n,m}(z)\right)\cdot\left(L'^{(\ell,avg)}_{n,m}\right)^{-1} \\ \left(\tilde{Y}'_{n,m}(z,s)\right) &\equiv s\left(C'_{n,m}(z)\right) \quad , \quad \left(\xi_{n,m}(z)\right) = \left(C'^{(\ell)}_{n,m}(z)\right)^{-1}\cdot\left(C'^{(\ell,avg)}_{n,m}(z)\right)\end{aligned} \tag{5.1}$$

Then we also have

$$\begin{aligned}\left(\tilde{\gamma}^{(\ell)}_{n,m}(s)\right) &= s\left(g^{(\ell)}_{n,m}\right) \\ \left(g^{(\ell)}_{n,m}\right) &= \left[\left(L'^{(\ell,avg)}_{n,m}\right)\cdot\left(C'^{(\ell,avg)}_{n,m}\right)\right]^{1/2} \text{ (all real nonnegative eigenvalues)} \\ \left(Z^{(\ell)}_{n,m}\right) &= \left(g^{(\ell)}_{n,m}\right)\cdot\left(C'^{(\ell,avg)}_{n,m}\right)^{-1} \\ &= \left(g^{(\ell)}_{n,m}\right)^{-1}\cdot\left(L'^{(\ell,avg)}_{n,m}\right) \\ \left(Y^{(\ell)}_{n,m}\right) &= \left(C'^{(\ell,avg)}_{n,m}\right)\cdot\left(g^{(\ell)}_{n,m}\right)^{-1} \\ &= \left(L'^{(\ell,avg)}_{n,m}\right)^{-1}\cdot\left(g^{(\ell)}_{n,m}\right)\end{aligned} \tag{5.2}$$

This removes the frequency dependence of the eigenvectors as

$$\begin{aligned}\left(g^{(\ell)}_{n,m}\right)^2\cdot\left(v^{(\ell)}_n\right)_\beta &= g^{(\ell)^2}_\beta\left(v^{(\ell)}_n\right)_\beta \\ \left(i_n\right)_\beta\cdot\left(g^{(\ell)}_{n,m}\right)^2 &= g^{(\ell)^2}_\beta\left(v^{(\ell)}_n\right)_\beta\end{aligned} \tag{5.3}$$

The $g^{(\ell)^2}_\beta$ are all real, nonnegative allowing the $g^{(\ell)}_\beta$ to be defined by the positive square root. Then we have

$$\tilde{\gamma}^{(\ell)}_\beta(s) = s\,g^{(\ell)}_\beta \tag{5.4}$$

and all other parameters in Sections 3 and 4 similarly simplify.

In particular we have

$$
\begin{aligned}
\left(\zeta_{n,m}(z)\right) &= \left(\Delta L'^{(\ell)}_{n,m}(z)\right) \bullet \left(L'^{(\ell,avg)}_{n,m}\right)^{-1} \\
\left(\xi_{n,m}(z)\right) &= \left(C'^{(\ell,avg)}_{n,m}(z)\right)^{-1} \bullet \left(\Delta C'^{(\ell)}_{n,m}(z)\right)
\end{aligned}
\tag{5.5}
$$

Furthermore the two definitions of average in (3.1) and (3.2) also carry over directly to the frequency-independent forms.

Applying the results of Sections 3 and 4 to the lossless case we have

$$
\begin{aligned}
&\left(\left(\tilde{G}^{(\ell,0)}_{n,m}(z, z_\ell; s)\right)_{\upsilon,\upsilon'}\right) \\
&= \sum_{\beta=1}^{N} \sum_{q=\pm 1} e^{-sg_\beta^{(\ell)} q[z-z_\ell]} \left[\left(\mathrm{v}_n^{(\ell)}\right)_\beta \otimes (b_\upsilon)_q\right]\left[\left(i_n^{(\ell)}\right)_\beta \otimes (b_\upsilon)_q\right]
\end{aligned}
\tag{5.6}
$$

Note that most of the terms are frequency independent allowing easy conversion to time domain as simple delays. Similarly (4.9) and (4.10) simplify showing the first-order correction proportional to s.

6. Case of Equal Modal Speeds

For further insight into the solution technique consider the case of equal modal speeds which occurs when $\left(\tilde{Y}'_{n,m}(z,s)\right)^{-1}$ is proportional to $\left(\tilde{Z}'_{n,m}(z,s)\right)$. One physical condition which gives this is that of perfectly conducting wires (varying in cross-section and spacing) in a uniform dielectric medium. In this case we have [2, 3] for the ℓ th section as in (3.4)

$$
\begin{aligned}
\left(\tilde{\gamma}_{n,m}^{(\ell)}(s)\right) &= \tilde{\gamma}^{(\ell)}(s)\left(1_{n,m}\right) = s\left(g_{n,m}^{(\ell)}\right) \\
\tilde{\gamma}^{(\ell)}(s) &= \frac{s}{\mathrm{v}^{(\ell)}} \\
\left(g_{n,m}^{(\ell)}\right) &= \frac{1}{\mathrm{v}^{(\ell)}}\left(1_{n,m}\right)
\end{aligned}
\tag{6.1}
$$

This simplifies things considerably since we do not need to be concerned with diagonalizing the identity.

Observe now that

$$
\begin{aligned}
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right) &= -\tilde{\gamma}^{(\ell)}(s)\left(1_{n,m}\right) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\left(\left(\tilde{G}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right) &= e^{-\tilde{\gamma}^{(\ell)}(s)\left(1_{n,m}\right) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}[z-z_\ell]} \\
&= \cosh\left(\tilde{\gamma}^{(\ell)}(s)\left[z-z_\ell\right]\right)\left(1_{n,m}\right)\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-\sinh\left(\tilde{\gamma}^{(\ell)}(s)\left[z-z_\ell\right]\right)\left(1_{n,m}\right)\otimes\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\left(\left(\tilde{G}_{n,m}^{(\ell,0)}(s)\right)_{\upsilon,\upsilon'}\right)^{-1} &= e^{-\tilde{\gamma}^{(\ell)}(s)\left(1_{n,m}\right) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}[z-z_\ell]} \\
&= \cosh\left(\tilde{\gamma}^{(\ell)}(s)\left[z-z_\ell\right]\right)\left(1_{n,m}\right)\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\sinh\left(\tilde{\gamma}^{(\ell)}(s)\left[z-z_\ell\right]\right)\left(1_{n,m}\right)\otimes\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(z,s)\right)_{\upsilon,\upsilon'}\right) &= -\tilde{\gamma}^{(\ell)}(s)\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{\zeta}_{n,m}^{(\ell)}(z,s)\right) \\ \left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix} \\
\left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(z,s)\right)_{\upsilon,\upsilon'}\right) &= -\tilde{\gamma}^{(\ell)}(s)\left[\left(\tilde{\zeta}_{n,m}^{(\ell)}(z,s)\right)\otimes\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}+\left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right)\otimes\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]
\end{aligned}
\tag{6.2}
$$

This, in turn, gives

$$
\begin{aligned}
&\left(\left(\tilde{H}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) \equiv \left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right)^{-1} \odot \left(\left(\tilde{\Gamma}_{n,m}^{(\ell,1)}(z,s)\right)_{\upsilon,\upsilon'}\right) \odot \left(\left(\tilde{G}_{n,m}^{(\ell,0)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right) \\
&= -\tilde{\gamma}^{(\ell)}(s)\left(\tilde{\zeta}_{n,m}^{(\ell)}(z,s)\right) \otimes \left[\left[\cosh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \sinh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\right. \\
&\qquad\qquad \left.\odot \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \odot \left[\cosh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sinh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\right] \\
&\quad -\tilde{\gamma}^{(\ell)}(s)\left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right) \otimes \left[\left[\cosh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \sinh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\right. \\
&\qquad\qquad \left.\odot \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \odot \left[\cosh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sinh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\right] \\
&= -\tilde{\gamma}^{(\ell)}(s)\left(\tilde{\zeta}_{n,m}^{(\ell)}(z,s)\right) \\
&\otimes \begin{pmatrix} \cosh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\sinh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right) & \cosh^2\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right) \\ -\sinh^2\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right) & -\cosh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\sinh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right) \end{pmatrix} \\
&\quad -\tilde{\gamma}^{(\ell)}(s)\left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right) \\
&\otimes \begin{pmatrix} -\cosh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\sinh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right) & -\sinh^2\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right) \\ \cosh^2\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right) & \cosh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\sinh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right) \end{pmatrix} \\
&= -\tilde{\gamma}^{(\ell)}(s) \begin{pmatrix} \cosh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\sinh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\left[\left(\tilde{\zeta}_{n,m}^{(\ell)}(z,s)\right)-\left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right)\right] & \cosh^2\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\left(\tilde{\zeta}_{n,m}^{(\ell)}(z,s)\right)\right)-\sinh^2\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right)\right) \\ -\sinh^2\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\left(\tilde{\zeta}_{n,m}^{(\ell)}(z,s)\right)\right)+\cosh^2\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right)\right) & -\cosh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\sinh\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\left[\left(\tilde{\zeta}_{n,m}^{(\ell)}(z,s)\right)-\left(\tilde{\xi}_{n,m}^{(\ell)}(z,s)\right)\right] \end{pmatrix}
\end{aligned}
\tag{6.3}
$$

Furthermore, let us consider the lossless assumption as in Section 5. Then (6.3) can be expanded for low frequencies as

$$\left(\left(\tilde{H}_{n,m}^{(\ell)}(z,z_\ell;s)\right)_{\upsilon,\upsilon'}\right)$$

$$= -\tilde{\gamma}^{(\ell)}(s)\begin{pmatrix} \begin{array}{l}\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\left[1+\mathrm{O}\left(\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)\right)\right]\left[\left(\zeta_{n,m}^{(\ell)}(z)\right)-\left(\xi_{n,m}^{(\ell)}(z)\right)\right] \\ \qquad \left(\zeta_{n,m}^{(\ell)}(z)\right) + \mathrm{O}\left(\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)^2\right)\left[\left(\zeta_{n,m}^{(\ell)}(z)\right)-\left(\xi_{n,m}^{(\ell)}(z)\right)\right] \\ \left(\xi_{n,m}^{(\ell)}(z)\right) + \mathrm{O}\left(\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)^2\right)\left[\left(\zeta_{n,m}^{(\ell)}(z)\right)-\left(\xi_{n,m}^{(\ell)}(z)\right)\right] \\ \qquad -\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\left[1+\mathrm{O}\left(\left(\tilde{\gamma}^{(\ell)}(s)[z-z_\ell]\right)^2\right)\right]\left[\left(\zeta_{n,m}^{(\ell)}(z)\right)-\left(\xi_{n,m}^{(\ell)}(z)\right)\right]\end{array}\end{pmatrix} \tag{6.4}$$

Using the perturbation approach and extending the integral over the entire ℓ th section we have

$$\begin{aligned}\left(\left(\tilde{G}_{n,m}^{(\ell,1)}(z_{\ell+1},z_\ell;s)\right)_{\upsilon,\upsilon'}\right) &= \left(\left(1_{n,m}\right)_{\upsilon,\upsilon'}\right) \\ &-\tilde{\gamma}^{(\ell)^2}(s)\begin{pmatrix} \int_{z_\ell}^{z_{\ell+1}}[z'-z_\ell]\left[\left(\zeta_{n,m}^{(\ell)}(z')\right)-\left(\xi_{n,m}^{(\ell)}(z')\right)\right]dz' & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & -\int_{z_\ell}^{z_{\ell+1}}[z'-z_\ell]\left[\left(\zeta_{n,m}^{(\ell)}(z')\right)-\left(\xi_{n,m}^{(\ell)}(z')\right)\right]dz'\end{pmatrix} \\ &+\begin{pmatrix}\mathrm{O}\left(\left(\tilde{\gamma}^{(\ell)}(s)[z_{\ell+1}-z_\ell]\right)^4\right) & \mathrm{O}\left(\left(\tilde{\gamma}^{(\ell)}(s)[z_{\ell+1}-z_\ell]\right)^3\right) \\ \mathrm{O}\left(\left(\tilde{\gamma}^{(\ell)}(s)[z_{\ell+1}-z_\ell]\right)^3\right) & \mathrm{O}\left(\left(\tilde{\gamma}^{(\ell)}(s)[z_{\ell+1}-z_\ell]\right)^4\right)\end{pmatrix} \\ &\text{as } \tilde{\gamma}^{(\ell)}(s)[z_{\ell+1}-z_\ell]\to 0\end{aligned} \tag{6.5}$$

Note that the leading off-diagonal blocks have integrated to zero provided we have used the definition of average as in (3.2) giving (4.2) applied to (4.3). This result gives some justification for use of this definition of average.

The dominant remaining term is

$$\begin{aligned}\left(\tilde{w}_{n,m}^{(\ell)}(s)\right) &= -\tilde{\gamma}^{(\ell)^2}(s)\int_{z_\ell}^{z_{\ell+1}}[z'-z_\ell]\left[\left(\zeta_{n,m}^{(\ell)}(z')\right) - \left(\xi_{n,m}^{(\ell)}(z')\right)\right]dz' \\ &= \mathrm{O}\left(\left(\tilde{\gamma}^{(\ell)}(s)[z_{\ell+1}-z_\ell]\right)^2\right) \text{ as } \tilde{\gamma}^{(\ell)}(s)[z_{\ell+1}-z_\ell]\to 0\end{aligned} \tag{6.6}$$

Integrating by parts we have an alternate expression as

$$\left(\tilde{w}_{n,m}^{(\ell)}(s)\right) = \tilde{\gamma}^{(\ell)^2}(s) \int_{z_\ell}^{z_{\ell+1}} \int_{z_\ell}^{z} \left[\left(\zeta_{n,m}^{(\ell)}(z')\right) - \left(\xi_{n,m}^{(\ell)}(z')\right)\right] dz' \tag{6.7}$$

again using the fact that the average makes the first integral zero. One can continue the matrizant series to higher order perturbations, but as long as the first order term is small the successive terms are progressively smaller.

Our correction term is now second order in frequency and block diagonal, so we can write

$$\begin{aligned} \left(\left(\tilde{G}_{n,m}^{(\ell,1)}(z_{\ell+1}, z_\ell; s)\right)_{\upsilon,\upsilon'}\right) &= \left[\left(1_{n,m}\right) + \left(\tilde{w}_{n,m}^{(\ell)}(s)\right)\right] \oplus \left[\left(1_{n,m}\right) - \left(\tilde{w}_{n,m}^{(\ell)}(s)\right)\right] \\ &= O\left(\left(\tilde{\gamma}^{(\ell)}(s)[z_{\ell+1} - z_\ell]\right)^3\right) \text{ as } \tilde{\gamma}^{(\ell)}(s)[z_{\ell+1} - z_\ell] \to 0 \end{aligned} \tag{6.8}$$

This also reduces the problem to the evaluation of a single N x N matrix.

Further insight can be gained for the lossless case by choosing frequencies on the $j\omega$ axis, for which

$$\begin{aligned} s &= j\omega \quad , \quad \tilde{\gamma}^{(\ell)}(s) = j\tilde{k}^{(\ell)}(j\omega)\left(\tilde{k} \text{ real}\right) \\ \cosh\left(\tilde{\gamma}^{(\ell)}(s)[z - z_\ell]\right) &= \cos\left(\tilde{k}(j\omega)[z - z_\ell]\right) \\ \sinh\left(\tilde{\gamma}^{(\ell)}(s)[z - z_\ell]\right) &= j\sin\left(\tilde{k}(j\omega)[z - z_\ell]\right) \end{aligned} \tag{6.9}$$

These sine and cosine terms are bounded in magnitude by 1.0. As ω increases, ω times these terms, when integrated over z, remain of this order. So the matrizant series is well approximated by the zeroth- and first-order terms as long as $\left(\zeta_{n,m}^{(\ell)}(z)\right)$ and $\left(\xi_{n,m}^{(\ell)}(z)\right)$ are small. In this case one needs to integrate the terms in (6.3) over z for each ω to obtain the first-order correction.

7. Case of N = 1

The case of a single conductor plus reference is a special case of equal modal speeds, since there is only one such speed. The problem involves 2 x 2 matrices, but the eigenvectors are only the $(a_\upsilon)_q$ or $(b_\upsilon)_q$ discussed in Section 4. The N x N matrices in Section 6 all reduce to scalars.

8. Concluding Remarks

We now have simplified the matrix diagonalization problem from a 2N x 2N problem to an N x N problem such as used previously for uniform multiconductor transmission lines. The perturbation analysis for the line nonuniformity is also improved by the judicious averaging of the per-unit-length impedance and admittance matrices.

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