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Source Dependent Transmission Line Parameters – Plane Wave vs TEM Excitation

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Abstract

Maxwell's equations for a thin, infinite and lossless wire above perfectly conducting ground are transformed into the form of the Telegrapher equations. The occurring line parameters are complex-valued functions and gauge-and source-dependent. Choosing plane wave excitation and grazing incidence, it is shown that the corresponding electromagnetic fields, Poynting vectors, currents and potentials assume TEM structure. This result is confirmed by a direct TEM solution of the Telegrapher equations.

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I. Introduction

The treatment of transmission lines at very high frequencies with the inclusion of radiation becomes an increasing topic in dealing with EMC effects. Since the classical transmission line equations (classical Telegrapher equations) do not include radiation, it is necessary to generalize these equations for higher frequencies and arbitrary modes. This is the purpose of the present paper where we demonstrate the single steps with the aid of a simple example: an infinite, lossless line above perfectly conducting ground excited by a plane wave. Firstly Maxwell's equations are transformed into the form of the Telegrapher equations with new, generalized complex-valued, source-and gauge-dependent line parameters. Then, in Section II we derive the general solution, using the theory of matrizants and product integrals [1]. On the basis of this solution we study, in Section III, a special case of excitation: grazing incidence of the plane wave. It is shown that this excitation leads to a pure TEM structure for the electromagnetic fields and the Poynting vector. Also the line parameters assume static values. Section IV compares the result of the previous section with a direct TEM ansatz to solve the classical transmission line equations. Finally, we finish this paper with a brief conclusion (Section V).

II. Maxwell -Telegrapher Equations for an Infinite Line

In two previous papers [2, 3] we have shown that Maxwell's equations for an infinite, lossless wire above perfectly conducting ground which is excited by an incident plane wave can be transformed into Telegrapher equations with generalized, frequency- and source-dependent line parameters. These equations read for the scalar potential $\varphi(z)$ and the current $I(z)$:

$$\begin{pmatrix} d\varphi(z)/dz \\ dI(z)/dz \end{pmatrix} = -j\omega \begin{pmatrix} 0 & L' \\ C' & 0 \end{pmatrix} \begin{pmatrix} \varphi(z) \\ I(z) \end{pmatrix} + \begin{pmatrix} E_z^{inc}(z) \\ 0 \end{pmatrix} \tag{1}$$

with the gauge-dependent line parameters, the inductance per-unit-length L' and the capacitance per-unit-length C'

$$L'(j\omega) = \frac{\mu_0}{4\pi} \begin{cases} G_k\left(\widetilde{k}_{inc}\right) & , \quad \text{Lorenz gauge} \\ \dfrac{k^2 - \widetilde{k}_{inc}^2}{k^2} G_k\left(\widetilde{k}_{inc}\right) + \dfrac{\widetilde{k}_{inc}^2}{k^2} G_0\left(\widetilde{k}_{inc}\right) & , \quad \text{Coulomb gauge} \end{cases} \tag{2 a,b}$$

and

$$C'(j\omega) = 4\pi\varepsilon_0 \begin{cases} G_k^{-1}\left(\widetilde{k}_{inc}\right) & , \quad \text{Lorenz gauge} \\ G_0^{-1}\left(\widetilde{k}_{inc}\right) & , \quad \text{Coulomb gauge} \end{cases} \tag{3 a,b}$$

Here the capital-letter G-functions are defined as

$$G_k\left(\widetilde{k}_{inc}\right) := -j\pi\left\{H_0^{(2)}(ka\sin\theta) - H_0^{(2)}(2kh\sin\theta)\right\} \tag{4a,b}$$

and

$$G_0\left(\widetilde{k}_{inc}\right) := 2\left\{K_0\left(ka\left|\cos\theta\right|\right) - K_0\left(2kh\left|\cos\theta\right|\right)\right\}$$

The function $H_0^{(2)}$ denotes a special Hankel function, and K_0 is a modified Bessel function [4]. The height of the wire above ground is denoted by h and its radius by a. The angle θ is the incident angle of the plane wave (see Fig. 1):

$$E_z^{inc}(z) = E_z^0(\theta, a, h, j\omega)e^{-jkz\cos\theta}, \qquad (\tilde{k}_{inc} := k\cos\theta) \tag{5}$$

The reflection of the plane wave at the ground plane is taken into account in the factor E_z^0.

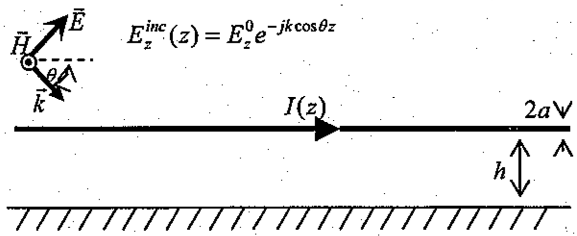

Fig. 1: Geometry of the problem.

The solution of equation (1) is easily expressed with the aid of the matrizant [1] $\mathbf{M}_{z_0}^z$

$$\begin{pmatrix}\varphi(z)\\ I(z)\end{pmatrix} = \mathbf{M}_{z_0}^z \begin{pmatrix}\varphi(z_0)\\ I(z_0)\end{pmatrix} + \int_{z_0}^{z} \mathbf{M}_{z_0}^z \begin{pmatrix}E_z^{inc}(z')\\ 0\end{pmatrix} dz' \tag{6}$$

where the matrizant can be given in closed form

$$\mathbf{M}_{z_0}^z = \begin{pmatrix}\cosh(\gamma(z-z_0)) & -Z_C\sinh(\gamma(z-z_0))\\ -Z_C^{-1}\sinh(\gamma(z-z_0)) & \cosh(\gamma(z-z_0))\end{pmatrix} \tag{7}$$

As usual in transmission line theory, we have defined the so-called propagation constant γ

$$\gamma := j\omega\sqrt{L'C'} \tag{8}$$

and the characteristic impedance Z_C

$$Z_C := \sqrt{\frac{L'}{C'}} \tag{9}$$

using the expressions for the line parameters.
The integral in eq.(6) yields the following contribution

$$\int_{z_0}^{z} \mathbf{M}_{z'}^z \begin{pmatrix}E_z^{inc}(z')\\ 0\end{pmatrix} dz' =$$

$$= \frac{E_z^0}{\left(\gamma^2 + \tilde{k}_{inc}^2\right)} \begin{pmatrix} j\tilde{k}_{inc} e^{-j\tilde{k}_{inc} z} + e^{-j\tilde{k}_{inc} z}\left(\gamma \sinh(\gamma(z-z_0)) - j\tilde{k}_{inc} \cosh((\gamma(z-z_0))\right) \\ Z_C^{-1}\left\{\gamma e^{-j\tilde{k}_{inc} z} - e^{-j\tilde{k}_{inc} z}\left(\gamma \cosh(\gamma(z-z_0)) - j\tilde{k}_{inc} \sinh(\gamma(z-z_0))\right)\right\} \end{pmatrix} \quad (10)$$

Now, taking into consideration the boundary values for the potential and the current

$$\varphi_0 \equiv \varphi(z_0) = \frac{j\tilde{k}_{inc} E_z^0}{\left(\gamma^2 + \tilde{k}_{inc}^2\right)} \quad (11)$$

and

$$I_0 \equiv I(z_0) = \frac{\gamma E_z^0}{Z_C\left(\gamma^2 + \tilde{k}_{inc}^2\right)} \quad (12)$$

which immediately can be obtained from equations (1) regarding the fact that $\varphi(z)$ and $I(z)$ can be written as

$$\varphi(z) = \varphi_0 e^{-j\tilde{k}_{inc} z} \quad \text{and} \quad I(z) = I_0 e^{-j\tilde{k}_{inc} z} \quad (13)$$

we can represent the solution of eq. (6) in a very compact form:

$$\begin{pmatrix} \varphi(z) \\ I(z) \end{pmatrix} = \frac{E_z^0}{\left(\gamma^2 + \tilde{k}_{inc}^2\right)} \begin{pmatrix} j\tilde{k}_{inc} \\ \gamma / Z_C \end{pmatrix} e^{-j\tilde{k}_{inc} z} \quad (14)$$

If we introduce the gauge-independent impedance function per-unit-length $Z'(j\omega)$ [2, 3]

$$Z'(j\omega) := j\omega L' + \frac{\tilde{k}_{inc}^2}{j\omega C'} \quad (16)$$

equation (14) may be rewritten as

$$\begin{pmatrix} \varphi(z) \\ I(z) \end{pmatrix} = \frac{E_z^0}{Z'(j\omega)} \begin{pmatrix} \tilde{k}_{inc} / (\omega C') \\ 1 \end{pmatrix} e^{-j\tilde{k}_{inc} z} \quad (16)$$

exhibiting very clearly the gauge-independency of the current and the gauge-dependency of the potential.

Remember, equation (16) is an exact solution of Maxwell's equations. Therefore radiation and higher modes are inherent in this solution (see also in this context refs. [2, 3]).

In concluding this section we calculate the relation between the potentials in both gauges. From eq. (16) we obtain a z-independent ratio

$$\left(\frac{\varphi_C(z)}{\varphi_L(z)}\right) = \left(\frac{\varphi_{C0}}{\varphi_{L0}}\right) = \frac{G_0(\tilde{k}_{inc})}{G_k(\tilde{k}_{inc})} \quad (17)$$

In particular, the boundary values for the potentials are in general not equal. In the next section we will show an example.

III Grazing Incidence of the Plane Wave

Grazing incidence $(\theta \to 0)$ seems to be a very special excitation of the line. On one hand one may expect that no current will be induced due to the TEM-wave configuration. On the other hand, throughout the following derivation of this limiting incidence case, from our results of the previous section we will show that there is a current flow, despite a TEM structure of the electromagnetic fields.

In a first step we show the TEM character of the EM wave for $\theta \to 0$. Here we rely on the result of ref. [3] where the fields and the corresponding Poynting vectors have been explicitly given. The electric field reads:

$$E_z(\vec{r}) = -\frac{1}{4}\eta_0 I_0 k \sin^2\theta \left[H_0^{(2)}(k\rho_1 \sin\theta) - H_0^{(2)}(k\rho_2 \sin\theta)\right] e^{-j\vec{k}_{inc} z} \tag{18}$$

and

$$\vec{E}_\rho(\vec{r}) = -\frac{1}{4}\eta_0 I_0 k \sin\theta\cos\theta \left[H_0^{(2)}(k\rho_1 \sin\theta)\vec{e}_{\rho_1} - H_0^{(2)}(k\rho_2 \sin\theta)\vec{e}_{\rho_2}\right] e^{-j\vec{k}_{inc} z} \tag{19}$$

with $\vec{e}_\rho := \vec{\rho}/\rho$ unit vector perpendicular to the z-direction,

$$\vec{\rho}_1 := (x-h)\vec{e}_x + y\vec{e}_y, \qquad \text{and} \qquad \vec{\rho}_2 := (x+h)\vec{e}_x + y\vec{e}_y \quad .$$

In the limit $\theta \to 0$ the field components become quasi-static, and the electric field vector is orthogonal to the wire direction.

$$\begin{aligned} E_z^{\theta\to 0} &= 0 \\ \vec{E}_\rho^{\theta\to 0} &= \left(\frac{\eta_0 I_0}{2\pi}\right)\left[\frac{1}{\rho_1}\vec{e}_{\rho_1} - \frac{1}{\rho_2}\vec{e}_{\rho_2}\right] e^{-jkz} \end{aligned} \tag{20}$$

The general expression for the magnetic field is (see ref. [3]):

$$\vec{H}(\vec{r}) = j\frac{I_0 k \sin\theta}{4}\left[H_1^{(2)}(k\rho_1 \sin\theta)\frac{1}{\rho_1}\left[\vec{e}_z, \vec{e}_{\rho_1}\right] - H_1^{(2)}(k\rho_2 \sin\theta)\frac{1}{\rho_2}\left[\vec{e}_z, \vec{e}_{\rho_2}\right]\right] e^{-j\vec{k}_{inc} z} \tag{21}$$

Again, for $\theta \to 0$ we get

$$\vec{H}^{\theta\to 0}(\vec{r}) = \left(\frac{I_0}{2\pi}\right)\left(\frac{1}{\rho_1}\left[\vec{e}_z, \vec{e}_{\rho_1}\right] - \frac{1}{\rho_2}\left[\vec{e}_z, \vec{e}_{\rho_2}\right]\right) e^{-jkz} \tag{22}$$

As expected, we have a field which is orthogonal to $\vec{E}_\rho$ and to the z-direction and also static in planes perpendicular to the transmission line. Thus, we have obtained the TEM-character of the wave for grazing incidence ($\vec{k}$ - vector parallel to the wire). Observe that at no place of our calculation the assumption of low frequencies (small k) was made. They remain arbitrarily.

In order to complete our vector-field calculation we also evaluate the Poynting vector $\vec{S}=\left[\vec{E},\vec{H}^*\right]$ for $\theta \to 0$. For the orthogonal component we obtain

$$\vec{S}_\rho(\vec{r})=\frac{1}{16}j\eta_0|I_0|^2 k^2 \sin^3\theta\left[H_0^{(2)}(k\rho_1\sin\theta)-H_0^{(2)}(k\rho_2\sin\theta)\right]\left[H_1^{(1)}(k\rho_1\sin\theta)\vec{e}_{\rho_1}-H_1^{(1)}(k\rho_2\sin\theta)\vec{e}_{\rho_2}\right] \tag{23}$$

or, when $\theta \to 0$

$$\vec{S}_\rho^{\theta\to 0}=-\frac{j\eta_0|I_0|^2}{4\pi^2}k\sin^2\theta\ln(2h/a)\left[\frac{1}{\rho_1}\vec{e}_{\rho_1}-\frac{1}{\rho_2}\vec{e}_{\rho_2}\right] \tag{24}$$

This is a pure imaginary term. It indicates that there is no radiation. For $\theta=0$ $\vec{S}_\rho(\theta=0)$ becomes zero too, and all the power density is conducted along the line in z-direction.
The z-component of the Poynting vector, $S_z\vec{e}_z$, is given by

$$S_z(\vec{r})\vec{e}_z=\frac{\eta_0|I_0|^2}{16}k^2\sin^2\theta\cos\theta\left|H_1^{(2)}(k\rho_1\sin\theta)\vec{e}_{\rho_1}-H_1^{(2)}(k\rho_2\sin\theta)\vec{e}_{\rho_2}\right|^2\vec{e}_z \tag{25}$$

For small angles θ this expression can be approximated by

$$\vec{S}_z(\vec{r})\underset{(\theta\to 0)}{\cong}\frac{\eta_0|I_0|^2}{4\pi^2}\vec{e}_z\left|\frac{1}{\rho_1}\vec{e}_{\rho_1}-\frac{1}{\rho_2}\vec{e}_{\rho_2}\right|^2\cos\theta \tag{26}$$

or

$$\vec{S}_z^{\theta=0}(\vec{r})=\frac{\eta_0|I_0|^2}{4\pi^2}\vec{e}_z\left|\frac{1}{\rho_1}\vec{e}_{\rho_1}-\frac{1}{\rho_2}\vec{e}_{\rho_2}\right|^2 \tag{27}$$

We found for $\theta=0$ a pure TEM-character of the electromagnetic fields and the Poynting vector, and observed that all energy is transported along the wire. It is now of interest to find the corresponding current, potentials and line parameter functions. This is done in our second step.
We begin with the current and the impedance per-unit-length. From eq. (16) we have

$$I(z)=\frac{E_z^0}{Z'(j\omega)}e^{-j\tilde{k}_{inc}z} \tag{28}$$

with (we consider an excitation by a vertically polarized plane wave, see Fig. 1):

$$E_z^0=2jE^i\sin(kh\sin\theta)\sin\theta\, e^{-jkz\cos\theta} \tag{29}$$

and

$$Z'(j\omega)=\frac{\eta_0}{4\pi}\frac{\tilde{k}_{inc}^2-k^2}{jk}G_k\left(\tilde{k}_{inc}\right) \tag{30}$$

Approximating eqs.(29) and (30) for small angles θ one gets

$$E_z^0 \cong 2jkhE^i \sin^2\theta\, e^{-jkz} \tag{31}$$

and

$$Z'(j\omega) \cong \frac{j\eta_0}{2\pi} k \sin^2\theta \ln(2h/a) \tag{32}$$

Therefore, in the ratio of both, the $\sin^2\theta$ cancels out and it remains a non-zero current amplitude:

$$I(z) = \left(\frac{4\pi h}{\eta_0 \ln(2h/a)} E^i \right) e^{-jkz} = I_0 e^{-jkz} \tag{33}$$

This result may be a little bit surprising, since it is not quite clear how the grazing field should induce this current. We will come back to this question at the end of this section.

A general result can be extracted from $Z'^{\theta=0}(j\omega) = 0$, using eq. (15). One obtains (independent of the gauge):

$$(L'C')^{\theta=0} = c^{-2} \tag{34}$$

(c - speed of light)

The line parameters become in the limit $\theta \to 0$:

$$L'(j\omega) = \frac{\mu_0}{2\pi} \begin{cases} \ln(2h/a) \ , & \text{Lorenz gauge} \\ K_0(ka) - K_0(2kh) \ , & \text{Coulomb gauge} \end{cases} \tag{35}$$

and

$$C'(j\omega) = 2\pi\varepsilon_0 \begin{cases} 1/\ln(2h/a) \ , & \text{Lorenz gauge} \\ 1/[K_0(ka) - K_0(2kh)] \ , & \text{Coulomb gauge} \end{cases} \tag{36}$$

With these parameters the characteristic impedance is derived as

$$Z_C^{\theta=0} = \left(\sqrt{\frac{L'}{C'}} \right)^{\theta=0} = \frac{\eta_0}{2\pi} \begin{cases} \ln(2h/a) \ , & \text{Lorenz gauge} \\ K_0(ka) - K_0(2kh) \ , & \text{Coulomb gauge} \end{cases} \tag{37}$$

This impedance is used to present the (scattered) potential

$$\varphi^{\theta=0}(z) = \left(I_0 Z_C^{\theta=0}\right) e^{-jkz} = \varphi_0 e^{-jkz} \tag{38}$$

It is obvious that φ_0 depends on the gauge. But we also recognize that for the small k-values (i.e. small frequencies) the expressions for $Z_C^{\theta=0}$ in eq. (37) become equal. Thus, if we in addition to the limit $\theta \to 0$ require that also $k \to 0$, the results for φ_0 coincide. In the Lorenz gauge this extra demand was not necessary, the wave number k and the $\sin\theta$ occur in the formulae as products $(k \cdot \sin\theta)$.

Is there a physical argument, which may lead us to an unique expression for $Z_C^{\theta=0}$? Let us require that at infinity (this may be the location $z=0$) the potential becomes

$$\varphi_0 = 2E^i h \tag{39}$$

The term $\left(2E^i h\right)$ represents the potential of the incident field in the plane perpendicular to the wire at infinity.
If we calculate φ_0 from eq. (38) in the Lorenz gauge we obtain the required result of eq. (39). Performing the equivalent calculation in the Coulomb gauge we find

$$\varphi_{0C} = \frac{2hE^i}{\ln(2h/a)}\left(K_0(ka) - K_0(2kh)\right) \tag{40}$$

This only gives the wanted result (39) in the limit $k \to 0$. Therefore, to obtain a gauge-independent TEM solution from our generalized Telegrapher equations, we have to consider grazing incidence *and* small frequencies. Finally, the amplitude I_0 of the current (see eq. (33)) can now be explained: It is that current that couples via the thought vertical part of the wire at infinity into the transmission line (acting as a generator) and is then conducted along the line obeying the formula (33).

IV TEM – Solution of the Telegrapher Equations

In this section we briefly compare our above results with those which we obtain from eq. (6) without any exterior source. The source is put into the boundary conditions I_0 and φ_0. The following results are well-known; however, they may elucidate our previous considerations. We assume TEM wave propagation with the corresponding line parameters

$$L' = \frac{\mu_0}{2\pi}\ln(2h/a) \quad , \qquad C' = \frac{2\pi\varepsilon_0}{\ln(2h/a)} \tag{41 a,b}$$

and the solution

$$\begin{pmatrix}\varphi(z)\\ I(z)\end{pmatrix} = \mathbf{M}_{z_0}^{z}\begin{pmatrix}\varphi_0\\ I_0\end{pmatrix} = \begin{pmatrix}\varphi_0\\ I_0\end{pmatrix}e^{-jkz} \tag{42}$$

with the matrizant

$$\mathbf{M}_{z_0}^{z} = \begin{pmatrix} \cos\left(\frac{\omega}{c}(z-z_0)\right) & -jZ_C\sin\left(\frac{\omega}{c}(z-z_0)\right) \\ -jZ_C^{-1}\sin\left(\frac{\omega}{c}(z-z_0)\right) & \cos\left(\frac{\omega}{c}(z-z_0)\right) \end{pmatrix} \tag{43}$$

and the relation

$$\varphi_0 = Z_C I_0 \tag{44}$$

between the boundary values. Thus it is sufficient to fix φ_0 in order get a completely determined solution. Looking at the equivalent circuit which is displayed in Fig. 2

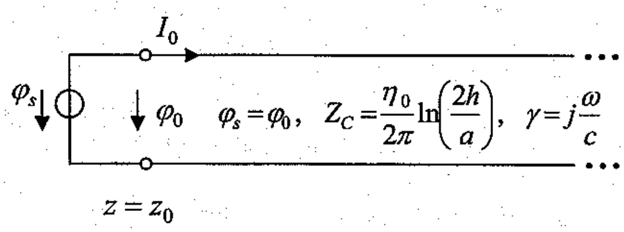

Fig. 2 : Equivalent circuit for the feeding part of the transmission line.

one can equate φ_0 with a source potential which feeds the line at the location z_0, a point which could be shifted to infinity. Then it becomes obvious that the choice

$$\varphi_S = \varphi_0 = 2hE^i \tag{45}$$

as source leads to the results of the foregoing section showing the consistency of our calculations.

V. Conclusion

An advantage of working with Telegrapher equations with generalized line parameters may exist in the fact that one can use all known solution procedures (and there is a host of literature on this subject) for transmission lines (e.g. ref. [5]). Also the new parameters have a physical meaning: Their imaginary parts constitute the radiation resistance of the line, whereas their real parts are connected with the stored reactive energy along the line. In case of a quasi - static approach [3], i.e. retardation is suppressed, the parameters become purely real, indicating that there is no radiation. A particular situation is obtained if we consider grazing incidence of the plane wave. Then we end up in a complete TEM configuration with static coefficients of the line (see also [6]). This limit requires small angles of incidence ($\theta \to 0$) and low frequencies ($k \to 0$), the usual assumptions in the derivation of the classical transmission line equations. Of course, our example describes a simple line configuration. However, it contains many new results in analytical form, especially appropriated for physical parameter studies. In our forthcoming investigations we will extend and apply the above formalism to multiconductor lines of finite length, even nonuniform ones, including ohmic and insulation losses.

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References

[1] F.R.Gantmacher, The Theory of Matrices, New-York, Chelsea Publishing Company, 1998.

[2] J.Nitsch, S.Tkachenko, Complex-Valued Transmission Line Parameters and their Relations to the Radiation-Resistance, Interaction Notes, Note 573, Sept. 2002.
[3] J.Nitsch, S.Tkachenko, Telegrapher Equations for Arbitrary Frequencies and Modes – Radiation of an Infinite, Lossless Transmission Line, Interaction Notes, Note 574, Oct. 2002.
[4] M.Abramowitz, I.A.Stegun (eds.), Handbook of Mathematical functions, Dover Publications, Inc., New-York.
[5] J.Nitsch, F.Gronwald, Analytical Solutions in Nonuniform Multiconductor Transmission Line Theory, IEEE Transaction on Electromagnetic Compatibility, Vol. 41, No. 4, November 1999, pp. 469-479.
[6] A.Reibiger, Field Theoretic Description of TEM waves on Lossless Multiconductor Transmission lines, Preprint, to be published in Kleinheubacher Berichte 2002, Band 46, 2003.